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Counting Triangulations of Configurations

Roland Bacher

December 12, 2003

1 Introduction

Given a finite subset $\mathcal{C} \subset \mathbf{R}^2$, calculating the number of triangulations of its convex hull $\text{Conv}(\mathcal{C})$ using only Euclidean triangles with vertices in \mathcal{C} seems to be difficult and has attracted some interest, both from an algorithmic and a theoretical point of view, see for instance [1], [2], [3], [4], [5], [7], [9], [10], [11].

The aim of this paper is to describe a class of configurations, *convex near-gons*, for which this problem can sometimes be solved in a satisfactory way. Loosely speaking, a convex near-gon is an infinitesimal perturbation of a *weighted convex polygon*, a convex polygon with edges subdivided by additional points according to weights. Our main result shows that the triangulation polynomial enumerating all triangulations of a convex near-polygon is defined in a straightforward way in terms of edge-polynomials associated to the “perturbed” edges of such a convex near-gone. These polynomials are difficult to compute in general except in a few special cases. We present a few algorithms related to them. One of these algorithms is a slightly more sophisticated version of an algorithm by Kaibel and Ziegler described in [9] and yields also a general purpose algorithm (unfortunately of exponential complexity), for computing arbitrary triangulation polynomials. This algorithm, based on a transfer matrix, is fairly simple and it would be interesting to compare its performance with existing algorithms, like for instance the algorithm of Aichholzer described in [1].

Figure 1 represents a weighted convex triangle and an infinitesimal generic perturbation of it. Figure 2 shows an integral realisation isotopic to this infinitesimal perturbation.

The complete triangulation polynomial of the weighted triangle Δ depicted on the left side of Figure 1 equals

$$901 s^{14} + 4825 s^{13} + 11734 s^{12} + 17130 s^{11} + 16710 s^{10} + 11466 s^9 \\ + 5670 s^8 + 2034 s^7 + 525 s^6 + 95 s^5 + 11 s^4 + s^3$$

showing that there are for instance 17130 triangulations of Δ using 11 vertices among the 14 vertices of Δ .

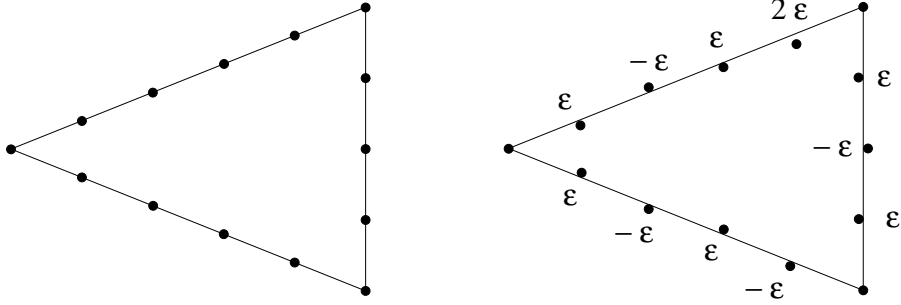


Figure 1: A weighted triangle and an associated generic perturbation

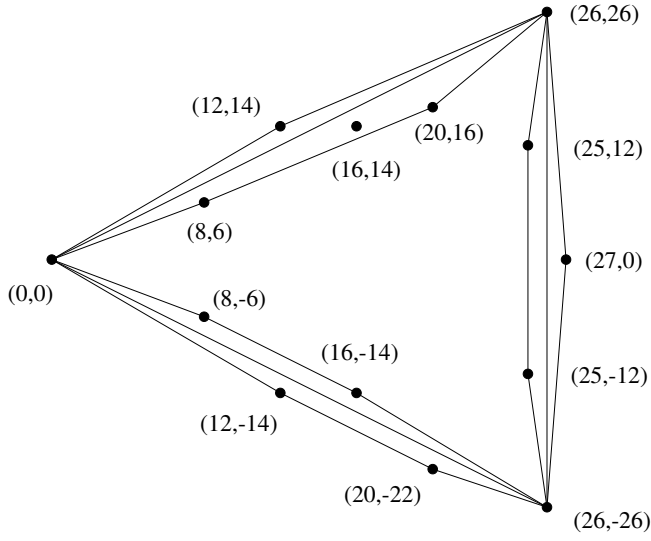


Figure 2: An integral representant of a convex near-polygon

This paper contains all details (except for some straightforward calculations involving polynomials where computers exceed human possibilities by orders of magnitudes) needed for computing the complete triangulation polynomial

$$194939 s^{14} + 338669 s^{13} + 263615 s^{12} + 119944 s^{11} \\ + 34773 s^{10} + 6522 s^9 + 748 s^8 + 42 s^7$$

of the near-triangle represented in Figure 2.

In the sequel of this paper, we recall first a few generalities concerning finite configurations of \mathbf{R}^2 .

We define then weighted convex polygons and give a formula for their complete triangulation polynomial in terms of their edge-weights.

After introducing convex near-gons, near-edges and edge-polynomials we state the main result of this paper expressing the complete triangulation

polynomial of a convex near-polygon in terms of the edge-polynomials of its edges.

We list then a few usefull facts and give some data concerning edge-polynomials.

Finally, we describe a transfer-matrix approach for computing edge-polynomials (and/or solving enumerative problems concerning triangulations).

We close this introduction by describing the following notation, suggested by the fact that polynomials and formal power series are each other's algebraic dual.

Notations. Given a polynomial $p(x) = \sum_{i=0}^d \alpha_i x^i \in \mathbf{k}[x]$ and a formal power series $s(x) = \sum_{i=0}^{\infty} \beta_i x^i \in \mathbf{k}[[x]]$ (over any commutative field or ring \mathbf{k}) we denote by

$$\langle p, s \rangle_x = \langle p(x), s(x) \rangle_x = \sum_{i=0}^d \alpha_i \beta_i$$

the obvious linear pairing.

This notation makes sense for polynomials and formal power series over several variables and we have

$$\langle p, s \, t \rangle_{x,y} = \langle \langle p, s \rangle_x, t \rangle_y = \langle \langle p, t \rangle_y, s \rangle_x$$

for a polynomial $p \in \mathbf{k}[x, y]$ and formal power series $s \in \mathbf{k}[[x]]$, $t \in \mathbf{k}[[y]]$.

Throughout the rest of this paper, p_n denotes always the polynomial

$$p_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} t^{n-k} \in \mathbf{Z}[t]$$

and $C_n = \binom{2n}{n} / (n+1)$ stands for the n -th Catalan number.

2 Configurations

A *configuration* of the oriented plane \mathbf{R}^2 is a finite set $\mathcal{C} = \{P_1, \dots, P_n\} \subset \mathbf{R}^2$ of n distinct points. Two configurations $\mathcal{C} = \{P_1, \dots, P_n\}$ and $\mathcal{C}' = \{P'_1, \dots, P'_n\}$ of n points are (*combinatorially*) *equivalent* or *isomorphic* if there exists a bijection $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ such that

$$\text{sign}(\det(P_j - P_i, P_k - P_i)) = \text{sign}(\det(\varphi(P_j) - \varphi(P_i), \varphi(P_k) - \varphi(P_i)))$$

for all $1 \leq i < j < k \leq n$ where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Two configurations $\mathcal{C} = \{P_1, \dots, P_n\}$ and $\mathcal{C}' = \{P'_1, \dots, P'_n\}$ of n points are *isotopic* if there exists a continuous path $t \mapsto \mathcal{C}(t) \in (\mathbf{R}^2)^n$ of equivalent configurations such that $\mathcal{C}(-1) = \mathcal{C}$ and $\mathcal{C}(1) = \mathcal{C}'$. Isotopic configurations are equivalent.

A configuration $\mathcal{C} \subset \mathbf{R}^2$ is *generic* if three distinct points of \mathcal{C} are never collinear.

Given a configuration \mathcal{C} , we denote by $\text{Conv}(\mathcal{C}) \subset \mathbf{R}^2$ its convex hull. An element $P \in \mathcal{C}$ is *extremal* if $\text{Conv}(\mathcal{C} \setminus \{P\}) \neq \text{Conv}(\mathcal{C})$. We denote by $\text{Extr}(\mathcal{C}) \subset \mathcal{C}$ the subset of extremal vertices. They are the vertices of the convex polygone $\text{Conv}(\mathcal{C})$.

A *triangulation* of a finite configuration \mathcal{C} is a finite set $\mathcal{T} = \{\Delta_1, \dots, \Delta_k\}$ of Euclidean triangles with vertices in \mathcal{C} such that $\text{Conv}(\mathcal{C}) = \cup_{\Delta_i \in \mathcal{T}} \Delta_i$ and $\Delta_i \cap \Delta_j$ is either empty, or a common vertex, or a common edge of two distinct triangles $\Delta_i, \Delta_j \in \mathcal{T}$.

The *complete triangulation polynomial* $\bar{p}(\mathcal{C})$ is the polynomial $\bar{p}(\mathcal{C}) = \sum \tau_k(\mathcal{C}) s^k \in \mathbf{Z}[s]$ whose coefficient $\tau_k(\mathcal{C})$ equals the number of triangulations of $\text{Conv}(\mathcal{C})$ using exactly k vertices of \mathcal{C} . The number of *maximal triangulations* using all vertices of \mathcal{C} is given by the leading coefficient of $\bar{p}(\mathcal{C})$. The lowest non-zero coefficient of $\bar{p}(\mathcal{C})$ is the $(m-2)$ -th Catalan number $C_{m-2} = \binom{2(m-2)}{m-2} / (m-1)$ where $m = \#(\text{Extr}(\mathcal{C}))$ is the number of vertices of the polygone $\text{Conv}(\mathcal{C})$, see for instance Exercice 6.19 of [13]. It is easy to check that the complete triangulation polynomial $\bar{p}(\mathcal{C})$ depends only of the equivalence class of a finite configuration \mathcal{C} .

Remark 2.1 One can also consider the triangulation polynomial defined by

$$\sum_{k_0, k_1} \tau_{k_0, k_1}(\mathcal{C}) s_0^{k_0} s_1^{k_1}$$

counting the number of triangulations using k_0 vertices and k_1 edges. The number k_2 of triangles can then be recovered using the Euler characteristic $k_0 - k_1 + k_2 = 1$ of a compact, simply connected triangulated polygonal region in \mathbf{R}^2 . This more general polynomial yields the same information as the complete polynomial considered above except if the boundary $\partial(\text{Conv}(\mathcal{C}))$ contains points of \mathcal{C} which are not extremal. Most of the results and algorithms of this paper can easily be modified in order to deal with this more general polynomial. For clarity and concision we stick to the simpler version $\bar{p}(\mathcal{C})$ defined above.

3 Weighted convex polygons

This section describes weighted convex polygons, a particular set of finite planar configurations, and gives formulae for their number of maximal triangulations and for their complete triangulation polynomials.

For $l \geq 3$ and natural numbers $a_1, \dots, a_l \geq 1$ we denote by $P(a_1 \cdot a_2 \cdots a_l) \subset \mathbf{R}^2$ a strictly convex polygon with l (counterclockwise) cyclically ordered edges subdivided respectively by $(a_1 - 1), (a_2 - 1), \dots, (a_l - 1)$ additional points. We call such a subdivided polygon a *weighted polygon* with (cyclical) weights a_1, \dots, a_l . Figure 3 displays two realisations of a weighted pentagon $P(1 \cdot 5 \cdot 2 \cdot 3 \cdot 4)$.

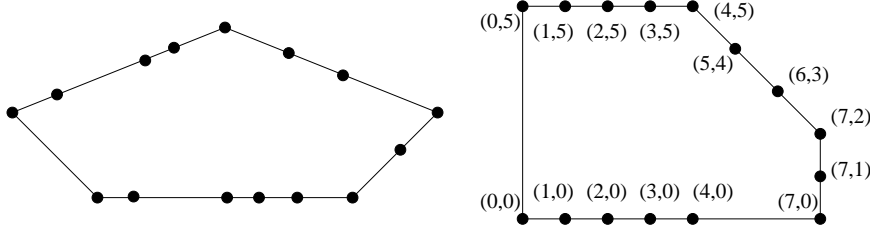


Figure 3: Two weighted pentagons with edge-weights 1, 5, 2, 3, 4.

The configurations given by the $(a_1 + \dots + a_l) = (b_1 + \dots + b_l)$ vertices of two weighted convex polygons $P(a_1 \cdots a_l)$ and $P(b_1 \cdots b_l)$ having the same number of edges and the same weights up to cyclical permutation, are isotopic and their sets of triangulations have the same combinatorics. We denote by $\tau_k(a_1 \cdots a_l)$ the number of triangulations which use k vertices of such a weighted convex polygon $P(a_1 \cdots a_l)$. Such triangulations exist of course only if $l \leq k \leq \sum_{i=1}^l a_i$.

For $n \geq 1$ we introduce the *maximal edge-polynomial* $p_n \in \mathbf{Z}[t]$ by setting

$$p_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} t^{n-k} \in \mathbf{Z}[t]$$

and the *complete edge-polynomial* $\bar{p}_n \in (\mathbf{Z}[s])[t]$ with indeterminate t over the ring $\mathbf{Z}[s]$ which is defined as

$$\bar{p}_n = \sum_{k=1}^n \binom{n-1}{k-1} p_k(t) s^k \in (\mathbf{Z}[s])[t].$$

The first few maximal edge-polynomials are

$$\begin{aligned} p_1 &= t & p_5 &= t^5 - 4t^4 + 3t^3 \\ p_2 &= t^2 - t & p_6 &= t^6 - 5t^5 + 6t^4 - t^3 \\ p_3 &= t^3 - 2t^2 & p_7 &= t^7 - 6t^6 + 10t^5 - 4t^3 \\ p_4 &= t^4 - 3t^3 + t^2 & p_8 &= t^8 - 7t^7 + 15t^6 - 10t^5 + t^4 \end{aligned}$$

and the first few complete edge-polynomials are

$$\begin{aligned} \bar{p}_1 &= p_1 s = t s, \\ \bar{p}_2 &= p_2 s^2 + p_1 s = (t^2 - t) s^2 + t s, \\ \bar{p}_3 &= p_3 s^3 + 2p_2 s^2 + p_1 s = (t^3 - 2t^2) s^3 + 2(t^2 - t) s^2 + t s, \\ \bar{p}_4 &= p_4 s^4 + 3p_3 s^3 + 3p_2 s^2 + p_1 s \\ &= (t^4 - 3t^3 + t^2) s^4 + 3(t^3 - 2t^2) s^3 + 3(t^2 - t) s^2 + t s. \end{aligned}$$

Theorem 3.1 *The number $\tau_{\max}(a_1 \cdot a_2 \cdots a_l)$ of maximal triangulations of the weighted convex polygon $P(a_1 \cdot a_2 \cdots a_l)$ is given by*

$$\langle \prod_{j=1}^l p_{a_j}(t), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t$$

where $C_n = \binom{2n}{n}/(n+1)$ is the n -th Catalan number.

Corollary 3.2 *The complete triangulation polynomial $\sum_k \tau_k(a_1 \cdots a_l) s^k$ of a weighted convex polygon $P(a_1 \cdot a_2 \cdots a_l)$ is given by*

$$\langle \prod_{j=1}^l \bar{p}_{a_j}(t), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t .$$

Corollary 3.3 *The complete triangulation polynomial (and hence the number of maximal triangulations) of a convex weighted polygon $P(a_1 \cdots a_l)$ depends only on the multi-set (set having perhaps multiple elements) $\{a_1, \dots, a_l\}$ defined by the weights and not on their particular cyclic order.*

Example. The two weighted convex polygons $P(1, 5, 2, 3, 4)$ of Figure 3 have

$$\begin{aligned} & \langle p_1 p_5 p_2 p_3 p_4, \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \\ &= \langle t(t^5 - 4t^4 + 3t^3)(t^2 - t)(t^3 - 2t^2)(t^4 - 3t^3 + t^2), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \\ &= \langle t^{15} - 10t^{14} + 39t^{13} - 75t^{12} + 74t^{11} - 35t^{10} + 6t^9, \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \\ &= C_{13} - 10C_{12} + 39C_{11} - 75C_{10} + 74C_9 - 35C_8 + 6C_7 \\ &= 7429000 - 10 \cdot 208012 + 39 \cdot 58786 - 75 \cdot 16796 \\ &\quad + 74 \cdot 4863 - 35 \cdot 1430 + 6 \cdot 429 \\ &= 8046 \end{aligned}$$

maximal triangulations.

Their complete triangulation polynomial $\sum_k \tau_k(P(1 \cdot 5 \cdot 2 \cdot 3 \cdot 4)) s^k$ equals

$$\begin{aligned} & \langle \bar{p}_1 \bar{p}_5 \bar{p}_2 \bar{p}_3 \bar{p}_4, \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \\ &= 8046 s^{15} + 37250 s^{14} + 77467 s^{13} + 95364 s^{12} + 77048 s^{11} \\ &\quad + 42776 s^{10} + 16584 s^9 + 4460 s^8 + 805 s^7 + 90 s^6 + 5 s^5 . \end{aligned}$$

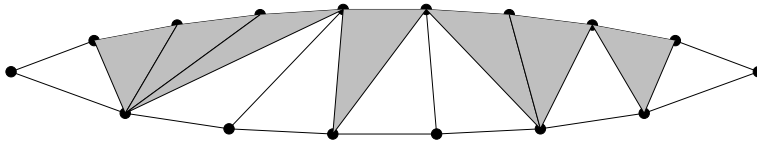


Figure 4: A maximal triangulation with 7 (shaded) up-triangles of the digon $P(9 \cdot 7)$.

Remark 3.4 *Theorem 3.1 and Corollary 3.2 remain valid for weighted “digons”, weighted “convex” polygons $P(a_1 \cdot a_2)$ having only two edges. Their “sides” or “edges” are in this case not straight lines but piecewise linear paths bending slightly outwards in order to enclose a non-empty interior if $a_1 a_2 \geq 2$ (see Figure 4 for an example). The weighted digon $P(1,1)$ has by convention one triangulation. $P(0,a) = P(a,0)$ has 1 triangulation for $a = 0$ and none otherwise. In all the remaining cases, only “triangulations” with all triangles meeting both edges are allowed. The number of such triangulations of $P(a_1 \cdot a_2)$ equals $\binom{a_1+a_2-4}{a_1-2}$: Indeed, every triangle except the first and the last one is either up (with two vertices on the upper edge) or down (with two vertices on the lower edge) and there are exactly $a_1 - 2$ such triangles which are up, cf. Figure 4 for an example.*

3.1 Proofs for weighted convex polygons

Let $P(a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdots a_l^{\alpha_l})$ denote the weighted convex polygon having α_1 edges of weight a_1 followed by α_2 edges of weight a_2 etc. We denote by $\tau_{\max}(a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdots a_l^{\alpha_l})$ the number of maximal triangulations of $P(a_1^{\alpha_1} \cdots a_l^{\alpha_l})$.

The main ingredient for proving Theorem 3.1 is the following illustration of the Inclusion-Exclusion Principle (cf. Chapter 2 of [12]).

Proposition 3.5 *(Inclusion-Exclusion Principle.) We have*

$$\tau_{\max}(a_1 \cdot a_2 \cdots a_l) = \sum_{k=0}^{\lfloor a_1/2 \rfloor} (-1)^k \binom{a_1-k}{k} \tau_{\max}(1^{a_1-k} \cdot a_2 \cdots a_l) .$$

Proof. Enumerate the sides of $P(1^{a_1} \cdot a_2 \cdots a_l)$ cyclically such that the first a_1 sides are of weight 1 and correspond to the initial factor 1^{a_1} of the product $1^{a_1} \cdot a_2 \cdots a_l$. Let T be a maximal triangulation of the weighted convex polygon $P(1^{a_1} \cdot a_2 \cdots a_l)$. An *initial boundary triangle* of T is a triangle of T having two edges contained among the first a_1 sides (having weight 1) of the convex weighted polygon $P(1^{a_1} \cdot a_2 \cdots a_l)$. A subset $\{\delta_1, \dots, \delta_k\}$, $0 \leq k \leq \lfloor a_1/2 \rfloor$ of k initial boundary triangles in T is called a Δ_k -*decoration* of T . Starving to death the k triangles $\delta_1, \dots, \delta_k$ of the Δ_k -decorated maximal triangulation $(T, \{\delta_1, \dots, \delta_k\})$, we get a maximal triangulation \tilde{T} of $P(1^{a_1-k} \cdot a_2 \cdots a_l)$ together with a graveyard $\{e_1, \dots, e_k\}$ of k marked edges in memory of the deceased triangles. We call the pair $(\tilde{T}, \{e_1, \dots, e_k\})$ a maximal E_k -*decorated* triangulation of $P(1^{a_1-k} \cdot a_2 \cdots a_l)$. Figure 5 displays a Δ_2 -decorated maximal triangulation of $P(1^7 \cdot 3 \cdot 2)$ and the corresponding E_2 -decorated maximal triangulation of $P(1^5 \cdot 3 \cdot 2)$.

Obviously, Δ_k -decorated maximal triangulations of $P(1^{a_1} \cdot a_2 \cdots a_l)$ and E_k -decorated maximal triangulations of $P(1^{a_1-k} \cdot a_2 \cdots a_l)$ are in bijection and since any maximal triangulation of $P(1^{a_1-k} \cdot a_2 \cdots a_l)$ has $\binom{a_1-k}{k}$ different

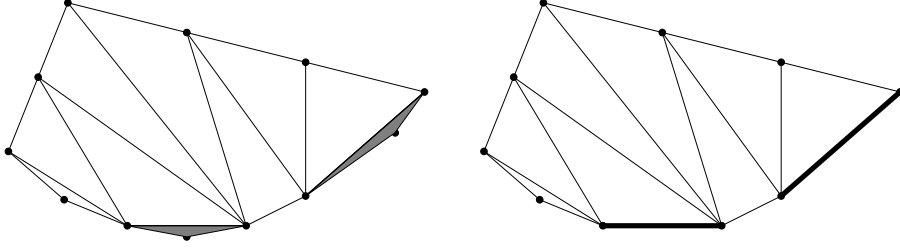


Figure 5: Decorated maximal triangulations of $P(1^7 \cdot 3 \cdot 2)$ and $P(1^5 \cdot 3 \cdot 2)$

E_k -decorations, there exist exactly

$$\binom{a_1 - k}{k} \tau_{\max}(1^{a_1 - k} \cdot a_2 \cdots a_l)$$

maximal Δ_k -decorated triangulations of $P(1^{a_1} \cdot a_2 \cdots a_l)$.

Consider now a maximal triangulation T of $P(1^{a_1} \cdot a_2 \cdots a_l)$ having exactly s initial boundary triangles. The triangulation T gives rise to $\binom{s}{k}$ different Δ_k -decorated maximal triangulations of $P(1^{a_1} \cdot a_2 \cdots a_l)$ and yields a contribution of

$$\sum_{k=0}^s \binom{s}{k} (-1)^k = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

to the alternating sum

$$\sum_{k=0}^{\lfloor a_1/2 \rfloor} (-1)^k \binom{a_1 - k}{k} \tau_{\max}(1^{a_1 - k} \cdot a_2 \cdots a_l)$$

counting Δ_k -decorated maximal triangulations of $P(1^{a_1} \cdot a_2 \cdots a_l)$ with sign $(-1)^k$. This alternating sum counts thus exactly the number of maximal triangulations of $P(1^{a_1} \cdot a_2 \cdots a_l)$ without initial boundary triangles. Such triangulations are in bijection with triangulations of $P(a_1 \cdot a_2 \cdots a_l)$: straight out the first a_1 edges of weight 1 in $P(1^{a_1} \cdot a_2 \cdots a_l)$ into a unique first edge of weight a_1 of $P(a_1 \cdot a_2 \cdots a_l)$. Figure 3.1 illustrates this: Its left side displays a maximal triangulation without initial boundary triangles of $P(1^5 \cdot 3 \cdot 2)$ and its right side shows the corresponding straightened maximal triangulation of $P(5 \cdot 3 \cdot 2)$. \square

Proof of Theorem 3.1 Given a polynomial $q(t) = \sum_{i=0}^d \alpha_i t^i$ we set

$$\tau(q(t); a_2 \cdots a_l) = \sum_{i=0}^d \alpha_i \tau_{\max}(1^i \cdot a_2 \cdots a_l) .$$

Since the polynomial

$$p_{a_i}(t) = \sum_{k=0}^{\lfloor a_i/2 \rfloor} (-1)^k \binom{n - k}{k} t^{n-k}$$

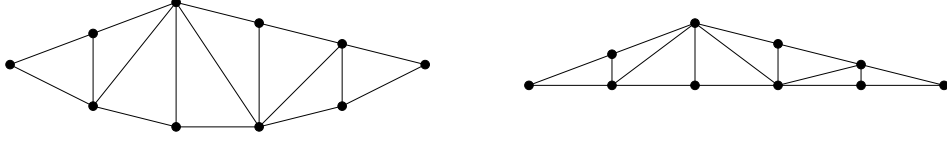


Figure 6: Straightening a maximal triangulation without initial boundary triangles

mimicks the formula of Proposition 3.5, we get by iterated application of Proposition 3.5

$$\tau_{\max}(a_1 \cdot a_2 \cdots a_l) = \tau\left(\prod_{i=1}^l p_{a_i}(t); \emptyset\right) = \sum_j \gamma_j \tau_{\max}(1^j)$$

where $\sum_j \gamma_j t^j = \prod_{i=1}^l p_{a_i}(t)$.

Since $P(1^n)$ is a convex polygon with n edges (of weight 1) and n extremal vertices, the number $\tau_{\max}(1^n)$ of maximal triangulations of $P(1^n)$ is given by the $(n-2)$ -th Catalan number $C_{n-2} = \binom{2(n-2)}{n-2}/(n-1)$, cf. for instance Exercise 6.19 of [13]. Hence the result. \square

Proof of Corollary 3.2. Non-maximal triangulations of $P(a_1 \cdots a_l)$ are in bijection with maximal triangulations of $P(b_1 \cdots b_l)$ where $1 \leq b_i \leq a_i$ and where the weighted polygons $P(b_1 \cdots b_l)$ are realised by choosing in all $\binom{a_i-1}{b_i-1}$ possible ways subsets of (b_i-1) points among the (a_i-1) interior points of the i -th edge having weight a_i of $P(a_1 \cdots a_l)$.

The complete triangulation polynomial of $P(a_1 \cdots a_l)$ is thus given by

$$\begin{aligned} & \sum_{1 \leq b_i \leq a_i} \left(\prod_{j=1}^l \binom{a_j-1}{b_j-1} \right) \left\langle \prod_{j=1}^l p_{b_j}(t), \sum_{n=2}^{\infty} C_{n-2} t^n \right\rangle_t s^{\sum_{j=1}^l b_j} \\ &= \left\langle \prod_{j=1}^l \sum_{b_j=1}^{a_j} \binom{a_j-1}{b_j-1} p_{b_j}(t) s^{b_j}, \sum_{n=2}^{\infty} C_{n-2} t^n \right\rangle_t \\ &= \left\langle \prod_{j=1}^l \bar{p}_{a_j}, \sum_{n=2}^{\infty} C_{n-2} t^n \right\rangle_t \end{aligned}$$

which proves the result. \square

Corollary 3.3 is obvious.

4 Near-gons

An n -near-edge E is a sequence of $(n+1)$ -points

$$(P_0, \dots, P_n) = \left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right)$$

in \mathbf{R}^2 such that $x_0 < x_1 < \dots < x_{n-1} < x_n$. We consider the complete order $P_0 < P_1 < P_2 < \dots, P_n$ on E and call P_0 , respectively P_n , the *initial*, respectively *final*, vertex of the near-edge E . We denote a near-edge E either by the sequence $E = (P_0, \dots, P_n)$ of its points or by the $2 \times (n+1)$ matrix

$$E = \begin{pmatrix} x_0 & \dots & x_n \\ y_0 & \dots & y_n \end{pmatrix}$$

satisfying $x_0 < x_1 < \dots < x_n$ whose columns contain the coordinates $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$ of P_i . We endow E with the total order $P_0 < P_1 < \dots < P_n$.

Two n -near-edges $E = \{P_0, \dots, P_n\}$ and $E' = \{P'_0, \dots, P'_n\}$ are *equivalent* if

$$\text{sign}(\det(P_j - P_i, P_k - P_i)) = \text{sign}(\det(P'_j - P'_i, P'_k - P'_i))$$

where $0 \leq i < j < k \leq n$ and where $\text{sign}(x) \in \{\pm 1, 0\}$ is defined by

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

A continuous path $t \mapsto E(t)$, $t \in [-1, 1]$ of equivalent near-edges is an *isotopy*. Two near-edges $E(-1)$ and $E(1)$ joined by an isotopy are *isotopic*. Isotopic near-edges are of course always equivalent.

A near-edge is *generic* if the underlying set of points is a generic configuration of \mathbf{R}^2 , i.e. is without three collinear points.

Given a near-edge $E = (P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix})_{i=0, \dots, n}$ and $\epsilon \in \mathbf{R}$, we denote by E^ϵ the near-edge

$$E^\epsilon = (P_i(\epsilon) = \begin{pmatrix} x_i \\ \epsilon y_i \end{pmatrix})_{i=0, \dots, n} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} P_i)_{i=0, \dots, n}.$$

For $\epsilon > 0$, the near-edges E and E^ϵ are isotopic and thus equivalent.

Let P be a convex polygon with l extremal vertices $\{V_0, V_1, V_2, \dots, V_{l-1}, V_l = V_0\}$ appearing in counterclockwise order around its boundary ∂P .

Given a sequence $E_1^{\epsilon_1}, \dots, E_l^{\epsilon_l}$ where $E_i = (P_{i,0}, P_{i,1}, \dots, P_{i,n_i})$ is a n_i -near-edges and $\epsilon_i \in \mathbf{R}$ a real number, we denote by

$$P(E_1^{\epsilon_1} \cdot E_2^{\epsilon_2} \dots E_l^{\epsilon_l})$$

the unique configuration obtained by gluing the n_i -near-edge $E_i^{\epsilon_i}$, rescaled by a suitable orientation-preserving similitude, along the oriented edge of P

which starts at V_{i-1} and ends at V_i . More precisely, the gluing similitude φ_i is the unique orientation-preserving similitude of \mathbf{R}^2 such that

$$\varphi_i(P_{i,0}(\epsilon_i)) = V_{i-1} \text{ and } \varphi_i(P_{i,n_i}(\epsilon_i)) = V_i .$$

The configuration $P(E_1^{\epsilon_1} \cdot E_2^{\epsilon_2} \cdots E_l^{\epsilon_l})$ is now the set of points $\cup_{i=1}^l \varphi_i(E_i^{\epsilon_i}) \subset \mathbf{R}^2$.

We have the following result which we state without proof.

Proposition 4.1 (i) *The configurations $P(E_1^{\epsilon_1} \cdots E_l^{\epsilon_l})$ are isotopic for $0 < \epsilon_i$ small enough.*

(ii) *Given a second convex polytope P' having l vertices $V'_0, V'_1, \dots, V'_l = V'_0$ in counterclockwise order, the configurations*

$$P(E_1^{\epsilon_1} \cdots E_l^{\epsilon_l}) \text{ and } P'(E_1^{\epsilon_1} \cdots E_l^{\epsilon_l})$$

associated to P and P' are isotopic for $0 < \epsilon_i$ small enough.

(iii) *Given pairs (E_i, E'_i) of equivalent near-edges, the configurations*

$$P(E_1^{\epsilon_1} \cdots E_l^{\epsilon_l}) \text{ and } P(E'_1{}^{\epsilon_1} \cdots E'_l{}^{\epsilon_l})$$

are equivalent for $0 < \epsilon_i$ small enough.

We call the equivalence class of the unique configuration described by Proposition 4.1 (with $0 < \epsilon_i$ small enough) the *convex l -near-gon* or *near-gon* with (cyclically oriented) near-edges E_1, \dots, E_l . We denote by $P(E_1 \cdots E_l)$ any configuration representing the equivalence class of this near-gon.

Theorem 4.2 *For every near-edge E there exist polynomials*

$$p(E) \in \mathbf{Z}[t] \text{ and } \bar{p}(E) \in \mathbf{Z}[s, t]$$

such that the convex near-polygon $P(E_1 \cdots E_l)$ has

$$\langle \prod_{j=1}^l p(E_j), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t$$

maximal triangulations and complete triangulation polynomial

$$\langle \prod_{j=1}^l \bar{p}(E_j), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t$$

where $C_n = \binom{2n}{n} / (n+1)$ are the Catalan numbers.

Corollary 4.3 *The number of maximal triangulations and the complete triangulation polynomial of a convex near-polygon $P(E_1, \dots, E_l)$ depend only on the multiset $\{E_1, \dots, E_l\}$ of (equivalence classes of) its near-edges.*

Example. The configuration of Figure 2 is equivalent to the convex near-gon $P(E_a \cdot E_b \cdot E_c)$ where

$$\begin{aligned} E_a &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix} \\ E_b &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix} \\ E_c &= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{pmatrix} \end{aligned}$$

are near-edges whose complete edge-polynomials

$$\begin{aligned} \bar{p}(E_a) &= (14p_3 + 7p_4 + p_5)s^5 + (10p_2 + 7p_3 + 2p_4)s^4 \\ &\quad + (2p_1 + 2p_2 + p_3)s^3, \\ \bar{p}(E_b) &= (5p_3 + p_4)s^4 + 2(2p_2 + p_3)s^3 + (p_1 + p_2)s^2 \\ \bar{p}(E_c) &= (10p_3 + 7p_4 + 2p_5)s^5 + (3p_2 + 13p_3 + 4p_4)s^4 \\ &\quad + 3(2p_2 + p_3)s^3 + (p_1 + p_2)s^2 \end{aligned}$$

(where $p_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} t^{n-k} \in \mathbf{Z}[t]$ as usual) will be computed in the sequel. The complete triangulation polynomial $\bar{p}(E_a \cdot E_b \cdot E_c)$ of this configuration is given by

$$\begin{aligned} &\langle \bar{p}(E_a) \bar{p}(E_b) \bar{p}(E_c), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \\ &= 194939 s^{14} + 338669 s^{13} + 263615 s^{12} + 119944 s^{11} \\ &\quad + 34773 s^{10} + 6522 s^9 + 748 s^8 + 42 s^7. \end{aligned}$$

4.1 Edge-polynomials

Given an n -near-edge

$$E = (P_0, \dots, P_n) = \begin{pmatrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{pmatrix}$$

we define the *lower convex hull* $\partial_- E$ as the piece-wise linear path joining P_0 to P_n formed by the “lower” edges of $\partial \text{Conv}(E)$. The set $V_-(E)$ of *lower extremal vertices* of E is defined as the increasing subsequence $\text{Extr}(E) \cap \partial_- E$ of all extremal vertices of $\text{Conv}(E)$ situated in the closed “lower” halfplane containing P_0 and P_n in its boundary and containing all points (x_0, y) with $y \leq y_0$.

A *roof* of E is a strictly increasing sub-sequence

$$R = (P_{j_0} = P_0, P_{j_1}, \dots, P_{j_{k-1}}, P_{j_k} = P_n) \subset E$$

starting at P_0 and ending at P_n of E . If a roof contains $k + 1$ elements we call $\text{length}(R) = k$ its *length*. The *skyline* $\text{Skyline}(R)$ of a roof R is the

piecewise-linear path from P_0 to P_n defined by joining consecutive elements of R using straight segments.

A roof R is *covering* if it “shelters” every element of E : for every point $P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in E$ we have either $P_i \in R$ or $\tilde{y}_i > y_i$ for $\begin{pmatrix} x_i \\ \tilde{y}_i \end{pmatrix} \in \text{Skyline}(R)$.

The *maximal edge-polynomial* $p(E)$ of the near-edge E is defined as

$$p(E) = \sum_{R \text{ covering roof of } E} \tau_{\max}(E, R) p_{\text{length}(R)} \in \mathbf{Z}[t]$$

where $\tau_{\max}(E, R)$ denotes the number of maximal triangulations of the (generally non-convex) compact polygonal region delimited by the two piecewise linear paths $\text{Skyline}(R)$ and $\partial_- E$.

A k -*sub-edge* of E is an increasing subsequence $E' \subset E$ of $k+1 \leq n+1$ elements in E such that $V_-(E') = V_-(E)$. In particular, any sub-edge E' of E has also initial vertex P_0 and final vertex P_n .

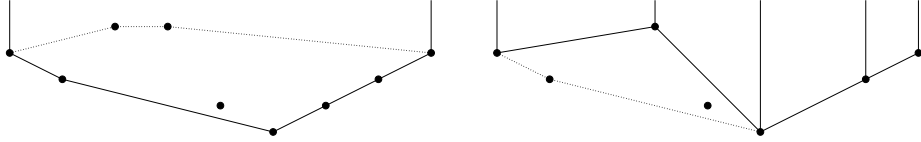


Figure 7: An 8-near-edge and a covering roof of weight 4 of a 6-sub-edge

Example. The left side of Figure 7 displays the 8-near edge

$$E = (P_0, \dots, P_8) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & -1 & 1 & 1 & -2 & -3 & -2 & -1 & 0 \end{pmatrix}.$$

We have $V_-(E) = (P_0, P_1, P_5, P_8)$ and E has 2^5 sub-edges obtained by removing any subset of vertices among $\{P_2, P_3, P_4, P_6, P_7\}$ from E . The right side of Figure 7 displays the covering roof $R = (P_0, P_3, P_5, P_7, P_8)$ of the sub-edge $E' = (P_0, P_1, P_3, P_4, P_5, P_7, P_8) \subset E$.

The *complete edge-polynomial* $\bar{p}(E)$ of an n -near-edge $E = (P_0, \dots, P_n)$ is defined as

$$\bar{p}(E) = \sum_{m=1}^n \sum_{E' \subset E \text{ } m\text{-sub-edge}} p(E') s^m$$

where $p(E')$ denotes the maximal edge-polynomial of the sub-edge $E' \subset E$.

Example. We compute the complete edge-polynomial $\bar{p}(E_a)$ of the near-edge

$$E_a = (P_0, \dots, P_5) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

involved in the near-gon $P(E_a \cdot E_b \cdot E_c)$ of Figure 2.

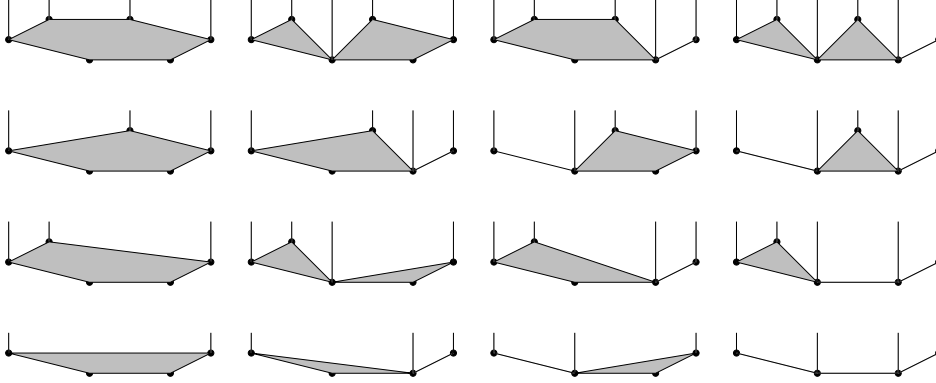


Figure 8: All covering roofs of sub-edges involved in $\bar{p}(E_a)$

Figure 8 contains all roofs of the four possible sub-edges of E_a obtained by removing any subset of points in $\{P_1, P_3\}$. Each possible sub-edge has four different covering roofs whose contributions to $\bar{p}(E_a)$ are given by

sub-edge				
$(P_0, P_1, P_2, P_3, P_4, P_5) :$	$14p_3s^5$	$2p_4s^5$	$5p_4s^5$	p_5s^5
$(P_0, P_2, P_3, P_4, P_5) :$	$5p_2s^4$	$2p_3s^4$	$2p_3s^4$	p_4s^4
$(P_0, P_1, P_2, P_4, P_5) :$	$5p_2s^4$	p_3s^4	$2p_3s^4$	p_4s^4
$(P_0, P_2, P_4, P_5) :$	$2p_1s^3$	p_2s^3	p_2s^3	p_3s^3

They sum up to the complete edge-polynomial

$$\bar{p}(E_a) = (14p_3 + 7p_4 + p_5)s^5 + (10p_2 + 7p_3 + 2p_4)s^4 + (2p_1 + 2p_2 + p_3)s^3$$

of E_a .

4.2 Proof of Theorem 4.2

Consider a triangulation \mathcal{T} of a convex near-polygon $P(E_1 \cdots E_l)$ with complete triangulation polynomial

$$\bar{p} = \sum_k \tau_k(P(E_1 \cdots E_l)) s^k .$$

A triangle $\Delta \in \mathcal{T}$ with vertices $\{a, b, c\}$ is of *edge-type* if there exists j such that $\{a, b, c\} \subset E_j$. Otherwise, the triangle Δ is *interior*. If Δ with vertices $\{a, b, c\}$ is of edge-type, we write $\Delta \in E_j$ if $\{a, b, c\} \subset E_j$.

The definition of a convex near-polygon implies that the set of all triangles $\Delta \in E_j \cap \mathcal{T}$ determines a covering roof R_j of a suitable k_j -sub-edge E'_j of E_j . More precisely, define E'_j as the set of all $(k_j + 1)$ points of E_j which appear as vertices in \mathcal{T} . The set of edge-type triangles in E_j triangulates now a (generally non-convex) polygonal region enclosed by two

(non-crossing) paths joining the endpoints of E_j . These paths are the lower convex hull $\partial_- E'_j = \partial_- E_j$ and the skyline $\text{Skyline}(R_j)$ of a unique covering roof $R_j \subset E'_j$ having length $\text{length}(R_j)$. The interior triangles of \mathcal{T} define (after straightening) a unique maximal triangulation of the weighted convex polygon $P(\text{length}(R_1) \cdots \text{length}(R_l))$. The left side of Figure 9 displays a maximal triangulation of a near-gon having four near-edges (of length 2, 3, 1 and 4). The inner triangles of the displayed triangulation yield a maximal triangulation of the convex weighted polygon $P(2 \cdot 2 \cdot 1 \cdot 3)$ depicted on the right side of Figure 9.

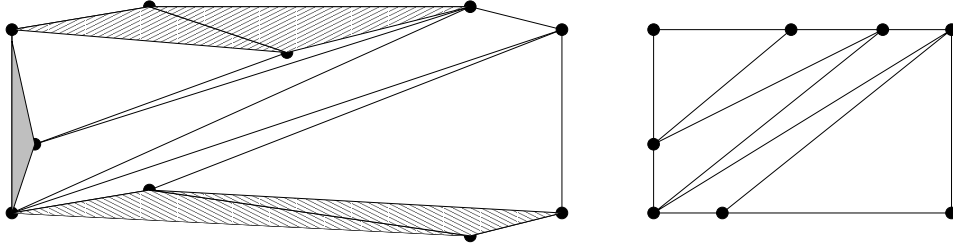


Figure 9: Edge-type and interior triangles of a triangulation

There are exactly

$$\tau_{\max}(\text{length}(R_1) \cdots \text{length}(R_l)) \prod_{j=1}^l \tau_{\max}(E'_j, R_j)$$

such triangulations contributing to the coefficient $s^{(k_1+k_2+\dots+k_l)}$ of \bar{p} for fixed covering roofs R_j of fixed k_j -sub-edges $E'_j \subset E_j$. Applying Theorem 3.1 and summing over all covering roofs in sub-edges yields the result. \square

5 Properties of edge-polynomials

A near edge $E = (P_0, \dots, P_n)$ factorises into near-edges E_1, E_2 if there exists a lower extremal vertex $P_k \in E \cap \partial_- E$ such that

$$E_1 = (P_0, P_1, \dots, P_{k-1}, P_k), \quad E_2 = (P_k, P_{k+1}, \dots, P_{n-1}, P_n)$$

and all points of $E \setminus E_i$ lie strictly above any line defined by two distinct points of E_i for $i = 1, 2$. We write $E = E_1 \cdot E_2$ if the near-edge E factorises with first factor E_1 and second factor E_2 . A near-edge is *prime* if it has no non-trivial factorisation. It is easy to show that every near-edge has a unique factorisation into prime near-edges.

Proposition 5.1 *Given a factorisation $E = E_1 \cdot E_2$ of a near-edge E we have*

$$p(E) = p(E_1) p(E_2) \text{ and } \bar{p}(E) = \bar{p}(E_1) \bar{p}(E_2) .$$

Proof. Since the near-polygons $P(1^k \cdot E)$ and $P(1^k \cdot E_1 \cdot E_2)$ are equivalent for all $k = 2, 3, \dots$ we have

$$\langle t^k \bar{p}(E), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t = \langle t^k \bar{p}(E_1) \bar{p}(E_2), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t .$$

This implies the result since $\det((C_{i+j+k})_{0 \leq i, j \leq n}) > 0$ for all $k \geq 0$ and $n \geq 1$ (this follows for instance easily from Exercice 6.26.b in [13]). \square

Since the so-called k -shifted Hankel matrices

$$H_{i,j} = C_{i+j+k}, \quad 0 \leq i, j < n$$

are all non-singular, the coefficients $\gamma_\alpha, \dots, \gamma_d$ (with $\alpha \geq 2$) of a polynomial $q(t) = \sum_{i=\alpha}^d \gamma_i t^i$ can be obtained from the $(d - \alpha + 1)$ “linear form evaluations”

$$\langle t^k q(t), \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t, \quad k = 0 \dots, d - \alpha .$$

This turns any method or algorithm for enumerating maximal (or all) triangulations of configurations into a method or algorithm for computing maximal (or complete) edge-polynomials.

Example. Using the online computing service [6] of O. Aichholzer, we compute the maximal triangulation polynomial $p(E_c)$ (which is of of course also the coefficient of s^5 in the complete triangulation polynomial $\bar{p}(E_c)$) of the near-edge

$$E_c = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{pmatrix}$$

involved in the near-gon of Figure 2.

Since the set $\partial_+(E_c)$ of all boundary vertices on the upper boundary of $\text{Conv}(E_c)$ consists of the 3 segments $\overline{P_0 P_1}$, $\overline{P_1 P_4}$ and $\overline{P_4 P_5}$, we have

$$p(E_c) = \alpha p_3 + \beta p_4 + \gamma p_5$$

where α, β and γ are unknown natural integers.

We identify now the near-edge $E_c = (P_0, \dots, P_5)$ with its underlying set

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\} .$$

Define

$$A_1 = \begin{pmatrix} 1 \\ 10 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 \\ 11 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$$

and check that the configurations

$$E_c \cup \{A_1\}, \quad E_c \cup \{A_1, A_2\}, \quad E_c \cup \{A_1, A_2, A_3\}$$

are near-polygons with near-edge factorisation

$$E_c \cdot E_1^2, E_c \cdot E_1^3, E_c \cdot E_1^4$$

where E_1 denotes the unique 1-edge with maximal edge polynomial $p(E_1) = p_1 = t$. Denoting by $\tau_{\max}(\mathcal{C})$ the number of maximal triangulations of a finite configuration \mathcal{C} and using for instance the on-line computing service of Aichholzer [6] mentionned above we get the linear system of equations

$$\begin{aligned} 19 &= \tau_{\max}(E_c \cup \{A_1\}) &= \langle p(E_c) t^2, \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \\ 87 &= \tau_{\max}(E_c \cup \{A_1, A_2\}) &= \langle p(E_c) t^3, \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \\ 175 &= \tau_{\max}(E_c \cup \{A_1, A_2, A_3\}) &= \langle p(E_c) t^4, \sum_{n=2}^{\infty} C_{n-2} t^n \rangle_t \end{aligned}$$

Substituting $p_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} t^{n-k}$, $C_n = \binom{2n}{n} / (n+1)$ and solving for the unknowns α, β, γ we get

$$p(E_c) = 10p_3 + 7p_4 + 2p_5 .$$

Remark. Since the computations of the coefficient $\alpha = \tau_{\max}(E_c) = 10$ of lowest order (in t) and of the leading coefficient

$$\begin{aligned} \gamma &= \tau_{\max}(\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}) \cdot \\ &\quad \cdot \tau_{\max}(\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\}) = 2 \cdot 1 = 2 \end{aligned}$$

are easy, we could have simplified the above computation by a considerable amount of work.

For completeness we mention also the following obvious fact:

Call two n -near-edges $E = (P_0, \dots, P_n)$ and $E' = (P'_0, \dots, P'_n)$ *vertical mirrors* if E' given by $E' = (\overline{P}_n, \overline{P}_{n-1}, \dots, \overline{P}_1, \overline{P}_0)$ where $\overline{P}_i = \begin{pmatrix} -x_i \\ y_i \end{pmatrix}$

is the Euclidean reflection of $P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ with respect to the vertical line $x = 0$.

The following result is obvious:

Proposition 5.2 *If a pair of near-edges (E, E') are vertical mirrors, then*

$$\overline{p}(E) = \overline{p}(E') .$$

5.1 Complete edge-polynomials for small near-edges

This subsection contains representants and edge-polynomials for all 1-, 2- and 3-near-edges.

5.1.1 1–near-edges

The unique 1–near-edge can be represented by $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is generic and prime and has complete edge-polynomial $\bar{p}(E_1) = p_1 s = s t$.

5.1.2 2–near-edges

There are two generic 2–near-edges, represented by

$$E_{2,1} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{2,2} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & -1 & 0 \end{pmatrix}.$$

$E_{2,1}$ is prime while $E_{2,2} = E_1 \cdot E_1$. They have complete edge-polynomials

$$\bar{p}(E_{2,1}) = p_2 s^2 + p_1 s, \quad \bar{p}(E_{2,2}) = (p_2 + p_1) s^2 = \bar{p}(E_1)^2.$$

Moreover, there is also a unique non-generic 2–near-edge represented for instance by

$$E_{2,3} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

with complete edge-polynomial given by

$$\bar{p}(E_{2,3}) = p_2 s^2 + p_1 s = \bar{p}(E_{2,1}).$$

5.1.3 3–near-edges

There are (up to equivalence) eight generic 3–near edges represented by

$$\begin{aligned} E_{3,1} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix}, & E_{3,2} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \\ E_{3,3} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 0 \end{pmatrix}, & E_{3,4} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \\ E_{3,5} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 0 \end{pmatrix}, & E_{3,6} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -3 & -1 & 0 \end{pmatrix}, \\ E_{3,7} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & -1 & 0 \end{pmatrix}, & E_{3,8} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & -3 & 0 \end{pmatrix}. \end{aligned}$$

The first five are prime. The last three have factorisations

$$E_{3,6} = E_1 E_{2,1}, \quad E_{3,7} = E_1^3, \quad E_{3,8} = E_{2,1} E_1.$$

The pairs $\{E_{3,1}, E_{3,3}\}$, $\{E_{3,4}, E_{3,5}\}$, $\{E_{3,6}, E_{3,8}\}$ are vertical mirrors. The prime near-edges have complete polynomials

$$\begin{aligned} \bar{p}(E_{3,1}) &= \bar{p}(E_{3,3}) = (p_2 + p_3) s^3 + 2p_2 s^2 + p_1 s, \\ \bar{p}(E_{3,2}) &= 2p_3 s^3 + 2p_2 s^2 + p_1 s, \\ \bar{p}(E_{3,4}) &= \bar{p}(E_{3,5}) = (2p_2 + p_3) s^3 + p_1^2 s^2. \end{aligned}$$

There are moreover nine more 3–near-edges which are not generic. They are represented for instance by

$$\begin{aligned}
E_{3,9} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \end{pmatrix}, & E_{3,10} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & 0 \end{pmatrix}, \\
E_{3,11} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & E_{3,12} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
E_{3,13} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & E_{3,14} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\
E_{3,15} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix}, & E_{3,16} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -2 & -1 & 0 \end{pmatrix}, \\
E_{3,17} &= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

The near-edges

$$E_{3,10} = E_{2,3} E_1, \quad E_{3,16} = E_1 E_{2,3}$$

factorise. The remaining prime near-edges have complete edge-polynomials

$$\begin{aligned}
\bar{p}(E_{3,9}) &= \bar{p}(E_{3,15}) = p_3 s^3 + 2p_2 s^2 + p_1 s, \\
\bar{p}(E_{3,11}) &= \bar{p}(E_{3,13}) = (p_2 + p_3) s^3 + 2p_2 s^2 + p_1 s, \\
\bar{p}(E_{3,12}) &= \bar{p}(E_{3,14}) = (p_2 + p_3) s^3 + p_2 s^2, \\
\bar{p}(E_{3,17}) &= \bar{p}_3 = p_3 s^3 + 2p_2 s^2 + p_1 s.
\end{aligned}$$

5.2 Enumeration of generic near-edges

This subsection is a digression sketching briefly a bijection between near-edges (up to equivalence) and orbits of suitable extremal vertices of configurations under automorphisms.

Let $P \in \text{Extr}(\mathcal{C})$ be an extremal vertex of a configuration $\mathcal{C} = \{P_1, \dots, P_{n+2}\} \subset \mathbf{R}^2$ consisting of $(n+2)$ points. Suppose that P is not collinear with two distinct elements of $\mathcal{C} \setminus \{P\}$. Let $P_-, P_+ \in \text{Extr}(\mathcal{C})$ be the immediate predecessor and successor (in counterclockwise order) of P on the boundary $\partial \text{Conv}(\mathcal{C})$.

A suitable projective transformation (sending a line outside $\text{Conv}(\mathcal{C})$ which misses P only nearly to the line at infinity and sending P close to the ideal point $(0, +\infty)$) transforms $\mathcal{C} \setminus \{P\}$ into an n –near-edge E with initial vertex P_+ and final vertex P_- . This near-edge E is well-defined, up to equivalence, and every n –near-edge can be obtained in this way.

In particular, generic n –near-edges are in bijection with orbits under automorphisms of extremal vertices in generic configurations of $(n+2)$ points. Figure 10 illustrates this for $n = 3$. It shows all three generic configurations

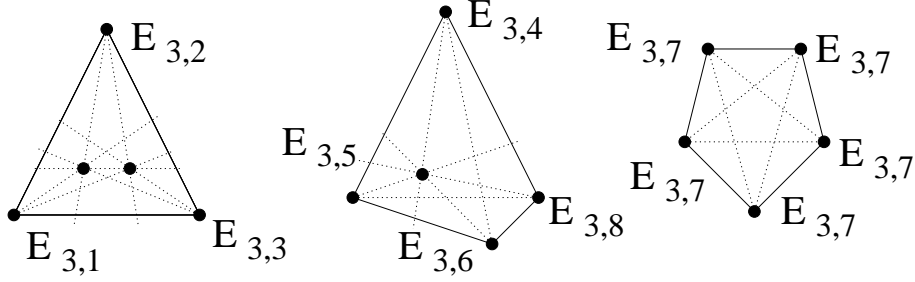


Figure 10: Generic configurations of 5 points and the associated generic 3-near-edges

of five points with extremal vertices labelled by the corresponding generic 3-near-edges (using the notations of the previous subsection).

Call two generic n -near-edges *related* if they are associated to two extremal vertices of the same generic configuration of $(n+2)$ points. Classes of related generic n -near-edges are in bijection with equivalence classes of generic configurations having $(n+2)$ points.

We say that two configurations $E = (P_0, \dots, P_n)$ and $E' = (P'_0, \dots, P'_n)$ are *related by a horizontal reflection* if $P'_i = \begin{pmatrix} x_i \\ -y_i \end{pmatrix}$ is the reflection with respect to the horizontal line $y = 0$ of $P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$.

Equivalence classes of related generic n -near-edges, up to horizontal reflections, are in bijection with generic projective configurations (up to equivalence) of $(n+2)$ points in the projective plane \mathbf{RP}^2 .

6 An algorithm for convex near-edges

An n -near-edge $E = \{P_0, \dots, P_n\}$ is *convex* if its points P_0, \dots, P_n are all extremal, i.e. vertices of a convex polygon with $(n+1)$ edges. This section describes an algorithm for computing maximal and complete edge-polynomials of convex near-edges.

A convex n -near-edge can be represented by the sequence of points

$$(P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots, P_i = \begin{pmatrix} i \\ \epsilon_i i(n-i) \end{pmatrix}, \dots, P_n = \begin{pmatrix} n \\ 0 \end{pmatrix})$$

where $\epsilon_1, \dots, \epsilon_{n-1} \in \{\pm 1\}$. There are 2^{n-1} equivalence classes, encoded by $(\epsilon_1, \dots, \epsilon_{n-1}) \in \{\pm 1\}^{n-1}$, of convex n -near-edges. The convex near-edge with $\epsilon_i = -1$, $i = 1, \dots, (n-1)$ has the factorisation E_1^n . All others are prime.

The following algorithm of polynomial complexity computes the complete (respectively maximal) edge-polynomial of a convex near-edge. It is

based on the fact that all connected components of the open polygonal region delimited by a covering roof R and $\partial_-(E)$ are open convex polygons if E is a convex near-edge. The exponent of u keeps track of the length of a “covering roof under construction” $(P_0, P_{i_1}, \dots, P_{i_j}, ?)$. The exponent of w keeps track of the number of vertices of the rightmost polygonal region which has not yet closed up.

Algorithm

Input: $n \in 1, 2, \dots$, and $(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}) \in \{\pm 1\}^{n-1}$.

Set

$$r = \begin{cases} s \cdot u & \text{(complete edge-polynomial)} \\ u & \text{(maximal edge-polynomial)} \end{cases}$$

For i from 1 to $n - 1$ do {

If $\epsilon_i = 1$ replace

$$r \mapsto \begin{cases} r \cdot (s \cdot u \cdot w + 1) & \text{(complete edge-polynomial)} \\ r \cdot u \cdot w & \text{(maximal edge-polynomial)} \end{cases}$$

If $\epsilon_i = -1$ replace

$$r \mapsto \begin{cases} r \cdot s \cdot w + \langle r, \left(\sum_{n=0}^{\infty} C_n w^n \right) \rangle_w \cdot s \cdot u & \text{(complete edge-polynomial)} \\ r \cdot w + \langle r, \left(\sum_{n=0}^{\infty} C_n w^n \right) \rangle_w u & \text{(maximal edge-polynomial)} \end{cases}$$

}

Result:

$$\langle r, \left(\sum_{n=0}^{\infty} C_n w^n \right) \left(\sum_{j=1}^{\infty} p_j u^j \right) \rangle_{u,w}$$

(where $p_n = \sum_k (-1)^k \binom{n-k}{k} t^{n-k}$).

Example. The first two near-edges

$$E_a = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

and

$$E_b = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$

involved in Figure 2 are convex. The near-edge E_a corresponds to $n = 5$ and the sequence $(1, -1, 1, -1)$. The near-edge E_b corresponds to $n = 4$ and $(1, -1, 1)$.

The above algorithm for E_a (with $n = 5$ and $(\epsilon_1, \dots, \epsilon_4) = (1, -1, 1, -1)$) yields

$i =$	$\epsilon_i =$	$r =$
1	1	su
2	-1	$s^2u^2w + su$
3	1	$s^3u^2w^2 + s^2uw + s^3u^3 + s^2u^2$
4	-1	$s^4u^3w^3 + s^3u^2w^2 + s^4u^4w + s^3u^3w +$ $+ s^3u^2w^2 + s^2uw + s^3u^3 + s^2u^2$ $s^5u^3w^4 + 2s^4u^2w^3 + s^5u^4w^2 + s^4u^3w^2 + s^3uw^2 + s^4u^3w$ $+ s^3u^2w + 5s^5u^4 + 4s^4u^3 + s^5u^5 + 2s^4u^4 + s^3u^2 + s^3u^3$

Result: $(14p_3 + 7p_4 + p_5)s^5 + (10p_2 + 7p_3 + 2p_4)s^4 + (2p_1 + 2p_2 + p_3)s^3$

which is the complete edge-polynomial $\bar{p}(E_a)$ of E_a .

Computing

$$\langle r, \left(\sum_{n=0}^{\infty} C_n w^n \right) \left(\sum_{j=1}^{\infty} p_j u^j \right) \rangle_{u,w}$$

with

$$r = s^4u^3w^3 + 2s^3u^2w^2 + s^4u^4w + s^3u^3w + s^2uw + s^3u^3 + s^2u^2$$

corresponding to $i = 3$ we get the complete edge-polynomial

$$\bar{p}(E_b) = (5p_3 + p_4)s^4 + (4p_2 + 2p_3)s^3 + (p_1 + p_2)s^2$$

of the convex near-edge E_b encoded by $n = 4$ and $(1, -1, 1)$.

7 Roof-sequences and a transfer matrix

In this section, we define the set of roof-sequences of a finite configuration (or near-edge). Such sequences are in bijection with triangulations. Using this bijection, we introduce a transfer matrix for computing the complete triangulation polynomial (or complete edge-polynomial) of a configuration (or near-edge).

We endow \mathbf{R}^2 with the total order relation defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} < \begin{pmatrix} x' \\ y' \end{pmatrix}$$

if either $x < x'$ or $x = x'$ and $y > y'$. This order is obtained by reflecting (orthogonally) the plane endowed with the familiar lexicographic order with respect to the line $y = 0$.

Consider a finite configuration $\mathcal{C} = \{P_0, \dots, P_n\} \subset \mathbf{R}^2$ consisting of $(n+1)$ totally ordered points $P_0 < P_1 < \dots < P_n$ for the above order. We identify \mathcal{C} with the strictly increasing sequence (P_0, \dots, P_n) of its $(n+1)$

points. The initial point P_0 and the final point P_n of \mathcal{C} are extremal vertices of $\text{Conv}(\mathcal{C})$ and are joined by two (generally distinct) piecewise-linear paths $\partial_+(\mathcal{C})$ and $\partial_-(\mathcal{C})$, called the *upper* and *lower* convex hull. The path $\partial_+(\mathcal{C})$ (respectively $\partial_-(\mathcal{C})$) is contained in the boundary $\partial\text{Conv}(\mathcal{C})$ and links P_0 to P_n by turning clockwise (respectively counterclockwise) around the convex hull $\text{Conv}(\mathcal{C})$ of \mathcal{C} .

A *roof* R of \mathcal{C} having length $w = \text{length}(R)$ is an increasing subsequence $R = (P_{i_0} = P_0, P_{i_1}, \dots, P_{i_{w-1}}, P_{i_w} = P_n)$ containing $(w + 1)$ elements.

The *skyline* of a (decorated) roof $R = (P_0, \dots, P_n)$ is the piece-wise linear path from P_0 to P_n obtained by joining consecutive points of R using rectilinear segments. Skylines and roofs are in bijection for generic configurations. If \mathcal{C} is non-generic, several roofs might share a common skyline.

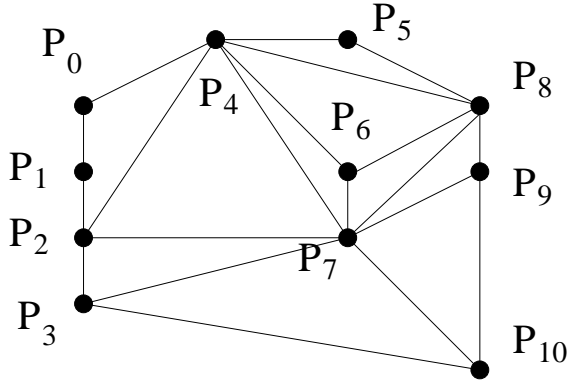


Figure 11: A triangulated configuration of $(10 + 1)$ points

Example. The configuration

$$\mathcal{C} = \{P_0, \dots, P_n\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 3 & 2 & 1 & 0 & 4 & 4 & 2 & 1 & 3 & 2 & -1 \end{pmatrix}$$

represented in Figure 11 has four roofs with skyline $\partial_-\mathcal{C}$:

$$\begin{aligned} & (P_0, P_3, P_{10}) , & (P_0, P_1, P_3, P_{10}) , \\ & (P_0, P_2, P_3, P_{10}) , & (P_0, P_1, P_2, P_3, P_{10}) \end{aligned}$$

and two roofs with skyline $\partial_+\mathcal{C}$:

$$(P_0, P_4, P_5, P_8, P_{10}) \quad (P_0, P_4, P_5, P_8, P_9, P_{10}) .$$

A *decorated roof* is a roof

$$R = (P_{i_0} = P_0, P_{i_1}, \dots, P_{i_{w-1}}, P_{i_w} = P_n) ,$$

decorated by one of the segments joining two consecutive points $P_{i_d}, P_{i_{d+1}}$ of R . We denote such a decorated roof by

$$R = (P_0, \dots, P_{i_{d-1}}, \overline{P_{i_d}, P_{i_{d+1}}}, P_{i_{d+2}}, \dots, P_{i_w} = P_n)$$

and call R *initially decorated* if $d = 0$, i.e. if the decoration starts at the initial point P_0 .

Three non-collinear points $A < B < C \in \mathcal{C}$ are vertices of a triangle *of type V* if B is “above” the line containing A and C . The non-collinear points A, B, C are vertices of a triangle *of type Λ* otherwise. In other terms, a triangle with (necessarily non-collinear) vertices $A < B < C$ is of type Λ if $\det(C - A, B - A) > 0$ and of type V if $\det(C - A, B - A) < 0$.

We call a triangle with non-collinear vertices A, B, C *minimal* if $\{A, B, C\} = \mathcal{C} \cap \text{Conv}(A, B, C)$. In particular, a triangulation is maximal if and only if all its triangles are minimal.

A decorated roof

$$R' = (P_{i_0} = P_0, \dots, P_{i_{k-1}}, \overline{P_{i_k}, P_{i_{k+1}}}, P_{i_{k+2}}, P_{i_{k+3}}, \dots, P_{i_w} = P_n)$$

of length w is a Λ -*successor* of R if R is a decorated roof of length $w - 1$ given by

$$R = (P_{i_0} = P_0, \dots, \overline{P_{i_d}, P_{i_{d+1}}}, \dots, P_{i_{k-1}}, P_{i_k}, P_{i_{k+2}}, P_{i_{k+3}}, \dots, P_{i_w} = P_n)$$

satisfying $P_{i_d} \leq P_{i_k}$ and the points $\{P_{i_k}, P_{i_{k+1}}, P_{i_{k+2}}\} \subset R'$ are vertices of a Λ -triangle.

Otherwise stated, R' is obtained from R by gluing a Λ -triangle with decorated left-upper edge onto an edge of R which is not situated to the left of the decorated edge in R .

Similarly, a decorated roof

$$R' = (P_{i_0} = P_0, \dots, P_{i_{k-1}}, \overline{P_{i_k}, P_{i_{k+2}}}, P_{i_{k+3}}, \dots, P_{i_w} = P_n)$$

of length $w - 1$ is a V -*successor* of R if R is the decorated roof of length w given by

$$R = (P_{i_0} = P_0, \dots, \overline{P_{i_d}, P_{i_{d+1}}}, \dots, P_{i_k}, P_{i_{k+1}}, P_{i_{k+2}}, P_{i_{k+3}}, \dots, P_{i_w} = P_n)$$

satisfying $P_{i_{d+1}} \leq P_{i_{k+2}}$ and the points $\{P_{i_k}, P_{i_{k+1}}, P_{i_{k+2}}\} \subset R$ are vertices of a V -triangle.

Otherwise stated, R' is obtained from R by gluing a V -triangle with decorated upper edge onto two consecutive edges of increasing slope of R and the decorated edge of R is not situated to the right of these two consecutive edges.

Example. Figure 12 displays the decorated roof

$$R = (P_0, P_1, \overline{P_3, P_6}, P_7, P_9, P_{10}, P_{12})$$

of the 12-near-edge

$$E = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

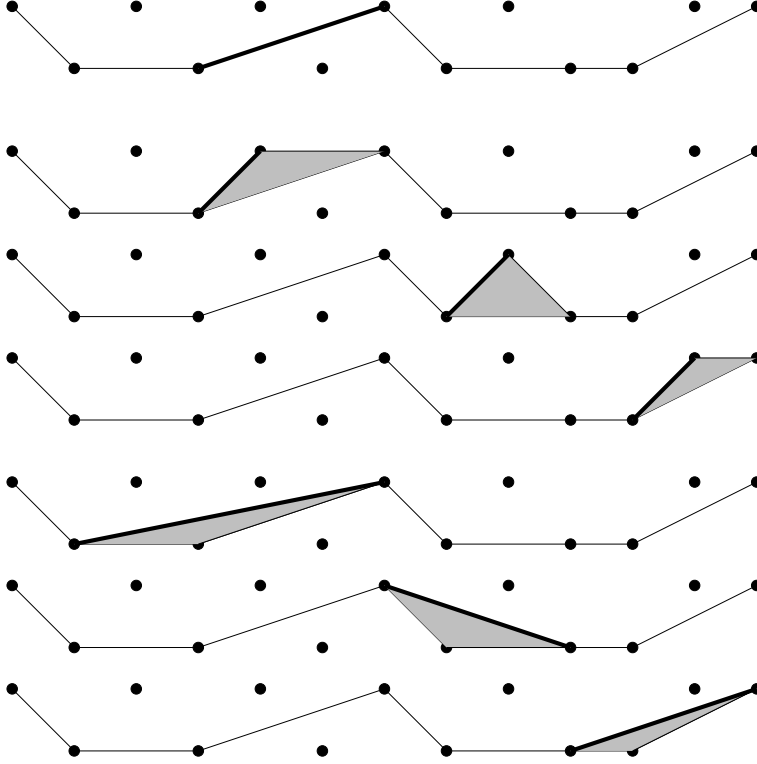


Figure 12: A decorated roof and all its Λ – and V –successors

and all its Λ – and V –successors

$$\begin{aligned}
 & (P_0, P_1, \overline{P_3, P_4}, P_6, P_7, P_9, P_{10}, P_{12}) \\
 & (P_0, P_1, P_3, P_6, \overline{P_7, P_8}, P_9, P_{10}, P_{12}) \\
 & (P_0, P_1, P_3, P_6, P_7, P_9, \overline{P_{10}, P_{11}}, P_{12}) \\
 & (P_0, \overline{P_1, P_6}, P_7, P_9, P_{10}, P_{12}) \\
 & (P_0, P_1, P_3, \overline{P_6, P_9}, P_{10}, P_{12}) \\
 & (P_0, P_1, P_3, P_6, P_7, \overline{P_9, P_{12}})
 \end{aligned}$$

A decorated roof R' is a *successor* of a decorated roof R if R' is either a Λ – or a V –successor of R . We call R' an *immediate successor* of R if the difference between R and R' is a minimal triangle.

Given a finite configuration $\mathcal{C} \subset \mathbf{R}^2$, a (*maximal*) *roof-sequence* is a sequence

$$R_{i_1}, \dots, R_{i_l}$$

of decorated roofs where R_{i_1} is initially decorated and where $R_{i_{j+1}}$ is a (immediate) successor of R_{i_j} for $j = 1, \dots, l-1$. The skyline of R_{i_1} (respectively of R_{i_l}) is the *initial* (respectively *final*) skyline of the roof-sequence.

Proposition 7.1 *The set of all (maximal) roof-sequences R_{i_1}, \dots, R_{i_l} with initial skyline S_α and final skyline S_ω is in bijection with the set of all (maximal) triangulations with vertices in \mathcal{C} of the polygonal region enclosed by S_α and S_ω .*

Proof. Given a roof-sequence $(R_{i_1}, \dots, R_{i_l})$, the union of all associated skylines triangulates (maximally) the polygonal region enclosed by S_{i_1} and S_{i_l} .

Given a (maximal) triangulation of a polygonal region enclosed by two skylines, remove vertically the rightmost triangle which has only “sky” above it (and which is a minimal triangle for a maximal triangulation). Iterating this algorithm one gets an undecorated roof-sequence in reversed order. The decorations can now be added only in a unique way, except for the lowest skyline which we decorate initially. \square

Example. The triangulation of the configuration shown in Figure 11 corresponds to the roof-sequence

$$\begin{aligned} &(\overline{P_0, P_2}, P_3, P_{10}), (P_0, \overline{P_2, P_3}, \overline{P_7}, P_{10}), (P_0, \overline{P_2, P_7}, P_{10}), \\ &(P_0, \overline{P_2, P_4}, P_7, P_{10}), (\overline{P_0, P_4}, P_7, P_{10}), (P_0, \overline{P_4, P_6}, P_7, P_{10}), \\ &(P_0, P_4, P_6, \overline{P_7, P_9}, P_{10}), (P_0, P_4, P_6, \overline{P_7, P_8}, P_9, P_{10}), \\ &(P_0, P_4, \overline{P_6, P_8}, P_9, P_{10}), (P_0, \overline{P_4, P_8}, P_9, P_{10}), (P_0, \overline{P_4, P_5}, P_8, P_9, P_{10}) . \end{aligned}$$

Consider the rational vector space $\mathcal{R} = \mathcal{R}(\mathcal{C})$ (only natural coordinates will in fact occur) spanned by the set of all decorated roofs of a finite configuration $\mathcal{C} \subset \mathbf{R}^2$. Identifying a decorated roof with the corresponding basis element, we define the *transfer-matrix* T as the linear application $T : \mathcal{R} \longrightarrow \mathcal{R}$ which sends a decorated roof to the sum of its successors. We introduce moreover the element $C_k \in \mathcal{R}(E)$ obtained by summing up all initially decorated roofs of length k with skyline $\partial_- \mathcal{C}$. Set $V_0 = 0$ and

$$V_i = C_i + T(V_{i-1}) .$$

We denote by $W : \mathcal{R} \longrightarrow \mathbf{Q}[s^{1/2}]$ the linear map which sends a decorated roof to 0 if its skyline is different from $\partial_+ \mathcal{C}$ and which sends a decorated roof R with skyline $\partial_+ \mathcal{C}$ to $s^{\text{length}(R)/2}$.

Proposition 7.2 *The complete triangulation polynomial of a finite configuration $\mathcal{C} = (P_0, \dots, P_n)$ is given by*

$$s \sum_{k=1}^n s^{k/2} W(V_k) .$$

The proof is a straightforward consequence of Proposition 7.1 and of the behaviour of lengths of successive roofs.

An obvious modification of the above proposition counts triangulations of a polygonal (not necessarily convex) region squeezed between two skylines.

If one is only interested in the number of maximal triangulations of $\text{Conv}(\mathcal{C})$, one considers the transfer matrix \tilde{T} which sends a decorated roof R to the sum of its immediate successors. The vector \tilde{C}_i is zero except for $i + 1 = \sharp(\partial_- \mathcal{C} \cap \mathcal{C})$ and \tilde{C}_i consists then of the initially decorated roof defined by the set $\partial_- \mathcal{C} \cap \mathcal{C}$. We define $\tilde{V}_0 = 0$ and $\tilde{V}_i = \tilde{C}_i + \tilde{T}(\tilde{V}_{i-1})$. Set now $\tilde{W}(R) = 0$ except if R is a decorated roof with underlying set $\partial_+ \mathcal{C} \cap \mathcal{C}$ where we set $\tilde{W}(R) = 1$. The number of maximal triangulations of $\text{Conv}(\mathcal{C})$ is now given by

$$\sum_i \tilde{W}(\tilde{V}_i)$$

(in fact, only one summand is non-zero).

Given a near-edge $E = (P_0, \dots, P_n)$, we introduce the vectors $C_i \in \mathcal{R}(E)$ and the transfer matrix $T \in \text{End}(\mathcal{R}(E))$ as above (where we consider E as a configuration) and compute the vectors $V_0 = 0$,

$$V_i = C_i + T(V_{i-1}), \quad 1 \leq i \leq n-1.$$

We introduce moreover the linear map $\tilde{W} : \mathcal{R}(E) \rightarrow \mathbf{Q}[s^{1/2}, t]$ defined by

$$\begin{aligned} \tilde{W}(R) &= s^{\text{length}(R)/2} p_{\text{length}(R)} \\ &= s^{\text{length}(R)/2} \sum_k (-1)^k \binom{\text{length}(R)-k}{k} t^{\text{length}(R)-k} \end{aligned}$$

for any decorated roof R of \mathcal{C} . One shows now easily the following result.

Proposition 7.3 *The complete edge-polynomial of a finite near-edge $E = (P_0, \dots, P_n)$ is given by*

$$\sum_{k=1}^n s^{k/2} \tilde{W}(V_k).$$

7.1 A few algorithmic remarks

Propositions 7.2 and 7.3 yield easily algorithms for computing complete triangulation and edge polynomials.

These algorithms are of exponential complexity: properly organized their memory and time requirements are of order $2^n f(n)$ where $f(n)$ is polynomial (of degree 1 for memory requirements, neglecting multiprecision problems, and of fairly small degree for time requirements) for a configuration of $n+1$ points. A nice feature of these algorithms is their simplicity: As an example, we run them by hand for the number of triangulations squeezed between two skylines of a configuration having 8 points and for the near-edge E_c of the near-polygon depicted in Figure 2.

For computational purposes, it is useful to have a simple representation for a decorated roof

$$R = (P_{i_0} = P_0, P_{i_1}, \dots, \overline{P_{i_d}, P_{i_{d+1}}}, \dots, P_{i_{w-1}}, P_{i_w} = P_n)$$

of a configuration (or near-edge) $\mathcal{C} = (P_0, \dots, P_n)$. Such a roof R is for instance encoded by the integer $l(R) \in \{0, \dots, n \cdot 2^{n-1} - 1\}$ defined by

$$l(R) = d \cdot 2^{n-1} + \sum_{k=1}^{w-1} 2^{i_k-1}.$$

This yields an injection from the set of decorated roofs (of a given finite configuration (P_0, \dots, P_n)) into a subset of $\sum_{k=1}^n k \binom{n-1}{k-1} = (n+1)2^{n-2}$ natural integers $< n \cdot 2^{n-1}$ for $n > 1$. A natural integer

$$m = \alpha \cdot 2^{n-1} + \sum_{i=0}^{n-2} \epsilon_i 2^i, \quad 0 \leq \alpha \leq n-1, \quad \epsilon_i \in \{0, 1\}$$

is of the form $l(R)$ if and only if either $\alpha = 0$ or $\epsilon_{\alpha-1} = 1$. The corresponding roof R is then given by

$$R = (P_0, \{P_{i+1} \mid 0 \leq i \leq n-2 \text{ and } \epsilon_i = 1\}, P_n)$$

with decoration beginning at P_α .

Let us also remark that many decorated roofs lead to dead ends when computing triangulation polynomials. Indeed, consider a decorated roof of the form

$$R = (P_{i_0}, \dots, P_{i_k}, \dots, \overline{P_{i_d}, P_{i_{d+1}}}, \dots)$$

where $P_{i_k} \in \partial_+ \mathcal{C}$ for $k \leq d$. If the partial skyline defined by the piecewise linear path with segments consecutive points of $(P_{i_0}, \dots, P_{i_k})$ is not contained in $\partial_+ \mathcal{C}$ then there exists no roof-sequence containing R which stops with a roof having skyline $\partial_+(\mathcal{C})$. Indeed, R hits the “ceiling” $\partial_+(\mathcal{C})$ at P_{i_k} before reaching the decorated edge. This freezes the partial skyline $(P_{i_0}, \dots, P_{i_k})$ in all successors (and iterated successors) of R .

If one is only interested in computing the number of maximal triangulations or the maximal edge-polynomial of a near-edge, a few simplifications take place:

For non-generic configurations, only decorated roofs which are *full* have to be considered where a roof is full if its sequence contains all points of \mathcal{C} lying on its skyline.

As already mentionned, the suitable transfer matrix in this case is defined by considering only immediate successors of a decorated roof (recall that a successor is immediate if it is obtained by adjoining a suitable triangle which is minimal, i.e. which intersects \mathcal{C} only in its three vertices).

Every decorated roof appears at most in one vector of V_1, \dots, V_n . This makes memory requirements somewhat smaller.

Let us also add that the algorithm which computes the complete triangulation polynomial of $\text{Conv}(\mathcal{C})$ can be modified for computing the number

of triangulations (with vertices in \mathcal{C}) for any polygonal region enclosed by two suitable non-crossing skylines S_α , S_ω : Replace $\partial_\pm(\mathcal{C})$ by S_α and S_ω .

The resulting algorithm for computing triangulation polynomials or edge-polynomials is straightforward and best illustrated by examples. This will be done in the 2 following subsections.

A few further tricks may be used:

Computing $V_i = C_i + T(V_{i-1})$ needs only C_i and V_{i-1} . This can be used to save memory by computing the contribution $s^{(i+1)/2}W(V_{i-1})$ or $s^{(i-1)/2}\tilde{W}(V_{i-1})$ (in the case of an edge polynomial) of V_{i-1} to the final result and by erasing V_{i-1} after determination of V_i .

Multiprecision problems can be avoided using the Chinese Remainder Theorem. Compute all coefficients of decorated roofs modulo several primes and use them to reconstruct the result (this needs however the computation of an approximation of the final result which can for instance be done by a computation using the type of floating reals).

Decorated roofs should always be totally ordered (using for instance the complete order coming from the labelling $l(R)$ for a decorated roof R). Since there are at most $\sim n2^{n-2}$ such roofs, the coefficient of a decorated roof can then be accessed using roughly $O(n \log(n))$ operations using a classical divide and conquer strategy.

7.2 A triangulation polynomial

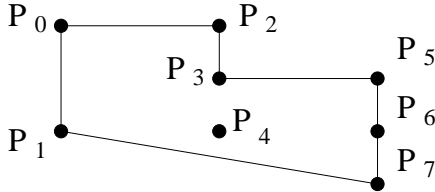


Figure 13: A configuration enclosed by a polygonal line

We want to compute the complete triangulation polynomial of the (non-convex) polygonal region enclosed by the two skylines S_α, S_ω defined by $S_\alpha = (P_0, P_1, P_7)$ and $S_\omega = (P_0, P_2, P_3, P_5, P_6, P_7)$ of the configuration

$$(P_0, \dots, P_7) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 1 & 2 & 1 & 0 \end{pmatrix}$$

depicted in Figure 13.

The following lists contain all relevant data. First we establish the list of all needed decorated roofs, enumerated and denoted using conventions and notations introduced in the previous subsection. For concision, we indicate a decorated roof $R = (P_0, \dots, \overline{P_d, P_{d+1}}, \dots, P_7)$ by the (decorated) sequence $(0 \ i_1 \ \dots \ \overline{i_d \ i_{d+1}} \ \dots 7)$ of the corresponding indices. For every roof R , we

indicate also its image $T(R)$ obtained by applying the transfer operator T , except that roofs in $T(R)$ leading to dead-ends (like $(01\overline{36}7)$ for instance) have been omitted. For a roof R with skyline S_ω , we indicate $s^{\text{length}(R)/2}$ instead of $T(R)$ which is always zero in this case.

Data appearing in boldface can be omitted if one is only interested in the number of maximal triangulations.

R_0	$(\overline{07})$	R_4
R_1	$(\overline{017})$	$\mathbf{R}_0 + \mathbf{R}_{69} + R_{73} + \mathbf{R}_{81} + \mathbf{R}_{97}$
R_4	$(\overline{037})$	R_6
R_6	$(\overline{0237})$	$\mathbf{R}_{150} + R_{166}$
R_8	$(\overline{047})$	$R_0 + R_{12}$
R_9	$(\overline{0147})$	$R_8 + R_{77} + \mathbf{R}_{153} + R_{169}$
R_{12}	$(\overline{0347})$	R_{14}
R_{14}	$(\overline{02347})$	$R_{134} + \mathbf{R}_{222} + R_{238}$
\mathbf{R}_{20}	$(\overline{0357})$	\mathbf{R}_{22}
\mathbf{R}_{22}	$(\overline{02357})$	s^2
R_{36}	$(\overline{0367})$	R_{38}
R_{38}	$(\overline{02367})$	R_{182}
R_{52}	$(\overline{03567})$	R_{54}
R_{54}	$(\overline{023567})$	$s^{5/2}$
R_{69}	$(01\overline{37})$	$R_4 + R_{71}$
R_{71}	$(01\overline{237})$	R_6
R_{73}	$(01\overline{47})$	$R_8 + R_{77} + \mathbf{R}_{153} + R_{169}$
R_{77}	$(01\overline{347})$	$R_{12} + R_{79}$
R_{79}	$(01\overline{2347})$	R_{14}
\mathbf{R}_{81}	$(01\overline{57})$	\mathbf{R}_{85}
\mathbf{R}_{85}	$(01\overline{357})$	$\mathbf{R}_{20} + \mathbf{R}_{87}$
\mathbf{R}_{87}	$(01\overline{2357})$	\mathbf{R}_{22}
\mathbf{R}_{97}	$(01\overline{67})$	$\mathbf{R}_{101} + \mathbf{R}_{113}$
R_{101}	$(01\overline{367})$	$R_{36} + R_{103}$
R_{103}	$(01\overline{2367})$	R_{38}
R_{113}	$(01\overline{567})$	R_{117}
R_{117}	$(01\overline{3567})$	$R_{52} + R_{119}$
R_{119}	$(01\overline{23567})$	R_{54}
R_{134}	$(02\overline{37})$	$\mathbf{R}_{150} + R_{166}$
\mathbf{R}_{150}	$(02\overline{357})$	s^2
\mathbf{R}_{153}	$(01\overline{457})$	\mathbf{R}_{81}
R_{166}	$(02\overline{367})$	R_{182}
R_{169}	$(01\overline{467})$	R_{185}
R_{182}	$(02\overline{3567})$	$s^{5/2}$
R_{185}	$(01\overline{4567})$	R_{113}

$$\begin{array}{lll}
\mathbf{R}_{222} & (023\overline{45}7) & \mathbf{R}_{150} \\
R_{238} & (023\overline{46}7) & R_{166} + R_{254} \\
R_{254} & (023\overline{45}67) & R_{182}
\end{array}$$

The vectors C_i are all 0 except $C_2 = R_1$. We have then $V_1 = 0$, $V_2 = C_2 = R_1$ and $V_i = T(V_{i-1})$, $i = 3, \dots, 9$ where T is the transfer operator.

$$\begin{aligned}
V_2 &= C_2 = R_1 \\
V_3 &= \mathbf{R}_0 + \mathbf{R}_{69} + R_{73} + \mathbf{R}_{81} + \mathbf{R}_{97} \\
V_4 &= \mathbf{R}_4 + (\mathbf{R}_4 + \mathbf{R}_{71}) + (R_8 + R_{77} + \mathbf{R}_{153} + R_{169}) + \mathbf{R}_{85} + (\mathbf{R}_{101} + \mathbf{R}_{113}) \\
&= 2\mathbf{R}_4 + R_8 + \mathbf{R}_{71} + R_{77} + \mathbf{R}_{85} + \mathbf{R}_{101} + \mathbf{R}_{113} + \mathbf{R}_{153} + R_{169} \\
V_5 &= 2\mathbf{R}_6 + (R_0 + R_{12}) + \mathbf{R}_6 + (R_{12} + R_{79}) + (\mathbf{R}_{20} + \mathbf{R}_{87}) \\
&\quad + (\mathbf{R}_{36} + \mathbf{R}_{103}) + \mathbf{R}_{117} + \mathbf{R}_{81} + R_{185} \\
&= R_0 + 3\mathbf{R}_6 + 2R_{12} + \mathbf{R}_{20} + \mathbf{R}_{36} + R_{79} + \mathbf{R}_{81} \\
&\quad + \mathbf{R}_{87} + \mathbf{R}_{103} + \mathbf{R}_{117} + R_{185} \\
V_6 &= R_4 + 3(\mathbf{R}_{150} + \mathbf{R}_{166}) + 2R_{14} + \mathbf{R}_{22} + \mathbf{R}_{38} + R_{14} + \mathbf{R}_{85} \\
&\quad + \mathbf{R}_{22} + \mathbf{R}_{38} + (\mathbf{R}_{52} + \mathbf{R}_{119}) + R_{113} \\
&= R_4 + 3R_{14} + 2\mathbf{R}_{22} + 2\mathbf{R}_{38} \\
&\quad + \mathbf{R}_{52} + \mathbf{R}_{85} + R_{113} + \mathbf{R}_{119} + 3\mathbf{R}_{150} + 3\mathbf{R}_{166} \\
V_7 &= R_6 + 3(R_{134} + \mathbf{R}_{222} + R_{238}) + 2 \cdot \mathbf{0} + 2\mathbf{R}_{182} \\
&\quad + \mathbf{R}_{54} + (\mathbf{R}_{20} + \mathbf{R}_{87}) + R_{117} + \mathbf{R}_{54} + 3 \cdot \mathbf{0} + 3\mathbf{R}_{182} \\
&= R_6 + \mathbf{R}_{20} + 2\mathbf{R}_{54} + \mathbf{R}_{87} + R_{117} \\
&\quad + 3R_{134} + 5\mathbf{R}_{182} + 3\mathbf{R}_{222} + 3R_{238} \\
V_8 &= (\mathbf{R}_{150} + R_{166}) + \mathbf{R}_{22} + 2 \cdot \mathbf{0} + \mathbf{R}_{22} + (R_{52} + R_{119}) \\
&\quad + 3(\mathbf{R}_{150} + R_{166}) + 5 \cdot \mathbf{0} + 3\mathbf{R}_{150} + 3(R_{166} + R_{254}) \\
&= 2\mathbf{R}_{22} + R_{52} + R_{119} + 7\mathbf{R}_{150} + 7R_{166} + 3R_{254} \\
V_9 &= 2 \cdot \mathbf{0} + R_{54} + R_{54} + 7 \cdot \mathbf{0} + 7R_{182} + 3R_{182} \\
&= 2R_{54} + 10R_{182} \\
V_{10} &= 0
\end{aligned}$$

We have $W(V_2) = \dots = W(V_5) = 0$ and

$$(W(V_6), \dots, W(V_9)) = (5s^2, 7s^{5/2}, 9s^2, 12s^{7/2})$$

and the complete triangulation polynomial of the polygonal region between S_α and S_ω is given by

$$s(s^3 5s^2 + s^{7/2} 7s^{5/2} + s^4 9s^2 + s^{9/2} 12s^{7/2}) = 12s^8 + 16s^7 + 5s^6.$$

7.3 An edge-polynomial

We consider the 5-near-edge $E = E_c$ represented in Figure 14 of the near-polygon appearing in Figure 2. This near-edge $E = (P_0, \dots, P_5)$ can be represented by

$$E = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{pmatrix}$$

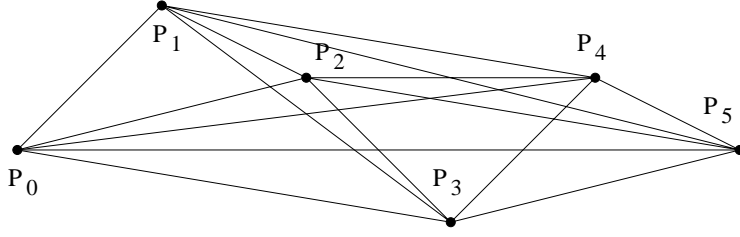


Figure 14: The near-edge E_c

In the following table we list all decorated roofs arising during the computations. The first two columns list all 23 decorated roofs needed for the computation, the third column yields the image $T(R_i)$ (where T denotes the transfer matrix) of the decorated roof R_i described by the previous columns. The 6–th line for instance lists the roof $(\overline{P_0}, \overline{P_1}, P_3, P_5)$ encoded by $(\overline{01}35)$ and wich we call R_5 using the labelling introduced previously. An easy computation shows that

$$T(R_5) = (0\overline{1}5) + (0\overline{1}235) + (013\overline{4}5) .$$

Using our numerotation of decorated roofs, we have thus

$$T(R_5) = R_{17} + R_{23} + R_{45} .$$

Data appearing in boldface is only involved in the computation of the complete edge-polynomial and can be omitted for the computation of the maximal edge-polynomial.

R_0	$(\overline{05})$	$(\overline{015})+(\overline{025})+(\overline{045})$	$\mathbf{R}_1+R_2+R_8$
\mathbf{R}_1	$(\overline{015})$	$(\overline{0145})$	R_{25}
R_2	$(\overline{025})$	$(\overline{0125})+(\overline{0245})$	R_3+R_{26}
R_3	$(\overline{0125})$	$(\overline{015})+(01245)$	$R_{17}+R_{43}$
R_4	$(\overline{035})$	$(\overline{05})+(\overline{0135})+(\overline{0235})+(\overline{0345})$	$R_0+R_5+R_6+R_{28}$
R_5	$(\overline{0135})$	$(\overline{015})+(\overline{01235})+(01345)$	$\mathbf{R}_{17}+R_{23}+R_{45}$
R_6	$(\overline{0235})$	$(\overline{025})+(\overline{01235})+(02345)$	$R_{18}+R_7+R_{46}$
R_7	$(\overline{01235})$	$(0125)+(012345)$	$R_{35}+R_{63}$
R_8	$(\overline{045})$	$(\overline{0145})+(\overline{0245})$	\mathbf{R}_9+R_{10}
\mathbf{R}_9	$(\overline{0145})$	0	0
R_{10}	$(\overline{0245})$	$(\overline{01245})$	R_{11}
R_{11}	$(\overline{01245})$	$(\overline{0145})$	R_{25}
R_{17}	$(\overline{015})$	$(\overline{0145})$	R_{25}
R_{18}	$(\overline{025})$	$(\overline{0245})$	R_{26}
R_{23}	$(\overline{01235})$	$(0125)+(012345)$	$R_{35}+R_{63}$
R_{25}	$(\overline{0145})$	0	0
R_{26}	$(\overline{0245})$	0	0
R_{28}	$(\overline{0345})$	$(\overline{045})$	R_8
R_{35}	(0125)	$(\overline{015})+(01245)$	$R_{17}+R_{43}$
R_{43}	(01245)	$(\overline{0145})$	R_{25}
R_{45}	(01345)	$(\overline{0145})$	\mathbf{R}_{25}
R_{46}	(02345)	$(\overline{0245})$	R_{26}
R_{63}	(012345)	(01245)	R_{43}

The vectors C_i are 0 except $C_2 = R_4$.

$$\begin{aligned}
V_1 &= C_1 = 0 \\
V_2 &= T(V_1) + C_2 = C_2 = R_4 \\
V_3 &= T(V_2) + C_3 = R_0 + R_5 + R_6 + R_{28} \\
V_4 &= T(V_3) + C_4 = (\mathbf{R}_1+R_2+R_8) + (\mathbf{R}_{17}+R_{23}+R_{45}) \\
&\quad + (R_{18}+R_7+R_{46}) + R_8 \\
&= \mathbf{R}_1+R_2+R_7+2R_8+\mathbf{R}_{17}+R_{18}+R_{23}+R_{45}+R_{46} \\
V_5 &= T(V_4) + C_5 = \mathbf{R}_{25}+(R_3+R_{26}) + (R_{35}+R_{63}) + 2(\mathbf{R}_9+R_{10}) \\
&\quad + \mathbf{R}_{25}+R_{26} + (R_{35}+R_{63})+\mathbf{R}_{25}+R_{26} \\
&= R_3+2\mathbf{R}_9+2R_{10}+3\mathbf{R}_{25}+3R_{26}+2R_{35}+2R_{63} \\
V_6 &= T(V_5) + C_6 = (R_{17}+R_{43})+2\cdot\mathbf{0}+2R_{11}+3\cdot\mathbf{0}+3\cdot\mathbf{0} \\
&\quad +2(R_{17}+R_{43})+2R_{43} \\
&= 2R_{11}+3R_{17}+5R_{43} \\
V_7 &= T(V_6) + C_7 = 3R_{25}+5R_{25}+2R_{25} \\
&= 10R_{25}
\end{aligned}$$

$$\begin{aligned}
\bar{p}(E) &= \sum_i s^{i/2} \tilde{W}(V_i) \\
&= p_2 s^2 + (p_1 s^2 + 3p_3 s^3) + (6p_2 s^3 + 4p_4 s^4) \\
&\quad + (13p_3 s^4 + 2p_5 s^5) + (3p_2 s^4 + 7p_4 s^5) + 10p_3 s^5 \\
&= (10p_3 + 7p_4 + 2p_5) s^5 + (3p_2 + 13p_3 + 4p_4) s^4 \\
&\quad + (6p_2 + 3p_3) s^3 + (p_1 + p_2) s^2
\end{aligned}$$

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