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Abstract. For a standard path of connections going to a generic point at infinity in the moduli space M_{DR} of connections on a compact Riemann surface, we show that the Laplace transform of the family of monodromy matrices has an analytic continuation with locally finite branching. In particular the convex subset representing the exponential growth rate of the monodromy is a polygon, whose vertices are in a subset of points described explicitly in terms of the spectral curve. Unfortunately we don't get any information about the size of the singularities of the Laplace transform, which is why we can't get asymptotic expansions for the monodromy.

1. Introduction

We study the asymptotic behavior of the monodromy of connections near a general point at ∞ in the space M_{DR} of connections on a compact Riemann surface X. We will consider a path of connections of the form $(E, \nabla + t\theta)$ which approaches the boundary divisor transversally at the point on the boundary of M_{DR} corresponding to a general Higgs bundle (E, θ). By some meromorphic gauge transformations in §5 we reduce to the case of a family of connections of the form $d + B + tA$. This is very similar to what was treated in [34] except that here our matrix B may have poles. We import the vast majority of our techniques directly from there. The difficulty posed by the poles of B is the new phenomenon which is treated here. We are not able to get results as good as the precise asymptotic expansions of [34]. We just show in Theorem 6.3 (p. 19) that if $m(t)$ denotes the family of monodromy or transport matrices for a given path, then the Laplace transform $f(\zeta)$ of m has an analytic continuation with locally finite singularities over the complex plane (see Definition 6.2, p. 19). The singularities are what determine the asymptotic behavior of $m(t)$. The upside of this situation

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is that since we are aiming for less, we can considerably simplify certain parts of the argument. What we don't know is the behavior of $f(\zeta)$ near the singularities: the main question left open is whether f has polynomial growth at the singularities, and if so, to what extent the generalized Laurent series can be calculated from the individual terms in our integral expression for f.

We can get some information about where the singularities are. Fix a general point (E, θ). Recall from [25] [26] [18] [29] that the *spectral curve* V is the subset of points in $T^*(X)$ corresponding to eigenforms of θ. We have a proper mapping $\pi : V \to X$. In the case of a general point, V is smooth and the mapping has only simple ramification points. Also there is a tautological one-form

$$\alpha \in H^0(V, \pi^*\Omega^1_X) \subset H^0(V, \Omega^1_V).$$

Finally there is a line bundle L over V such that $E \cong \pi_*(L)$ and θ corresponds to the action of α on the direct image bundle. This is all just a geometric version of the diagonalization of θ considered as a matrix over the function field of X.

Let $\mathcal{R} \subset X$ denote the subset of points over which the spectral curve is ramified, that is the image of the set of branch points of π. It is the set of *turning points* of our singular perturbation problem. Suppose p and q are points in X joined by a path γ. A *piecewise homotopy lifting* of γ to the spectral curve V consists of a collection of paths

$$\tilde{\gamma} = \{\tilde{\gamma}_i\}_{i=1,\ldots,k}$$

such that each $\tilde{\gamma}_i$ is a continuous path in V, and such that if we denote by $\gamma_i := \pi \circ \tilde{\gamma}_i$ the image paths in V, then γ_1 starts at p, γ_k ends at q, and for $i = 1, \ldots, k-1$, the endpoint of γ_i is equal to the starting point of γ_{i+1} and this is a point in $\mathcal{R}$. Among these there is a much more natural class of paths which are the *continuous homotopy liftings*, namely those where the starting point of $\tilde{\gamma}_i$ is equal to the starting point of $\tilde{\gamma}_{i+1}$ (which is not necessarily the case for a general piecewise lifting).

Denote by $\Sigma(\gamma) \subset \mathbb{C}$ the set of integrals of the tautological form α along piecewise homotopy liftings of γ, i.e. the set of complex numbers of the form

$$\sigma = \int_{\tilde{\gamma}} \alpha := \sum_{i=1}^{k} \int_{\tilde{\gamma}_i} \alpha.$$

Let $\Sigma^{\mathrm{cont}}(\gamma)$ be the subset of integrals along the continuous homotopy liftings. The following is the statement of Theorem 6.3 augmented with a little bit of information about where the singularities are.

Theorem 1.1. *Let p, q be two points on X, and let γ denote a path from p to q. Let $\{(E, \nabla + t\theta)\}$ denote a curve of connections cutting the divisor P_{DR} at a general point (E, θ) and let (V, α, L) denote the spectral data for this Higgs bundle. Let $m(t)$ be the function (with values in $Hom(E_p, E_q)$) whose value at $t \in \mathbb{C}$ is the transport matrix for the connection $\nabla + t\theta$ from p to q along the path γ. Let $f(\zeta)$ denote the Laplace transform of m. Then f has an analytic continuation with locally finite singularities over the complex plane. The set of singularities which are ever encountered is a subset of the set $\Sigma(\gamma) \subset \mathbb{C}$ of integrals of the tautological form along piecewise homotopy liftings defined above.*

It would have been much nicer to be able to say that the set of singularities is contained in $\Sigma^{\rm cont}(\gamma)$, however I don't see that this is necessarily the case. However, it might be that the singularities in $\Sigma^{\rm cont}(\gamma)$ have a special form different from the others. This is an interesting question for further research.

Ths first singularities which are encountered in the analytic continuation of f determine the growth rate of $m(t)$ in a way which we briefly formalize. Suppose that $m(t)$ is an entire function with exponentially bounded growth. We say that $m(t)$ is *rapidly decreasing in a sector*, if for some (open) sector of complex numbers going to ∞, there is $\varepsilon > 0$ giving a bound of the form $|m(t)| \leq e^{-\varepsilon |t|}$. Define the *hull* of m by

$$\mathbf{hull}\,(m) := \{\zeta \in \mathbb{C}\,\text{s.t.}\; e^{-\zeta t} m(t) \text{not rapidly decreasing in any sector}\}.$$

It is clear from the definition that the set of ζ such that $e^{-\zeta t}m(t)$ is rapidly decreasing in some sector, is open. Therefore $\mathbf{hull}\,(m)$ is closed. It is also not too hard to see that it is convex (see §13). Note that the hull is defined entirely in terms of the growth rate of the function m.

Corollary 1.2. *In the situation of Theorem 1.1, the hull of m is a finite convex polygon with at least two vertices, and all of its vertices are contained in $\Sigma(\gamma)$.*

The above results fall into the realm of *singular perturbation theory* for systems of ordinary differential equations, which goes back at least to Liouville. A steady stream of progress in this theory has led to a vast literature which we don't attempt completely to cover here (and which the reader can explore by using internet and database search techniques, starting for example from the authors mentionned in the bibliography).

Recall that following [4], Voros and Ecalle looked at these questions from the viewpoint of "resurgent functions" [41] [43] [42] [20] [19] [21] [7] [9] [14]. In the terminology of Ecalle's article in [7], the singular perturbation problem we are considering here is an example of *co-equational resurgence*. Our approach is very related to this viewpoint, though self-contained. We use a notion of analytic continuation of the Laplace transform 6.2 which is a sort of weak version of resurgence, like that used in [14] and [9]. The elements of our expansion 6.1 are what Ecalle calls the "elementary monomials" and the trees which appear in §8 are related to *(co)moulds (co)arborescents*, see [7]. Conversion properties related to the trees have been discussed in [22] (which is on the subject of KAM theory [23]). The relationship with integrals on a spectral curve was explicit in [13], [14]. The works [42], Ecalle's article in [7], and [14], raise a number of questions about how to prove resurgence for certain classes of singular perturbation problems notably some arising in quantum mechanics. A number of subsequent articles treat these questions; I haven't been able to include everything here but some examples are [22], [15], [16], . . . (and apparently [45]). In particular [16] discuss extensively the way in which the singularities of the Laplace transform determine the asymptotic behavior of the original function, specially in the case of the kinds of integrals which appear as terms in the decomposition 6.1.

There are a number of other currents of thought about the problem of singular perturbations. It is undoubtedly important to pursue the relationship with all of these. For example, the study initiated in [6] and continuing with several articles in [7], as well as the more modern [1] (also Prof. Kawai's talk at this conference) indicates that there is an intricate and fascinating geometry in the propagation of the Stokes phenomenon. And on the other hand it would be good to understand the relationship with the local study of turning points such as in [8], [40]. The article [15] incorporates some aspects of all of these approaches, and one can see [5] for a physical perspective. Also works on Painlevé's equations and isomonodromy such as [11] [27] [33] [44] are probably relevant.

Even though he doesn't appear in the references of [34], the ideas of J.-P. Ramis indirectly had a profound influence on that work (and hence on the present note). This can be traced to at least two inputs as follows:

(1) I had previously followed G. Laumon's course about ℓ-adic Fourier transform, which was partly inspired by the corresponding notions in complex function theory, a subject in which Ramis (and Ecalle, Voros, . . .) had a great influence; and

(2) at the time of writing [34] I was following N. Katz's course about exponential sums, where again much of the inspiration came from Ramis' work (which Katz mentionned very often) on irregular singularities.

Thus I would like to take this opportunity to thank Jean-Pierre for inspiring such a rich mathematical context.

I would also like to thank the several participants who made interesting remarks and posed interesting questions. In particular F. Pham pointed out that it would be a good idea to look at what the formula for the location of the singularities actually said, leading to the statement of Theorem 6.3 in its above form. I haven't been able to treat other suggestions (D. Sauzin, ...), such as looking at the differential equation satisfied by $f(\zeta)$.

2. The compactified moduli space of connections

Let X be a smooth projective curve over the complex numbers $\mathbb{C}$. Fix r and suppose E is a vector bundle of rank r over X. A *connection* (by which we mean an algebraic one) on E is a $\mathbb{C}$-linear morphism of sheaves $\nabla : E \rightarrow E \otimes_{\mathcal{O}} \Omega^1_X$ satisfying the Leibniz rule $\nabla(ae) = (da)e + a\nabla(e)$. If p and q are points joined by a path $\gamma : [0,1] \rightarrow X$, $\gamma(0) = P$, $\gamma(1) = Q$ then local solutions of $\nabla(e) = 0$ continue along γ, giving a *transport matrix* $m_\gamma(E, \nabla) : E_P \rightarrow E_Q$. The transport matrix, our main object of study, is the fundamental solution of a linear system of ODE's. If E is a trivial bundle (which will always be the case at least on a Zariski open subset of X containing γ) then there is a formula for the transport matrix as a sum of iterated integrals [10] [24]. A modified version of this formula is basic to the argument below, although we mostly refer to [34] for the details of that part of the argument.

Recall that we have a moduli space M_{DR} of rank r vector bundles with integrable connection on X [36], which has a compactification $M_{DR} \subset \overline{M}_{DR}$ constructed as follows. A *Higgs bundle* is a pair (E, θ) where $\theta : E \rightarrow E \otimes_{\mathcal{O}} \Omega^1_X$ is an $\mathcal{O}_X$-linear bundle map (rather than a connection) [25] [26] [35], which is *semistable of degree* 0 if E has degree zero and if any sub-Higgs bundle has degree ≤ 0. In fact for any $\lambda \in \mathbb{A}^1_{\mathbb{C}}$ we can look at the notion of *vector bundle with λ-connection* [17]—related in an obvious way to the notion of singular perturbation—which is a pair (E, ∇) of a bundle plus a connection-like operator satisfying Leibniz' rule with a factor of λ in front of the first term. For $\lambda = 0$ this is just a Higgs bundle and for any $\lambda \neq 0$ the operator $\lambda^{-1}\nabla$ is a connection.

With these definitions, there is a moduli space [37] [39] [36] $M_{Hod} \rightarrow \mathbf{A}^1$ for vector bundles with λ-connection, $\lambda \in \mathbf{A}^1$. The fiber over $\lambda = 0$ is the moduli space M_{Dol} for semistable Higgs bundles of degree zero, whereas for any $\lambda \neq 0$ the fiber is isomorphic to M_{DR}.

The Higgs-bundle moduli space has a subvariety $M^{\rm nil}_{Dol}$ parametrizing the Higgs bundles (E,θ) such that θ is nilpotent as an Ω^1_X-valued endomorphism of E. Let M^*_{Dol} denote the complement of $M^{\rm nil}_{Dol}$ in M_{Dol} and let M^*_{Hod} denote the complement of $M^{\rm nil}_{Dol}$ in M_{Hod}. Then the algebraic group $\mathbf{G}_m$ acts on M_{Hod} preserving all of the above subvarieties, and the compactification is obtained as the quotient [37] [39]

$$\overline{M}_{DR} := M^*_{Hod}/\mathbf{G}_m.$$

The complement of M_{Dol} in M_{Hod} (which is also the complement of M^*_{Dol} in M^*_{Hod}) is isomorphic to $M_{DR}\times \mathbf{G}_m$ and this gives the embedding $M_{DR}\hookrightarrow \overline{M}_{DR}$. The complementary divisor is given by

$$P_{DR} = M^*_{Dol}/\mathbf{G}_m.$$

In conclusion, this means that the points at ∞ in $\overline{M}_{DR}$ correspond to equivalence classes of semistable, degree 0, non-nilpotent Higgs bundles (E,θ) under the equivalence relation

$$(E,\theta)\cong (E,u\theta)$$

for any $u\in \mathbf{G}_m$.

Recall that the moduli space M_{Dol} is an irreducible algebraic variety [36], so P_{DR} is also irreducible. The general point therefore corresponds to a "general" Higgs bundle (E,θ) (in what follows we often forget to add the adjectives "semistable, degree 0"). For a general point, the *spectral curve* of θ (described in more detail in the section after next) is an irreducible curve with ramified map to X, such that the ramification points are all of the simplest type.

We should note that Arinkin [2] [3] has defined a finer compactification by modifying the notion of λ-connection, and this is taken up by Inaba, Iwasaki and Saito [27].

3. Curves going to infinity

The moduli spaces considered above are coarse only. In an etale neighborhood of the generic point, though, they are fine and smooth. At a general point of the divisor P_{DR}, both $\overline{M}_{DR}$ and P_{DR} are smooth. Thus we can look for a curve cutting P_{DR} transversally at a general point. Such a curve may be obtained by taking the projection of a curve in M_{Hod} cutting M_{Dol} at a general point. In turn, this amounts to giving a family (E_c,∇_c) where ∇_c is a $\lambda(c)$-connection, parametrized by $c\in C$ for some curve C. In an etale neighborhood of the point $\lambda=0$, the function $\lambda(c)$ should be etale. Note also that (E_0,∇_0) should be a general semistable Higgs bundle of degree zero.

The easiest way to obtain such a curve is as follows: let (E, θ) be a general Higgs bundle, stable of degree zero. The bundle E is stable as a vector bundle (since stability is an open condition and it certainly holds on the subset of Higgs bundles with $\theta = 0$, so it holds at general points). In particular E supports a connection ∇ and we can set

$$\nabla_\lambda := \lambda \nabla + \theta$$

for $\lambda \in \mathbf{A}^1$. Here the parameter is λ itself. The subset $\mathbf{G}_m \subset \mathbf{A}^1$ corresponds to points which are mapped into M_{DR}, and indeed the vector bundle with connection corresponding to the above λ-connection is

$$(E, \nabla + t\theta) \quad , \; t = \lambda^{-1}.$$

The map actually extends to a map from $\mathbf{A}^1$ into M_{DR} for the other coordinate chart $\mathbf{A}^1$ providing a neighborhood at ∞ in $\mathbf{P}^1$. In conclusion, the family of connections $\{(E, \nabla + t\theta)\}$ corresponds to a morphism

$$\mathbf{P}^1 \to \overline{M}_{DR}$$

sending $t \in \mathbf{A}^1$ into M_{DR}, sending the point $t = \infty$ to a general point in the divisor P_{DR}, and the curve is transverse to the divisor at that point. This type of curve was called a *pencil of connections* by Losev and Manin [30].

We will look only at curves of the above form. It should be possible to obtain similar results for other curves cutting P_{DR} transversally at a general point, but that is left as a problem for future study.

We will investigate the asymptotic behavior of the monodromy representations of the connections $(E, \nabla + t\theta)$ as $t \to \infty$. Recall that the *Betti moduli space* M_B is the moduli space for representations of $\pi_1(X)$ up to conjugation, and we have an analytic isomorphism $M_{DR}^{\rm an} \cong M_B^{\rm an}$ sending a connection to its monodromy representation. We will look at the asymptotics of the resulting analytic curve $\mathbf{A}^1 \to M_B$.

In order to set things up it will be useful to fix a basepoint $p \in X$ and a trivialization $\tau : E_p \cong \mathbb{C}^r$. Then for any $\gamma \in \pi_1(X, x)$ we obtain the monodromy matrix

$$\rho(E, \nabla + t\theta, \tau, \gamma) \in GL(r, \mathbb{C}).$$

Of course the monodromy matrices don't directly give functions on the moduli space M_B of representations, because they depend on the choice of trivialization τ. However, one has the Procesi coordinates (see Culler and Shalen [12] and Procesi [32]) which are certain polynomials in the monodromy matrices (for several γ at once) which are invariant under change of trivialization and give an embedding of the Betti moduli space M_B into an affine space. We will look at the asymptotic behavior of the monodromy matrices, but the resulting asymptotic information will also hold for any polynomials (see Corollary 14.2), and in particular for the Procesi coordinates. This will give asymptotic information about the image curve in M_B.

Notationally it is easier to start right out considering the transport matrices between points p and q. In any case, the functions we shall consider, be they the matrix coefficients of the monodromy ρ or some other polynomials in these or the transport matrices, will be entire functions $m(t)$ on the complex line $t \in \mathbb{C}$. We will be looking to characterize their asymptotic properties.

The method we will use is the same as the method already used in [34] to treat exactly this question, for a more special class of curves going to infinity in M_{DR}. In that book was treated the case of families of connections $(E, \nabla + t\theta)$ where

$$E = \mathcal{O}_X^r, \;\; \nabla = d + B, \;\; \theta = A$$

with A and B being $r \times r$ matrices of one-forms on X such that A is diagonal and B contains only zeros on the diagonal. In [34], a fairly precise description of the asymptotic behavior of the monodromy was obtained. It was also indicated how one should be able to reduce to this case in general; we shall explain that below. The only problem is that in the course of this reduction, one obtains the special situation but with B being a matrix of one-forms which has some poles on X. In this case the exact method used in [34] breaks down.

The purpose of the present paper is to try to remedy this situation as far as possible. We change very slightly the method (essentially by taking the more canonical gradient flows of the functions $\Re g_{ij}$ rather than the flows defined in Chapter 3 of [34], and also stopping the flows before arriving at the poles of B). However, we don't obtain the full results of [34], namely we can show an analytic continuation result for the Laplace transform of $m(t)$ (this Laplace transform is explained in more detail below), however we don't get good bounds or information about the singularities of the Laplace transform other than that they are locally finite sets of points. In particular we obtain information about the growth rate of $m(t)$ but not asymptotic expansions.

Even in order to obtain the analytic continuation, a much more detailed examination of the dynamics generated by the general method of [34] is necessary. This is the main body of the present paper (see Theorem 12.5). For the remainder of the technique we mostly refer to [34].

Thus while we treat a much more general type of curve going to infinity than was treated in [34], we obtain a weaker set of results for these curves. This leaves open the difficult question of what kinds of singularities the Laplace transforms have, and thus what type of asymptotic expansion we can get for $m(t)$.

4. Genericity results for the spectral data

Before beginning to look more closely at the monodromy representations, we will consider some properties of general points (E, θ) on P_{DR}, best expressed in terms of the *spectral curve* [25] [26] [18] [29] [13] [33].

Suppose (E, θ) is a Higgs bundle. Suppose $P \in X$ and $v \in T_P X$; then we obtain the fiber E_P which is a vector space of rank r, with an endomorphism $\theta_P(v) \in End(E_P)$. We say that P is *singular* if $\theta_P(v)$ has an eigenvalue (i.e. zero of the characteristic polynomial) of multiplicity ≥ 2. It is more natural to look at the *eigenforms* of θ obtained by dividing out the vector v. The eigenforms are elements of the cotangent space $T^*_P X = (\Omega^1_X)_P$.

We say that a singular point P is *generic* if there is exactly one eigenform of multiplicity ≥ 2; if it has multiplicity exactly 2; and if the two eigenforms $\alpha^{\pm}$ of θ which come together at P, may be expressed in a neighborhood with coordinate z as

$$\alpha^{\pm} = cdz \pm az^{1/2}dz + \ldots .$$

The condition that all singular points are generic is a Zariski open condition on the moduli space of Higgs bundles.

Suppose P is a generic singular point. The eigenforms give a set of $r-1$ distinct elements of $T^*_P X$, consisting of the values of the multiplicity-one eigenvalues of θ at P, plus the leading term cdz for the pair $\alpha^{\pm}$. Call this set EF_P. We say that P is *non-parallel* if EF_P, viewed as a subset of the real two-dimensional space $T^*_P X$, doesn't have any colinear triples, nor any quadruples of points defining two parallel lines.

In terms of a coordinate z at P we can write the elements of EF_P as

$$\alpha_i(P) = a_i dz$$

with a_i being distinct complex numbers, and say $a_1 = c$ in the previous formulation. Then P is non-parallel if and only if the set of $a_i \in \mathbb{C} \cong \mathbb{R}^2$ doesn't have any colinear triples or parallel quadruples. In turn this is equivalent to saying that the angular coordinates of the complex numbers $a_i - a_j$ are distinct.

Lemma 4.1. *The set of Higgs bundles (E, θ) such that the singularities are generic and satisfy the non-parallel condition, is a dense real Zariski-open subset of the moduli space.*

Proof. The condition of being non-parallel is a real Zariski open condition. In particular, the condition that all singular points be generic and non-colinear, holds in the complement of a closed real algebraic subset of the moduli space. Therefore, if there is one such point then the set of such points is a dense real Zariski open subset.

To show that there is one point (E, θ) such that all of the singular points are generic and non-parallel, we can restrict to the case where $E = \mathcal{O}^{\oplus r}$ is a trivial bundle. In this case, θ corresponds to a matrix of holomorphic one-forms on X. We will consider a matrix of the form $A + \lambda B$ with A diagonal having entries α_i, and B is off-diagonal with λ small. The singular points are perturbations of the points where $\alpha_i(P) = \alpha_j(P)$. A simple calculation with a 2×2 matrix shows that the singularities are generic in this case. In order to obtain the non-colinear condition, it suffices to have that for a point P where $\alpha_i(P) = \alpha_j(P)$, the subset of $r-1$ valuse of all the $\alpha_k(P)$ is non-parallel.

For a general choice of the α_k, this is the case. Suppose we are at a point P where $\alpha_1(P) = \alpha_2(P)$ for example. Then moving the remaining α_k for $k \geq 3$ shows that the remaining points are general with respect to the first one. A set of $r-1$ points such that the last $r-2$ are general with respect to the first one (whatever it is), satisfies the non-parallel condition. □

Lemma 4.2. *If (E, θ) is generic in the sense of the previous lemma, then the spectral curve V is actually an irreducible smooth curve sitting in the cotangent bundle T^*X. There is a line bundle L on V such that $E \cong \pi_*(L)$ and θ is given by multiplication by the tautological one-form over V.*

Proof. The genericity condition on the way the eigenforms come together at any point where the multiplicity is ≥ 2, guarantees that at any point where the projection $\pi : V \to X$ is not locally etale, the curve V is a smooth ramified covering of order 2 in the usual standard form. This shows that V is smooth. It is irreducible, because this is so for at least some points (for example the deformations used in the previous proof) and Zariski's connectedness implies that in a connected family of smooth projective curves if one is irreducible then all are. For connectedness of the family we use the irreducibility of the moduli space of Higgs bundles cf [36]. The last statement is standard in theory of spectral curves [25] [26] [18] [29]. □

Remark: Once p and q are fixed, then for general θ the endpoints p, q will not be contained in the set $\mathcal{R}$ of turning points.

5. Pullback to a ramified covering and gauge transformations

Fix a general Higgs bundle (E, θ) on X. By taking a Galois completion of the spectral curve of θ and Galois-completing a further two-fold ramified covering if necessary, we can obtain a ramified Galois covering

$$\varphi : Y \to X$$

such that the pullback Higgs field φ^* has a full set of eigen-one-forms defined on Y; and such that the ramification powers over singular points of θ are divisible by 4.

We have one-forms $\alpha_1, \ldots, \alpha_r$ and line sub-bundles

$$L_1, \ldots, L_r \subset \varphi^* E$$

such that at a general point of Y we have

$$\psi : \varphi^* E \cong L_1 \oplus \ldots \oplus L_r$$

with $\varphi^*\theta$ represented by the diagonal matrix with entries α_i. Note that $\varphi^*\theta$ preserves L_i (acting there by multiplication by α_i) globally on Y. However, the isomorphism ψ will only be meromorphic, and also the L_i are of degree < 0. Choose modifications L'_i of L_i (see Lemma 5.1 below, also the modifications are made only over singular points) such that L'_i is of degree zero, and set

$$E' := L'_1 \oplus \ldots \oplus L'_r.$$

Let θ' denote the diagonal Higgs field with entries α_i on E'. Let ∇' be a diagonal flat connection on E'. We have a meromorphic map

$$\psi : E \to E',$$

and

$$\psi \circ \varphi^*\theta \circ \psi^{-1} = \theta'.$$

Suppose now that ∇ was a connection on E, giving a connection $\varphi^*\nabla$ on φ^*E. We can write

$$\psi \circ \varphi^*\nabla \circ \psi^{-1} = \nabla' + \beta$$

with β a meromorphic section of $End(E') \otimes_{\mathcal{O}} \Omega^1_Y$.

A transport matrix of $(E, \nabla + t\theta)$ may be recovered as a transport matrix for the pullback bundle on Y. Indeed if γ is a path in X going from p to q then it lifts to a path going from a lift p' of p to a lift q' of q. Thus it suffices to look at the problem of the asymptotics for transport matrices for the pullback family

$$\{(\varphi^*E, \varphi^*\nabla + t\varphi^*\theta)\}.$$

We may assume that p and q are not singular points of θ, so p' and q' will not be singular points of $\varphi^*\theta$. Then the transport matrices for this family are conjugate (by a conjugation which is constant in t) to the transport matrices for the family

$$\{(E', \nabla' + \beta + t\theta')\}.$$

Lemma 5.1. *In the above situation, the modifications L_i' of L_i may be chosen so that the diagonal entries of β are holomorphic. Furthermore the poles of the remaining entries of β are restricted to the points lying over singular points in X for the original Higgs field θ (the "turning points").*

Proof. Note first that, by definition, away from the singular points of θ the eigen-one-forms are distinct so the eigenvectors form a basis for E, in other words the direct sum decomposition ψ is an isomorphism at these points. Thus ψ only has poles over the singular points of θ (hence the same for β).

We will describe a choice of L_i' locally at a singular point.

Look now in a neighborhood of a point $P' \in Y$, lying over a singular point $P \in X$. Let z' denote a local coordinate at P' on Y, with z a local coordinate at P on X and with

$$z = (z')^m.$$

Our assumption on Y was that m is divisible by 4. In fact we may as well assume that $m = 4$ since raising to a further power doesn't modify the argument. Thus we can write

$$z' = z^{1/4}.$$

There are two eigenforms of θ which come together at P. Suppose that their lifts are α_1 and α_2. Then near P' we can write

$$\varphi^*E = U \oplus L_3 \oplus \ldots \oplus L_r$$

where U is the rank two subbundle of φ^*E corresponding to eigenvalues α_1 and α_2. The direct sum decomposition is holomorphic at P' because the other eigenvalues of θ were distinct at P and different from the two singular ones (of course after the pullback all of the eigenforms have a value of zero at P' but the decomposition still holds nonetheless).

Now we use a little bit more detailed information about spectral curves for Higgs bundles: the general (E, θ) is obtained as the direct image of a line bundle on the spectral curve (Lemma 4.2). This means that locally near P there is a two-fold branched covering with coordinate $u = z^{1/2}$ such that the rank 2 subbundle of E corresponding to the singular values looks like the direct image of the trivial bundle on the covering, and the 2×2 piece of θ looks like the action of multiplication by $udz = 2u^2du$. The direct image, considered as a module over the series in z, is just the series in u. One can obtain a basis by looking at the odd and even powers of u: the basis vectors are $e_1 = 1$ and $e_2 = u$. In these terms we have

$$\theta e_1 = e_2 dz; \quad \theta e_2 = z e_1 dz.$$

Thus the 2×2 singular part of θ has matrix

$$\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} dz.$$

Pulling back now to the covering Y which is locally 4-fold, we have a basis for U in which

$$\varphi^*\theta|_U = \begin{pmatrix} 0 & (z')^7 \\ (z')^3 & 0 \end{pmatrix} dz'.$$

On the other hand, since up until now our decomposition is holomorphic, the pullback connection $\varphi^*\nabla$ may be written (in terms of our basis for U plus trivializations of the L_i for $i \geq 3$) as $d + B'$ where B' is a holomorphic matrix of one-forms. Since the basis can be pulled back from downstairs, we can even say that B' consists of one-forms pulled back from X.

To choose the modifications L_i' (for $i = 1, 2$) locally at P' we have to find a meromorphic change of basis for the bundle U, which diagonalizes $\varphi^*\theta|_U$. The eigenforms of the matrix are $\pm(z')^5dz'$ and we can choose eigenvectors

$$e_\pm := \begin{pmatrix} z' \\ \pm(z')^{-1} \end{pmatrix}.$$

Note by calculation that

$$(\varphi^*\theta|_U)e_\pm = (\pm(z')^5dz')e_\pm.$$

Choose the line bundles L_1' and L_2' to be spanned by the meromorphic sections e_+ and e_- of U. These are indeed eigen-subbundles for $\varphi^*\theta$. We just have to calculate the connection $\varphi^*\nabla$ on the bundle $U' = L_1' \oplus L_2'$. Which is the same as the modification of U given by the meromorphic basis $z'e_1, (z')^{-1}e_2$.

Note first that the matrix B' of one-forms pulled back from X consists of one-forms which have zeros at least like $(z')^3dz'$. Thus B' transported to U' is still a matrix of holomorphic one-forms so it doesn't affect our lemma. In particular we just have to consider the transport to U' of the connection d_U constant with respect to the basis (e_1, e_2) on the bundle U.

Calculate

$$d_U(a_+e_+ + a_-e_-) = d_U \begin{pmatrix} (a_+ + a_-)z' \\ (a_+ - a_-)(z')^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (da_+ + da_-)z' \\ (da_+ - da_-)(z')^{-1} \end{pmatrix} + \begin{pmatrix} (a_+ + a_-)(d\log z')z' \\ -(a_+ - a_-)(d\log z')(z')^{-1} \end{pmatrix}$$

and with the notation $d_{U'}$ for the constant connection on the bundle U' with respect to its basis $e_\pm$, this is equal to

$$= d_{U'}(a_+e_+ + a_-e_-) + a_+(d\log z')e_- + a_-(d\log z')e_+.$$

We conclude that the connection matrix β is, up to a holomorphic piece, just the 2×2 matrix

$$\begin{pmatrix} 0 & (z')^{-1} \\ (z')^{-1} & 0 \end{pmatrix} dz'.$$

In particular the diagonal terms of β are holomorphic, as desired for the lemma.

These local modifications piece together to give global modifications L'_i of the L_i. We have to show that the L'_i are of degree zero.

In general, given a meromorphic connection on a bundle which is a direct sum of line bundles, we can extract its "diagonal" part, which in terms of a local framing compatible with the direct sum is just the connection given by the diagonal entries of the original connection matrix. Denote this operation by $(\ )_{\rm diag}$. Note that for any diagonal connection ∇' and meromorphic endomorphism-valued one-form β, the diagonal connection is given by $(\nabla' + \beta)_{\rm diag} = \nabla' + \beta_{\rm diag}$ where $\beta_{\rm diag}$ is the matrix of diagonal entries of β.

Setting $E' := \bigoplus L'_i$ we have a meromorphic map $\psi : E \to E'$. We obtain a meromorphic connection $\psi\circ\varphi^*\nabla\circ\psi^{-1}$ on E', and by the above choice of L'_i the associated diagonal connection is holomorphic at the singularities. On the other hand, $\psi\circ\varphi^*\nabla\circ\psi^{-1}$ is holomorphic away from the singularities, so its diagonal part is holomorphic there too. Therefore the global diagonal connection $(\psi\circ\varphi^*\nabla\circ\psi^{-1})_{\rm diag}$ on $\bigoplus L'_i$ is holomorphic. This proves that the L'_i are of degree zero. In particular, our choice of modification is allowable for the argument given at the start of the present section. This proves the lemma. □

Remarks:

(i) The above proof gives further information: the only terms with poles in the matrix β are the off-diagonal terms corresponding to the two eigenvalues which came together originally downstairs in X; and these terms have exactly logarithmic (i.e. first-order) poles with residue 1. This information might be useful in trying to improve the current

results in order to obtain precise expansions at the singularities of the Laplace transform of the monodromy.
(ii) This gauge transformation is probably not new, but I don't currently have a good reference. It looks related to [28], [44] and [33], and indeed may go back to [13] [41].
(iii) The fact that we had to go to a covering whose ramification power is divisible by 4 rather than just 2 (as would be sufficient for diagonalizing θ) is somewhat mysterious; it probably indicates that we (or some of us at least) don't fully understand what is going on here.

Let β^{diag} denote the matrix of diagonal entries of β. Let $Z = \widetilde{Y}$ be the universal covering. Over Z we can use the diagonal connection $\nabla' + \beta^{\mathrm{diag}}$ to trivialize

$$E'|_Z \cong \mathcal{O}_Z^r.$$

With respect to this trivialization , our family now has the form of a family of connections

$$\{(\mathcal{O}_Z^r, d + B + tA)\}$$

where A (corresponding to the pullback of θ' to Z) is the diagonal matrix whose entries are the pullbacks of the α_i; and where B is a matrix whose diagonal entries are zero, and whose off-diagonal entries are meromorphic with poles at the points lying over singular points for θ.

We can now apply the method developped in [34] to this family of connections. Note that it is important to know that the diagonal entries of A come from forms on the compact Riemann surface Y; on the other hand the fact that B is only defined over the universal covering Z is not a problem. The next two sections will constitute a brief discussion of how the method of [34] works; however the reader is refered back there for the full details.

6. Laplace transform of the monodromy operators

We now look at a family of connections of the form $d + B + tA$ on the trivial bundle $\mathcal{O}^r$ on the universal covering Z of the ramified cover Y, where A is a diagonal matrix with one-forms α_i along the diagonal, and B is a matrix of meromorphic one-forms with zeros on the diagonal. We assume that the poles of B are at points $P \in \mathcal{R}$ coming from the original singular points of the Higgs field θ on X. We make no assumption about the order of poles, in spite of the additional information given by Remark (i) after the proof of Lemma 5.1 above.

Assume that p and q are two points in Z, not on the singular points. Choose a path γ from p to q not passing through the singular points.

We obtain the *transport matrix* $m(t)$ for continuing solutions of the ordinary differential equation $(d + B + tA)f = 0$ from p to q along the path γ. Note that $m(t)$ is a holomorphic $r \times r$-matrix-valued function defined for all $t \in \mathbb{C}$.

Denote by Z^* (resp. Z^ϵ) the complement of the inverse image of $\mathcal{R}$ (resp. the complement of the union of open discs of radius ϵ around points in the inverse image of $\mathcal{R}$). The poles of B force us to work in Z^* rather than Z, and in the course of the argument an ϵ will be chosen so that we really work in Z^ϵ. Actually it turns out that the fact of staying inside these regions will be guaranteed by our choice of vector fields, so we don't need to worry about any modification of the procedure of [34] because of this difference.

Recall that after a gauge transformation and an expansion as a sum of iterated integrals, we obtain a formula for the transport matrix. One way of thinking of this formula is to look at the transport for the connection $d + sB + tA$ and expand in a Taylor series in s about the point $s = 0$, then evaluate at $s = 1$. The terms in the expansion are the higher derivatives in s, at $s = 0$, which are functions of t. A concrete derivation of the formula is given in [34]. It says

$$m(t) = \sum_I \int_{\eta_I} b_I e^{tg_I}$$

where:
—the sum is taken over multi-indices of the form $I = (i_0, i_1, \ldots, i_k)$ where we note $k = |I|$;
—for a multi-index I we denote by Z_I^* the product of $k = |I|$ factors $Z \times \ldots \times Z$;
—in Z_I^* we have a cycle

$$\eta_I := \{(\gamma(t_1), \ldots, \gamma(t_k))\}$$

for $0 \leq t_1 \leq \ldots \leq t_k \leq 1$ where γ is viewed as a path parametrized by $t \in [0, 1]$;
—the cycle η_I should be thought of as representing a class in a relative homology group of Z_I^* relative to the simplex formed by points where $Z_I^* = z_{i+1}$ or at the ends $z_1 = p$ or $z_k = q$;
—the matrix B leads to a (now meromorphic) matrix-valued k-form b_I on Z_I^* defined as follows: if the entries of B are denoted $b_{ij}(z)dz$ then

$$b_I = b_{i_k i_{k-1}}(z_k)dz_k \wedge \ldots \wedge b_{i_1 i_0}(z_1)dz_1 \mathbf{e}_{i_k i_0}$$

where $\mathbf{e}_{i_k i_0}$ denotes the elementary matrix with zeros everywhere except for a 1 in the $i_k i_0$ place;

—and finally g_I is a holomorphic function $Z_I^* \to \mathbb{C}$ defined by integrating the one-forms α_i as follows:

$$g_I(z_1, \ldots, z_k) = \int_p^{z_1} \alpha_{i_0} + \ldots + \int_{z_k}^q \alpha_{i_k}.$$

The terms in the above expression correspond to what Ecalle calls the *elementary monomials* som, see his article in [7].

The fact that b_I is meromorphic rather than holomorphic is the only difference between our present situation and the situation of [34]. Note that because our path γ misses the singular points and thus the poles of B, the cycle η_I is supported away from the poles of b_I. We will be applying essentially the same technique of moving the cycle of integration η, but we need to do additional work to make sure it stays away from the poles of b_I.

It is useful to have the formula

$$g_I(z_1, \ldots, z_k) = g_{i_0 i_1}(z_1) + \ldots + g_{i_{k-1} i_k}(z_k) + \int_p^q \alpha_{i_k},$$

where

$$g_{ij}(z) := \int_p^z \alpha_i - \alpha_j.$$

Our formula for m gives a preliminary bound of the form

$$|m(t)| \le Ce^a|t|.$$

Indeed, along the path γ the one-forms b_{ij} are bounded, so

$$|b_I| \le C^k$$

on η_I; also we have a bound $|g_I(z)| \le a$ for $z \in \eta_I$, uniform in I; and finally the cycle of integration η_I has size $(k!)^{-1}$. Putting these together gives the bound for $m(t)$ (and, incidentally, shows why the formula for m converged in the first place).

Recall now that the *Laplace transform* of a function $m(t)$ which satisfies a bound such as the above, is by definition the integral

$$f(\zeta) := \int_0^\infty m(t) e^{-\zeta t} dt$$

where $\zeta \in \mathbb{C}$ with $|\zeta| > a$ and the path of integration is taken in a suitably chosen direction so that the integrand is rapidly decreasing at infinity. In our case since $m(t)$ is a matrix, $f(\zeta)$ is also a matrix. We can recover $m(t)$ by the inverse transform

$$m(t) = \frac{1}{2\pi i} \oint f(\zeta) e^{\zeta t} d\zeta$$

with the integral being taken over a loop going around once counterclockwise in the region $|\zeta| > a$.

The singularities of $f(\zeta)$ are directly related to the asymptotic behavior of $m(t)$. This is a classical subject which we discuss a little bit more in §14. One can note for example that by the inverse transform, there exist functions $m(t)$ satisfying the preliminary bound $|m(t)| \leq Ce^a|t|$ but such that the Laplace transforms $f(\zeta)$ have arbitrarily bad singularities in the region $|\zeta| \leq a$. Thus getting any nontrivial restrictions on the singularities of f amounts to a restriction on which types of functions $m(t)$ can occur.

In our case, the expansion formula for $m(t)$ leads to a similar formula for the Laplace tranform, which we state as a lemma. Define the image support of a collection $\eta = \{\eta_I\}$ by the collection of functions $g = \{g_I\}$ to be the closure of the union of the images of the component pieces:

$$g(\eta) := \overline{\bigcup_I g_I(|\eta_I|)} \subset \mathbb{C},$$

where $|\eta_I| \subset Z_I$ is the usual support of the chain η_I.

Lemma 6.1. *With the functions g_I, the forms b_I, and the chains η_I intervening above, for any ζ in the complement of the region $g(\eta)$ the formula*

$$f(\zeta) = \sum_I \int_{\eta_I} \frac{b_I}{g_I - \zeta}$$

converges, and gives an analytic continuation of the Laplace transform in the (unique) unbounded connected component of the complement of $g(\eta)$.

Proof. The convergence comes from the same bounds on b_I and the size of η_I which allowed us to bound m. The fact that this formula gives the Laplace transform is an exercise in complex path integrals. □

The terms in this expansion correspond to Ecalle's elementary monomials "soc" in [7].

A first approach would be to try to move the path γ so as to move the union of images $g(\eta)$ and analytically continue f to a larger region. This works quite well for rank 2, where one can get an analytic continuation to a large region meeting the singularities [13]. In higher rank, the 3×3 example at the end of [34] shows that this approach cannot be optimal. In fact, we should instead move each cycle of integration η_I individually. Unfortunately this has to be done with great care in order to maintain control of the sizes of the individual terms so that the infinite sum over I still converges.

Now we get to the main definition. It is a weak version of resurgence, see [14], [9].

Definition 6.2. *A function such as* $f(\zeta)$ *defined on* $|\zeta| > a$ *is said to have an* analytic continuation with locally finite branching *if for every* $M > 0$ *there is a finite set of points* $S_M \subset \mathbb{C}$ *such that if* σ *is any piecewise linear path in* $\mathbb{C} - S_M$ *starting at a point where* $|\zeta| > a$ *and such that the length of* σ *is* $\leq M$*, then* $f(\zeta)$ *can be analytically continued along* σ*.*

And the statement of the main theorem.

Theorem 6.3. *Suppose* $m(t)$ *is the transport matrix from* p *to* q *for a family of connections on the trivial bundle* $\mathcal{O}_Z^r$ *of the form* $\{d+B+tA\}$*. Suppose that* A *is diagonal with one-forms* α_i*, coming from the pullback of a general Higgs field* θ *over the original curve* X*, and suppose that* B *is a meromorphic matrix of one-forms with poles only at points lying over the singular points of* θ*. Let* $f(\zeta)$ *denote the Laplace transform of* $m(t)$*. Then* f *has an analytic continuation with locally finite branching.*

Most of the remainder of these notes is devoted to explaining the proof.

7. Analytic continuation of the Laplace transform

We now recall the basic method of [34] for moving the cycles η_I to obtain an analytic continuation of $f(\zeta)$. We refer there for most details and concentrate here just on stating what the end result is. Still we need a minimal amount of notation. Before starting we should refer to [16] (and the references therein) for an extensive discussion of this process for each individual integral in the sum, including numerical results on how the singularities of the analytic continuations determine the asymptotics of the pre-transformed integrals.

We work with *pro-chains* which are formal sums of the form $\eta = \sum_I \eta_I$ of chains on the Z_I^*. We have a boundary operator denoted $\partial + A$ where ∂ is the usual boundary operator on each η_I individually, and A (different from the matrix of one-forms considered above) is a signed sum of face maps corresponding to the inclusions $Z_{I'}^* \to Z_I^*$ obtained when some $z_i = z_{i+1}$. Our original pro-chain of integration in the integral expansion satisfies $(\partial + A)\eta = 0$. We can write the expansion formula of Lemma 6.1 as an integral over the pro-chain $\eta = \sum_I \eta_I$,

$$f(\zeta) = \int_\eta \frac{b}{g - \zeta}$$

where b is the collection of forms b_I on Z_I^* and g is collection of functions g_I. Such a formula is of course subject to the condition that the infinite sum of integrals converges.

In a formal way (i.e. element-by-element in the infinite sums implicit in the above notation), if we add to η a boundary term of the form $(\partial + A)\kappa$ then the integral doesn't change:

$$\int_{\eta+(\partial+A)\kappa} \frac{b}{g-\zeta} = \int_{\eta} \frac{b}{g-\zeta}.$$

This again is subject to the condition that the infinite sums on both sides converge absolutely and in fact that the individual terms in the rearrangement (i.e. separating ∂ and A) converge absolutely. Whenever we use this, we will be refering (perhaps without mentionning it further) to the work on convergence which was done in [34].

Our analytic continuation procedure rests upon consideration of the locations of the images by the function g, of the pro-chains of integration. Recall the notation

$$g(\eta) := \overline{\bigcup_I g_I(|\eta_I|)}$$

where $|\eta_I|$ is the usual support of the chain η_I.

If f is defined by the right-hand integral over η in a neighborhood of a point ζ_0, meaning that the image $g(\eta)$ misses an open neighborhood of ζ_0, and if the image $g(\eta + (\partial + A)\kappa)$ misses an entire segment going from ζ_0 to ζ_1, then the integral over $\eta + (\partial + A)\kappa$ defines an analytic continuation of f along the segment. The procedure can be repeated with η replaced by $\eta + (\partial + A)\kappa$.

At this point we let our notation slide a little bit, and denote by η any pro-chain which would be obtained from the original chain of integration by a sequence of modifications of the kind we are presently considering, such that the integral over η serves to define an analytic continuation of $f(\zeta)$ to a neighborhood of a point $\zeta_0 \in \mathbb{C}$. The original pro-chain η of Lemma 6.1 is the initial case. Our assumption on η says among other things that the image $g(\eta)$ doesn't meet a disc around ζ_0. Fix a line segment S going from ζ_0 to another point ζ_1; we would like to continue f in a neighborhood of S. By making a rotation in the complex plane (which can be seen as a rotation of the original Higgs field) we may without loss of generality assume that the segment S is parallel to the real axis and the real part of z_1 is smaller than the real part of ζ_0. Let u be a cut-off function for a neighborhood of S and write

$$\eta = \eta' + \eta'', \quad \eta' = g^*(u) \cdot \eta.$$

We will apply the method of [34] to move the piece η' (this piece corresponds to what was called η in Chapter 4 of [34]).

The first step is to choose flows. This corresponds to Chapter 3 of [34]. In our case, we will use flows along vector fields W_{ij} which are $\mathcal{C}^\infty$ multiples of the gradient vector fields of the real parts $\Re g_{ij}$. To link up with the terminology of [34], these vector fields determine flowing functions $f_{ij}(z,t)$ (for $z \in Z$ and $t \in \mathbb{R}^+$ taking values in Z) by the equations

$$\frac{\partial}{\partial t} f_{ij}(z,t) = W_{ij}(f_{ij}(z,t)), \quad f_{ij}(z,0) = z.$$

Note that this choice is considerably simpler than that of [34]. The choice of vector fields will be discussed in detail below, and will in particular be subject to the following constraints.

Condition 7.1. *(i) the vector fields W_{ij} are lifts to Z of vector fields defined on the compact surface Y;*
(ii) the differential $d\Re g_{ij}$ applied to W_{ij} at any point, is a real number ≤ 0;
(iii) there exists ϵ such that the flows preserve Z^ϵ i.e. the vector fields W_{ij} are identically zero in the discs of radius ϵ around the singular points; and
(iv) the W_{ii} are identically zero.

The flows given by our vector fields lead to a number of operators F, K and H defined as in Chapters 4 and 5 of [34]. These give pro-chains

$$F\tau = \sum_{r,s} F(-KA)^r H(AK)^s \eta',$$

$$F\psi = \sum_r F(-KA)^r K(\partial + A)\eta',$$

$$FK\varphi = \sum_r FK(AK)^r \eta'.$$

The reader can get a fairly good idea of these definitions from our discussion of the points on $|F\tau|$ in §8 below.

Lemma 7.2. *With these notations, and assuming that the vector fields satisfy the constraints marked above, we can write*

$$\eta + (\partial + A)FK\varphi = \eta'' + F\tau - F\psi.$$

On the right, the images $g(\eta'')$ and $g(F\psi)$ miss a neighborhood of the segment S. Assuming *we can show that the image $g(F\tau)$ also misses a* *neighborhood of the segment S, then*

$$f(\zeta) = \int_{\eta+(\partial+A)FK\varphi} \frac{b}{g-\zeta}$$

gives an analytic continuation of f from ζ_0 to ζ_1 along the segment S.

Proof. The operator K corresponds to applying the flows defined by W_{ij} in the various coordinates. This has the effect of decreasing the real part $\Re g$. The fact that in our case we use flows along vector fields which are positive real multiples of $-\mathbf{grad}\,\Re g_{ij}$ (this is the second of the constraints on W_{ij}) implies that the flows strictly respect the imaginary part of g. This differs from the case of [34] and means we can avoid discussion of "angular sectors" such as on pages 52-53 there. Thus, in our case, when we apply a flow to a point, the new point has the same value of $\Im g$, and the real part $\Re g$ is decreased.

The operator F is related to the use of buffers; we refer to [34] for that discussion and heretofore ignore it. The operator A is the boundary operator discussed above; and the operator H is just the result of doing the flows K after unit time. In particular, A doesn't affect the value of g. And H decreases $\Re g$ while fixing $\Im g$ just as K did (this point will perhaps become clearer with the explicit description of points in the supports of $F\tau$ and $FK\varphi$ in the next section).

The proof of the first formula is the same as in [34] Lemma 4.4, and we refer there for it.

To show that the supports of $g(\eta'')$ and $g(F\psi)$ miss a neighborhood of S, it is useful to be a little bit more precise about the neighborhoods which are involved. Let N_1 be the support of u, which is a neigborhood of S (we assume it is convex), and let N_2 be the support of du which is an oval going around S but not touching it. Let N_3 be the neighborhood of S where u is identically 1. Let D be a disc around ζ_0, such that $g(\eta)$ misses D, and which we may assume has radius bigger than the width of N_1. Then

$$g(\eta') \subset N_1 - (N_1 \cap D),$$
$$g(\eta'') \subset \mathbb{C} - (N_3 \cup D),$$

and $(\partial A)\eta' = -(\partial A)\eta''$ with

$$g((\partial A)\eta') \subset N_2 - (N_2 \cap D).$$

In particular the support of $g(\eta'')$ misses the neighborhood N_3 of S. Also, given that the boundary term $(\partial + A)\eta'$ is supported in the U-shaped region N_2, the effect of our operators on $\Re g$ and $\Im g$ described above implies that $g(F\psi)$ is supported away from N_3. This completes the proof of the second statement of the lemma.

For the last statement, assume that we have chosen things such that the support of $g(F\tau)$ also misses S. This is certainly what we hope, because of the inclusion of the operator H applying all the flows for unit time. The only possible problem would be if we get too close to singular points; that is the technical difficulty which is to be treated in the remainder of the paper. For now, we assume that this is done.

Formally speaking, the first equation of the lemma means that

$$\int_{\eta} \frac{b}{g-\zeta} = \int_{\eta''+F\tau-F\psi} \frac{b}{g-\zeta}.$$

By our starting assumption $f(\zeta)$ is defined by the integral on the left, in a neighborhood of ζ_0. On the other hand, the integral on the right defines an analytic continuation along the segment S.

An important part of justifying the argument of the preceding paragraph (and indeed, of showing that the integral on the right is convergent) is to bound the sizes and numbers of all the chains appearing here. This was done in [34].

The only difference in our present case is the poles in the integrand b. However, thanks to the third constraint on the vector fields W_{ij}, everything takes place in $Z^{\epsilon}_I := Z^{\epsilon} \times \ldots \times Z^{\epsilon}$, and on Z^{ϵ} there is a uniform bound on the size of b_{ij}. Also, everything takes place inside a relatively compact subset of Z, see §9. Thus the integrand in the multivariable integral is bounded by

$$\sup_{Z^{\epsilon}_I} |b| \leq C^k$$

for $k = |I|$. With this information the remainder of the argument of [34] works identically the same way (it is too lengthy to recall here). This justifies the formal argument of two paragraphs ago and completes the proof of the lemma. □

Remark: It is clear from the end of the proof that the bounds depend on ϵ, which in turn will depend on how close we want to get to a singularity. This is the root of why we don't get any good information about the order of growth of the Laplace transform at its singularities.

8. Description of cells using trees

As was used in [34], the chains defined above can be expressed as sums of cells. We are most interested in the chain $F\tau$ although what we say also applies to the other ones such as $FK\varphi$. These chains are unions of cells which have the form of a family of cubes parametrized by points in one of the original cells η'_I. We call these things just cubes.

In the cubes which occur the points are parametrized by "trees" furnished with lots of additional information. [1] We make this precise as follows: a *furnished tree* is:
—a binary planar tree T sandwiched between a top horizontal line and a bottom horizontal line;
—with leftmost and rightmost vertical strands whose edges are called the *side edges*;
—for each top vertex of the tree (i.e. where an edge meets the top horizontal line) we should specify a point $z \in Z^*$ (the point corresponding to the left resp. right side edge is p resp. q);
—for each region in the complement of the tree between the top and bottom horizontal lines and between the side edges we should specify an index, so that each (non-side) edge of the tree is provided with left and right indices which will be denoted i_e and j_e below; and
—each edge e is assigned a "length" $s(e) \in [0, 1]$.

Suppose T is a furnished tree. By looking at the indices assigned to the regions meeting the top and bottom horizontal lines we obtain multi-indices $I^{\rm top}$ and $I^{\rm bot}$, so the collection of points $(z_1, \ldots, z_k)$ attached to the top vertices gives a point $z^{\rm top} \in Z^*_{I^{\rm top}}$.

We can now explain how a furnished tree leads to a point $z^{\rm bot} \in Z_{I^{\rm bot}(T)}$. This depends on a choice of vector fields W_{ij} for each pair of indices i, j, which we now assume as having been made. A *flowing map* $\Phi : T \to Z$ is a map from the topological realization of the tree, into Z, satisfying the following properties:
(i) if v is a top vertex which is assigned a point z in the information contained in T, then $\Phi(v) = z$;
(ii) the side edges are mapped by constant maps to the points p or q respectively; and
(iii) if e is an edge with left and right indices i_e and j_e and with initial vertex v and terminal vertex v', then $\Phi(e)$ is the flow curve for flowing along the vector field $W_{i_e j_e}$ from $\Phi(v)$ to $\Phi(v')$, where the flow is done for time $s = s(e)$. This determines $\Phi(v')$ as a function of $\Phi(v)$ and the information in the tree. Thus by recursion we determine the $\Phi(v)$ for all vertices, as well as the paths $\Phi(e)$ for the edges e (the map Φ on the edges is only well determined up to reparametrization because we

[1]The occurence of trees here is certainly related to and probably the same as Ecalle's notions of *(co)mould (co)arborescent* cf [21]. In another direction, John Conway pointed out at the time of [34] that cubes parametrized by trees in this way glue together into Stasheff polytopes. I didn't know what those were at the time, but retrospectively this still remains mysterious since we are dealing with representations of the fundamental group and it isn't clear what that has to do with homotopy-associativity. This is certainly a good subject for further thought.

don't fix a parametrization of the edges; the length s is abstract, since it is convenient to picture even edges assigned $s = 0$ as being actual edges).

For a given choice of vector fields W_{ij} and of information attached to the tree T, the flowing map exists and is unique. This determines a point given by the values z at the bottom vertices,

$$z^{\mathrm{bot}}(W, T) \in Z_{I^{\mathrm{bot}}}.$$

Now go back to the situation of the previous section. Starting from a chain η' we obtained a chain $F\tau$.

Lemma 8.1. *The points in the support of $F\tau$ are described as the $z^{\mathrm{bot}}(W, T)$, where $W = \{W_{ij}\}$ is the collection of vector fields used to define the flows K and H, and where T is a furnished tree such that $z^{\mathrm{top}}(T)$ is in the support of η' and satisfying the following auxiliary condition:*
(*) there exists (up to reparametrization of the planar embedding) a horizontal line which cuts the tree along a sequence of edges, such that all of these edges are assigned the fixed length value $s = 1$.

Proof. See [34], pages 54-55. The auxiliary condition comes from the term H in the formula for τ. □

Remark: For the chain $FK\varphi$ the same statement holds except that the furnished trees T might not necessarily satisfy the auxiliary condition.

We finish this section by pointing out the relationship between $g(z^{\mathrm{top}})$ and $g(z^{\mathrm{bot}})$. This is the key point in our discussion, because z^{top} is the input point coming from the chain η' and z^{bot} is the output point which goes into the resulting chain $F\tau$. We want to prove that the real part of $g(z^{\mathrm{bot}})$ can be moved down past the end of the segment S.

Lemma 8.2. *If T is a furnished tree and W a choice of vector fields, then*

$$g(z^{\mathrm{bot}}(T)) = g(z^{\mathrm{top}}(T)) + \sum_e \int_{\Phi(e)} dg_{i_e j_e},$$

In particular if W saitsfies Condition 7.1 then

$$g(z^{\mathrm{bot}}(T)) - g(z^{\mathrm{top}}(T)) \in \mathbb{R}_{\leq 0}.$$

Proof. If e is an edge of T and $s' \in [0, s(e)]$ then we can define the tree T' obtained by *pruning* T at (e, s'). This is obtained by cutting off everything below e and sending the bottom vertex of e to the line at the bottom. The indices associated to regions in the complement follow accordingly. Finally we set $s(e) := s'$ in the new tree T'.

Suppose for the same edge e we also pick $s'' \in [s', s(e)]$. Then we obtain a different pruning denoted T'' (which has almost all the same information except for the length of the edge e). Let v' (resp. v'') denote the bottom vertices corresponding to e in the trees T' (resp. T''). Let Φ' (resp. Φ'') denote the flowing map for T' (resp. T''). These coincide and coincide with Φ on the parts of the trees that are in common (the unpruned parts). We have

$$g(z^{\mathrm{bot}}(T'')) = g(z^{\mathrm{bot}}(T')) + \int_{\Phi'(v')}^{\Phi''(v'')} dg_{i_e j_e}.$$

Note that the segment of $\Phi(e)$ going from $\Phi'(v')$ to $\Phi''(v'')$ is a flow curve for the vector field $W_{i_e j_e}$, and it flows for time $s'' - s'$.

If we prune at an edge e with $s' = s(e)$ then it amounts to cutting off the tree at the lower vertex of e. If furthermore all of the length vectors assigned to edges below e are 0, then $g(z^{\mathrm{bot}}(T')) = g(z^{\mathrm{bot}}(T))$.

By recurrence we obtain the first statement in the lemma.

Recall that one of the constraints was the condition that the vector fields $W_{i_e j_e}$ be negative multiples of the gradient vector fields for the real functions $\Re g_{i_e j_e}$. With this condition we get that the integral of $dg_{i_e j_e}$ along a flow curve for $W_{i_e j_e}$ is a negative real number, so this gives at each stage of the recurrence

$$g(z^{\mathrm{bot}}(T'')) - g(z^{\mathrm{bot}}(T')) \in \mathbb{R}_{\leq 0}.$$

Putting these together gives the second statement of the lemma. □

There is also another way to prune a tree: if e is an edge such that $i_e = j_e$ then we can cut off e and all of the edges below it, and consolidate the two edges above and to the side of e into one edge. The only difficulty here is that the consolidated edge might have total length > 1 but this doesn't affect the remainder of our argument (since at this point we can ignore questions about the sizes of the cells). Let T' denote the pruned tree obtained in this way. We again have

$$g(z^{\mathrm{bot}}(T)) - g(z^{\mathrm{bot}}(T')) \in \mathbb{R}_{\leq 0}.$$

In general we will be trying to show for the trees which arise in $F\tau$, that the real part of $g(z^{\mathrm{bot}}(T))$ is small enough. If we can show it for T' then it follows also for T. In this way we can reduce for the remainder of the argument, to the case where $i_e \neq j_e$ for all edges of T. This is the content of the following lemma. For its statement, recall the neighborhood $S \subset N_1$ appearing in the proof of Lemma 7.2.

Lemma 8.3. *Let $|S| = \zeta_0 - \zeta_1$ denote the length of the segment along which we want to continue f. In order to show that the image $g(F\tau)$*

misses a neighborhood, say N_1, of the segment S it suffices to choose our vector fields W (satisfying Condition 7.1) so that if T is any furnished tree satisfying:
(i) the auxiliary condition (*) of Lemma 8.1;
(ii) that $i_e \neq j_e$ for all edges e of T; and
(iii) that $z^{\mathrm{top}}(T)$ is in the support of η';
then $g(z^{\mathrm{bot}}(T))$ lies outside of our neighborhood N_1 of S.

Proof. Assume that we have chosen the vector fields to give the reduced condition of this statement. Suppose z is a point on the support of $F\tau$. Then there is a furnished tree T^1 as in Lemma 8.1 such that $z = z^{\mathrm{bot}}(T^1)$ and such that $z^{\mathrm{top}}(T^1)$ is on the support of η'. Let $T := (T^1)'$ be the pruning of T^1 described directly above. It still satisfies (i), i.e. the condition (*) of Lemma 8.1, and by the pruning process it automatically satisfies (ii). Also $z^{\mathrm{top}}(T) = z^{\mathrm{top}}(T^1)$ is on the support of η', so our condition gives that $g(z^{\mathrm{bot}}(T))$ lies outside of N_1. On the other hand,

$$g(z^{\mathrm{bot}}(T^1)) - g(z^{\mathrm{bot}}((T^1)')) \in \mathbb{R}_{\leq 0}, \quad g(z^{\mathrm{bot}}(T)) - g(z^{\mathrm{top}}(T)) \in \mathbb{R}_{\leq 0}.$$

Thus $g(z) = g(z^{\mathrm{bot}}(T^1))$, then $g(z^{\mathrm{bot}}(T)) = g(z^{\mathrm{bot}}((T^1)'))$, and then $g(z^{\mathrm{top}}(T))$ lie in order on a line segment parallel to the real axis. Given that $g(z^{\mathrm{top}}(T)) \in N_1$ but $g(z^{\mathrm{bot}}(T)) \notin N_1$, and that N_1 is a convex, we obtain $g(z) \notin N_1$ as desired. □

Remark: The condition of the lemma will not be possible, of course, when the segment S passes through a turning point. Finding out the conditions on S to make it possible will tell us where the turning points are.

9. Remoteness of points

One of the important facets of the statements of theorems 6.3 and 1.1 is the local finiteness of the set of singularities. We describe here briefly how this works. It reproduces the discussion of [34], but with considerable simplification due to Condition 7.1 (iv) which says that when $i = j$ the flow $f_{ij}(z, t)$ is constant.

It should be noted that the local finiteness notion 6.2 is fairly strong in that one can wind arbitrarily many times around a given singularity for an arbitrarily small cost in terms of length of the path. In our mechanism, this is achieved by analytically continuing along a large number of very small segments.

We can choose a metric $d\sigma$ on Z^* (and which is a singular but finite metric on Z) with the property that for any distinct pair of indices

$i \neq j$, if $\xi : [0,1] \to Z$ is a path whose derivative is a negative real multiple of $\mathbf{grad}\, \Re g_{ij}$ then

$$\int_{\xi} d\sigma \leq \Re(g_{ij}(\xi(0)) - g_{ij}(\xi(1))).$$

Now suppose $z = (z_i) \in Z_I$, and suppose T is a binary planar tree embedded in Z, with one top vertex at p and whose bottom vertices are the z_i. Let

$$\mathbf{r}_T(z) := \int_T d\sigma$$

be the total length of the tree with respect to our metric. Define the *remoteness* $\mathbf{r}(z)$ to be the infimum of $\mathbf{r}_T(z)$ over all such trees.

Lemma 9.1. *Suppose T is a furnished tree, and use flows defined by vector fields satisfying Condition 7.1 to define $z^{\mathrm{bot}}(T)$. Then*

$$\mathbf{r}(z^{\mathrm{bot}}(T)) \leq \mathbf{r}(z^{\mathrm{top}}(T)) + g(z^{\mathrm{top}}(T)) - g(z^{\mathrm{bot}}(T)).$$

Proof. If T^1 is any tree as in the definition of remoteness for $z^{\mathrm{top}}(T)$ then we can add T to T^1 (the top vertices of T being the same as the bottom vertices of T^1) to obtain a tree T^2 as in the definition of remoteness for $z^{\mathrm{bot}}(T)$. The formula

$$\mathbf{r}_{T^2}(z^{\mathrm{bot}}(T)) \leq \mathbf{r}_{T^1}(z^{\mathrm{top}}(T)) + g(z^{\mathrm{top}}(T)) - g(z^{\mathrm{bot}}(T))$$

is immediate from Lemma 8.2 and the property of $d\sigma$; use Condition 7.1 (iv) to deal with edges of T having $i_e = j_e$. □

Lemma 9.2. *Let γ be a path from p to q, which leads to the original pro-chain η appearing in Lemma 6.1. Suppose M_0 is the length of γ in the metric $d\sigma$. Then for any point z on the support of η we have $\mathbf{r}(z) \leq M_0$.*

Proof. For any point z on the support of η, we have $z_i = \gamma(t_i)$ for $t_1 \leq \ldots \leq t_k \leq 1$. The path γ can be considered as a tree (of total length M_0) starting at p with one spine and k edges of length 0 coming off at the points z_i. □

In our procedure for analytic continuation along a path of length $\leq M$, we obtain chains whose support consists only of points with $\mathbf{r}(z) \leq M_0 + 2M$ (see §13 below). In particular each z_i is at distance $\leq M_0 + 2M$ from p with respect to $d\sigma$. Thus everything we do takes place in a relatively compact subset of Z (and concerns only a finite number of singular points $P \in Z$).

10. Calculations of gradient flows

We express the gradient of the real part of a holomorphic function, as a vector field in a usual coordinate and in logarithmic coordinates. This is of course elementary but we do the calculation just to get the formula right. Suppose z is a coordinate in a coordinate patch on X. The metric on X may be expressed by the real-valued positive function

$$h(z) := \frac{|dz|^2}{2}.$$

Write $z = x + iy$. Note that dx and dy are perpendicular and have the same length, so

$$h(z) = |dx|^2.$$

The real tangent space has orthogonal basis

$$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$$

and the formula

$$1 = |\frac{\partial}{\partial x} \cdot dx| = h^{1/2} |\frac{\partial}{\partial x}|$$

yields

$$|\frac{\partial}{\partial x}| = h^{-1/2}.$$

In particular an orthonormal basis for the real tangent space is given by

$$\{h^{1/2}\frac{\partial}{\partial x}, h^{1/2}\frac{\partial}{\partial y}\}.$$

Thus we have the formula, for any function a:

$$\mathbf{grad}\, a = h\frac{\partial a}{\partial x}\frac{\partial}{\partial x} + h\frac{\partial a}{\partial y}\frac{\partial}{\partial y}.$$

Now suppose $g = a + ib$ is a holomorphic function (with a, b real), and pose $f(z) := \frac{\partial g}{\partial z}$ so that $dg = f(z)dz$. Write $f(z) = u + iv$ with u, v real, and expand:

$$(u + iv)(dx + idy) = \frac{\partial a}{\partial x}dx + \frac{\partial a}{\partial y}dy + i\frac{\partial b}{\partial x}dx + i\frac{\partial b}{\partial y}dy.$$

Comparing both sides we get

$$u = \frac{\partial a}{\partial x}, \quad v = -\frac{\partial a}{\partial y}.$$

Note that $a = \Re g$ is the real part of g, so finally we have the formula

$$\mathbf{grad}\, \Re g = h(z)\left((\Re\frac{\partial g}{\partial z})\frac{\partial}{\partial x} - (\Im\frac{\partial g}{\partial z})\frac{\partial}{\partial y}\right).$$

Suppose now that w is a local coordinate at a point P, and consider

$$g = a_m w^m.$$

Let $z = -i \log w$ so $w = e^{iz}$, and writing $z = x + iy$ we have $w = e^{ix-y}$. Then

$$g(z) = a_m e^{imz}; \quad \frac{\partial g}{\partial z} = mia_m e^{imz}.$$

If we write $mia_m = e^{r+is}$ then

$$\frac{\partial g}{\partial z} = e^{r-my+i(s+mx)},$$

so

$$\mathbf{grad}\, \Re g = h(z) e^{r-my} \left(\cos(s+mx)\frac{\partial}{\partial x}, \sin(s+mx)\frac{\partial}{\partial y} \right).$$

The *asymptotes* are the values $x = B$ where $\cos(s + mx) = 0$. At these points, the gradient flow vector field is vertical (going either up or down, depending on the sign of $\sin(s + mx)$). If the flow goes up, then it stays on the vertical line until $y = \infty$.

Note that the gradient of $\Re g$ is perpendicular to the level curves of $\Re g$, so it is parallel to the level curves of $\Im g$. Which is to say that the level curves of $\Im g$ are the flow lines. This gives an idea of the dynamics of the flow. We have

$$\Im g = \Im(im^{-1} e^{(r-my)+i(s+mx)}) = -m^{-1} e^{r-my} \sin(s + mx).$$

Thus a curve $\Im g = C$ is given by

$$e^{-my} = \frac{-mC}{e^r \sin(s + mx)}$$

or (noting that the sign of C must be chosen so that the right hand side is positive)

$$y = m^{-1} r \log|\sin(s + mx)| - m^{-1} \log|mC|.$$

In particular the level curves are all vertical translates of the same curve; this curve $y = m^{-1} r \log|\sin(s+mx)|$ has vertical asymptotes at the points where $\sin(s+mx) = 0$. Note however that at the asymptotes, we get $y \to -\infty$; whereas our coordinate patch corresponds to a region $y > y_0$. Thus, every gradient flow except for the inbound (i.e. upward) flows directly on the asymptotes, eventually turns around and exits the coordinate patch. This of course corresponds to what the classical picture looks like in terms of the original coordinate w.

Also we can calculate the second derivative (which depends only on x and not on which level curve we are on, since they are all vertical translates). Consider for example points where $\sin(s+mx) > 0$. There

$$\frac{dy}{dx} = \frac{r\cos(s+mx)}{\sin(s+mx)}$$

and

$$\frac{d^2y}{dx^2} = \frac{-rm}{\sin^2(s+mx)}$$

In particular note that we have a uniform bound everywhere:

$$\frac{d^2y}{dx^2} \leq -\gamma,$$

here with $\gamma = rm$.

Suppose now more generally that g is a holomorphic function with Taylor expansion

$$g = a_m w^m + a_{m+1}w^{m+1} + \ldots.$$

Then we will get

$$h(z)^{-1}e^{my-r}\mathbf{grad}\,\Re g = \left(\cos(s+mx)\frac{\partial}{\partial x}, \sin(s+mx)\frac{\partial}{\partial y}\right) + O(e^{-}y).$$

In particular, the direction of the gradient flow for g is determined, up to an error term in $O(e^{-y})$, by the vector $(\cos(s+mx), \sin(s+mx))$.

The asymptotes are no longer vertical curves, but they remain in bands $x \in B_{ij,a}$. Also we can choose A in the definition of steepness, so that at non-steep parts of the level curves we still have a bound

$$\frac{d^2y}{dx^2} \leq -\gamma.$$

11. Choice of the vector fields W_{ij}

The only thing left to be determined in order to fix our procedure for moving the cycle of integration is to choose the vector fields. Before going further, fix a smooth metric h on Z, for example coming from the pullback of a smooth metric on Y. Use this to calculate gradients. Suppose ϵ is given. Let ρ denote a cutoff function which is identically 0 in the discs $D_{\epsilon/2}(P)$ (for all points P in the inverse image of $\mathcal{R}$), and is identically 1 outside the (closed) discs $D_\epsilon(P)$. Of course ϵ will be small enough that the discs don't intersect. Consider also a positive real constant $\mu \in \mathbb{R}_{>0}$. Then we put

$$W_{ij} := \mu\mathbf{grad}\,\Re g_{ij},$$

and

$$W'_{ij} := \rho W_{ij}.$$

The vector fields W'_{ij} satisfy Condition 7.1 (with $\epsilon/2$ in place of ϵ). We will use these vector fields for our choice of flows, and apply the criterion of Lemma 8.3.

The point we want to make in the present section is that the flow curves for the cut-off gradient vector field W'_{ij} are the same as those of the true gradient flow along W_{ij}, up until any point where they enter some $D_\epsilon(P)$. This will allow the notational simplification of looking at W_{ij} rather than W'_{ij} in the next section.

Let $\nu > 0$ be the radius used to define the oval neighborhood N_1, i.e. choose N_1 equal to the set of points of distance $< \nu$ from S. Once ϵ is given, choose μ large enough so that the following property holds:

Condition 11.1. *If $z(t) = f_{ij}(z_0, t)$ is a flow curve for W_{ij} (for distinct indices $i \neq j$) which never enters into any $D_\epsilon(P)$ flowing for $t \in [0, s]$ with $s \geq 1$, then*

$$g_{ij}(f_{ij}(z_0, s)) - g_{ij}(z_0) < \zeta_1 - \zeta_0 - 2\nu.$$

Recall that ζ_0, ζ_1 were the endpoints of the segment S with $\zeta_1 - \zeta_0$ a negative real number.

It is possible to choose μ (we only need to do it over a relatively compact subset of $z_0 \in Z$ by the remark at the end of §9, but in any case everything involved is pulled back from the compact Y so the choice of μ is uniform in z_0).

The next lemma formalizes the following reduction: the trees which show up in Lemma 8.3 have a horizontal line of edges assigned length 1. If the flow for at least one of these edges stays outside of all the $D_\epsilon(P)$ then by Condition 11.1 the value of g is decreased sufficiently to get us out of N_1. Thus the only case which poses a problem is when every downward branch of the tree ends up flowing into some $D_\epsilon(P)$. In this case we prune the tree at the points where it enters these discs.

Lemma 11.2. *Suppose ϵ is given, and μ chosen to satisfy Condition 11.1. Use the vector fields W'_{ij} to define the flows. In order to show that the image $g(F\tau)$ misses our neighborhood N_1 of the segment S, it suffices to show that if T is any furnished tree satisfying the following conditions:*
(i) that $z^{\rm top}(T)$ lies on the support of η';
(ii) that $i_e \neq j_e$ for any edge of T;
(iii) that for each bottom vertex v of T there is a singular point $P(v)$ such that $\Phi(v) \in D_\epsilon(P(v))$; and

(iv) that all other points of $\Phi(T)$ *are outside the discs* $D_\epsilon(P)$*,*
then $g(z^{\rm bot}(T))$ *is not in the neighborhood* N_1 *of* S*.*

Proof. Suppose T is a furnished tree as in the reduction of Lemma 8.3. Prune T at any point where the flowing map Φ enters into one of the closed discs $D_\epsilon(P)$. If this prunes all branches of the tree, then by an argument using 8.2 similar to the previous reductions, that puts us in the case described here so we are done.

Thus we may assume that there is at least one branch which is not pruned. By condition 8.3 (i) which is the same as Condition (*) of Lemma 8.1, the branch going to the bottom has at least one edge assigned length 1. This edge has $i_e \neq j_e$. By Condition 11.1 we have for this edge

$$\int_{\Phi(e)} dg_{i_e j_e} < \zeta_1 - \zeta_0 - 2\nu.$$

Therefore, by the formula of Lemma 8.2 we have

$$g(z^{\rm bot}(T)) - g(z^{\rm top}(T)) < \zeta_1 - \zeta_0 - 2\nu.$$

Given that $g(z^{\rm top}(T)) \in g(\eta') \subset N_1$ but N_1 is an oval with largest diameter $2\nu + \zeta_0 - \zeta_1$, we get $g(z^{\rm bot}(T)) \notin N_1$. □

Corollary 11.3. *Define the chain* $F\tau$ *using the vector fields* W'_{ij}*. Then, in order to show that* $g(F\tau)$ *misses* N_1 *it suffices to show that for any furnished tree* T *satisfying the conditions (i)-(iv) of 11.2 with respect to the flowing map* Φ *defined by the vector fields* W_{ij} *(rather than* W'_{ij}*), we have* $g(z^{\rm bot}(T)) \notin N_1$*.*

Proof. The two flowing maps coincide, in view of condition (iv). □

In view of this corollary, we can in the next section ignore the cutoff functions ρ and look directly at the gradient flows W_{ij}.

12. Results on the dynamics of our flowing maps

We will consider a system of discs centered at our singular points P:

$$D_\epsilon(P) \subset D_\xi(P) \subset D_u(P) \subset D_w(P).$$

We will first fix u and w so that certain things are true in a coordinate system for $D_w(P)$ (and say $u = w/2$). Then once u and w are fixed we will let $\epsilon \to 0$. Finally $\xi > \epsilon$ will be a function of ϵ with $\xi \to 0$ when $\epsilon \to 0$.

The innermost discs $D_\epsilon(P)$ are those which will enter into the reduction of Lemma 11.2. Recall that μ is chosen after ϵ. In view of the Corollary 11.3, we henceforth look directly at the gradient flows $W_{ij} = \mu \mathbf{grad}\, \Re g_{ij}$.

Our first lemma bounds the number of outgoing subtrees.

Lemma 12.1. *If T is a furnished tree with one top edge e, and if $\Phi : T \to X$ is a flowing map such that the images of all bottom vertices are contained in some $D_\epsilon(P_i)$, and if $\Phi(e)$ exits from $D_u(P)$ then T contains a strand σ such that $\Phi(\sigma)$ exits from $D_w(P)$ also.*

Our next lemma gives a normal form for any subtree which stays entirely within $D_u(P)$.

Lemma 12.2. *If T is a furnished tree with one edge e at the top, and if Φ is a flowing map from T into $D_u(P) \subset X$ such that all of the bottom vertices are mapped into $D_\epsilon(P)$, then the curve $\Phi(e)$ passes into $D_\xi(P)$, and flows along a vector field $W_{i_e j_e}$ in an ingoing sector near an ingoing curve $G_{i_e j_e}$.*

The last of our preliminary lemmas bounds the number of subtrees having the previous normal form.

Lemma 12.3. *There is a number K (depending on u, w, A but independent of ϵ, ξ and μ) such that if T is a furnished tree consisting of one edge strand κ plus a number of sub-trees coming out of κ, and if Φ is a flowing map from T into $D_u(P)$ with the property that all the sub-trees coming out of κ are covered by Lemma 12.2, then there are $\leq K$ of these sub-trees.*

For the proofs of these lemmas, we will use a logarithmic coordinate system for $D_w(P)$. If z_D denotes the coordinate in the disc then we introduce $z_L = -i \log z_D$ and write $z_L = x + iy$ $z_D = e^{ix-y}$.

The disc $D_w(P)$ is given by $y > y_0$.

The vector fields $W_{ij} = \mu \mathbf{grad}\, \Re g_{ij}$ are approximately equal (up to a term smaller by a factor of $O(\mu e^{-y})$) to the standard vector fields $W'_{ij} = \mu \mathbf{grad}\, \Re g'_{ij}$ where g'_{ij} is the leading term in the Taylor expansion for g_{ij} at P.

Because of this, we obtain the following facts. The asymptotic directions (which are close to vertical lines) occur in bands of the form $x \in B_{ij,a}$ where $B_{ij,a} \subset \mathbb{R}$ are intervals which can be made as small as we like by modifying y_0. These intervals are disjoint, except for the asymptotes of the pairs $\{W_{ik}, W_{jk}\}$ or $\{W_{ki}, W_{kj}\}$, where i, j are the two indices attached to P, and k is any index different from these two. In those cases the pairs share the same values $B_{ij,a}$ and the same bands. We say that a vector field W_{ij} is *attached* to an interval B if $B = B_{ij,a}$. The only intervals with more than one vector field attached to them are those described above.

It is worth mentionning why we have this disjointness property. It is because of the non-parallel condition on the eigenforms of θ at the singular points. The non-parallel condition implies that the bands, which are the solutions of $s + mx = 0$ modulo π, are distinct, because the values of s (which are the angular coordinates of the constants attached to the leading terms of g_{ij} as explained in the preceding section) are different exactly because of it. Notice that the exponents m are the same for all of the values ij except the two attached to the singular point; for those which are attached the value m' is bigger. The non-parallel condition gives disjointness for all of the bands except the ones corresponding to the attached indices ij and ji. For those, note that if we make a general rotation of everything, the asymptotic solutions of $s + m'x$ move differently than the solutions of $s + mx$, so those bands are disjoint from all the other ones. The general rotation of everything corresponds to a condition that the line segments in the complex plane along which we analytically continue, might be constrained to not be parallel to a certain finite number of directions. This doesn't hurt our ability to analytically-continue the function.

We can fix a number $A > 0$ with the following properties: outside of an asymptotic band for W_{ij} or W_{ji}, the slope of the vector W_{ij} satisfies

$$\left| \frac{dy}{dx}(W_{ij}) \right| \leq A.$$

Inside an asymptotic band B, only the vector fields W_{ij} which are attached to B can have slope bigger than A or less than $-A$.

Suppose now that $(x(t), y(t))$ is a flow along one of the vector fields W_{ij}. We say that the path is *steep* if

$$\left| \frac{dy}{dx} \right| > A,$$

and we say that it is *not steep* otherwise. We say that the path is *ingoing* if $\frac{dy}{dx} > 0$ and outgoing otherwise. Note that with our logarithmic coordinate system, outgoing is downward and ingoing is upward. The coordinate patch (i.e. choice of y_0) and the choice of A can be made so that all of the paths satisfy the following property:

—once the path is steep and outgoing, it remains steep and outgoing for the remainder of the time of definition, and ends up leaving the region $y > y_0$.

This is true even though the vector field is not exactly equal to the standard model but only close to it.

On the other hand, the direction, i.e. the sign of $\frac{dx}{dt}$ remains the same throughout the interval where the path is not steep. Call this sign

$(-1)^m$. In particular we can think of the path as being parametrized by x. Define the *slope* to be the signed derivative $(-1)^m \frac{dy}{dx}$.

We have a bound, in the region where the path is not steep:

$$\frac{d^2y}{dx^2} \leq -\gamma$$

with $\gamma > 0$ a positive constant. Note that the second derivative is also the variation of the slope with respect to x when we go in the direction of the path.

In particular, once the path is outgoing it remains outgoing for the remainder of its period of definition. This is because of the second derivative when it is not steep, and the fact that when it becomes steep and outgoing then it stays that way.

We now note the *additive relation* for the vector fields at vertices of a tree.

Lemma 12.4. *Suppose we are in the situation of a flowing map* $\Phi : T \to X$ *defined by vector fields* $W_{ij} = \mu \mathbf{grad} \Re g_{ij}$. *At any vertex* v *of* T *with edges noted* e_1, e_2, e_3 *(say* e_1 *ingoing and* e_2, e_3 *outgoing), we have three indices* i, j, k *such that*

$$i_{e_1} = i_{e_2} = i;\ j_{e_2} = i_{e_3} = j;\ j_{e_1} = j_{e_3} = k.$$

For the three vector fields W_{ik}, W_{ij}, W_{jk} *corresponding to the edges* e_1, e_2, e_3 *we have the relation*

$$W_{ik}(\Phi(v)) = W_{ij}(\Phi(v)) + W_{jk}(\Phi(v)).$$

Proof. The vector fields W_{ij} are all the same multiple of the gradients $\mathbf{grad} \Re g_{ij}$. The fact that $dg_{ij} = \alpha_i - \alpha_j$ implies that $dg_{ij} + dg_{jk} = dg_{ik}$ giving the relation in question. □

Proof of Lemma 12.1. The disc $D_u(P)$ will be determined by $y > y_1$ for some y_1 fixed as a function of y_0 (and in fact one could take $y_1 = y_0 + 1$ for example). A consequence of the additive relation is that if $W_{ik}(\Phi(v))$ is outgoing (i.e. $\frac{dy}{dt} \leq 0$ along this vector) then one of the other two $W_{ij}(\Phi(v))$ or $W_{jk}(\Phi(v))$ will also be outgoing. As we have noted above, if the flow along any edge is outgoing at some point then it is outgoing for all further points. In particular if at any point in the tree the flow is outgoing then we can choose a strand going down to the bottom, along which the flow is always outgoing. If there is an edge which crosses out of $D_u(P)$, at the crossing point it has $\frac{dy}{dt} \leq 0$, so we get a strand which maintains $\frac{dy}{dt} \leq 0$ as long as it stays inside $D_w(P)$. In particular the strand cannot go back to $D_\epsilon(P)$ so it must exit from $D_w(P)$ (here using the hypothesis that any strand must end in some $D_\epsilon(P_i)$). This completes the proof of Lemma 12.1. □

Now we come to the proofs of Lemmas 12.2 and 12.3. Fix notations $L := -\log \epsilon$ and $L_1 := -\log \xi$. Thus we will let $L \to \infty$ and we have to specify L_1 as a function of L such that $L_1 \to \infty$ too. Our discs $D_\epsilon(P)$ and $D_\xi(P)$ respectively become the regions $y > L$ and $y > L_1$. We will specify L_1 as a function of L so as to make the proofs of Lemmas 12.2 and 12.3 work.

In both lemmas, we lift the maps Φ into maps into the coordinate chart for the logarithmic coordinates.

Proof of Lemma 12.2. At any point where the flow is not steep, the second derivative is bounded above by $-\gamma$. In particular the flow becomes outgoing before it becomes steep again. Furthermore, if v is a vertex with indices i, j, k as above, such that the vector field $W_{ik}(\Phi(v))$ is not steep but is ingoing, then the additive relation insures that one of the other two flows $W_{ij}(\Phi(v))$ or $W_{jk}(\Phi(v))$ has slope less than or equal to the slope of $W_{ik}(\Phi(v))$. For this, draw a line through the first vector, and note that one of the two other vectors has to lie below or on the line. Note that this gives two cases: either the new vector changes direction (i.e. the sign $(-1)^m$ changes) and the new vector is in fact outgoing; or else the direction stays the same and the slope decreases. Thus if t_0 is any point in T where the flow is ingoing but not steep, then we can choose a strand σ below t_0 with the property that at the end of the strand the flow becomes outgoing; and along the strand the direction stays the same and the second derivative satisfies

$$\frac{d}{dx}((-1)^m \frac{dy}{dx}) \leq -\gamma$$

in a distributional sense. Then (noting by $x(t), y(t)$) the coordinates of the image point $\Phi(t)$ for $t \in \sigma$) we have

$$y(t) \leq y(t_0) + A(-1)^m(x(t) - x(t_0)) - \frac{\gamma}{2}(x(t) - x(t_0))^2$$

for any $t \geq t_0$. In particular there is a number N such that

$$y(t) \leq y(t_0) + N$$

further along the strand. We will choose $L_1 = L - N$.

Recall now that in the hypotheses of the lemma, we suppose that all strands in the tree remain inside $D_u(P)$ and also finish in $D_\epsilon(P)$. However, we construct above a strand which eventually becomes outgoing; therefore the strand must enter the region corresponding to $D_\epsilon(P)$ before it becomes outgoing (and notice also that it could simply stop inside this region before becoming outgoing, a case not mentionned above). In particular, if there is any point t_0 corresponding to a non-steep ingoing flow, or of course to any sort of outgoing flow, then we have to have $y(t_0) + N > L$ or $y(t_0) > L_1$.

Now we can complete the proof of the lemma. If v is any vertex, such that the incoming edge is steep and ingoing, then one of the two outgoing edges has to be either non-steep and ingoing, or outgoing. This is verified from the fact that at most two different vector fields can be attached as ingoing asymptotic vector fields for the same band B. From what was said above, the bottom vertex of the first edge e of the tree must satisfy $y(\Phi(v)) > L_1$, in other words the first edge continues all the way until $D_\xi(P)$. Also the part of the edge e which is outside of $D_\xi(P)$ must be contained in an ingoing asymptotic band for its vector field W_{ij} and the flow is steep at all points of $\Phi(e)$ which are outside of $D_\xi(P)$. This completes the proof of Lemma 12.2. $\square$

Proof of Lemma 12.3. Consider a vertex v along κ where a subtree in the normal form of Lemma 12.2 comes off. Use the same notation as previously for the edjes and indices adjoining v. For the sake of simplicity we assume that κ corresponds to the two leftmost edges e_1 and e_2 at v. The upper edge of the subtree is thus e_3 with indices jk.

Note from the proof of 12.2 that W_{jk} is ingoing and steep at $\Phi(v)$.

As a first case, note that if the anterior edge e_1 of κ has W_{ik} which is outgoing and steep, then the subsequent edge e_2 of κ is also outgoing and steep. In particular at any point where κ becomes outgoing and steep, it remains that way and in fact will leave the region $y > y_1$ before it goes into any other band B. By looking at the possible combinatorics of the indices one sees, even in the case of two vector fields sharing the same band, that there can be no further normal-form vertices on κ.

In view of the previous paragraph we may restrict our attention to the places where κ is either not steep, or else steep but ingoing. However, if it is steep but ingoing then again at most one vertex with a normal-form subtree can correspond to the current band; thus at some point κ leaves this band and must become non-steep. On the other hand, once κ is non-steep, it doesn't change to become steep and ingoing. It doesn't do this in the middle of an edge, because of the second derivative condition. It doesn't do it at a vertex because the edge e_3 which comes off is steep and ingoing, and a W_{ik} which is not steep couldn't be the sum of two steep and ingoing vectors.

The two previous paragraphs show that we may (at the price of at most two extra normal-form subtrees) restrict our attention to the region where κ is non-steep. Now one sees again from the additive relation that if W_{ik} and W_{ij} are non-steep, whereas W_{jk} is steep and ingoing, then the directions of W_{ik} and W_{ij} must be the same. Indeed, if not then we would have $W_{jk} = W_{ik} + (-W_{ij})$ which would be a sum of two vectors in the same non-steep quadrant, so W_{jk} in a steep quadrant would be impossible.

Since the sign $(-1)^m$ of $\frac{dx}{dt}$ doesn't change, we can use x to parametrize κ. Furthermore the slope $(-1)^m \frac{dy}{dx}$ is decreasing along κ (note that at any vertices where a subtree in normal form comes off, the remaining outgoing edge of κ has a smaller slope than the ingoing edge, because of the additive relation).

In other words, the second derivative is distributionally less than the constant $-\gamma$, so at some time t with $|x(t) - x_0| \leq 2A/\gamma$ we get to $(-1)^m \frac{dy}{dx} \leq -A$, i.e. κ becomes steep and outgoing. We get that the non-steep part of the path κ is parametrized by an interval in the x-coordinate, of length $\leq 2A/\gamma$. There is a bound K so that such an interval can cross (or go near) at most $K-2$ asymptotic bands. A band is attached to at most two pairs of indices, but only one of these can lead correspond to a normal-form subtree. Thus (counting the two we may have missed above) the number of normal-form subtrees attached to γ is $\leq K$. This completes the proof of Lemma 12.3. ☐

We now come to the main result of this section. Fix u, w as above, and let $L_1 := L + N$ be the function determined by the above proofs. For any ϵ put $L := \log \epsilon$ and set $\xi := e^{L_1} = e^N \epsilon$. Note that $\xi \to 0$ as $\epsilon \to 0$.

Theorem 12.5. *There is a bound K depending on u, w and a real constant F (which will be $\zeta_0 + 2\nu - \zeta_1$ later on), but with K independent of ϵ, ξ and μ, with the following properties. Suppose T is a furnished tree and $\Phi : T \to X$ is a flowing map such that the top vertices are outside of any $D_w(P_i)$ and such that the bottom vertices are each mapped into some $D_\epsilon(P_i)$. Suppose furthermore that $g(z^{\mathrm{bot}}) \geq g(z^{\mathrm{top}}) - F$. Then we can cut T into a tree T' onto which are attached subtrees, such that Φ maps the bottom vertices of T' into various $D_\xi(P_i)$ and such that the number of bottom vertices of T' is bounded by K.*

Proof. Among the subtrees that we strip off are any ones starting with edges e for which $i_e = j_e$. In particular we may assume from the start that T has no such edges.

Next group the bottom vertices into series connected by intervals where the bounding loop of the interval is mapped into $D_u(P)$. There is a bound K_1 for the number of such series, because any loop which goes out of $D_u(P)$ has to contribute at least a certain fixed amount to $g(z^{\mathrm{top}}) - g(z^{\mathrm{bot}})$. Next we can look at a specific series. It is the set of bottom vertices of a subtree T_1 obtained by taking the union of all of the loops joining the bottom vertices together. Note in particular that $\Phi(T_1) \subset D_u(P)$. Let κ denote the boundary path of T_1. Note that the subtree T_1 doesn't necessarily include all strands emanating from all of its vertices. However, if v is a vertex on κ corresponding to an adjoining edge e not in κ, then either e goes into the interior of the region bounded by κ, in which case e starts a subtree mapped into $D_u(P)$ and such that all bottom edges go into $D_\epsilon(P)$; or else it goes out of the region bounded by κ in which case e is not a part of the tree T_1. In the former case, the normal form of Lemma 12.2 applies to the subtree starting at e. In the latter case, the subtree starting at e could be in normal form or not. However, if e is an edge going out of κ such that the subtree starting at e is not in the normal form of Lemma 12.2, then this subtree contains at least one strand which goes out of $D_u(P)$. By Lemma 12.1 it also contains a strand which goes out of $D_w(P)$ and there is a global bound K_2 on the number of such edges e. If we cut κ at vertices v where such edges e go out, then it is cut into $\leq K_2$ strands κ' and each little strand has only vertices corresponding to normal-form subtrees. Finally, by the bound of Lemma 12.3 there are no more than K_3 such vertices on each little strand κ'. Each of these normal-form subtrees can be cut at the point where it goes into $D_\xi(P)$, and there is only one such point for each subtree. Thus if we trim off the tree T_1 at all of the points where the strands enter $D_\xi(P)$, there are at most K_2K_3 bottom vertices. Finally, since there were at most K_1 subtrees T_1 corresponding to series of bottom vertices, we can trim off T to a tree T' where there are at most $K_1K_2K_3$ bottom vertices, all going inside some $D_\xi(P_i)$. This proves the theorem. □

13. Proofs

By a *multisingular point* we mean a point $y = (y_1, \ldots, y_k) \in Z_I^*$ such that the y_n are singular points of the functions $g_{i_n i_{n+1}}$. Note that in our situation the singular points in Z are the preimages of the turning points $P \in \mathcal{R}$ corresponding to the places where the Higgs field θ has singular eigenvalues.

If z is a point in Z_I^* with $\mathbf{r}(z) \leq M$ then in particular each z_i is at distance $\leq M$ from p with respect to $d\sigma$ (using the notations of §9). This defines a relatively compact subset of Z, containing a finite number of singular points. It is improved with the lemma below.

Define S_M to be the set of complex values of the form $g(y)$ where y are multisingular points with $\mathbf{r}(y) \leq M_0+2M$. This is the subset which is to enter into the definition of analytic continuation with locally finite branching for $f(\zeta)$.

Lemma 13.1. *For each M, the set S_M is finite.*

Proof. There is a positive constant c such that if P_1 and P_2 are distinct singular points, then the distance from P_1 to P_2 using the metric $d\sigma$ is at least c. Suppose $y = (y_1, \ldots, y_k)$ is a multisingular point, so each y_i is a singular point. If $\mathbf{r}(y) \leq M_0 + 2M$ then, in view of the definition of $\mathbf{r}$ there are at most $(M_0 + 2M)/c$ indices i such that $y_i \neq y_{i+1}$. Let $y' := (y'_1, \ldots, y'_{k'})$ be the sequence of distinct different points in the sequence y. Define a new multi-index I' by setting $i'_a = i_{b(a)}$ where $b(a)$ is the place with $y_{b(a)} = y'_a$ and $y_{b(a)+1} = y'_{a+1}$. Then $y' \in Z_{I'}$ and $g_{I'}(y') = g_I(y)$. Now $k' \leq (M_0 + 2M)/c$ so there are only a finite number of possibilities for y' (the singular points themselves occuring in a fixed relatively compact subset of Z as pointed out above). Thus there are only a finite number of possible values. □

Proof of Theorem 6.3. Suppose we have already analytically continued f along a piecewise linear path of length $\leq M_1$. Inductively we may assume that the points of η have remoteness $\leq M_0 + 2M_1$. If we add a segment S then the total length of the path is $\leq M$ where $M = M_1 + |S|$. We assume that S doesn't meet any of the points in S_M.

Fix a number $\nu > 0$ so that the segment S stays at a distance $> 2\nu$ away from the points of S_M. Choose our neighborhoods N_i with N_1 being the oval around S of radius ν, so N_1 stays at a distance $> \nu$ away from the points of S_M. Let K be the bound of Theorem 12.5. Choose ϵ small enough so that if $z \in D_\xi(P)$ then for any i

$$\left| \int_P^z \alpha_i \right| < \frac{\nu}{K}.$$

We show that all points of the chain $F\tau$ are sent (by g) outside of N_3. Suppose on the contrary that we had a point, corresponding to a tree T, such that $g(z^{\mathrm{bot}}(T)) \in N_3$.

By Theorem 12.5 there exists a pruning T' of T with $\leq K$ bottom vertices, such that for every bottom vertex v of T' we have $\Phi(v) \in D_\xi(P(v))$ for some singular point $P(v)$. In particular, the point $z^{\mathrm{bot}}(T')$ which is the vector of these $\Phi(v)$ is near to a point $y = (\ldots, P(v), \ldots)$. More precisely we obtain from $k \leq K$ and the bound above,

$$|g(y) - g(z^{\mathrm{bot}}(T'))| < \nu.$$

On the other hand, if $g(z^{\mathrm{bot}}(T'))$ were inside N_3 then the singular point y would occur below points of η at distance $\leq |S|$, and hence below points of p at distance $\leq M_0 + 2M_1 + |S| < M$, therefore $g(y)$ must be included in S_M. On the other hand, the point $g(z^{\text{bot}}(T'))$ occurs on the real segment between $g(z^{\text{bot}}(T))$ and some point of $g(\eta')$. This contradicts the assumption that the neighborhood N_1 stays away from S_M by distance at least ν. This shows that all points of $g(F\tau)$ are outside of N_3, and completes the proof that we can analytically continue $f(\zeta)$ along the segment S.

Finally in order to maintain the inductive hypothesis we note that, cutting everything off fairly close to the segment S we can insure that the points of the new cycle of integration $F\tau$ (and also $F\psi$) are remote from points of η at distance $\leq 2|S|$, hence they have remoteness $\leq M_0 + 2M$ as required. □

Proof of Theorem 1.1. — The statement is essentially contained in that of Theorem 6.3, but we need to show that $S_M \subset \Sigma(\gamma)$. In other words, if $z \in Z_I$ is a multisingular point, we need to show that $g(z)$ is the integral of the tautological form on a piecewise homotopy lifting $\tilde{\gamma}$. Recall the formula

$$g_I(z) = \int_p^{z_1} \alpha_{i_0} + \ldots + \int_{z_k}^{q} \alpha_{i_k}.$$

Let $\tilde{\gamma}'_i$ be the path joining z_i to z_{i+1} where by convention $z_0 = p$ and $z_{k+1} = q$. These paths are unique up to homotopy because we are working in the contractible universal cover Z. Composing the main projection $Z \to Y$ with Galois automorphisms of Y and then the projection $Y \to V$, gives projections $\tau_i : Z \to V$ which commute with the projection to X, such that α_i is the pullback of the tautological form α on V, i.e. $\alpha_i = \tau_i^*(\alpha)$. We can put $\tilde{\gamma}_i := \tau_i \circ \tilde{\gamma}'_i$. The collection $\tilde{\gamma} = \{\tilde{\gamma}_i\}$ is a piecewise homotopy lifting of γ. To see this, note that the projections to X of the $\tilde{\gamma}_i$ are equal to the projections of the original $\tilde{\gamma}'_i$, so these join together to give a path homotopic to the projection of the path from p to q in Z. Since the lifts $p, q \in Z$ were chosen to correspond to our original path γ in X, so the composite path in X is homotopic to γ. Our formula for $g_I(z)$ becomes

$$g_I(z) = \sum_i \int_{\tilde{\gamma}_i} \alpha = \int_{\tilde{\gamma}} \alpha.$$

This shows that S_M is a subset of $\Sigma(\gamma)$. □

14. Conclusion

We close with a few more general remarks about the consequences of Theorem 6.3. The first is to note that it also applies to any polynomials in the transport matrix coefficients, in particular to the Procesi coordinates for M_B.

Lemma 14.1. *Suppose f_1 and f_2 have analytic continuations with locally finite branching, then the same is true for their convolution $f_1 * f_2$.*

Proof. This was proven in [9]. See also the proof of [34] Lemma 11.1. There, the proof of locally finite branching for the convolutions uses only locally finite branching for the two functions. □

Corollary 14.2. *If $P(t)$ is a polynomial in the transport matrices for various paths, then then the Laplace transform of $P(t)$ has an analytic continuation with locally finite branching.*

Proof. The Laplace transform of a product of functions $m_1(t)m_2(t)$ is the convolution of their Laplace transforms, so Lemma 14.1 and Theorem 6.3 give the result. □

The next remark is about the growth rate of $m(t)$. This is measured by the hull **hull** (m) defined in the introduction.

For reference we indicate first an elementary argument showing that **hull** (m) is convex. Indeed, if ζ_0 is a point which is not in **hull** (m), then by definition there is an angular sector **s** in which $m(t)e^{-\zeta_0 t}$ is rapidly decreasing. Suppose u is a complex number such that $\zeta_0 + u$ is in **hull** (m). Again by the definition of **hull** (m) this implies that $m(t)e^{-\zeta_0 t}e^{-ut}$ is no longer rapidly decreasing in any part of **s**. This means that **s** is contained in the half-plane $\Re ut \leq 0$. In particular, for any vector u' which is a negative real multiple of u, we have that $\Re u't \geq 0$ so $m(t)e^{-\zeta_0 t}e^{-u't}$ is rapidly decreasing on **s**, therefore $\zeta_0 + u'$ is not in **hull** (m). This proves the convexity.

Next we can characterize **hull** (m) as the intersection of all closed half-planes $H \subset \mathbb{C}$ such that the Laplace transform f of m admits an analytic continuation over the complementary open half-plane (this would give another proof of convexity). Indeed, if a point ζ is in the complement of **hull** (m) then the sector along which $m(t)e^{-\zeta t}$ is rapidly decreasing provides an open half-plane containing ζ over which f can be analytically continued. This shows one inclusion. The other inclusion is clear from the inverse Laplace transform.

The hull is related to growth rates as follows. If **hull** (m) is a single point, then some multiplicative translate of the form $m(t)e^{\zeta t}$ has subexponential growth. If **hull** (m) contains at least a line segment, then we say m is *semistrictly exponential*: for sectors covering all but two directions we have a lower bound of the form $|m(t)| \geq ce^{a|t|}$, and in particular there is a positive lower bound for the possible exponents a which can enter into bounds of the form $|m(t)| \leq Ce^{a|t|}$. If $\mathbf{hull}\,(m)$ contains a nonempty interior then we say m is *strictly exponential*: there is a lower bound of the form $|m(t)| \geq ce^{a|t|}$ valid in all directions.

Unfortunately we are only able to show that some monodromy matrix is semistrictly exponential in the generic case of Corollary 1.2 of the introduction.

Proof of Corollary 1.2. By Theorem 1.1, the Laplace transform has locally finite branching (Definition 6.2). Choose M big enough so that one goes all the way around $\mathbf{hull}\,(m)$ with a path of length $\leq M$. Let $S_M^{\rm real} \subset S_M$ be the subset of non-removable singularities of the Laplace transform attainable by a path of length $\leq M$ (which is finite because S_M is finite). Then f admits an analytic continuation to an open half-plane if and only if this half-plane doesn't meet $S_M^{\rm real}$. Therefore $\mathbf{hull}\,(m)$ is a polygon.

We show by specialization that for some fundamental group elements at least, $\mathbf{hull}\,(m)$ is not reduced to a single point. General considerations using Hartogs' theorem show that if the monodromy is semistrictly exponential for a special curve going to infinity, then the same will be true away from a piecewise holomorphic real codimension 2 divisor.

We choose as special curve the family of connections on the trivial bundle of the form $d + B + tA$ with A diagonal and B off-diagonal, everything being holmorphic on X, that was originally considered in [34]. In that case, we get asymptotic expansions whose coefficients can be calculated. One route is to note that for generic values of A and B, calculation of the coefficients gives nonzero coefficients at more than one singular point. Another route would be to note that if there were only one singularity for the monodromy matrices for this family, then the monodromy representation would actually have polynomial growth. That possibility is ruled out by specializing again to a direct sum of a 2×2 system and trivial systems, and noting that for 2×2 systems we have proven (in the paper [38]) that the monodromy representation always has growth at least $e^{t^{1/k}}$ for some integer k.

In any case by either of these two routes we can conclude that the Laplace transform for at least one monodromy matrix has at least two singularities. □

It is perhaps more interesting to note that the same thing also works for the Procesi coordinates. This improves, at least for certain generic points at infinity approached from certain sectors, the bound given in [38].

Corollary 14.3. *For each family* $(E, \nabla + t\theta)$ *going to infinity at a generic Higgs bundle* (E, θ), *let* ρ_t *denote the family of monodromy representations, thought of as a point in* M_B. *Let* $R_i : M_B \to \mathbb{C}$ *denote a set of Procesi coordinates giving an affine embedding. Write by abuse of notation* $R_i(t) := R_i(\rho_t)$. *Then each* **hull** (R_i) *is a polygon, and for general* (E, θ) *(in a dense open set) at least one* R_i *is semistrictly exponential (i.e. its hull has at least two vertices). If we define* $|\rho_t| := \sup_i |R_i(t)|$ *then for general* (E, θ) *and for a family of sectors of* $t \to \infty$ *covering all but possibly two opposite directions, we have bounds of the form*

$$|\rho_t| \geq ce^{a|t|}$$

with $a > 0$.

Proof. The same proof as for Corollary 1.2 works here too. □

Lastly it is important to reiterate that, in spite of the above consequences, the result of Theorem 6.3 is highly unsatisfactory in that it doesn't say anything about the behavior of the Laplace transform $f(\zeta)$ near the singularities. It doesn't even seem clear what the answer will be: on the one hand one can imagine that an improvement of the present analysis, potentially based on Remark (i) following the proof of Lemma 5.1, might lead to a polynomial bound for the singularities. On the other hand, a crude look at the present argument yields no such bound, and it is also quite concievable that the poles in the matrix B lead unavoidably to more complicated singularities of $f(\zeta)$. This is undoubtedly true in the general case where B has poles of order > 1.

This problem also leads to the unsatisfactory statement of Corollary 14.3: if we could calculate exactly where the singularities were we could probably show that for generic values of (E, θ) the singularities would span a convex hull with nonempty interior, in other words that the monodromy families ρ_t would be strictly exponential. This would be a more significant improvement of the result of [38].

The result of Theorem 6.3 should be thought of as a weak form of "resurgence" for the monodromy function $m(t)$ and its Laplace transform. The problem of getting more precise information about this behaviour is probably most naturally attacked using new ideas and techniques for resummation such as have been developped by the school of J.-P. Ramis.

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