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Pierre Grangé, Ernst Werner

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Pierre Grangé $^{(a)}$ *and Ernst Werner* $^{(b)}$
(a) Laboratoire de Physique Mathématique UMR-CNRS 5825,
Université de Montpellier II,34060 Montpellier-Cedex, France
(b) Institut für Theoretische Physik, Universität Regensburg,
D-93040 Regensburg, Germany

Abstract
In Continuum Light Cone Quantization (CLCQ) the treatment of scalar fields as operator valued distributions and properties of the accompanying test functions are recalled. Due to the paracompactness property of the Euclidean manifold these test functions appear as decomposition of unity. The approach is extended to QED Dirac fields in a gauge invariant way. With such test functions the usual triangle anomalies are calculated in a simple and transparent way.

1. INTRODUCTION

Over the years the interest in Light Cone (LC) formulation of field theories keeps growing mainly because of the varieties of physical processes amenable to direct evaluation, as reported in this volume. An important issue still under debate is the treatment of LC induced infrared (IR) divergencies. Compactification in one LC direction, say $x^- = t - x^3$, with appropriate boundary conditions, permits an *ad hoc* elimination of the problematic zero mode of the field operator. But it is well recognized by now that it is precisely this zero mode which carries the important non-perturbative informations which, in the equal time formalism, are present in the existence of a non-trivial vacuum. For many purposes, in particular to study critical properties of a given field theory, a non-compact formulation is necessary. It uses the notion of fields as operator valued distributions (OPVD) as developped in [1]. These studies focussed on Φ^4 scalar field theory in $1+1$ dimension. Here we want to extend this approach to gauge theories. Due to the paracompactness property of an Euclidean manifold we show that the OPVD formulation permits a simple and transparent evaluation of the QED triangle anomalies.

2. FIELDS AS OPVD

The Klein-Gordon (KG) equation for the free scalar field in D-dimension , $(\Box_x + m^2)\varphi(x) = 0$, writes, after a Fourier transform, $(p^2 - m^2)\tilde{\varphi}(p) = 0$. The solution is a distribution $\tilde{\varphi}(p) = \delta^{(D)}(p^2 - m^2)\chi(p)$, with $\chi(p)$ arbitrary. The solution of the KG-eqution is therefore also a distribution, *ie* an OPVD, which defines a functional with respect to a test function $\rho(x)$, which is C^∞ with compact support,

$$\Phi(\rho) \equiv < \varphi, \rho > = \int d^{(D)}y \varphi(y)\rho(y). \tag{1}$$

$\Phi(\rho)$ is a C-number with the possible interpretation of a more general functional $\Phi(x, \rho)$ evaluated at $x = 0$. Indeed the translated functional is a well defined object [2] such that

$$T_x\Phi(\rho) = < T_x\varphi, \rho > = < \varphi, T_{-x}\rho > = \int d^{(D)}y \varphi(y)\rho(x - y) \tag{2}$$

Now the test function $\rho(x - y)$ has a well defined Fourier decomposition

$$\rho(x - y) = \int \frac{d^{(D)}q}{(2\pi)^D} \exp^{iq(x-y)} f(q) \tag{3}$$

It follows that

$$T_x\Phi(\rho) = \int \frac{d^{(D)}p}{(2\pi)^D} e^{-ipx}\delta(p^2 - m^2)\chi(p)f(p). \tag{4}$$

Due to the properties of ρ , $T_x\Phi(\rho)$ obeys the KG equation and is taken as the physical field with quantized form

$$\varphi_1(x) = \int \frac{d^{(D-1)}p}{(2\pi)^{(D-1)}} \frac{1}{\sqrt{2\omega_p}} [a_p^+ e^{ipx} + a_p e^{-ipx}] f(p, \omega_p). \tag{5}$$

$f(p, \omega_p)$ acts as regulator with very specific properties [1][4]. This expression for $\varphi_1(x)$ is particurlarly useful on the LC because the Haag serie can be used and is well defined in terms of products of $\varphi_1(x_i)$.

3. PARACOMPACT MANIFOLD: TEST FUNCTIONS AS DECOMPOSITION OF UNITY

Consider a topological space $\mathcal{M}$. An open covering [3] of $\mathcal{M}$ is a family of open subspaces Ω_i, $i \in I$, with the property $\mathcal{M} = \bigcup_{i\in I} \Omega_i$. Paracompactness is the property that for each Ω_i there exists a C^∞ function $\beta_i(x)$ such that $\beta_i(x) = 1$ if $x \in \omega_i \subset \Omega_i$, $0 < \beta_i(x) < 1$ in the boundary region $\omega_i \subset \mathcal{B}_i \subset \Omega_i$, and $\beta_i(x) = 0$ *outside* Ω_i. For all $x \in \mathcal{M}$ there is only a finite number of $\beta_j(x) \neq 0$. Let $\alpha_j = \frac{\beta_j}{\Sigma_j \beta_j}$. Now $\Sigma_j \beta_j$ is always non zero and $\Sigma_j \alpha_j = 1$. $\{\alpha_j\}$ is therefore a decomposition of unity on $\mathcal{M}$ [2]. The important theorem is that: "An Euclidean manifold is paracompact" [3]. We shall therefore work in Euclidean metric. Then $f(p)$ is 1 except in the boundary region where it is C^∞ and goes to zero with all its derivatives.

4. QED: CONSTRUCTION AND GAUGE TRANSFORMATION OF THE OPVD FERMIONIC FIELD

Let $\psi(x)$ be the Dirac massive free field, then $(i \not{\partial} - m)\psi(x) = 0 \Longrightarrow \Psi(x) \equiv < T_x\psi, \rho > = \int d^{(D)}y \psi(y)\rho(y - x)$. For QED the fermionic field obeys $(i \not{\partial} - \not{A} - m)\psi(x) = 0$, and it is clear that the translation in $\Psi(x)$ must be done in a gauge invariant way, that is

$$\Psi_\gamma(x) = \int d^{(D)}y \rho(y - x) \exp[ie \int_x^y dz^\mu A_\mu(z)]\psi(y). \tag{6}$$

In a gauge transformation $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\Lambda(x)$,$\psi(y) \to e^{i\Lambda(y)}\psi(y)$ and then $\Psi_\gamma(x) \to e^{i\Lambda(x)}\Psi_\gamma(x)$. Due to the presence of the gauge phase factor in (6) $\Psi_\gamma(x)$ is path dependant. Let $\gamma(s)$ be a parametrization of the path from x to y, $\gamma(0) = x, \gamma(1) = y$. Then

$$\begin{aligned} \int_x^y dz^\mu A_\mu(z) &= \int_0^1 ds\dot{\gamma}^\mu(s) A_\mu(\gamma(s)) = \int d^{(D)}z [\int_0^1 ds\dot{\gamma}^\mu(s)\delta^{(D)}(\gamma(s) - z)] A_\mu(z) \\ &\equiv \int d^{(D)}z \mathfrak{C}^\mu(\gamma; x, y, z) A_\mu(z) \end{aligned} \tag{7}$$

It is easy to see that$\mathfrak{C}^\mu(\gamma; x, y, z)$ obeys the differential equation $\partial_z^\mu \mathfrak{C}_\mu(\gamma; x, y, z) = \delta(x - z) - \delta(y - z)$, the solution of which is known only after a choice of path and boundary condition on z [3]. With $y - x = \epsilon$, the OPVD Dirac field is now $\Psi_\gamma(x) = \int d^{(D)}\epsilon\rho(\epsilon) \exp[ie \int_x^{x+\epsilon} dz^\mu A_\mu(z)]\psi(x + \epsilon)$. One expects that if the extent of the ball $\mathcal{B}(\epsilon)$, support of $\rho(\epsilon)$, is "small" the straight path is the good choice. This is corroborated when evaluating the change $\Delta\Psi_\gamma(x) = \Psi_{\gamma+\delta\gamma}(x) - \Psi_\gamma(x)$ for a change $\delta\gamma$ of the path γ. Indeed $\Delta\Psi_\gamma(x) \propto (\Delta\gamma^\nu(s)\Delta\dot{\gamma}^\mu(s) - \Delta\gamma^\mu(s)\Delta\dot{\gamma}^\nu(s)) F_{\nu,\mu}(\gamma(s))\Psi_\gamma(x)$, which is zero for a straight path $\Delta\gamma(s) = f(s)(y - x), f(0) = 0, f(1) = 1$.

[1] f(p) is also C^∞ with fast decrease in the sense of L. Schwartz [2].

[2] An explicit construction involves the characteristic function $\chi_{\omega_j}(x) = 1(0)$ if $x \in (\notin)\ \omega_j$ and Schwartz's test function $\rho_\epsilon(x)$ in the ball $\mathcal{B}(\epsilon)$, $\beta_j(x) = \int \chi_{\omega_j}(t)\rho_\epsilon(x - t)dt$.

[3] Solution of the form [4]$\mathfrak{C}^\mu(\gamma; x, y, z) = \partial^\mu c(\gamma; x, y, z)$ are excluded, for then $\int d^{(D)}z(\partial_z^\mu c)A_\mu = -\int d^{(D)}zc(\partial_z^\mu A_\mu)$ which would be zero in the Lorentz gauge.

5. QED ANOMALIES

We consider the usual QED triangle diagrams with Ryder's convention [5] and Euclidean metric. Let $I^1_{\kappa,\lambda,\mu}$ and $I^2_{\lambda,\kappa,\mu}$ be the direct and exchange contributions respectively. The direct axial current contribution writes, after performing the traces on $\gamma-$matrices

$$(p_1+p_2)^\mu I^1_{\kappa,\lambda,\mu} = 4e^2\epsilon_{\sigma,\lambda,\delta,\kappa}\int\frac{d^4k}{(2\pi)^4}\Big[\frac{p_2^\sigma k^\delta}{(k+p_2)^2k^2}-\frac{k^\delta p_1^\sigma}{k^2(k-p_1)^2}\Big]f(k^2)f((k+p_2)^2)f((k-p_1)^2), \tag{8}$$

where the f's factors come from the test fuctions present in the fermionic propagators to lowest order in e and $\epsilon_{\sigma,\lambda,\delta,\kappa}$ is the usual antisymmetric tensor. The exchange axial contribution is obtained with the changes $(\kappa\leftrightarrow\lambda),(p1\leftrightarrow p2)$. Due to the f's the integrals are finite: one may change k to $k-p_1$ in the first integral and k to $k+p_2$ in the second. Regrouping terms the total axial contribution is now

$$\begin{aligned}(p_1+p_2)^\mu(I^1_{\kappa,\lambda,\mu}+I^2_{\lambda,\kappa,\mu}) &= 4e^2\epsilon_{\sigma,\lambda,\delta,\kappa}\int\frac{d^4k}{(2\pi)^4}\frac{k^\delta}{k^2}\Big\{\frac{p_1^\sigma}{(k-p_1)^2}\Big[f((k-p_1-p_2)^2)-f((k+p_2)^2)\Big]\\ f((k-p_1)^2) &- \frac{p_2^\sigma}{(k+p_2)^2}\Big[f((k+p_1+p_2)^2)-f((k-p_1)^2)\Big]f((k+p_2)^2)\Big\}f(k^2).\end{aligned} \tag{9}$$

It is seen that if $f=1$ everywhere the axial contribution would be zero, but the variable change in this case is not legitimate for the integrals are linearly divergent. However $f=1$ *almost* everywhere except in the vicinity of the boundary of its support. Its generic shape in the k_δ direction (in dimensionless units) is shown in FIG.1. Clearly the situation of interest is the large Λ limit

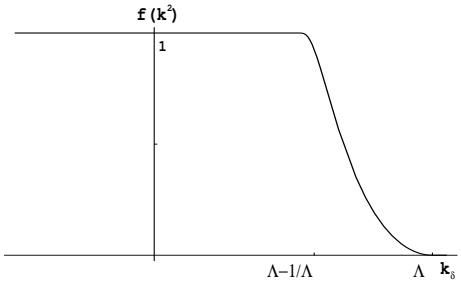

FIG. 1 Generic shape of $f(k)$ as a function of k.

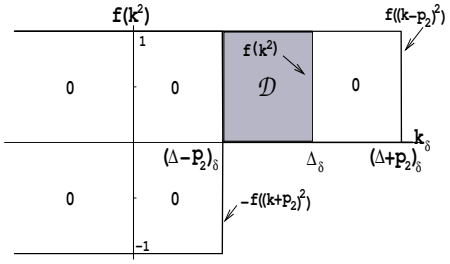

FIG. 2 Domain $\mathcal{D}$ where $\Delta f\neq 0$

and we can look at cases where $p_1,p_2\ll\Lambda$ and all f's shrink to step functions.Then Eq.(9) reduces to

$$\begin{aligned}(p_1+p_2)^\mu(I^1_{\kappa,\lambda,\mu}+I^2_{\lambda,\kappa,\mu}) = 4e^2\epsilon_{\sigma,\lambda,\delta,\kappa}\int\frac{d^4k}{(2\pi)^4}\frac{k^\delta}{k^4}f^2(k^2)\Big\{p_1^\sigma\Big[f((k-p_2)^2)-f((k+p_2)^2)\Big]\\ -p_2^\sigma\Big[f((k+p_1)^2)-f((k-p_1)^2)\Big]\Big\}.\end{aligned} \tag{10}$$

Consider the quantity $\Delta f=f^2(k^2)\Big[f((k-p_2)^2)-f((k+p_2)^2)\Big]$ in the direction of k_δ. The situation is depicted in FIG.2. Δf is different from zero in the shaded area $\mathcal{D}$ of amplitude $(p_2)_\delta,\forall\Lambda$. Hence

$$\int_{(\Lambda-p_2)_\delta}^{\Lambda_\delta}\frac{d^4k}{(2\pi)^4}\frac{k^\delta}{k^4}=\frac{2\pi^2}{(2\pi)^4}\int_{(\Lambda-p_2)_\delta}^{\Lambda_\delta}dk\frac{k_\delta}{k}=\frac{1}{8\pi^2}\int_{(\Lambda-p_2)_\delta}^{\Lambda_\delta}dk_\delta=\frac{(p_2)_\delta}{8\pi^2}, \tag{11}$$

and we have the result $(p_1+p_2)^\mu(I^1_{\kappa,\lambda,\mu}+I^2_{\lambda,\kappa,\mu})=\frac{e^2}{2\pi^2}\epsilon_{\sigma,\lambda,\delta,\kappa}\Big[p_1^\sigma p_2^\delta+p_2^\sigma p_1^\delta\Big]=0$, because of the antisymmetry of $\epsilon_{\sigma,\lambda,\delta,\kappa}$. The axial current is therefore conserved. Consider now the vector current.

After tracing over the $\gamma-$matrices the potentially divergent contribution is

$$p_1^\kappa(I^1_{\kappa,\lambda,\mu}+I^2_{\lambda,\kappa,\mu})=-4e^2\epsilon_{\lambda,\kappa,\mu,\alpha}p_1^\alpha\int\frac{d^4k}{(2\pi)^4}\frac{k^\alpha}{k^4}f(k^2)\Big[f((k-p_1)^2)f((k+p_2)^2)$$
$$-f((k+p_1)^2f((k-p_2)^2)\Big]. \qquad (12)$$

Denote Δf the test function factor and let $e^\nu(\psi_k,\varphi_k,\theta_k)=\frac{k^\nu}{k}=\{\sin(\psi_k)\sin(\varphi_k)\cos(\theta_k),etc\}$ Performing the analysis of Δf in terms of step functions gives, using $\cos\theta_{kp_i}=p_i^\nu e_\nu/p_i$, $\int dk\Delta f=2(p_1\cos\vartheta_{kp_1}-p_2\cos\vartheta_{kp_2})=2(p_1-p_2)^\nu e_\nu(\psi_k,\varphi_k,\theta_k)$. The integral over $d\Omega_k$ is now straightforward with the result $p_1^\kappa(I^1_{\kappa,\lambda,\mu}+I^2_{\lambda,\kappa,\mu})=\frac{e^2}{4\pi^2}\epsilon_{\kappa,\lambda,\mu,\alpha}p_1^\kappa p_2^\alpha$. The vector current (charge) conservation is therefore restored with the correction $\delta I_{\kappa,\lambda,\mu}=\frac{e^2}{4\pi^2}\epsilon_{\kappa,\lambda,\mu,\alpha}(p_1-p_2)^\alpha$, resulting in the standard axial anomaly $(p_1+p_2)^\mu(I^1_{\kappa,\lambda,\mu}+I^2_{\lambda,\kappa,\mu}+\delta I_{\kappa,\lambda,\mu})=\frac{e^2}{2\pi^2}\epsilon_{\kappa,\lambda,\mu,\alpha}p_2^\mu p_1^\alpha$.

6. CONCLUSIONS

Treating scalar fields as OPVD gives a consistent LCQ in the continuum which permits the study of critical properties. It is achieved because IR-induced divergences are handled by the test function present in the regularized field which, in the limit $k^+\to 0$, goes to zero faster than any inverse power of k^+. An essential feature is also the possible use of the Haag serie, for its construction is well defined in terms of the regularized scalar field. In going to gauge theories the definition of the regularized fermionic field from its OPVD counterpart faces the problem of gauge invariance. Taking into account the necessity that the original OPVD fermionic field must be translated in a gauge invariant manner leads to a regularization scheme which does not suffer the general illness of a straight momentum cut-off. It is examplified in the field equation, $\partial_\mu F^{\mu,\nu}(x)=j^\nu(x)=\bar\Psi_\gamma(x)\gamma^\nu\Psi_\gamma(x)$, since by construction the regularized fermionic field renders the current $j^\nu(x)$ gauge invariant. The important property of paracompactness of the Euclidean manifold permits using test functions which are decomposition of unity. They lead to a transparent analysis of the QED anomalies, in complete agreement with the standard results. This is a strong incitation to pursue further the investigations on the merits and possible illnesses of this regularisation scheme in the context of the LC formalism of field theories.

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