



Fractional supersymmetric Quantum Mechanics as a set of replicas of ordinary supersymmetric Quantum Mechanics

M. Daoud, M. Kibler

► To cite this version:

M. Daoud, M. Kibler. Fractional supersymmetric Quantum Mechanics as a set of replicas of ordinary supersymmetric Quantum Mechanics. *Physics Letters A*, 2004, 321, pp.147-151. 10.1016/j.physleta.2003.12.027 . hal-00000926

HAL Id: hal-00000926

<https://hal.science/hal-00000926>

Submitted on 6 Dec 2003

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fractional supersymmetric Quantum Mechanics as a set of replicas of ordinary supersymmetric Quantum Mechanics¹

M. Daoud², M. Kibler³

Institut de Physique Nucléaire de Lyon,
IN2P3-CNRS et Université Claude Bernard,
43 Bd du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

Abstract

A connection between fractional supersymmetric quantum mechanics and ordinary supersymmetric quantum mechanics is established in this Letter.

0. Although *ordinary* supersymmetric Quantum Mechanics (sQM) was introduced more than 20 years ago, its extension as *fractional* sQM is still the object of numerous works. The parentage between ordinary sQM and fractional sQM needs to be clarified. In particular, we may ask the question: Can fractional sQM be reduced to ordinary sQM as far as spectral analyses are concerned? It is the aim of this work to study a connection between fractional sQM of order k and ordinary sQM corresponding to $k = 2$. We consider here the case where the number of supercharges is equal to 1 (corresponding to 2 supercharges related via Hermitean conjugation).

1. Our definition of *fractional* sQM of order k , with $k \in \mathbf{N} \setminus \{0, 1\}$, is as follows. Following Refs. [1-4], a doublet of linear operators $(H, Q)_k$, with H a self-adjoint operator and Q a supersymmetry operator, acting on a separable Hilbert space and satisfying the relations

$$Q_- = Q, \quad Q_+ = Q^\dagger \quad (\Rightarrow \quad Q_-^\dagger = Q_+), \quad Q_\pm^k = 0 \quad (1a)$$

$$Q_-^{k-1}Q_+ + Q_-^{k-2}Q_+Q_- + \dots + Q_+Q_-^{k-1} = Q_-^{k-2}H \quad (1b)$$

$$[H, Q_\pm] = 0 \quad (1c)$$

is said to define a k -fractional supersymmetric quantum-mechanical system (see also Refs. [5-8]). The operator H is the Hamiltonian of the system spanned by the two (dependent) supercharge operators Q_- and Q_+ . In the special case $k = 2$, the system described by a doublet of type $(H, Q)_2$ is referred to as an ordinary supersymmetric

¹Accepted for publication in Physics Letters A.

²Permanent address: Laboratoire de Physique de la Matière Condensée, Faculté des Sciences, Université Ibn Zohr, BP 28/S, Agadir, Morocco.

³Correspondence to: M. Kibler; e-mail: m.kibler@ipnl.in2p3.fr.

quantum-mechanical system ; it corresponds to a Z_2 -grading with fermionic and bosonic states.

2. We now introduce a generalized Weyl-Heisenberg algebra W_k , with $k \in \mathbf{N} \setminus \{0, 1\}$, from which we can construct k -fractional supersymmetric quantum-mechanical systems. The algebra W_k is spanned by four linear operators, *viz.*, X_- (annihilation operator), X_+ (creation operator), N (number operator) and K (Z_k -grading operator). The operators X_- and X_+ are connected via Hermitean conjugation; N is a self-adjoint operator and K is a unitary operator. The four operators satisfy the relationships

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N) \Pi_s, \quad [N, X_{\pm}] = \pm X_{\pm}, \quad [K, X_{\pm}]_{q^{\pm 1}} = 0, \quad [K, N] = 0, \quad K^k = 1 \quad (2)$$

Here, the functions $f_s : N \mapsto f_s(N)$ are arbitrary functions subjected to the constraints $f_s(N)^{\dagger} = f_s(N)$. Furthermore, the Hermitean operators Π_s are defined by

$$\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-st} K^t$$

where

$$q = \exp\left(\frac{2\pi i}{k}\right)$$

is a root of unity, so that they are projection operators for the cyclic group C_k . Finally, $[K, X_{\pm}]_{q^{\pm 1}}$ stands for the deformed commutator $KX_{\pm} - q^{\pm 1}X_{\pm}K$.

3. The operators X_- , X_+ and K can be realized in terms of k pairs $(b(s)_-, b(s)_+)$ of deformed bosons with

$$[b(s)_-, b(s)_+] = f_s(N)$$

and one pair (f_-, f_+) of k -fermions with

$$[f_-, f_+]_q = 1, \quad f_{\pm}^k = 0$$

The f 's commute with the b 's. Of course, we have $b(s)_+ = b(s)_-^{\dagger}$ but $f_+ \neq f_-^{\dagger}$ except for $k = 2$. The k -fermions introduced in [9] and recently discussed in [10] are objects interpolating between fermions and bosons (the case $k = 2$ corresponds to ordinary fermions and the case $k \rightarrow \infty$ to ordinary bosons); the k -fermions also share some features of the anyons introduced in [11,12]. For k arbitrary in $\mathbf{N} \setminus \{0, 1\}$, the realization

$$\begin{aligned} K &= [f_-, f_+] \\ X_- &= \left(f_- + \frac{f_+^{k-1}}{[k-1]_q!} \right) \sum_{s=0}^{k-1} b(s)_- \Pi_s \\ X_+ &= \left(f_- + \frac{f_+^{k-1}}{[k-1]_q!} \right)^{k-1} \sum_{s=0}^{k-1} b(s)_+ \Pi_s \end{aligned}$$

has been discussed in Ref. [8]. Here, we have $[n]_q! = [1]_q [2]_q \cdots [n]_q$ (with $[0]_q! = 1$) and the symbol $[\]_q$ is defined by

$$[n]_q = \frac{1 - q^n}{1 - q}$$

where $n \in \mathbf{N}$.

4. An Hilbertean representation of W_k can be constructed in the following way. Let \mathcal{F} be the Hilbert-Fock space on which the generators X_- , X_+ , N and K act. Since K is a cyclic operator of order k , the space \mathcal{F} can be graded as

$$\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s$$

where the subspace $\mathcal{F}_s = \{|n, s\rangle : n = 1, 2, \dots, d\}$ is a d -dimensional space (d can be finite or infinite). The representation is given by

$$\begin{aligned} K|n, s\rangle &= q^s|n, s\rangle, \quad N|n, s\rangle = n|n, s\rangle \\ X_-|n, s\rangle &= \sqrt{F_s(n)} \begin{cases} |n-1, s-1\rangle & \text{if } s \neq 0 \\ |n-1, k-1\rangle & \text{if } s = 0 \end{cases} \\ X_+|n, s\rangle &= \sqrt{F_{s+1}(n+1)} \begin{cases} |n+1, s+1\rangle & \text{if } s \neq k-1 \\ |n+1, 0\rangle & \text{if } s = k-1 \end{cases} \end{aligned}$$

where the function F is a structure function such that

$$F_{s+1}(n+1) - F_s(n) = f_s(n) \quad (3)$$

with $F_s(0) = 0$.

5. We are now in a position to associate a k -fractional supersymmetric quantum-mechanical system to the algebra W_k characterized by a given set of functions $\{f_s : s = 0, 1, \dots, k-1\}$. We define the supercharge Q via

$$Q \equiv Q_- = X_-(1 - \Pi_1) \Leftrightarrow Q^\dagger \equiv Q_+ = X_+(1 - \Pi_0) \quad (4)$$

There are k equivalent definitions of Q corresponding to the k circular permutations of $1, 2, \dots, k-1$; our choice, which is such that $Q|n, 1\rangle = 0$, is adapted to the sequence H_k, H_{k-1}, \dots, H_1 to be considered below. By making repeated use of Eqs. (1), (2) and (4), we can derive the operator

$$H = (k-1)X_+X_- - \sum_{s=3}^k \sum_{t=2}^{s-1} (t-1) f_t(N-s+t) \Pi_s - \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k) f_t(N-s+t) \Pi_s \quad (5)$$

which is self-adjoint and commutes with Q_- and Q_+ . (Equation (5) and some other relations below include Π_k . Indeed, in view of the cyclic character of K , we have

$\Pi_k = \Pi_0$ so that the action of terms involving Π_k is quite well-defined on the space \mathcal{F} .) As a result, the doublet $(H, Q)_k$ associated to W_k satisfies Eq. (1) and thus defines a k -fractional supersymmetric quantum-mechanical system.

6. In order to establish a connection between *fractional* sQM (of order k) and *ordinary* sQM (of order $k = 2$), it is necessary to construct subsystems from the doublet $(H, Q)_k$ that correspond to ordinary supersymmetric quantum-mechanical systems. This may be achieved in the following way. Equation (5) can be rewritten as

$$H = \sum_{s=1}^k H_s \Pi_s \quad (6)$$

where

$$H_s \equiv H_s(N) = (k-1)F(N) - \sum_{t=2}^{k-1} (t-1) f_t(N-s+t) + (k-1) \sum_{t=s}^{k-1} f_t(N-s+t) \quad (7)$$

It can be shown that the operators $H_k \equiv H_0, H_{k-1}, \dots, H_1$ turn out to be isospectral operators. By introducing

$$X(s)_- = \sum_n [H_s(n)]^{\frac{1}{2}} |n-1, s-1\rangle \langle n, s|$$

$$X(s)_+ = \sum_n [H_s(n+1)]^{\frac{1}{2}} |n+1, s\rangle \langle n, s-1|$$

it is possible to factorize H_s as

$$H_s = X(s)_+ X(s)_-$$

modulo the omission of the ground state $|0, s\rangle$ (which amounts to subtract the corresponding eigenvalue from the spectrum of H_s). Let us now define: (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \Pi_s, \quad q(s)_+ = X(s)_+ \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- X(s)_+ \Pi_{s-1} + X(s)_+ X(s)_- \Pi_s \quad (8)$$

It is then a simple matter of calculation to prove that $h(s)$ is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0, \quad h(s) = q(s)_- q(s)_+ + q(s)_+ q(s)_-, \quad [h(s), q(s)_\pm] = 0$$

Consequently, the doublet $(h(s), q(s))_2$, with $q(s) \equiv q(s)_-$, satisfies Eq. (1) with $k = 2$ and thus defines an ordinary supersymmetric quantum-mechanical system (corresponding to $k = 2$).

7. The Hamiltonian $h(s)$ is amenable to a form more appropriate for discussing the link between ordinary sQM and fractional sQM. Indeed, we can show that

$$X(s)_- X(s)_+ = H_s(N+1) \quad (9)$$

Then, by combining Eqs. (2), (3), (7) and (9), Eq. (8) leads to the important relation

$$h(s) = H_{s-1} \Pi_{s-1} + H_s \Pi_s \quad (10)$$

to be compared with the expansion of H in terms of supersymmetric partners H_s (see Eq. (6)).

8. To close this Letter, let us sum up the obtained results and offer some conclusions.

Starting from a Z_k -graded algebra W_k , characterized by a set $\{f_s : s = 0, 1, \dots, k-1\}$, it was shown how to associate a k -fractional supersymmetric quantum-mechanical system $(H, Q)_k$ characterized by an Hamiltonian H and a supercharge Q .

The extended Weyl-Heisenberg algebra W_k covers numerous algebras describing exactly solvable one-dimensional systems. The particular system corresponding to a given set $\{f_s : s = 0, 1, \dots, k-1\}$ yields, in a Schrödinger picture, a particular dynamical system with a specific potential. Let us mention two interesting cases. The case

$$\forall s \in \{0, 1, \dots, k-1\} : f_s(N) = f_s \text{ independent of } N$$

corresponds to systems with cyclic shape-invariant potentials (in the sense of Ref. [13]) and the case

$$\forall s \in \{0, 1, \dots, k-1\} : f_s(N) = aN + b \text{ where } (a, b) \in \mathbf{R}^2$$

to systems with translational shape-invariant potentials (in the sense of Ref. [14]). For instance, the case $(a = 0, b > 0)$ corresponds to the harmonic oscillator potential, the case $(a < 0, b > 0)$ to the Morse potential and the case $(a > 0, b > 0)$ to the Pöschl-Teller potential. For these various potentials, the part of W_k spanned by X_- , X_+ and N can be identified with the ordinary Weyl-Heisenberg algebra for $(a = 0, b \neq 0)$, with the $\text{su}(1,1)$ Lie algebra for $(a > 0, b > 0)$ and with the $\text{su}(2)$ Lie algebra for $(a < 0, b > 0)$. These matters shall be the subject of a forthcoming paper.

The Hamiltonian H for the system $(H, Q)_k$ was developed as a superposition of k isospectral supersymmetric partners H_0, H_1, \dots, H_{k-1} .

The system $(H, Q)_k$ itself, corresponding to k -fractional sQM, was expressed in terms of $k-1$ sub-systems $(h(s), q(s))_2$, corresponding to ordinary sQM. The Hamiltonian $h(s)$ is given as a sum involving the supersymmetric partners H_{s-1} and H_s (see Eq. (10)). Since the supercharges $q(s)_\pm$ commute with the Hamiltonian $h(s)$, it follows that

$$H_{s-1}X(s)_- = X(s)_-H_s, \quad H_sX(s)_+ = X(s)_+H_{s-1} \quad (11)$$

As a consequence, the operator $X(s)_+$ (respect. $X(s)_-$) makes it possible to pass from the spectrum of H_{s-1} (respect. H_s) to the one of H_s (respect. H_{s-1}). This result is quite familiar for ordinary sQM (corresponding to $s = 2$). Note that Eq. (11) is reminiscent of the intertwining method based on the Darboux transformation and on the factorization method which are useful for studying superintegrability of quantum systems.

For $k = 2$, the operator $h(1)$ is nothing but the total Hamiltonian H corresponding to ordinary sQM. For arbitrary k , the other operators $h(s)$ are simple replicas (except for the ground state of $h(s)$) of $h(1)$. It is in this sense that k -fractional sQM can be considered as a set of $k - 1$ replicas of ordinary sQM typically described by $(h(s), q(s)_\pm)_2$. Along this vein, it is to be emphasized that

$$H = q(2)_- q(2)_+ + \sum_{s=2}^k q(s)_+ q(s)_-$$

which can be identified to $h(2)$ for $k = 2$.

Thanks are due to the referee for pertinent and constructive remarks.

References

- [1] E. Witten, Nucl. Phys. B 138 (1981) 513.
- [2] V.A. Rubakov, V.P. Spiridonov, Mod. Phys. Lett. A 3 (1988) 1337.
- [3] A. Khare, J. Phys. A 25 (1992) L749; J. Math. Phys. 34 (1993) 1277.
- [4] A.T. Filippov, A.P. Isaev, A.B. Kurdikov, Mod. Phys. Lett. A 7 (1992) 2129; Int. J. Mod. Phys. A 8 (1993) 4973.
- [5] S. Durand, Mod. Phys. Lett. A 7 (1992) 2905; Phys. Lett. B 312 (1993) 115; Mod. Phys. Lett. A 8 (1993) 2323.
- [6] A. LeClair and C. Vafa, Nucl. Phys. B 401 (1993) 413.
- [7] M. Rausch de Traubenberg, M.J. Slupinski, J. Math. Phys. 41 (2000) 4556.
- [8] M. Daoud, M. Kibler, Phys. Part. and Nuclei (Suppl. 1) 33 (2002) S43; Int. J. Quantum Chem. 91 (2003) 551.
- [9] M. Daoud, Y. Hassouni, M. Kibler, in: B. Gruber, M. Ramek (Eds.), Symmetries in Science X, Plenum, New York, 1998; Yad. Fiz. 61 (1998) 1935.
- [10] H.-Y. Pan, Z.S. Zhao, Phys. Lett. A 312 (2003) 1.
- [11] J.M. Leinaas, J. Myrheim, Nuovo Cimento B 37 (1977) 1.

- [12] G.A. Goldin, R. Menikoff, D.H. Sharp, J. Math. Phys. 21 (1980) 650; G.A. Goldin, D.H. Sharp, Phys. Rev. Lett. 76 (1996) 1183.
- [13] U.P. Sukhatme, C. Rasinariu, A. Khare, Phys. Lett. A 234 (1997) 401.
- [14] G. Junker, Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin, 1996.