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# Fractional supersymmetric Quantum Mechanics as a set of replicas of ordinary supersymmetric Quantum Mechanics<sup>1</sup>

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## Abstract

A connection between fractional supersymmetric quantum mechanics and ordinary supersymmetric quantum mechanics is established in this Letter.

0. Although *ordinary* supersymmetric Quantum Mechanics (sQM) was introduced more than 20 years ago, its extension as *fractional* sQM is still the object of numerous works. The parentage between ordinary sQM and fractional sQM needs to be clarified. In particular, we may ask the question: Can fractional sQM be reduced to ordinary sQM as far as spectral analyses are concerned? It is the aim of this work to study a connection between fractional sQM of order  $k$  and ordinary sQM corresponding to  $k = 2$ . We consider here the case where the number of supercharges is equal to 1 (corresponding to 2 supercharges related via Hermitean conjugation).

1. Our definition of *fractional* sQM of order  $k$ , with  $k \in \mathbf{N} \setminus \{0, 1\}$ , is as follows. Following Refs. [1-4], a doublet of linear operators  $(H, Q)_k$ , with  $H$  a self-adjoint operator and  $Q$  a supersymmetry operator, acting on a separable Hilbert space and satisfying the relations

$$Q_- = Q, \quad Q_+ = Q^\dagger \quad (\Rightarrow \quad Q_-^\dagger = Q_+), \quad Q_\pm^k = 0 \quad (1a)$$

$$Q_-^{k-1}Q_+ + Q_-^{k-2}Q_+Q_- + \dots + Q_+Q_-^{k-1} = Q_-^{k-2}H \quad (1b)$$

$$[H, Q_\pm] = 0 \quad (1c)$$

is said to define a  $k$ -fractional supersymmetric quantum-mechanical system (see also Refs. [5-8]). The operator  $H$  is the Hamiltonian of the system spanned by the two (dependent) supercharge operators  $Q_-$  and  $Q_+$ . In the special case  $k = 2$ , the system described by a doublet of type  $(H, Q)_2$  is referred to as an ordinary supersymmetric

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quantum-mechanical system ; it corresponds to a  $Z_2$ -grading with fermionic and bosonic states.

2. We now introduce a generalized Weyl-Heisenberg algebra  $W_k$ , with  $k \in \mathbf{N} \setminus \{0, 1\}$ , from which we can construct  $k$ -fractional supersymmetric quantum-mechanical systems. The algebra  $W_k$  is spanned by four linear operators, *viz.*,  $X_-$  (annihilation operator),  $X_+$  (creation operator),  $N$  (number operator) and  $K$  ( $Z_k$ -grading operator). The operators  $X_-$  and  $X_+$  are connected via Hermitean conjugation;  $N$  is a self-adjoint operator and  $K$  is a unitary operator. The four operators satisfy the relationships

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N) \Pi_s, \quad [N, X_{\pm}] = \pm X_{\pm}, \quad [K, X_{\pm}]_{q^{\pm 1}} = 0, \quad [K, N] = 0, \quad K^k = 1 \quad (2)$$

Here, the functions  $f_s : N \mapsto f_s(N)$  are arbitrary functions subjected to the constraints  $f_s(N)^{\dagger} = f_s(N)$ . Furthermore, the Hermitean operators  $\Pi_s$  are defined by

$$\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-st} K^t$$

where

$$q = \exp\left(\frac{2\pi i}{k}\right)$$

is a root of unity, so that they are projection operators for the cyclic group  $C_k$ . Finally,  $[K, X_{\pm}]_{q^{\pm 1}}$  stands for the deformed commutator  $KX_{\pm} - q^{\pm 1}X_{\pm}K$ .

3. The operators  $X_-$ ,  $X_+$  and  $K$  can be realized in terms of  $k$  pairs  $(b(s)_-, b(s)_+)$  of deformed bosons with

$$[b(s)_-, b(s)_+] = f_s(N)$$

and one pair  $(f_-, f_+)$  of  $k$ -fermions with

$$[f_-, f_+]_q = 1, \quad f_{\pm}^k = 0$$

The  $f$ 's commute with the  $b$ 's. Of course, we have  $b(s)_+ = b(s)_-^{\dagger}$  but  $f_+ \neq f_-^{\dagger}$  except for  $k = 2$ . The  $k$ -fermions introduced in [9] and recently discussed in [10] are objects interpolating between fermions and bosons (the case  $k = 2$  corresponds to ordinary fermions and the case  $k \rightarrow \infty$  to ordinary bosons); the  $k$ -fermions also share some features of the anyons introduced in [11,12]. For  $k$  arbitrary in  $\mathbf{N} \setminus \{0, 1\}$ , the realization

$$\begin{aligned} K &= [f_-, f_+] \\ X_- &= \left( f_- + \frac{f_+^{k-1}}{[k-1]_q!} \right) \sum_{s=0}^{k-1} b(s)_- \Pi_s \\ X_+ &= \left( f_- + \frac{f_+^{k-1}}{[k-1]_q!} \right) \sum_{s=0}^{k-1} b(s)_+ \Pi_s \end{aligned}$$

has been discussed in Ref. [8]. Here, we have  $[n]_q! = [1]_q [2]_q \cdots [n]_q$  (with  $[0]_q! = 1$ ) and the symbol  $[\ ]_q$  is defined by

$$[n]_q = \frac{1 - q^n}{1 - q}$$

where  $n \in \mathbf{N}$ .

4. An Hilbertean representation of  $W_k$  can be constructed in the following way. Let  $\mathcal{F}$  be the Hilbert-Fock space on which the generators  $X_-$ ,  $X_+$ ,  $N$  and  $K$  act. Since  $K$  is a cyclic operator of order  $k$ , the space  $\mathcal{F}$  can be graded as

$$\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s$$

where the subspace  $\mathcal{F}_s = \{|n, s\rangle : n = 1, 2, \dots, d\}$  is a  $d$ -dimensional space ( $d$  can be finite or infinite). The representation is given by

$$\begin{aligned} K|n, s\rangle &= q^s|n, s\rangle, \quad N|n, s\rangle = n|n, s\rangle \\ X_-|n, s\rangle &= \sqrt{F_s(n)} \begin{cases} |n-1, s-1\rangle & \text{if } s \neq 0 \\ |n-1, k-1\rangle & \text{if } s = 0 \end{cases} \\ X_+|n, s\rangle &= \sqrt{F_{s+1}(n+1)} \begin{cases} |n+1, s+1\rangle & \text{if } s \neq k-1 \\ |n+1, 0\rangle & \text{if } s = k-1 \end{cases} \end{aligned}$$

where the function  $F$  is a structure function such that

$$F_{s+1}(n+1) - F_s(n) = f_s(n) \quad (3)$$

with  $F_s(0) = 0$ .

5. We are now in a position to associate a  $k$ -fractional supersymmetric quantum-mechanical system to the algebra  $W_k$  characterized by a given set of functions  $\{f_s : s = 0, 1, \dots, k-1\}$ . We define the supercharge  $Q$  via

$$Q \equiv Q_- = X_-(1 - \Pi_1) \Leftrightarrow Q^\dagger \equiv Q_+ = X_+(1 - \Pi_0) \quad (4)$$

There are  $k$  equivalent definitions of  $Q$  corresponding to the  $k$  circular permutations of  $1, 2, \dots, k-1$ ; our choice, which is such that  $Q|n, 1\rangle = 0$ , is adapted to the sequence  $H_k, H_{k-1}, \dots, H_1$  to be considered below. By making repeated use of Eqs. (1), (2) and (4), we can derive the operator

$$H = (k-1)X_+X_- - \sum_{s=3}^k \sum_{t=2}^{s-1} (t-1) f_t(N-s+t) \Pi_s - \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k) f_t(N-s+t) \Pi_s \quad (5)$$

which is self-adjoint and commutes with  $Q_-$  and  $Q_+$ . (Equation (5) and some other relations below include  $\Pi_k$ . Indeed, in view of the cyclic character of  $K$ , we have

$\Pi_k = \Pi_0$  so that the action of terms involving  $\Pi_k$  is quite well-defined on the space  $\mathcal{F}$ .) As a result, the doublet  $(H, Q)_k$  associated to  $W_k$  satisfies Eq. (1) and thus defines a  $k$ -fractional supersymmetric quantum-mechanical system.

6. In order to establish a connection between *fractional* sQM (of order  $k$ ) and *ordinary* sQM (of order  $k = 2$ ), it is necessary to construct subsystems from the doublet  $(H, Q)_k$  that correspond to ordinary supersymmetric quantum-mechanical systems. This may be achieved in the following way. Equation (5) can be rewritten as

$$H = \sum_{s=1}^k H_s \Pi_s \quad (6)$$

where

$$H_s \equiv H_s(N) = (k-1)F(N) - \sum_{t=2}^{k-1} (t-1) f_t(N-s+t) + (k-1) \sum_{t=s}^{k-1} f_t(N-s+t) \quad (7)$$

It can be shown that the operators  $H_k \equiv H_0, H_{k-1}, \dots, H_1$  turn out to be isospectral operators. By introducing

$$X(s)_- = \sum_n [H_s(n)]^{\frac{1}{2}} |n-1, s-1\rangle \langle n, s|$$

$$X(s)_+ = \sum_n [H_s(n+1)]^{\frac{1}{2}} |n+1, s\rangle \langle n, s-1|$$

it is possible to factorize  $H_s$  as

$$H_s = X(s)_+ X(s)_-$$

modulo the omission of the ground state  $|0, s\rangle$  (which amounts to subtract the corresponding eigenvalue from the spectrum of  $H_s$ ). Let us now define: (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \Pi_s, \quad q(s)_+ = X(s)_+ \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- X(s)_+ \Pi_{s-1} + X(s)_+ X(s)_- \Pi_s \quad (8)$$

It is then a simple matter of calculation to prove that  $h(s)$  is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0, \quad h(s) = q(s)_- q(s)_+ + q(s)_+ q(s)_-, \quad [h(s), q(s)_\pm] = 0$$

Consequently, the doublet  $(h(s), q(s))_2$ , with  $q(s) \equiv q(s)_-$ , satisfies Eq. (1) with  $k = 2$  and thus defines an ordinary supersymmetric quantum-mechanical system (corresponding to  $k = 2$ ).

7. The Hamiltonian  $h(s)$  is amenable to a form more appropriate for discussing the link between ordinary sQM and fractional sQM. Indeed, we can show that

$$X(s)_- X(s)_+ = H_s(N+1) \quad (9)$$

Then, by combining Eqs. (2), (3), (7) and (9), Eq. (8) leads to the important relation

$$h(s) = H_{s-1} \Pi_{s-1} + H_s \Pi_s \quad (10)$$

to be compared with the expansion of  $H$  in terms of supersymmetric partners  $H_s$  (see Eq. (6)).

8. To close this Letter, let us sum up the obtained results and offer some conclusions.

Starting from a  $Z_k$ -graded algebra  $W_k$ , characterized by a set  $\{f_s : s = 0, 1, \dots, k-1\}$ , it was shown how to associate a  $k$ -fractional supersymmetric quantum-mechanical system  $(H, Q)_k$  characterized by an Hamiltonian  $H$  and a supercharge  $Q$ .

The extended Weyl-Heisenberg algebra  $W_k$  covers numerous algebras describing exactly solvable one-dimensional systems. The particular system corresponding to a given set  $\{f_s : s = 0, 1, \dots, k-1\}$  yields, in a Schrödinger picture, a particular dynamical system with a specific potential. Let us mention two interesting cases. The case

$$\forall s \in \{0, 1, \dots, k-1\} : f_s(N) = f_s \text{ independent of } N$$

corresponds to systems with cyclic shape-invariant potentials (in the sense of Ref. [13]) and the case

$$\forall s \in \{0, 1, \dots, k-1\} : f_s(N) = aN + b \text{ where } (a, b) \in \mathbf{R}^2$$

to systems with translational shape-invariant potentials (in the sense of Ref. [14]). For instance, the case  $(a = 0, b > 0)$  corresponds to the harmonic oscillator potential, the case  $(a < 0, b > 0)$  to the Morse potential and the case  $(a > 0, b > 0)$  to the Pöschl-Teller potential. For these various potentials, the part of  $W_k$  spanned by  $X_-$ ,  $X_+$  and  $N$  can be identified with the ordinary Weyl-Heisenberg algebra for  $(a = 0, b \neq 0)$ , with the  $\text{su}(1,1)$  Lie algebra for  $(a > 0, b > 0)$  and with the  $\text{su}(2)$  Lie algebra for  $(a < 0, b > 0)$ . These matters shall be the subject of a forthcoming paper.

The Hamiltonian  $H$  for the system  $(H, Q)_k$  was developed as a superposition of  $k$  isospectral supersymmetric partners  $H_0, H_1, \dots, H_{k-1}$ .

The system  $(H, Q)_k$  itself, corresponding to  $k$ -fractional sQM, was expressed in terms of  $k-1$  sub-systems  $(h(s), q(s))_2$ , corresponding to ordinary sQM. The Hamiltonian  $h(s)$  is given as a sum involving the supersymmetric partners  $H_{s-1}$  and  $H_s$  (see Eq. (10)). Since the supercharges  $q(s)_\pm$  commute with the Hamiltonian  $h(s)$ , it follows that

$$H_{s-1}X(s)_- = X(s)_-H_s, \quad H_sX(s)_+ = X(s)_+H_{s-1} \quad (11)$$

As a consequence, the operator  $X(s)_+$  (respect.  $X(s)_-$ ) makes it possible to pass from the spectrum of  $H_{s-1}$  (respect.  $H_s$ ) to the one of  $H_s$  (respect.  $H_{s-1}$ ). This result is quite familiar for ordinary sQM (corresponding to  $s = 2$ ). Note that Eq. (11) is reminiscent of the intertwining method based on the Darboux transformation and on the factorization method which are useful for studying superintegrability of quantum systems.

For  $k = 2$ , the operator  $h(1)$  is nothing but the total Hamiltonian  $H$  corresponding to ordinary sQM. For arbitrary  $k$ , the other operators  $h(s)$  are simple replicas (except for the ground state of  $h(s)$ ) of  $h(1)$ . It is in this sense that  $k$ -fractional sQM can be considered as a set of  $k - 1$  replicas of ordinary sQM typically described by  $(h(s), q(s)_\pm)_2$ . Along this vein, it is to be emphasized that

$$H = q(2)_- q(2)_+ + \sum_{s=2}^k q(s)_+ q(s)_-$$

which can be identified to  $h(2)$  for  $k = 2$ .

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