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Raphaël Rouquier

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# DIMENSIONS OF TRIANGULATED CATEGORIES

RAPHAËL ROUQUIER

ABSTRACT. We define a dimension for a triangulated category. We prove a representability Theorem for a certain class of functors on finite dimensional triangulated categories. We study the dimension of the bounded derived category of an algebra or a scheme and we show in particular that the bounded derived category of coherent sheaves over a variety has a finite dimension. For a self-injective algebra, a lower bound for Auslander's representation dimension is given by the dimension of the stable category. We use this to compute the representation dimension of exterior algebras. This provides the first known examples of representation dimension  $> 3$ . We deduce that the Loewy length of the group algebra over  $\mathbf{F}_2$  of a finite group is strictly bounded below by 2-rank of the group (a conjecture of Benson).

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## 1. INTRODUCTION

In his 1971 Queen Mary College notes [Au], Auslander introduced an invariant of finite dimensional algebras, the representation dimension. It was meant to measure how far an algebra is to having only finitely many classes of indecomposable modules. Whereas many upper bounds have been found for the representation dimension, lower bounds were missing. In particular, it wasn't known whether the representation dimension could be greater than 3. A proof that all algebras have representation dimension 3 would have led for example to a solution of the finitistic dimension conjecture [IgTo].

We prove here that the representation dimension of the exterior algebra of a finite dimensional vector space is one plus the dimension of that vector space — in particular, the representation dimension can be arbitrarily large. Thus, the representation dimension is a useful invariant of finite dimensional algebras of infinite representation type, confirming the hope of Auslander. The case of algebras with infinite global dimension is particularly interesting.

As a consequence of our results, we prove the characteristic  $p = 2$  case of a conjecture of Benson asserting that the  $p$ -rank of a finite group is less than the Loewy length of its group algebra over a field of characteristic  $p$ .

Our approach to these problems is to define and study a “dimension” for triangulated categories. This is inspired by Bondal and Van den Bergh's work [BoVdB] and we generalize some of their main results.

Whereas Brown's representability Theorem for cohomological functors deals with triangulated categories with infinite direct sums (typically, the unbounded derived category of an algebra or a scheme), we prove here a representability Theorem for finite dimensional triangulated categories (typically, the bounded derived category of (finitely generated) modules over a (noetherian) algebra or of (quasi-)coherent sheaves over a quasi-projective scheme).

We study the dimension for derived categories of algebras and schemes, providing lower and upper bounds in various cases. The exact computation is achieved only in few cases.

Let us review the content of the chapters. In a first part §3, we review various types of generation of triangulated categories and we define a dimension for triangulated categories. Part §4 is a study of analogs of bounded complexes of finitely generated objects in a triangulated category. We analyze the corresponding cohomological functor. The finiteness condition we introduce turns out to provide a new representability Theorem of Brown type, for triangulated categories without infinite direct sums but with a strong generator (Theorem 4.14). This extends a main result of [BoVdB] which dealt only with Ext-finite categories. We define a notion of (homological) regularity for a triangulated category.

In §5, we analyze the dimension of derived categories in algebra and geometry. We show that the derived category of coherent sheaves  $D^b(X\text{-coh})$  for a separated scheme  $X$  of finite type over a perfect field has finite dimension (Theorem 5.38). We deduce that  $D^b(X\text{-coh})$  is equivalent to the category of perfect complexes over a dg-algebra  $A$  such that  $D(A)$  is regular. In the smooth case, the finiteness is a result of Kontsevich [BoVdB]. We give here an upper bound in the smooth quasi-projective case :  $\dim D^b(X\text{-coh}) \leq 2 \dim X$  (Proposition 5.8).

We prove that the dimension of the scheme is a lower bound for the dimension of the derived category (Proposition 5.36) and that the dimensions coincide in the smooth affine case (Theorem 5.37), as well as for some particular classes of varieties, including  $\mathbf{P}^n$  where our result takes the much more precise form of Beilinson's Theorem (§5.1.2). We prove that the derived category

of a ring is regular if and only if the ring has finite global dimension and the derived category of quasi-coherent sheaves over a quasi-projective scheme over a field is regular if and only if the scheme is regular (Propositions 5.23 and 5.34).

Finally, in §6, we analyze the dimension of the stable category of a self-injective algebra, in relation with Auslander's representation dimension. Via Koszul duality, we compute these dimensions for the exterior algebra of a finite dimensional vector space (Theorem 6.10). This enables us to settle the characteristic 2 case of a conjecture of Benson (Theorem 6.14).

Preliminary results have been obtained and exposed at the conference "Twenty years of tilting theory" in Fraueninsel in November 2002. I wish to thank the organizers for giving me the opportunity to report on these early results and the participants for many useful discussions, particularly Thorsten Holm for introducing me to Auslander's work.

The geometric part of this work was motivated by lectures given by A. A. Beilinson at the University of Chicago and by discussions with A. Bondal.

## 2. NOTATIONS AND TERMINOLOGY

For  $\mathcal{C}$  an additive category and  $\mathcal{I}$  a subcategory of  $\mathcal{C}$ , we denote by  $\text{add}(\mathcal{I})$  (resp.  $\overline{\text{add}}(\mathcal{I})$ ) the smallest additive full subcategory of  $\mathcal{C}$  containing  $\mathcal{I}$  and closed under taking direct summands (resp. and closed under direct sums). We say that  $\mathcal{I}$  is *dense* if every object of  $\mathcal{C}$  is isomorphic to a direct summand of an object of  $\mathcal{I}$ .

We denote by  $\mathcal{C}^\circ$  the category opposite to  $\mathcal{C}$ . We identify a set of objects of  $\mathcal{C}$  with the full subcategory with the corresponding set of objects.

Let  $\mathcal{T}$  be a triangulated category. A *thick* subcategory of  $\mathcal{T}$  is a dense triangulated subcategory. Given  $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow$  a distinguished triangle, then  $Z$  is called a *cone* of  $f$  and  $X$  a *cocone* of  $g$ .

Let  $A$  be a differential graded (=dg) algebra. We denote by  $D(A)$  the derived category of dg  $A$ -modules and by  $A\text{-perf}$  the category of *perfect complexes*, *i.e.*, the smallest thick subcategory of  $D(A)$  containing  $A$ .

Let  $A$  be a ring. We denote by  $A\text{-Mod}$  the category of left  $A$ -modules, by  $A\text{-mod}$  the category of finitely generated left  $A$ -modules, by  $A\text{-Proj}$  the category of projective  $A$ -modules and by  $A\text{-proj}$  the category of finitely generated projective  $A$ -modules. We denote by  $\text{gldim } A$  the global dimension of  $A$ . For  $M$  an  $A$ -module, we denote by  $\text{pdim } M$  the projective dimension of  $M$ . We denote by  $A^\circ$  the opposite ring to  $A$ . For  $A$  an algebra over a commutative ring  $k$ , we put  $A^{\text{en}} = A \otimes_k A^\circ$ .

Let  $X$  be a scheme. We denote by  $X\text{-coh}$  (resp.  $X\text{-qcoh}$ ) the category of coherent (resp. quasi-coherent) sheaves on  $X$ . A complex of sheaves of  $\mathcal{O}_X$ -modules is *perfect* if it is locally isomorphic to a bounded complex of vector bundles (=locally free sheaves of finite rank).

Let  $C$  be a complex of objects of an additive category and  $i \in \mathbf{Z}$ . We put  $\sigma^{\leq i} C = \cdots \rightarrow C^{i-1} \rightarrow C^i \rightarrow 0$  and  $\sigma^{\geq i} C = 0 \rightarrow C^i \rightarrow C^{i+1} \rightarrow \cdots$ . Let now  $C$  be a complex of objects of an abelian category. We put  $\tau^{\geq i} C = 0 \rightarrow C^i / \text{im } d^{i-1} \rightarrow C^{i+1} \rightarrow C^{i+2} \rightarrow \cdots$  and  $\tau^{\leq i} C = \cdots \rightarrow C^{i-2} \rightarrow C^{i-1} \rightarrow \ker d^i \rightarrow 0$ .

## 3. DIMENSION

### 3.1. Dimension for triangulated categories.

3.1.1. We review here various types of generation of triangulated categories, including the crucial “strong generation” due to Bondal and Van den Bergh.

Let  $\mathcal{T}$  be a triangulated category.

Let  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be two subcategories of  $\mathcal{T}$ . We denote by  $\mathcal{I}_1 * \mathcal{I}_2$  the full subcategory of  $\mathcal{T}$  consisting of objects  $M$  such that there is a distinguished triangle  $M_1 \rightarrow M \rightarrow M_2 \rightsquigarrow$  with  $M_i \in \mathcal{I}_i$ .

Let  $\mathcal{I}$  be a subcategory of  $\mathcal{T}$ . We denote by  $\langle \mathcal{I} \rangle$  the smallest full subcategory of  $\mathcal{C}$  containing  $\mathcal{I}$  and closed under finite direct sums, direct summands and shifts. We denote by  $\overline{\mathcal{I}}$  the smallest full subcategory of  $\mathcal{C}$  containing  $\mathcal{I}$  and closed under direct sums and shifts.

We put  $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$ .

We put  $\langle \mathcal{I} \rangle_0 = 0$  and we define by induction  $\langle \mathcal{I} \rangle_i = \langle \mathcal{I} \rangle_{i-1} \diamond \langle \mathcal{I} \rangle$  for  $i \geq 1$ . We put  $\langle \mathcal{I} \rangle_\infty = \bigcup_{i \geq 0} \langle \mathcal{I} \rangle_i$ .

The objects of  $\langle \mathcal{I} \rangle_i$  are the direct summands of the objects obtained by taking a  $i$ -fold extension of finite direct sums of shifts of objects of  $\mathcal{I}$ .

We will also write  $\langle \mathcal{I} \rangle_{\mathcal{T}, i}$  when there is some ambiguity about  $\mathcal{T}$ .

We say that

- $\mathcal{I}$  *generates*  $\mathcal{T}$  if given  $C \in \mathcal{T}$  with  $\text{Hom}_{\mathcal{C}}(D[i], C) = 0$  for all  $D \in \mathcal{I}$  and all  $i \in \mathbf{Z}$ , then  $C = 0$
- $\mathcal{I}$  is a  *$d$ -step generator* of  $\mathcal{T}$  if  $\mathcal{T} = \langle \mathcal{I} \rangle_d$  (where  $d \in \mathbf{N} \cup \{\infty\}$ )
- $\mathcal{I}$  is a *complete  $d$ -step generator* of  $\mathcal{T}$  if  $\mathcal{T} = \langle \overline{\mathcal{I}} \rangle_d$  (where  $d \in \mathbf{N} \cup \{\infty\}$ ).

We say that  $\mathcal{T}$  is

- *finitely generated* if there exists  $C \in \mathcal{T}$  which generates  $\mathcal{T}$  (such a  $C$  is called a *generator*)
- *classically finitely (completely) generated* if there exists  $C \in \mathcal{T}$  which is a (complete)  $\infty$ -step generator of  $\mathcal{T}$  (such a  $C$  is called a *classical (complete) generator*)
- *strongly finitely (completely) generated* if there exists  $C \in \mathcal{T}$  which is a (complete)  $d$ -step generator of  $\mathcal{T}$  for some  $d \in \mathbf{N}$  (such a  $C$  is called a *strong (complete) generator*).

Note that  $C$  is a classical generator of  $\mathcal{T}$  if and only if  $\mathcal{T}$  is the smallest thick subcategory of  $\mathcal{T}$  containing  $C$ . Note also that if  $\mathcal{T}$  is strongly finitely generated, then every classical generator is a strong generator.

It will also be useful to allow only certain infinite direct sums. We define  $\widetilde{\mathcal{I}}$  to be the smallest full subcategory of  $\mathcal{T}$  closed under finite direct sums and shifts and containing multiples of objects of  $\mathcal{I}$  (i.e., for  $X \in \mathcal{I}$  and  $E$  a set such that  $X^{(E)}$  exists in  $\mathcal{T}$ , then  $X^{(E)} \in \widetilde{\mathcal{I}}$ ).

3.1.2. We now define a dimension for a triangulated category.

**Definition 3.1.** *The dimension of  $\mathcal{T}$ , denoted by  $\dim \mathcal{T}$ , is the minimal integer  $d \geq 0$  such that there is  $M$  in  $\mathcal{T}$  with  $\mathcal{T} = \langle M \rangle_{d+1}$ .*

*We define the dimension to be  $\infty$  when there is no such  $M$ .*

The following Lemmas are clear.

**Lemma 3.2.** *Let  $\mathcal{T}'$  be a dense full triangulated subcategory of  $\mathcal{T}$ . Then,  $\dim \mathcal{T} = \dim \mathcal{T}'$ .*

**Lemma 3.3.** *Let  $F : \mathcal{T} \rightarrow \mathcal{T}'$  be a triangulated functor with dense image. If  $\mathcal{T} = \langle \mathcal{I} \rangle_d$ , then  $\mathcal{T}' = \langle F(\mathcal{I}) \rangle_d$ . So,  $\dim \mathcal{T}' \leq \dim \mathcal{T}$ .*

*In particular, let  $\mathcal{I}$  be a thick subcategory of  $\mathcal{T}$ . Then,  $\dim \mathcal{T}/\mathcal{I} \leq \dim \mathcal{T}$ .*

**Lemma 3.4.** *Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be two triangulated subcategories of  $\mathcal{T}$  such that  $\mathcal{T} = \mathcal{T}_1 \diamond \mathcal{T}_2$ . Then,  $\dim \mathcal{T} \leq 1 + \dim \mathcal{T}_1 + \dim \mathcal{T}_2$ .*

### 3.2. Remarks on generation.

#### 3.2.1.

**Remark 3.5.** One can strengthen the notion of generation of  $\mathcal{T}$  by  $\mathcal{I}$  by the stronger requirement that  $\mathcal{T}$  is the smallest triangulated subcategory containing  $\mathcal{I}$  and closed under direct sums. Cf Theorem 4.20 for a case where both notions coincide.

**Remark 3.6.** Let  $\langle \mathcal{I} \rangle'$  be the smallest full subcategory of  $\mathcal{T}$  containing  $\mathcal{I}$  and closed under finite direct sums and shifts. Define similarly  $\langle \mathcal{I} \rangle'_d$ . Then,  $\mathcal{I}$  is a classical generator of  $\mathcal{T}$  if and only if the triangulated subcategory  $\langle \mathcal{I} \rangle'_\infty$  of  $\mathcal{T}$  is dense. By Thomason's characterization of dense subcategories (Theorem 4.26 below), if  $\mathcal{I}$  classically generates  $\mathcal{T}$  and the classes of objects of  $\mathcal{I}$  generate the abelian group  $K_0(\mathcal{T})$ , then  $\mathcal{T} = \langle \mathcal{I} \rangle'_\infty$ .

A similar statement does not hold in general for  $d$ -step generation,  $d \in \mathbf{N}$  : take  $\mathcal{T} = D^b((k \times k)\text{-mod})$ , where  $k$  is a field. Let  $\mathcal{I}$  be the full subcategory containing  $k \times k$  and  $k \times 0$  (viewed as complexes concentrated in degree 0). Then,  $\mathcal{T} = \langle \mathcal{I} \rangle$  and  $K_0(\mathcal{T}) = \mathbf{Z} \times \mathbf{Z}$  is generated by the classes of objects of  $\mathcal{I}$ , but  $\langle \mathcal{I} \rangle'$  is not a triangulated subcategory of  $\mathcal{T}$ .

Note the necessity of allowing direct summands when  $K_0(\mathcal{T})$  is not a finitely generated group (eg., when  $\mathcal{T} = D^b(X\text{-coh})$  and  $X$  is an elliptic curve).

**Remark 3.7.** It would be interesting to study the ‘‘Krull dimension’’ as well. We say that a thick subcategory  $\mathcal{I}$  of  $\mathcal{T}$  is irreducible if given two thick subcategories  $\mathcal{I}_1$  and  $\mathcal{I}_2$  of  $\mathcal{I}$  such that  $\mathcal{I}$  is classically generated by  $\mathcal{I}_1 * \mathcal{I}_2$ , then  $\mathcal{I}_1 = \mathcal{I}$  or  $\mathcal{I}_2 = \mathcal{I}$ . We define the Krull dimension of  $\mathcal{T}$  as the maximal integer  $n$  such that there is a chain of thick irreducible subcategories  $0 \neq \mathcal{I}_0 \subset \mathcal{I}_1 \subset \dots \subset \mathcal{I}_n = \mathcal{T}$  with  $\mathcal{I}_i \neq \mathcal{I}_{i+1}$ .

By Hopkins-Neeman's Theorem [Nee1], given a commutative noetherian ring  $A$ , the Krull dimension of the category of perfect complexes of  $A$ -modules is the Krull dimension of  $A$ .

By [BeCaRi], given a finite  $p$ -group  $P$ , the Krull dimension of the stable category of finite dimensional representations of  $P$  over a field of characteristic  $p$  is the  $p$ -rank of  $P$  minus 1.

Another approach would be to study the maximal possible value for the transcendence degree of the field of fractions of the center of  $\bigoplus_{i \in \mathbf{Z}} \text{Hom}(\text{Id}_{\mathcal{T}/\mathcal{I}}, \text{Id}_{\mathcal{T}/\mathcal{I}}[i])$ , where  $\mathcal{I}$  runs over finitely generated thick subcategories of  $\mathcal{T}$ .

**Remark 3.8.** When  $\mathcal{T}$  has finite dimension, every classical generator is a strong generator. It would be interesting to study the supremum, over all classical generators  $M$  of  $\mathcal{T}$ , of  $\min\{d \mid \mathcal{T} = \langle M \rangle_{1+d}\}$ .

**Remark 3.9.** One can study also, as a dimension, the minimal integer  $d \geq 0$  such that there is  $M$  in  $\mathcal{T}$  with  $\mathcal{T} = \langle \overline{M} \rangle_{d+1}$  or  $\mathcal{T} = \langle \widetilde{M} \rangle_{d+1}$ . This is of interest for  $D^b(A)$  or  $D^b(X\text{-qcoh})$ .

3.2.2. We often obtain dévissages of objects in the following functorial way (yet another notion of dimension...):

Assume there is  $M \in \mathcal{T}$ , triangulated functors  $F_i : \mathcal{T} \rightarrow \mathcal{T}$  with image in  $\langle \overline{M} \rangle$  for  $1 \leq i \leq d$ , triangulated functors  $G_i : \mathcal{T} \rightarrow \mathcal{T}$  for  $0 \leq i \leq d$  with  $G_0 = \text{Id}$ ,  $G_d = 0$  and distinguished triangles  $F_i \rightarrow G_i \rightarrow G_{i-1} \rightsquigarrow$  for  $1 \leq i \leq d$ . Then,  $\mathcal{T} = \langle \overline{M} \rangle_d$ .

### 3.3. Compact objects.

3.3.1. Let  $\mathcal{C}$  be an additive category. We say that  $\mathcal{C}$  is *cocomplete* if arbitrary direct sums exist in  $\mathcal{C}$ .

An object  $C \in \mathcal{C}$  is *compact* if for every set  $\mathcal{F}$  of objects of  $\mathcal{C}$  such that  $\bigoplus_{F \in \mathcal{F}} F$  exists, then the canonical map  $\bigoplus_{F \in \mathcal{F}} \text{Hom}(C, F) \rightarrow \text{Hom}(C, \bigoplus_{F \in \mathcal{F}} F)$  is an isomorphism. We denote by  $\mathcal{C}^c$  the set of compact objects of  $\mathcal{C}$ .

A triangulated category is *compactly generated* if it is generated by a set of compact objects.

3.3.2. Let  $\mathcal{T}$  be a triangulated category. Let  $X_0 \xrightarrow{s_0} X_1 \xrightarrow{s_1} \dots$  be a sequence of objects and maps of  $\mathcal{T}$ . If  $\bigoplus_{i \geq 0} X_i$  exists, then the *homotopy colimit* of the sequence, denoted by  $\text{hocolim } X_i$ , is a cone of the morphism  $\sum_i \text{id}_{X_i} - s_i : \bigoplus_{i \geq 0} X_i \rightarrow \bigoplus_{i \geq 0} X_i$ .

We have a canonical map

$$\text{colim } \text{Hom}_{\mathcal{T}}(Y, X_i) \rightarrow \text{Hom}_{\mathcal{T}}(Y, \text{hocolim } X_i)$$

that makes the following diagram commutative

$$\begin{array}{ccccccccc} \text{Hom}(Y, \bigoplus X_i) & \longrightarrow & \text{Hom}(Y, \bigoplus X_i) & \longrightarrow & \text{Hom}(Y, \text{hocolim } X_i) & \longrightarrow & \text{Hom}(Y, \bigoplus X_i[1]) & \longrightarrow & \text{Hom}(Y, \bigoplus X_i[1]) \\ \uparrow & & \uparrow & & \uparrow \text{ (dotted)} & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \bigoplus \text{Hom}(Y, X_i) & \longrightarrow & \text{colim } \text{Hom}(Y, X_i) & \xrightarrow{0} & \bigoplus \text{Hom}(Y, X_i[1]) & \longrightarrow & \bigoplus \text{Hom}(Y, X_i[1]) \end{array}$$

Since the horizontal sequences of the diagram above are exact, we deduce (cf e.g. [Nee2, Lemma 1.5]) :

**Lemma 3.10.** *The canonical map  $\text{colim } \text{Hom}_{\mathcal{T}}(Y, X_i) \rightarrow \text{Hom}_{\mathcal{T}}(Y, \text{hocolim } X_i)$  is an isomorphism if  $Y$  is compact.*

We now combine the commutation of  $\text{Hom}(Y, -)$  with colimits and with direct sums in the following result (making more precise a classical result [Nee2, Lemma 2.3]) :

**Proposition 3.11.** *Let  $0 = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots$  be a directed system in  $\mathcal{T}$ , let  $\mathcal{F}_i$  be a set of compact objects such that  $\bigoplus_{C \in \mathcal{F}_i} C$  exists and let  $X_{i-1} \rightarrow X_i \rightarrow \bigoplus_{C \in \mathcal{F}_i} C \rightsquigarrow$  be distinguished triangles, for  $i \geq 1$ .*

*Let  $Y$  be a compact object and  $f : Y \rightarrow \text{hocolim } X_i$ . Then, there is an integer  $r \geq 1$ , a finite subset  $\mathcal{F}'_i$  of  $\mathcal{F}_i$  for  $1 \leq i \leq r$  and a commutative diagram*

$$\begin{array}{ccccccccccc} 0 = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \dots & \longrightarrow & X_r \\ \uparrow & & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & & & \uparrow \\ & & \bigoplus_{\mathcal{F}_1} C & & \bigoplus_{\mathcal{F}_2} C & & \bigoplus_{\mathcal{F}_3} C & & & & h \\ & & \uparrow & & \uparrow & & \uparrow & & & & \\ & & \bigoplus_{\mathcal{F}'_1} C & & \bigoplus_{\mathcal{F}'_2} C & & \bigoplus_{\mathcal{F}'_3} C & & & & \\ & & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & & & \\ 0 = X'_0 & \longrightarrow & X'_1 & \longrightarrow & X'_2 & \longrightarrow & X'_3 & \longrightarrow & \dots & \longrightarrow & X'_r \end{array}$$

such that  $f$  factors through  $X'_r \xrightarrow{h} X_r \xrightarrow{\text{can}} \text{hocolim } X_i$ .

*Proof.* By Lemma 3.10, there is  $d \geq 1$  such that  $f$  factors through the canonical map  $X_d \rightarrow \text{hocolim } X_i$ . We proceed now by induction on  $d$ . The composite map  $Y \rightarrow X_d \rightarrow \bigoplus_{C \in \mathcal{F}_d} C$  factors through the sum indexed by a finite subset  $\mathcal{F}'_d$  of  $\mathcal{F}_d$ . Let  $Z$  be the cocone of the corresponding map  $Y \rightarrow \bigoplus_{C \in \mathcal{F}'_d} C$  and  $X''_d$  the cocone of the composite map  $\bigoplus_{C \in \mathcal{F}'_d} C \rightarrow \bigoplus_{C \in \mathcal{F}_d} C \rightarrow X_{d-1}[1]$ . The composite map  $X''_d \rightarrow \bigoplus_{C \in \mathcal{F}'_d} C \rightarrow \bigoplus_{C \in \mathcal{F}_d} C$  factors through  $X_d$ . The map  $Y \rightarrow X_d$  factors through  $X''_d$  and the composite map  $Z \rightarrow Y \rightarrow X''_d$  factors through  $X_{d-1}$ . Summarizing, we have a commutative diagram

$$\begin{array}{ccccc}
 X_{d-1} & \longrightarrow & X_d & \longrightarrow & \bigoplus_{C \in \mathcal{F}_d} C \rightsquigarrow \\
 \parallel & & \uparrow & & \uparrow \\
 X_{d-1} & \longrightarrow & X''_d & \longrightarrow & \bigoplus_{C \in \mathcal{F}'_d} C \rightsquigarrow \\
 \uparrow & \curvearrowright & \uparrow a & & \parallel \\
 Z & \longrightarrow & Y & \dashrightarrow & \bigoplus_{C \in \mathcal{F}'_d} C \rightsquigarrow
 \end{array}$$

By induction, we have already a commutative diagram as in the proposition for the corresponding map  $Z \rightarrow X_{d-1}$ . We define now  $X'_d$  to be the cocone of the composite map  $\bigoplus_{C \in \mathcal{F}'_d} C \rightarrow Z[1] \rightarrow X'_{d-1}[1]$ . There is a commutative diagram

$$\begin{array}{ccccccc}
 X_{d-1} & \longrightarrow & X''_d & \longrightarrow & \bigoplus_{C \in \mathcal{F}'_d} C & \longrightarrow & X_{d-1}[1] \\
 \uparrow & & \uparrow & & \parallel & \searrow & \uparrow \\
 X'_{d-1} & \longrightarrow & X'_d & \longrightarrow & \bigoplus_{C \in \mathcal{F}'_d} C & \longrightarrow & X'_{d-1}[1] \\
 \uparrow & \curvearrowright & \uparrow a & & \parallel & \searrow & \uparrow \\
 Z & \longrightarrow & Y & \longrightarrow & \bigoplus_{C \in \mathcal{F}'_d} C & \longrightarrow & Z[1]
 \end{array}$$

The composite map  $Z \rightarrow Y \rightarrow X''_d$  factors through  $X_{d-1}$ , hence through  $X'_{d-1}$ . It follows that  $a$  factors through  $X'_d$  and we are done.  $\square$

We deduce the following result [BoVdB, Proposition 2.2.4] :

**Corollary 3.12.** *Let  $\mathcal{I}$  be a subcategory of  $\mathcal{T}^c$  and let  $d \in \mathbf{N} \cup \{\infty\}$ . Then,  $\mathcal{T}^c \cap \langle \overline{\mathcal{I}} \rangle_d = \langle \mathcal{I} \rangle_d$ .*

*Proof.* Let  $Y$  be a compact object and  $f : Y \rightarrow X_n$  be a split injection where  $X_n$  is obtained by taking a  $n$ -fold extension of objects of  $\langle \mathcal{I} \rangle$ . The conclusion is now given by Proposition 3.11 applied to a system  $(X_i)$  constant for  $i \geq n$ .  $\square$

**3.4. Relation with dg algebras.** Following Keller, we say that a triangulated category  $\mathcal{T}$  is *algebraic* if it is the stable category of a Frobenius exact category [GeMa, Chapter 5, §2.6] (for example,  $\mathcal{T}$  can be the derived category of an abelian category).



Recall the construction of [Ke, §4.3]. Let  $\mathcal{T} = \mathcal{E}\text{-stab}$  be the stable category of a Frobenius exact category  $\mathcal{E}$ . Let  $\mathcal{E}'$  be the category of acyclic complexes of projective objects of  $\mathcal{E}$  and  $Z^0 : \mathcal{E}' \rightarrow \mathcal{E}\text{-stab}$  be the functor that sends  $C$  to  $\text{coker } d_C^{-1}$ .

Given  $X$  and  $Y$  two complexes of objects of  $\mathcal{E}$ , we denote by  $\text{Hom}^\bullet(X, Y)$  the total Hom complex (*i.e.*,  $\text{Hom}^\bullet(X, Y)^i = \prod_{j \in \mathbf{Z}} \text{Hom}_{\mathcal{E}}(X^j, Y^{i+j})$ ).

Let  $M \in \mathcal{E}\text{-stab}$  and  $\tilde{M} \in \mathcal{E}'$  with  $Z^0(\tilde{M}) \xrightarrow{\sim} M$ . Let  $A = \text{End}^\bullet(\tilde{M})$  be the dg algebra of endomorphisms of  $\tilde{M}$ . The functor  $\text{Hom}^\bullet(\tilde{M}, -) : \mathcal{E}' \rightarrow D(A)$  factors through  $Z^0$  and induces a triangulated functor  $R\text{Hom}^\bullet(M, -) : \mathcal{E}\text{-stab} \rightarrow D(A)$ . That functor restricts to an equivalence  $\langle M \rangle_\infty \xrightarrow{\sim} A\text{-perf}$ . In particular, if  $M$  is a classical generator of  $\mathcal{T}$ , then we get the equivalence  $\mathcal{T} \xrightarrow{\sim} A\text{-perf}$ .

So,

**Proposition 3.13.** *Let  $\mathcal{T}$  be an algebraic triangulated category. Then,  $\mathcal{T}$  is classically finitely generated if and only if it is equivalent to the category of perfect complexes over a dg algebra.*

This should be compared with the following result.

Assume now  $\mathcal{E}$  is a cocomplete Frobenius category (*i.e.*, all direct sums exist and are exact). If  $M$  is compact, then  $R\text{Hom}^\bullet(M, -)$  restricts to an equivalence between the smallest full triangulated subcategory of  $\mathcal{T}$  containing  $M$  and closed under direct sums and  $D(A)$  (cf Theorem 4.20 (2) and Corollary 4.29 below). So, using Theorem 4.20 (2) below, we deduce [Ke, Theorem 4.3] :

**Theorem 3.14.** *Let  $\mathcal{E}$  be a cocomplete Frobenius category and  $\mathcal{T} = \mathcal{E}\text{-stab}$ . Then,  $\mathcal{T}$  has a compact generator if and only if it is equivalent to the derived category of a dg algebra.*

## 4. FINITENESS CONDITIONS AND REPRESENTABILITY

**4.1. Finiteness for cohomological functors.** We introduce a class of “locally finitely presented” cohomological functors that includes the representable functors, inspired by Brown’s representability Theorem. It extends the class of locally finite functors, of interest only for Ext-finite triangulated categories.

4.1.1. Let  $k$  be a commutative ring.

Let  $\mathcal{T}$  be a  $k$ -linear triangulated category. Let  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  be a ( $k$ -linear) functor. We say that  $H$  is *cohomological* if for every distinguished triangle  $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow$ , then the associated sequence  $H(Z) \xrightarrow{H(g)} H(Y) \xrightarrow{H(f)} H(X)$  is exact.

For  $C \in \mathcal{T}$ , we denote by  $h_C$  the cohomological functor  $\text{Hom}_{\mathcal{T}}(-, C) : \mathcal{T}^\circ \rightarrow k\text{-Mod}$ .

We will repeatedly use Yoneda’s Lemma :

**Lemma 4.1.** *Let  $X \in \mathcal{T}$  and  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  a functor. Then, the canonical map  $\text{Hom}(h_C, H) \rightarrow H(C), f \mapsto f(C)(\text{id}_C)$  is an isomorphism.*

Let  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  be a functor. We say that  $H$  is

- *locally bounded* (resp. *bounded above*, resp. *bounded below*) if for every  $X \in \mathcal{T}$ , we have  $H(X[i]) = 0$  for  $|i| \gg 0$  (resp. for  $i \ll 0$ , resp. for  $i \gg 0$ )
- *locally finitely generated* if for every  $X \in \mathcal{T}$ , there is  $D \in \mathcal{T}$  and  $\alpha : h_D \rightarrow H$  such that  $\alpha(X[i])$  is surjective for all  $i$ .

- *locally finitely presented* if it is locally finitely generated and the kernel of any map  $h_E \rightarrow H$  is locally finitely generated.

Let  $X \in \mathcal{T}$ . We introduce two conditions :

- (a) there is  $D \in \mathcal{T}$  and  $\alpha : h_D \rightarrow H$  such that  $\alpha(X[i])$  is surjective for all  $i$
- (b) for every  $\beta : h_E \rightarrow H$ , there is  $f : F \rightarrow E$  such that  $\beta h_f = 0$  and  $h_F(X[i]) \xrightarrow{h_f} h_E(X[i]) \xrightarrow{\beta} H(X[i])$  is an exact sequence for all  $i$ .

Note that  $H$  is locally finitely presented if and only if for every  $X \in \mathcal{T}$ , then conditions (a) and (b) are fulfilled.

**Lemma 4.2.** *For  $C \in \mathcal{T}$ , then  $h_C$  is locally finitely presented.*

*Proof.* We take  $D = C$  and  $\alpha = \text{id}$  for condition (a). For (b), a map  $\beta : h_E \rightarrow h_C$  comes from a map  $g : E \rightarrow C$  and we pick a distinguished triangle  $F \xrightarrow{f} E \xrightarrow{g} C \rightsquigarrow$ .  $\square$

**Proposition 4.3.** *Let  $H_0 \rightarrow H_1 \rightarrow H \rightarrow H_2 \rightarrow H_3$  be an exact sequence of functors  $\mathcal{T}^\circ \rightarrow k\text{-Mod}$ .*

*If  $H_1$  and  $H_2$  are locally finitely generated and  $H_3$  is locally finitely presented, then  $H$  is locally finitely generated.*

*If  $H_0$  is locally finitely generated and  $H_1, H_2$  and  $H_3$  are locally finitely presented, then  $H$  is locally finitely presented.*

*Proof.* Let us name the maps :  $H_0 \xrightarrow{t_0} H_1 \xrightarrow{t_1} H \xrightarrow{t_2} H_2 \xrightarrow{t_3} H_3$ . Let  $X \in \mathcal{T}$ .

Let  $\alpha_2 : h_{D_2} \rightarrow H_2$  as in (a). Let  $\beta_3 = t_3 \alpha_2 : h_{D_2} \rightarrow H_3$ . Let  $f_3 : E \rightarrow D_2$  as in (b). Since  $H(E) \rightarrow H_2(E) \rightarrow H_3(E)$  is exact, the composite map  $\alpha_2 h_{f_3} : h_E \rightarrow H_2$  factors as  $h_E \xrightarrow{\gamma} H \xrightarrow{t_2} H_2$ . Let  $\alpha_1 : h_{D_1} \rightarrow H_1$  as in (a).

Let  $a : h_X \rightarrow H$ . The composite  $t_2 a : h_X \rightarrow H_2$  factors as  $t_2 a : h_X \xrightarrow{b} h_{D_2} \xrightarrow{\alpha_2} H_2$ . The composition  $t_3(t_2 a) : h_X \rightarrow H_3$  is zero, hence  $b$  factors as  $b : h_X \xrightarrow{c} h_E \xrightarrow{h_{f_3}} h_{D_2}$ . Now, we have  $t_2 \gamma c = \alpha_2 h_{f_3} c = \alpha_2 b = t_2 a$ . Since the composite  $t_2(a - \gamma c) : h_X \rightarrow H_2$  is zero, it follows that  $a - \gamma c$  factors as  $h_X \xrightarrow{a_1} H_1 \xrightarrow{t_1} H$ . Now,  $a_1$  factors through  $\alpha_1$ . So, we have shown that  $a$  factors through  $\gamma + t_1 \alpha_1 : h_E \oplus h_{D_1} \rightarrow H$ , hence  $H$  satisfies (a).

Let  $\alpha_0 : h_{D_0} \rightarrow H_0$  as in (a). Let  $\beta' : h_E \rightarrow \ker t_2$ . Then, there is  $\beta_1 : h_E \rightarrow H_1$  such that  $\beta' = t_1 \beta_1$ . Since  $H_1$  is locally finitely presented, there are  $u : h_F \rightarrow h_{D_0}$  and  $v : h_F \rightarrow h_E$  such that  $(\beta_1 + t_0 \alpha_0)(u - v) = 0$  and  $h_F(X[i]) \xrightarrow{u-v} h_E(X[i]) \oplus h_{D_0}(X[i]) \xrightarrow{\beta_1 + t_0 \alpha_0} H_1(X[i])$  is exact for every  $i$ . Summarizing, we have a commutative diagram

$$\begin{array}{ccccc}
 h_F & & & & \\
 \downarrow u & \searrow v & & & \\
 h_{D_0} & & h_E & & \\
 \downarrow \alpha_0 & \searrow & \downarrow \beta_1 & \searrow \beta' & \\
 H_0 & \xrightarrow{t_0} & H_1 & \xrightarrow{t_1} & \ker t_2
 \end{array}$$

It follows that  $\beta v = 0$  and  $h_F(X[i]) \xrightarrow{v} h_E(X[i]) \xrightarrow{\beta} (\ker t_2)(X[i])$  is exact for every  $i$ , hence  $\ker t_2$  satisfies (b).

Let now  $\beta : h_E \rightarrow H$ . Let  $G = \ker \beta$  and  $G_2 = \ker(t_2\beta)$ . Now, we have exact sequences  $0 \rightarrow G \rightarrow G_2 \rightarrow \ker t_2$  and  $0 \rightarrow G_2 \rightarrow h_E \rightarrow H_2$ . The first part of the Proposition together with Lemma 4.2 shows that  $G$  is finitely generated. Consequently,  $H$  is locally finitely generated.  $\square$

4.1.2. We will now study conditions (a) and (b) in the definition of locally finitely presented functors.

**Lemma 4.4.** *Let  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  be a  $k$ -linear functor and  $X \in \mathcal{T}$ . Let  $\beta_r : h_{E_r} \rightarrow H$  for  $r \in \{1, 2\}$  such that (b) holds for  $\beta = \beta_1 + \beta_2 : h_{E_1 \oplus E_2} \rightarrow H$ . Then, (b) holds for  $\beta_1$  and  $\beta_2$ .*

*Assume (a) holds. If (b) holds for those  $\beta : h_E \rightarrow H$  such that  $\beta(X[i])$  is surjective for all  $i$ , then (b) holds for all  $\beta$ .*

*Proof.* Let  $E = E_1 \oplus E_2$ . Denote by  $i_r : E_r \rightarrow E$  and  $p_r : E \rightarrow E_r$  the injections and projections. There is  $f : F \rightarrow E$  such that  $\beta h_f = 0$  and  $h_F(X[i]) \xrightarrow{h_f} h_E(X[i]) \xrightarrow{\beta} H(X[i])$  is an exact sequence for all  $i$ .

Fix a distinguished triangle  $F_1 \xrightarrow{f'_1} F \xrightarrow{p_2 f} E_2 \rightsquigarrow$  and let  $f_1 = p_1 f f'_1 : F_1 \rightarrow E_1$ . We have  $\beta_1 h_{f_1} = 0$  since  $\beta_1 h_{p_1 f} = -\beta_2 h_{p_2 f}$ .

For all  $i$ , the horizontal sequences and the middle vertical sequence in the following commutative diagram are exact

$$\begin{array}{ccccccc}
& & h_{F_1}(X[i]) & \xrightarrow{h_{f'_1}} & h_F(X[i]) & \xrightarrow{h_{p_2 f}} & h_{E_2}(X[i]) \\
& & \downarrow h_{f_1} & & \downarrow h_f & & \parallel \\
0 & \longrightarrow & h_{E_1}(X[i]) & \xrightarrow{h_{i_1}} & h_E(X[i]) & \xrightarrow{h_{p_2}} & h_{E_2}(X[i]) \longrightarrow 0 \\
& & \downarrow \beta_1 & & \downarrow \beta & & \\
& & H(X[i]) & \xlongequal{\quad} & H(X[i]) & & \\
& & & & \downarrow & & \\
& & & & 0 & & 
\end{array}$$

hence the left vertical sequence is exact as well.

Let us now prove the second part of the Lemma. Let  $\beta : h_E \rightarrow H$ . Since (a) holds, there is  $D \in \mathcal{T}$  and  $\alpha : h_D \rightarrow H$  such that  $\alpha(X[i])$  is surjective for all  $i$ . Let  $E' = D \oplus E$  and  $\beta' = \alpha + \beta : h_{E'} \rightarrow H$ . Then, (b) holds for  $\beta'$ , hence it holds for  $\beta$  by the first part of the Lemma.  $\square$

**Remark 4.5.** For the representability Theorem (cf Lemma 4.12), only the surjective case of (b) is needed, but the previous Lemma shows that this implies that (b) holds in general.

**Lemma 4.6.** *Let  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  be a cohomological functor.*

*The full subcategory of  $X$  in  $\mathcal{T}$  such that (a) and (b) hold is a thick triangulated subcategory of  $\mathcal{T}$ .*

*Proof.* Let  $\mathcal{I}$  be the full subcategory of those  $X$  such that (a) and (b) hold. It is clear that  $\mathcal{I}$  is closed under shifts and under taking direct summands. So, we are left with proving that  $\mathcal{I}$  is stable under extensions.

Let  $X_1 \xrightarrow{u} X \rightarrow X_2 \rightsquigarrow$  be a distinguished triangle in  $\mathcal{T}$  with  $X_1, X_2 \in \mathcal{I}$ . Pick  $D_r \in \mathcal{T}$  and  $\alpha_r : h_{D_r} \rightarrow H$  such that  $\alpha_r(X_r[i])$  is surjective for all  $i$ . Put  $E = D_1 \oplus D_2$  and  $\beta = \alpha_1 + \alpha_2 : h_E \rightarrow H$ . There is  $F_r \in \mathcal{T}$  and  $f_r : F_r \rightarrow E$  such that  $\beta h_{f_r} = 0$  and  $h_{F_r}(X_r[i]) \xrightarrow{h_f} h_E(X_r[i]) \xrightarrow{\beta} H(X_r[i])$  is an exact sequence for all  $i$ . Put  $F = F_1 \oplus F_2$  and  $f = f_1 + f_2 : F \rightarrow E$ . Let  $F \xrightarrow{f} E \xrightarrow{t} E' \rightsquigarrow$  be a distinguished triangle. We have an exact sequence  $H(E') \rightarrow H(E) \rightarrow H(F)$ . The image in  $H(F)$  of the element of  $H(E)$  corresponding to  $\beta$  is 0, since  $\beta h_f = 0$ . Hence,  $\beta$  factors as  $h_E \xrightarrow{ht} h_{E'} \xrightarrow{\gamma} H$ . Let  $D = E \oplus E'$  and  $\alpha = \beta + \gamma : h_D \rightarrow H$ .

Let  $a : h_X \rightarrow H$ . Then, there is a commutative diagram where the top horizontal sequence is exact

$$\begin{array}{ccccc} h_{X_2[-1]} & \longrightarrow & h_{X_1} & \xrightarrow{h_u} & h_X \\ \vdots & & \downarrow c & & \downarrow a \\ h_F & \xrightarrow{h_f} & h_E & \xrightarrow{\beta} & H \end{array}$$

The composite  $h_{X_2[-1]} \rightarrow h_{X_1} \xrightarrow{c} h_E \xrightarrow{ht} h_{E'}$  is zero, hence  $h_t c : h_{X_1} \rightarrow h_{E'}$  factors as  $h_{X_1} \xrightarrow{h_u} h_X \xrightarrow{b} h_{E'}$ . We have  $ah_u = \beta c = \gamma h_t c = \gamma b h_u$ , hence the composite  $h_{X_1} \rightarrow h_X \xrightarrow{a'} H$  is zero, where  $a' = a - \gamma b$ . So,  $a'$  factors through a map  $h_{X_2} \rightarrow H$ . Such a map factors through  $\beta$ , hence  $a'$  factors through  $\beta$  and  $a$  factors through  $\alpha$ . The same conclusion holds for  $a$  replaced by any map  $h_{X[i]} \rightarrow H$  for some  $i \in \mathbf{Z}$ . So, every map  $h_{X[i]} \rightarrow H$  factors through  $\alpha$ , *i.e.*, (a) holds for  $X$ .

Consider now a map  $\beta' : h_{E'} \rightarrow H$ . Let  $\beta'' : h_{E''} \rightarrow H$  such that  $\beta''(X_1[i])$  is surjective for all  $i$ . Let  $\beta = \beta' + \beta'' : h_E \rightarrow H$ , where  $E = E' \oplus E''$ . In order to prove that  $\beta'$  satisfies (b), it suffices to prove that  $\beta$  satisfies (b), thanks to Lemma 4.4.

There is  $F_1 \in \mathcal{T}$  and  $f_1 : F_1 \rightarrow E$  such that  $\beta h_{f_1} = 0$  and  $h_{F_1}(X_1[i]) \xrightarrow{h_{f_1}} h_E(X_1[i]) \xrightarrow{\beta} H(X_1[i])$  is an exact sequence for all  $i$ . Let  $E_1$  be the cone of  $f_1$ . As in the discussion above,  $\beta$  factors through a map  $\gamma : h_{E_1} \rightarrow H$ . Let  $F_2 \in \mathcal{T}$  and  $f_2 : F_2 \rightarrow E_1$  such that  $\gamma h_{f_2} = 0$  and  $h_{F_2}(X_2[i]) \xrightarrow{h_{f_2}} h_{E_1}(X_2[i]) \xrightarrow{\gamma} H(X_2[i])$  is an exact sequence for all  $i$ . Let  $F$  be the cocone of the sum map  $E \oplus F_2 \rightarrow E_1$ . The composition  $h_F \rightarrow h_E \xrightarrow{\beta} H$  is zero. We have a commutative diagram

$$\begin{array}{ccccc} & & h_F & \longrightarrow & h_{F_2} \\ & & \downarrow & & \downarrow h_{f_2} \\ h_{F_1} & \xrightarrow{h_{f_1}} & h_E & \longrightarrow & h_{E_1} \\ & & \searrow \beta & & \swarrow \gamma \\ & & & & H \end{array}$$

In the diagram, the square is homotopy cartesian, *i.e.*, given  $Y \in \mathcal{T}$  and  $u : Y \rightarrow E$ ,  $v : Y \rightarrow F_2$  such that the compositions  $Y \xrightarrow{u} E \rightarrow E_1$  and  $Y \xrightarrow{v} F_2 \rightarrow E_1$  are equal, then there is  $w : Y \rightarrow F$  such that  $u$  is the composition  $Y \xrightarrow{w} F \rightarrow E$  and  $v$  the composition  $Y \xrightarrow{w} F \rightarrow F_2$ .

Let  $a : h_X \rightarrow h_E$  such that  $\beta a = 0$ . The composite  $h_{X_1} \rightarrow h_X \xrightarrow{a} h_E$  factors through  $h_{F_1}$ . It follows that the composition  $h_{X_1} \rightarrow h_X \xrightarrow{a} h_E \rightarrow h_{E_1}$  is zero. Hence, the composite  $h_X \xrightarrow{a} h_E \rightarrow h_{E_1}$  factors through a map  $b : h_{X_2} \rightarrow h_{E_1}$ . The composite  $b' : h_{X_2} \xrightarrow{b} h_{E_1} \rightarrow H$  factors through a map  $c : h_{X_1[1]} \rightarrow H$ , since  $h_X \rightarrow h_{X_2} \xrightarrow{b'} H$  is zero. Now,  $c$  factors as  $h_{X_1[1]} \xrightarrow{d} h_{E_1} \xrightarrow{\gamma} H$ . Summarizing, we have a diagram all of whose squares and triangles but the one marked “ $\neq$ ” are commutative and where the horizontal sequences are exact

$$\begin{array}{ccccccc}
h_{X_1} & \longrightarrow & h_X & \longrightarrow & h_{X_2} & \longrightarrow & h_{X_1[1]} \\
\downarrow \text{dotted} & & \downarrow a & & \downarrow b & \neq & \downarrow d \\
h_{F_1} & \longrightarrow & h_E & \longrightarrow & h_{E_1} & & \\
& & \downarrow \beta & \nearrow \gamma & & & \\
& & H & & & & 
\end{array}$$

Let  $d'$  be the composition  $h_{X_2} \rightarrow h_{X_1[1]} \xrightarrow{d} h_{E_1}$ . Then, the composition  $h_{X_2} \xrightarrow{b-d'} h_{E_1} \xrightarrow{\gamma} H$  is zero. The map  $b - d'$  factors as  $h_{X_2} \xrightarrow{d''} h_{F_2} \xrightarrow{h_{f_2}} h_{E_1}$ . It follows that  $h_X \xrightarrow{a} h_E \rightarrow h_{E_1}$  factors as  $h_X \rightarrow h_{X_2} \xrightarrow{d''} h_{F_2} \xrightarrow{h_{f_2}} h_{E_1}$ . Using the homotopy cartesian square above, we deduce that  $a$  factors through  $a' : h_X \rightarrow h_F$ . So, the sequence  $h_F(X) \rightarrow h_E(X) \rightarrow H(X)$  is exact. The same holds for all  $i$ , hence (b) holds for  $X$ .  $\square$

**Remark 4.7.** All the results concerning locally finitely generated and presented functors remain valid if we replace the conditions “for all  $i \in \mathbf{Z}$ ” by “for all  $i \geq 0$ ”, by “for all  $i \leq 0$ ” or by “for  $i = 0$ ” in (a) and (b). The only change is in Lemma 4.6 : the full subcategory of those  $Y$  such that (a) and (b) hold for  $X = Y[r]$  for all  $r \in \mathbf{Z}$  is thick.

4.1.3. Assume  $k$  is noetherian. We say that  $\mathcal{T}$  is *Ext-finite* if  $\bigoplus_i \text{Hom}(X, Y[i])$  is a finitely generated  $k$ -module, for every  $X, Y \in \mathcal{T}$ .

Assume now  $\mathcal{T}$  is Ext-finite and let  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  be a functor. We say that  $H$  is *locally finite* if for every  $X \in \mathcal{T}$ , the  $k$ -module  $\bigoplus_i H(X[i])$  is finitely generated.

**Proposition 4.8.** *Let  $H$  be a locally finite functor. Then,  $H$  is locally bounded and locally finitely presented.*

*Proof.* It is clear that  $H$  is locally bounded. Let  $X \in \mathcal{T}$ . Let  $I_i$  be a minimal (finite) family of generators of  $H(X[i])$  as a  $k$ -module. We have  $I_i = \emptyset$  for almost all  $i$ , since  $H$  is locally bounded. Put  $D = \bigoplus_i X[i] \otimes_k k^{I_i}$  and let  $\alpha : h_D \rightarrow H$  be the canonical map. The map  $\alpha(X[i])$  is surjective for all  $i$ . So, every locally finite functor is locally finitely generated.

Let now  $\beta : h_E \rightarrow H$ . Let  $G = \ker \beta$ . Since  $\mathcal{T}$  is Ext-finite,  $G$  is again locally finite, hence locally finitely generated.  $\square$

## 4.2. Locally finitely presented functors.

4.2.1. Let us start with some remarks on cohomological functors.

Given  $0 \rightarrow H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow 0$  an exact sequence of functors  $\mathcal{T}^\circ \rightarrow k\text{-Mod}$ , if two of the functors amongst the  $H_i$ 's are cohomological, then the third one is cohomological as well. The category of cohomological functors  $\mathcal{T}^\circ \rightarrow k\text{-Mod}$  is closed under direct sums.

Given  $H_1 \rightarrow H_2 \rightarrow \dots$  a directed system of cohomological functors  $\mathcal{T}^\circ \rightarrow k\text{-Mod}$ , we have an exact sequence  $0 \rightarrow \bigoplus H_i \rightarrow \bigoplus H_i \rightarrow \text{colim } H_i \rightarrow 0$ . This shows that  $\text{colim } H_i$  is a cohomological functor.

**Lemma 4.9.** *Let  $H_1, \dots, H_{n+1}$  be cohomological functors on  $\mathcal{T}$  and  $f_i : H_i \rightarrow H_{i+1}$  for  $1 \leq i \leq n$ . Let  $\mathcal{I}_i$  be a subcategory of  $\mathcal{T}$  on which  $f_i$  vanishes. Then,  $f_n \cdots f_1$  vanishes on  $\mathcal{I}_1 \diamond \cdots \diamond \mathcal{I}_n$ .*

*Proof.* Note first that if a morphism between cohomological functors vanishes on a subcategory  $\mathcal{I}$ , then it vanishes on  $\langle \mathcal{I} \rangle$ .

By induction, it is enough to prove the Lemma for  $n = 2$ . Let  $X_1 \rightarrow X \rightarrow X_2 \rightsquigarrow$  be a distinguished triangle with  $X_i \in \mathcal{I}_i$ . The map  $f_1(X)$  factors through  $H_2(X_2)$ , *i.e.*, we have a commutative diagram with exact horizontal sequences

$$\begin{array}{ccccc} H_1(X_2) & \longrightarrow & H_1(X) & \longrightarrow & H_1(X_1) \\ \downarrow & \swarrow \text{dotted} & \downarrow & & \downarrow 0 \\ H_2(X_2) & \longrightarrow & H_2(X) & \longrightarrow & H_2(X_1) \\ 0 \downarrow & & \downarrow & & \downarrow \\ H_3(X_2) & \longrightarrow & H_3(X) & \longrightarrow & H_3(X_1) \end{array}$$

This shows that  $f_2 f_1(X) = 0$ . □

**Remark 4.10.** Let  $M \in \mathcal{T}$  be a complete classical generator. Let  $f : \bigoplus_i \text{Hom}(\text{Id}_{\mathcal{T}}, \text{Id}_{\mathcal{T}}[i]) \rightarrow \bigoplus_i \text{Hom}(M, M[i])$  be the canonical map. Let  $\zeta \in \ker f$ . It follows from Lemma 4.9 that  $\zeta$  is locally nilpotent. If  $\mathcal{T} = \langle \overline{M} \rangle_d$ , then  $(\ker f)^d = 0$ .

4.2.2. In this part, we study convergence conditions on directed systems. This builds on [BoVdB, §2.3].

Let  $V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \dots$  be a system of abelian groups. We say that the system  $(V_i)$  is *almost constant* if one of the following equivalent conditions is satisfied :

- $V_i = \text{im } f_{i-1} \cdots f_2 f_1 + \ker f_i$  and  $\ker f_{i+r} \cdots f_i = \ker f_i$  for any  $r \geq 0$  and  $i \geq 1$ .
- Denote by  $\alpha_i : V_i \rightarrow V = \text{colim } V_i$  the canonical map. Then,  $\alpha_i$  induces an isomorphism  $V_i / \ker f_i \xrightarrow{\sim} V$ .

Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{I}$  a subcategory of  $\mathcal{T}$ . Let  $H_1 \rightarrow H_2 \rightarrow \dots$  be a directed system of functors  $\mathcal{T}^\circ \rightarrow k\text{-Mod}$ . We say that  $(H_i)_{i \geq 1}$  is *almost constant* on  $\mathcal{I}$  if for every  $X \in \mathcal{I}$ , the system  $H_1(X) \rightarrow H_2(X) \rightarrow \dots$  is almost constant.

Given  $1 \leq r_1 < r_2 < \dots$ , we denote by  $(H_{r_i})$  the system  $H_{r_1} \xrightarrow{f_{r_2-1} \cdots f_{r_1+1} f_{r_1}} H_{r_2} \xrightarrow{f_{r_3-1} \cdots f_{r_2+1} f_{r_2}} H_{r_3} \rightarrow \dots$ .

**Proposition 4.11.** *Let  $(H_i)_{i \geq 1}$  be a directed system of cohomological functors on  $\mathcal{T}$ .*

- (i) *If  $(H_i)_{i \geq 1}$  is almost constant on  $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n$ , then, for any  $r > 0$ , the system  $(H_{ni+r})_{i \geq 0}$  is almost constant on  $\mathcal{I}_1 \diamond \cdots \diamond \mathcal{I}_n$ .*

*Assume now  $(H_i)_{i \geq 1}$  is almost constant on  $\mathcal{I}$ . Then,*

- (ii)  *$(H_i)_{i \geq 1}$  is almost constant on  $\text{add}(\mathcal{I})$ . If in addition the functors  $H_i$  commute with products, then  $(H_i)_{i \geq 1}$  is almost constant on  $\overline{\text{add}}(\mathcal{I})$ .*

- (iii)  $(H_{ir+s})_{i \geq 0}$  is almost constant on  $\mathcal{I}$  for any  $r, s > 0$ .
- (iv) the canonical map  $H_{n+1} \rightarrow \operatorname{colim} H_i$  is a split surjection, when the functors are restricted to  $\langle \mathcal{I} \rangle_n$ .

*Proof.* Let  $H = \operatorname{colim} H_i$  and let  $K_i = \ker(H_i \rightarrow H)$ . Take  $\mathcal{I}$  and  $\mathcal{I}'$  such that  $(H_i)$  is almost constant on  $\mathcal{I}$  and  $\mathcal{I}'$ . Let  $I \rightarrow J \rightarrow I'$  be a distinguished triangle with  $I \in \mathcal{I}$  and  $I' \in \mathcal{I}'$ .

Given  $i \geq 1$ , we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
& & & & H_i(I') & \longrightarrow & H(I') & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & H_i(J) & \longrightarrow & H(J) & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_i(I) & \longrightarrow & H_i(I) & \longrightarrow & H(I) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_i(I'[-1]) & \longrightarrow & H_i(I'[-1]) & \longrightarrow & H(I'[-1]) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_i(J[-1]) & \longrightarrow & H_i(J[-1]) & & & & 
\end{array}$$

This shows that  $H_i(J) \rightarrow H(J)$  is onto. By induction, we deduce that  $H_i(X) \rightarrow H(X)$  is onto for any  $i \geq 1$  and any  $X \in \mathcal{I}_1 \diamond \cdots \diamond \mathcal{I}_n$ . It follows from Lemma 4.9 that the composition  $K_i \xrightarrow{f_i} K_{i+1} \rightarrow \cdots \rightarrow K_{i+n}$  vanishes on  $\mathcal{I}_1 \diamond \cdots \diamond \mathcal{I}_n$ . We deduce that (i) holds.

The assertions (ii) and (iii) are clear.

By (i), it is enough to prove (iv) for  $n = 1$ . The map  $f_1 : H_1 \rightarrow H_2$  factors through  $H_1/K_1$  as  $\bar{f}_1 : H_1/K_1 \rightarrow H_2$ . We have a commutative diagram

$$\begin{array}{ccc}
H_1/K_1 & \xrightarrow{\bar{f}_1} & H_2 \\
& \searrow & \swarrow \\
& & H
\end{array}$$

When restricted to  $\mathcal{I}$ , the canonical map  $H_1/K_1 \rightarrow H$  is an isomorphism, hence the canonical map  $H_2 \rightarrow H$  is a split surjection. This proves (iv).  $\square$

We say that a direct system  $(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots)$  of objects of  $\mathcal{T}$  is *almost constant* on  $\mathcal{I}$  if the system  $(h_{A_i})$  is almost constant on  $\mathcal{I}$ .

4.2.3. We study now approximations of locally finitely presented functors.

**Lemma 4.12.** *Let  $\mathcal{T}$  be a triangulated category and  $G \in \mathcal{T}$ . Let  $H$  be a locally finitely presented cohomological functor. Then, there is a directed system  $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots$  in  $\mathcal{T}$  that is almost constant on  $\{G[i]\}_{i \in \mathbb{Z}}$  and a map  $\operatorname{colim} h_{A_i} \rightarrow H$  that is an isomorphism on  $\langle G \rangle_\infty$ .*

*Proof.* Since  $H$  is locally finitely presented, there is  $A_1 \in \mathcal{T}$  and  $\alpha_1 : h_{A_1} \rightarrow H$  such that  $\alpha_1(G[r])$  is onto for all  $r$ .

We now construct the system by induction on  $i$ . Assume  $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_{i-1}} A_i$  and  $\alpha_1, \dots, \alpha_i$  have been constructed.

Since  $H$  is locally finitely presented, there is  $g : B \rightarrow A_i$  with  $\text{im } h_g(G[r]) = \ker \alpha_i(G[r])$  for all  $r$  and with  $h_g \alpha_i = 0$ . Let  $B \xrightarrow{g} A_i \xrightarrow{f_i} A_{i+1} \rightsquigarrow$  be a distinguished triangle. We have an exact sequence  $h_B \xrightarrow{h_g} h_{A_i} \xrightarrow{h_{f_i}} h_{A_{i+1}}$ , hence, there is  $\alpha_{i+1} : h_{A_{i+1}} \rightarrow H$  with  $\alpha_i = \alpha_{i+1} f_i$ . We have a surjection  $h_g(G[r]) : h_B(G[r]) \rightarrow \ker \alpha_i(G[r])$ , hence  $\ker \alpha_i(G[r]) \subseteq \ker f_i(G[r])$ . So, the system is almost constant on  $\{G[i]\}_{i \in \mathbf{Z}}$ . It follows from Proposition 4.11 (iv) that the canonical map  $H \rightarrow \text{colim } h_{A_i}$  is an isomorphism on  $\langle G \rangle_\infty$ .  $\square$

**Proposition 4.13.** *Let  $\mathcal{T}$  be a triangulated category classically generated by an object  $G$ . Let  $H$  be a cohomological functor. Then,  $H$  is locally finitely presented if and only if there is a directed system  $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$  in  $\mathcal{T}$  that is almost constant on  $\{G[i]\}_{i \in \mathbf{Z}}$  and an isomorphism  $\text{colim } h_{A_i} \xrightarrow{\sim} H$ .*

*Proof.* The first implication is given by Lemma 4.12. Let us now show the converse.

Since  $\mathcal{T}$  is classically generated by  $G$ , it is enough to show conditions (a) and (b) for  $X = G$  (cf Lemma 4.6). Condition (a) is obtained with  $\alpha_1 : h_{A_1} \rightarrow H$ . Fix now  $\beta : h_E \rightarrow H$ . There is an integer  $i$  such that  $E \in \langle G \rangle_i$ . By Proposition 4.11 (iii) and (iv), the restriction of  $\alpha_{i+1}$  to  $\langle G \rangle_i$  has a right inverse  $\rho$ . We obtain a map  $\rho\beta$  between the functors  $h_E$  and  $h_{A_{i+1}}$  restricted to  $\langle G \rangle_i$ . It comes from a map  $f : E \rightarrow A_{i+1}$ . Let  $F$  be the cocone of  $f$ . The kernel of  $h_f(G[r]) : h_E(G[r]) \rightarrow h_{A_{i+1}}(G[r])$  is the same as the kernel of  $\beta(G[r])$ . So, the exact sequence  $h_F(G[r]) \rightarrow h_E(G[r]) \rightarrow h_{A_{i+1}}(G[r])$  induces an exact sequence  $h_F(G[r]) \rightarrow h_E(G[r]) \rightarrow H(G[r])$  and (b) is satisfied.  $\square$

### 4.3. Representability.

4.3.1. We can now state a representability Theorem for strongly finitely generated triangulated categories.

**Theorem 4.14.** *Let  $\mathcal{T}$  be a strongly finitely generated triangulated category and  $H$  be a cohomological functor.*

*Then,  $H$  is locally finitely presented if and only if it is a direct summand of a representable functor.*

*Proof.* Let  $G$  be a  $d$ -step generator of  $\mathcal{T}$  for some  $d \in \mathbf{N}$ . Let  $(A_i)$  be a directed system as in Lemma 4.12. Then,  $\alpha_{d+1} : h_{A_{d+1}} \rightarrow H$  is a split surjection by Proposition 4.11 (iv). The converse follows from Lemmas 4.2 and 4.6.  $\square$

Recall that an additive category is *Karoubian* if for every object  $X$  and every idempotent  $e \in \text{End}(X)$ , there is an object  $Y$  and maps  $i : Y \rightarrow X$  and  $p : X \rightarrow Y$  such that  $pi = \text{id}_Y$  and  $ip = e$ .

**Corollary 4.15.** *Let  $\mathcal{T}$  be a strongly finitely generated Karoubian triangulated category. Then, every locally finitely presented cohomological functor is representable.*

Via Proposition 4.8, Theorem 4.14 generalizes the following result of Bondal and Van den Bergh [BoVdB, Theorem 1.3].



**Corollary 4.16.** *Let  $\mathcal{T}$  be an Ext-finite strongly finitely generated Karoubian triangulated category. A cohomological functor  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  is representable if and only if it is locally finite.*

The following Lemma is classical:

**Lemma 4.17.** *Let  $\mathcal{T}$  be a triangulated category closed under countable multiples. Then,  $\mathcal{T}$  is Karoubian.*

*Proof.* Given  $X \in \mathcal{T}$  and  $e \in \text{End}(X)$  an idempotent, then  $\text{hocolim}(X \xrightarrow{e} X \xrightarrow{e} X \rightarrow \dots)$  is the image of  $e$ .  $\square$

We have a variant of Theorem 4.14, with a similar proof :

**Theorem 4.18.** *Let  $\mathcal{T}$  be a triangulated category that has a strong complete generator and  $H$  be a cohomological functor that commutes with products.*

*Then,  $H$  is locally finitely presented if and only if it is a direct summand of a representable functor.*

*If  $\mathcal{T}$  is closed under countable multiples, then  $H$  is locally finitely presented if and only if it is representable.*

4.3.2. Let us now consider cocomplete and compactly generated triangulated categories — the “classical” setting.

**Lemma 4.19.** *Assume  $\mathcal{T}$  is cocomplete. Then, every functor is locally finitely presented.*

*Proof.* Let  $H$  be a functor and  $X \in \mathcal{T}$ . Let  $D = \bigoplus_i X[i]^{|H(X[i])|}$  and  $\alpha : h_D \rightarrow H$  the canonical map. Then,  $\alpha(X[i])$  is surjective for every  $i$ . It follows that  $H$  is locally finitely generated.

Now, the kernel of a map  $h_E \rightarrow H$  will also be locally finitely generated, hence  $H$  is locally finitely presented.  $\square$

So, we can derive the classical representability Theorem ([Nee3, Theorem 3.1], [Ke, Theorem 5.2], [Nee2, Lemma 2.2]) :

**Theorem 4.20.** *Let  $\mathcal{T}$  be a cocomplete triangulated category generated by a set  $\mathcal{S}$  of compact objects. Then,*

- (1) *a cohomological functor  $\mathcal{T}^\circ \rightarrow k\text{-Mod}$  is representable if and only if it commutes with products*
- (2) *every object of  $\mathcal{T}$  is a homotopy colimit of a system  $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots$  almost constant on  $\langle \overline{\mathcal{S}} \rangle$  and such that  $A_1$  and the cone of  $f_i$  for all  $i$  are in  $\overline{\mathcal{S}}$ . In particular,  $\mathcal{T}$  is the smallest full triangulated subcategory containing  $\mathcal{S}$  and closed under direct sums.*
- (3)  *$\mathcal{S}$  classically generates  $\mathcal{T}^c$ .*

*Proof.* Let  $G = \bigoplus_{S \in \mathcal{S}} S$ . Let  $H : \mathcal{T}^\circ \rightarrow k\text{-Mod}$  be a cohomological functor that commutes with products. Let  $(A_i, f_i)$  be a directed system constructed as in Lemma 4.12 and  $C = \text{hocolim } A_i$ . Note that we can assume that  $A_1$  and the cone of  $f_i$  are direct sums of shifts of  $G$  (cf Lemmas 4.12 and 4.19). By Proposition 4.11 (ii), the system is almost constant on  $\langle \overline{\mathcal{S}} \rangle$ .

The distinguished triangle  $\bigoplus A_i \rightarrow \bigoplus A_i \rightarrow C \rightsquigarrow$  induces an exact sequence  $H(C) \rightarrow \prod H(A_i) \rightarrow \prod H(A_i)$ , since  $H$  takes direct sums in  $\mathcal{T}$  to products. Consequently, there is a

map  $f : h_C \rightarrow H$  that makes the following diagram commutative

$$\begin{array}{ccc} & h_C & \\ & \nearrow & \searrow f \\ \operatorname{colim} h_{A_i} & \longrightarrow & H \end{array}$$

where the canonical maps from  $\operatorname{colim} h_{A_i}$  are isomorphisms when the functors are restricted to  $\langle \mathcal{S} \rangle$  (cf Lemma 3.10). So, the restriction of  $f$  to  $\langle \mathcal{S} \rangle$  is an isomorphism. Consequently,  $f$  is an isomorphism on the smallest full triangulated subcategory  $\mathcal{T}'$  of  $\mathcal{T}$  containing  $\mathcal{S}$  and closed under direct sums. To conclude, it is enough to show that  $\mathcal{T}' = \mathcal{T}$  and we will prove the more precise assertion (2) of the Theorem.

We take  $X \in \mathcal{T}$  and  $H = h_X$ . Then,  $f$  comes from a map  $g : C \rightarrow X$ . The cone  $Y$  of  $g$  is zero, since  $\operatorname{Hom}(S[i], Y) = 0$  for all  $S \in \mathcal{S}$  and  $i \in \mathbf{Z}$ . Hence,  $g$  is an isomorphism, so (2) holds.

Assume finally that  $X \in \mathcal{T}^c$ . Then,  $g^{-1} : X \xrightarrow{\sim} C$  factors through some object of  $\langle \mathcal{S} \rangle_i$  by Proposition 3.11, hence  $X \in \langle \mathcal{S} \rangle_i$ .  $\square$

#### 4.4. Finiteness for objects.

4.4.1. We say that  $C$  is *cohomologically locally bounded* (resp. *bounded above*, resp. *bounded below*, resp. *finitely generated*, resp. *finitely presented*, resp. *finite*) if the restriction of  $h_C$  to  $\mathcal{T}^c$  has that property.

From Lemma 4.2, we deduce

**Lemma 4.21.** *Let  $C \in \mathcal{T}^c$ . Then,  $C$  is cohomologically locally finitely presented.*

From Proposition 4.3, we deduce

**Proposition 4.22.** *The full subcategory of  $\mathcal{T}$  of cohomologically locally finitely presented objects is a thick subcategory.*

Note that the full subcategory of cohomologically locally bounded (resp. bounded above, resp. bounded below) is also a thick subcategory.

From Theorem 4.14, we deduce

**Corollary 4.23.** *Let  $\mathcal{T}$  be a triangulated category such that  $\mathcal{T}^c$  is strongly finitely generated. Then,  $C \in \mathcal{T}^c$  if and only if  $C$  is cohomologically locally finitely presented.*

**Remark 4.24.** Not all cohomological functor on  $\mathcal{T}^c$  are isomorphic to the restriction of  $h_C$ , for some  $C \in \mathcal{T}$ . This question has been studied for example in [Nee4, Bel, ChKeNee]. Let us mention the following result [ChKeNee, Lemma 2.13] : let  $\mathcal{T}$  be a cocomplete and compactly generated triangulated category. Assume  $k$  is a field. Let  $H$  be a cohomological functor on  $\mathcal{T}^c$  with value in the category  $k\text{-mod}$  of finite dimensional vector spaces. Then there is  $C \in \mathcal{T}$  such that  $H$  is isomorphic to the restriction of  $h_C$  to  $\mathcal{T}^c$ .

4.4.2. We put  $\mathcal{T}^f = \langle \widetilde{\mathcal{T}^c} \rangle_\infty$ . Note that if  $E$  is a classical generator of  $\mathcal{T}^c$ , then  $\mathcal{T}^f = \langle \widetilde{E} \rangle_\infty$ .

If  $\mathcal{T}^f = \langle \widetilde{E} \rangle_d$  for some  $d \in \mathbf{N}$  and some  $E \in \mathcal{T}^c$ , then  $\mathcal{T}^c = \langle E \rangle_d$  by Corollary 3.12.

We say that  $X \in \mathcal{T}$  is *cohomologically locally presented* if the restriction of  $h_X$  to  $\mathcal{T}^f$  is locally finitely presented.

Note that the objects of  $\mathcal{T}^f$  are cohomologically locally presented (Lemma 4.2) and that the full subcategory of  $\mathcal{T}$  of cohomologically locally presented objects is a thick subcategory (Proposition 4.3).

We say that a triangulated category is *regular* if  $\mathcal{T}^f = \langle \widetilde{E} \rangle_d$  for some  $E \in \mathcal{T}^c$  and  $d \in \mathbf{N}$ . The justification for this terminology comes from Proposition 5.23 below.

Note that this is equivalent to requiring that  $\mathcal{T}^f = \langle \widetilde{E} \rangle_d$  for some  $E \in \mathcal{T}^f$  and  $d \in \mathbf{N}$ .

If  $\mathcal{T}$  is regular, then  $\mathcal{T}^c$  is strongly finitely generated by Corollary 3.12.

From Theorem 4.18, we deduce

**Corollary 4.25.** *Let  $\mathcal{T}$  be a regular triangulated category. Then,  $C \in \mathcal{T}^f$  if and only if  $C$  is cohomologically locally presented.*

4.4.3. Let us recall Thomason's classification of dense subcategories [Th, Theorem 2.1] :

**Theorem 4.26.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{I}$  a dense full triangulated subcategory. Then, an object of  $\mathcal{T}$  is isomorphic to an object of  $\mathcal{I}$  if and only if its class is in the image of the canonical map  $K_0(\mathcal{I}) \rightarrow K_0(\mathcal{T})$ .*

We now state a version of Thomason-Trobaugh-Neeman's Theorem useful for our purposes [Nee2, Theorem 2.1].

**Theorem 4.27.** *Let  $\mathcal{T}$  be a cocomplete and compactly generated triangulated category. Let  $\mathcal{I}$  a full triangulated subcategory closed under direct sums and generated by a set of objects of  $\mathcal{I} \cap \mathcal{T}^c$ . Denote by  $F : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{I}$  the quotient functor.*

(i) *Let  $X \in \mathcal{T}^c$  and  $Y \in \mathcal{T}$ . Then, the canonical map*

$$\lim \mathrm{Hom}_{\mathcal{T}}(X', Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}/\mathcal{I}}(FX, FY)$$

*is an isomorphism, where the limit is taken over the maps  $X' \rightarrow X$  whose cone is in  $\mathcal{I} \cap \mathcal{T}^c$ . Also, given  $Y \in \mathcal{T}$  such that  $F(Y)$  is in  $F(\mathcal{T}^c)$ , then, there is  $C \in \mathcal{T}^c$  and  $f : C \rightarrow Y$  such that  $F(f)$  is an isomorphism.*

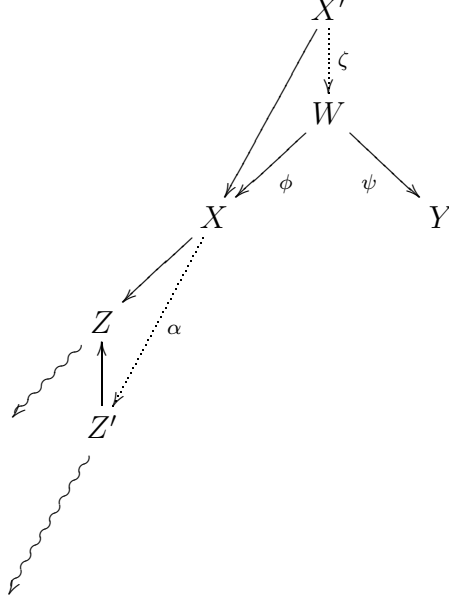
(ii)  *$F$  commutes with direct sums and the canonical functor  $\mathcal{T}^c/(\mathcal{I} \cap \mathcal{T}^c) \rightarrow \mathcal{T}/\mathcal{I}$  factors through a fully faithful functor  $G : \mathcal{T}^c/(\mathcal{I} \cap \mathcal{T}^c) \rightarrow (\mathcal{T}/\mathcal{I})^c$ .*

(iii) *An object of  $(\mathcal{T}/\mathcal{I})^c$  is isomorphic to an object in the image of  $G$  if and only if its class is in the image of  $K_0(G)$ . Furthermore,  $F$  induces an equivalence  $\mathcal{T}^f/(\mathcal{I} \cap \mathcal{T}^f) \xrightarrow{\sim} (\mathcal{T}/\mathcal{I})^f$ .*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}/\mathcal{I} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{I} \cap \mathcal{T}^f & \longrightarrow & \mathcal{T}^f & \longrightarrow & (\mathcal{T}/\mathcal{I})^f & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{I} \cap \mathcal{T}^c & \longrightarrow & \mathcal{T}^c & \longrightarrow & (\mathcal{T}/\mathcal{I})^c & & \end{array}$$

*Proof.* Let  $X \in \mathcal{T}^c$  and  $Y \in \mathcal{T}$ . Let  $\phi : W \rightarrow X$  and  $\psi : X \rightarrow Y$ . Let  $Z$  be a cone of  $\phi$  and assume  $Z \in \mathcal{I}$ . By Theorem 4.20 (2) and Proposition 3.11,  $X \rightarrow Z$  factors through a map  $\alpha : X \rightarrow Z'$  for some  $Z' \in \mathcal{I} \cap \mathcal{T}^c$ . Let  $X'$  be the cocone of  $\alpha$ . The map  $X' \rightarrow X$  factors as a

composition  $\phi\zeta$ . This shows (i).



Since  $\mathcal{T}$  is cocomplete and the direct sum in  $\mathcal{T}$  of objects of  $\mathcal{I}$  is in  $\mathcal{I}$ , it follows from [BöNee, Lemma 1.5] that  $F$  commutes with direct sums.

Let now  $X \in \mathcal{T}^c$  and  $\{Z_i\}$  be a family of elements of  $\mathcal{T}$ . Let  $f : F(X) \rightarrow \bigoplus_i F(Z_i) = F(\bigoplus_i Z_i)$ . There is  $\phi : X' \rightarrow X$  and  $\psi : X' \rightarrow \bigoplus_i Z_i$  with the cone of  $\phi$  in  $\mathcal{I} \cap \mathcal{T}^c$  and  $f = F(\psi)F(\phi)^{-1}$ . Since  $X'$  is compact,  $\psi$  factors through a finite sum of  $Z_i$ 's, hence  $f$  factors through a finite sum of  $F(Z_i)$ 's. Consequently,  $F(X)$  is compact. The fully faithfulness of  $G$  comes from (i).

Let us finally prove (iii). By Theorem 4.20 (3),  $(\mathcal{T}/\mathcal{I})^c$  is classically generated by  $F(\mathcal{T}^c)$ . Since  $F(\mathcal{T}^c)$  is a full triangulated subcategory of  $(\mathcal{T}/\mathcal{I})^c$ , it is dense. The result about  $\mathcal{T}^c$  follows now from Theorem 4.26.

It follows that  $F(\mathcal{T}^f)$  is dense in  $(\mathcal{T}/\mathcal{I})^f$ . Since  $\mathcal{T}^f$  is closed under taking multiples, it follows from Lemma 4.17 that  $F(\mathcal{T}^f) \xrightarrow{\sim} (\mathcal{T}/\mathcal{I})^f$ .  $\square$

**Corollary 4.28.** *Let  $\mathcal{T}$  be a cocomplete and compactly generated triangulated category. Let  $\mathcal{I}$  be a full triangulated subcategory closed under direct sums and generated by a set of objects of  $\mathcal{I} \cap \mathcal{T}^c$ .*

*If  $\mathcal{T}$  is regular, then  $\mathcal{T}/\mathcal{I}$  is regular.*

## 4.5. Algebras and schemes.

4.5.1. From Theorem 4.20 (3), we deduce the following result [Ke, §5.3] :

**Corollary 4.29.** *Let  $A$  be a dg algebra. Then,  $D(A)^c = \langle A \rangle_\infty$  and  $D(A)^f = \langle \tilde{A} \rangle_\infty$ .*

**Proposition 4.30.** *Let  $A$  be a dg algebra and  $C \in D(A)$ . Then,  $C$  is cohomologically locally bounded (resp. bounded above, resp. bounded below) if and only if  $H^i(C) = 0$  for  $|i| \gg 0$  (resp. for  $i \gg 0$ , resp. for  $i \ll 0$ ).*

*Proof.* We have  $D(A)^c = \langle A \rangle_\infty$  (Corollary 4.29). Hence,  $C$  is cohomologically locally bounded (resp. bounded above, resp. bounded below) if and only if  $h_C(A[i]) = 0$  for  $|i| \gg 0$  (resp. for  $i \ll 0$ , resp. for  $i \gg 0$ ). Since  $h_C(A[i]) \xrightarrow{\sim} H^{-i}(C)$ , the result follows.  $\square$

4.5.2. For  $A$  an algebra, we denote by  $K^{-,b}(A\text{-proj})$  (resp.  $K^{-,b}(A\text{-Proj})$ ) the homotopy category of right bounded complexes of finitely generated projective  $A$ -modules (resp. projective  $A$ -modules) with bounded cohomology.

**Proposition 4.31.** *Let  $A$  be an algebra. The canonical functors induce equivalences between*

- $K^b(A\text{-proj})$  and  $D(A)^c$
- $K^b(A\text{-Proj})$  and  $D(A)^f$
- $K^{-,b}(A\text{-proj})$  and the full subcategory of  $D(A)$  of cohomologically locally finitely presented objects
- $D^b(A)$  and the full subcategory of  $D(A)$  of cohomologically locally presented objects.

*Proof.* The first two assertions are immediate consequences of Corollary 4.29.

Recall that the canonical functor  $K^{-,b}(A\text{-Proj}) \rightarrow D^b(A)$  is an equivalence.

We now prove the third assertion. Let  $C \in D(A)$ . By Corollary 4.29 and Lemma 4.6,  $C$  is cohomologically locally finitely presented if and only if conditions (a) and (b) hold for  $X = A$ .

Let  $C$  be a right bounded complex of finitely generated projective  $A$ -modules with bounded cohomology. Pick  $r$  such that  $H^i(C) = 0$  for  $i \leq r$ . The canonical map from the stupid truncation  $\sigma^{\geq r}C$  to  $C$  is surjective on cohomology, so  $C$  satisfies (a).

Let  $D$  be a bounded complex of finitely generated projective  $A$ -modules and  $f : D \rightarrow C$ . Take  $s \leq r$  such that  $D^i = 0$  for  $i \leq s$ . Then,  $f$  factors through the canonical map  $\sigma^{\geq s}C \rightarrow C$  in a map  $f' : D \rightarrow \sigma^{\geq s}C$ . Pick a distinguished triangle  $E \xrightarrow{g} D \xrightarrow{f'} \sigma^{\geq s}C \rightsquigarrow$ . Then, we have an exact sequence  $H^i E \rightarrow H^i D \rightarrow H^i C$  for all  $i$ , hence  $C$  satisfies (b). So,  $C$  is cohomologically locally finitely presented.

Let  $C$  be a cohomologically locally finitely generated object. Then,  $C$  has bounded cohomology. Let  $i$  maximal such that  $H^i(C) \neq 0$ . Up to isomorphism, we can assume  $C^j = 0$  for  $j > i$ . By assumption, there is a bounded complex  $D$  of finitely generated projective  $A$ -modules and  $f : D \rightarrow C$  a morphism of complexes such that  $H(f)$  is onto. In particular, we have a surjection  $D^i \rightarrow C^i \rightarrow H^i(C)$ , hence  $H^i(C)$  is finitely generated.

Let  $C$  be cohomologically locally finitely presented.

Assume first  $C = M$  is a complex concentrated in degree 0. Let  $f : D^0 \rightarrow M$  be a surjection, with  $D^0$  finitely generated projective. Then,  $\ker f$  is cohomologically locally finitely presented (Proposition 4.22), hence is the quotient of a finitely generated projective module. By induction, it follows that  $M$  has a left resolution by finitely generated projective  $A$ -modules.

We take now for  $C$  an arbitrary cohomologically locally finitely presented object. We know that  $C$  has bounded cohomology (see above) and we now prove by induction on  $\sup\{i | H^i(C) \neq 0\} - \min\{i | H^i(C) \neq 0\}$  that  $C$  is isomorphic to an object of  $K^{-,b}(A\text{-proj})$ .

Let  $i$  be maximal such that  $H^i(C) \neq 0$ . As proven above, there is a finitely generated projective  $A$ -module  $P$  and a morphism of complexes  $f : P[-i] \rightarrow C$  such that  $H^i(f)$  is surjective. Let  $C'$  be the cone of  $f$ . By Proposition 4.22,  $C'$  is again cohomologically locally finitely presented. By induction,  $C'$  is isomorphic to an object of  $K^{-,b}(A\text{-proj})$  and we are done.

The last assertion has a similar (easier) proof.  $\square$

**Corollary 4.32.** *Let  $A$  be a noetherian algebra. Then, the full subcategory of cohomologically locally finitely presented objects of  $\mathcal{T} = D(A)$  is equivalent to  $D^b(A\text{-mod})$ .*

**Remark 4.33.** For the derived category of an algebra, the bounded derived category is also the full subcategory of cohomologically locally bounded objects. For a dg algebra, there might be no non-zero cohomologically locally bounded objects (e.g., for  $k[x, x^{-1}]$  with  $x$  in degree 1 and differential zero). The notion of cohomologically locally presented objects is more interesting for our purposes.

4.5.3.

**Proposition 4.34.** *Let  $\mathcal{A}$  be an abelian category with exact filtered colimits and a set  $\mathcal{F}$  of generators (i.e., a Grothendieck category). Assume that for any  $G \in \mathcal{F}$ , the subobjects of  $G$  are compact.*

*Then,  $(D^b(\mathcal{A}))^c = \langle \mathcal{A}^c \rangle_\infty$ .*

*Proof.* An object  $I$  of  $\mathcal{A}$  is injective if for any  $G \in \mathcal{F}$  and any subobject  $G'$  of  $G$ , the canonical map  $\text{Hom}_{\mathcal{A}}(G', I) \rightarrow \text{Hom}_{\mathcal{A}}(G, I)$  is surjective [Ste, Proposition V.2.9]. Note that  $G'$  is compact. It follows that a direct sum of injectives is injective.

Let  $M \in \mathcal{A}^c$ . Let  $\mathcal{F}$  be a family of objects of  $D^b(\mathcal{A})$ . Then,  $\bigoplus_{F \in \mathcal{F}} F$  exists in  $D^b(\mathcal{A})$  if and only if the direct sum, computed in  $D(\mathcal{A})$ , has bounded homology, i.e., if and only if, there are integers  $r$  and  $s$  such that for any  $F \in \mathcal{F}$ , we have  $H^i(F) = 0$  for  $i < r$  and for  $i > s$ . Given  $F \in \mathcal{F}$ , let  $I_F$  be a complex of injectives quasi-isomorphic to  $F$  with zero terms in degrees less than  $r$ . Since  $\bigoplus_F I_F^j$  is injective, we have  $\text{Ext}^i(M, \bigoplus_F I_F^j) = 0$  for all  $j$  and  $i > 0$ . Hence,

$$\begin{aligned} \text{Hom}_{D(\mathcal{A})}(M, \bigoplus_F F) &\xrightarrow{\sim} H^0 \text{Hom}_{\mathcal{A}}^\bullet(M, \bigoplus_F I_F) \xrightarrow{\sim} H^0 \bigoplus_F \text{Hom}_{\mathcal{A}}^\bullet(M, I_F) \xrightarrow{\sim} \\ &\xrightarrow{\sim} \bigoplus_F H^0 \text{Hom}_{\mathcal{A}}^\bullet(M, I_F) \xrightarrow{\sim} \bigoplus_F \text{Hom}_{D(\mathcal{A})}(M, F). \end{aligned}$$

It follows that  $M \in D^b(\mathcal{A})^c$ .

Let  $C \in D^b(\mathcal{A})^c$ . We prove by induction on  $\max\{i | H^i C \neq 0\} - \min\{i | H^i C \neq 0\}$  that  $C \in \langle \mathcal{A}^c \rangle_\infty$ .

Take  $i$  maximal such that  $H^i C \neq 0$ . Then,  $\text{Hom}_{D^b(\mathcal{A})}(C, M[-i]) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(H^i C, M)$  for any  $M \in \mathcal{A}$ . It follows that  $H^i C \in \mathcal{A}^c$ . As proven above, we deduce that  $H^i C[-i] \in D^b(\mathcal{A})^c$ , hence  $\tau^{\leq i-1} C \in D^b(\mathcal{A})^c$ . By induction,  $\tau^{\leq i-1} C \in \langle \mathcal{A}^c \rangle_\infty$  and we are done.  $\square$

**Corollary 4.35.** *Let  $A$  be a noetherian ring. Then,  $D^b(A\text{-mod}) \xrightarrow{\sim} D^b(A)^c$ .*

*Let  $X$  be a noetherian scheme. Then,  $D^b(X\text{-coh}) \xrightarrow{\sim} D^b(X\text{-qcoh})^c$ .*

*Proof.* In the ring case, we take  $\mathcal{F} = \{A\}$ . In the geometric case, we take for  $\mathcal{F}$  the set of coherent sheaves, cf [ThTr, Appendix B, §3].  $\square$

## 5. DIMENSION FOR DERIVED CATEGORIES OF RINGS AND SCHEMES

5.1. **Resolution of the diagonal.** Let  $k$  be a field.

5.1.1.

**Lemma 5.1.** *Let  $A$  be a noetherian  $k$ -algebra such that  $\text{pdim}_{A^{\text{en}}} A < \infty$ . Then,  $D^b(A) = \langle \tilde{A} \rangle_{1+\text{pdim}_{A^{\text{en}}} A}$  and  $D^b(A\text{-mod}) = \langle A \rangle_{1+\text{pdim}_{A^{\text{en}}} A}$ . In particular,  $\dim D^b(A\text{-mod}) \leq \text{pdim}_{A^{\text{en}}} A$ .*

*Proof.* By §3.2.2, we deduce that  $D^b(A) = \langle \tilde{A} \rangle_{1+\text{pdim}_{A^{\text{en}}} A}$ . Now, we have  $D^b(A\text{-mod}) \simeq D^b(A)^c$  (Corollary 4.35) and the result follows from Corollary 3.12.  $\square$

We say that a commutative  $k$ -algebra  $A$  is *essentially of finite type* if it is the localization of a commutative  $k$ -algebra of finite type over  $k$ .

Recall the following classical result :

**Lemma 5.2.** *Let  $A$  be a finite dimensional  $k$ -algebra or a commutative  $k$ -algebra essentially of finite type. Assume that given  $V$  a simple  $A$ -module, then  $Z(\text{End}_A(V))$  is a separable extension of  $k$ . Then,  $\text{pdim}_{A^{\text{en}}} A = \text{gldim } A$ .*

*Proof.* Note that under the assumptions,  $A^{\text{en}}$  is noetherian. In the commutative case,  $\text{gldim } A = \sup\{\text{gldim } A_{\mathfrak{m}}\}_{\mathfrak{m}}$  and  $\text{pdim}_{A^{\text{en}}} A = \sup\{\text{pdim}_{(A_{\mathfrak{m}})^{\text{en}}} A_{\mathfrak{m}}\}_{\mathfrak{m}}$  where  $\mathfrak{m}$  runs over the maximal ideals of  $A$ . It follows that it is enough to prove the Lemma for  $A$  local.

So, let us assume now  $A$  is finite dimensional or is a commutative local  $k$ -algebra essentially of finite type.

Let  $0 \rightarrow P^{-r} \rightarrow \dots \rightarrow P^0 \rightarrow A \rightarrow 0$  be a minimal projective resolution of  $A$  as an  $A^{\text{en}}$ -module. So, there is a simple  $A^{\text{en}}$ -module  $U$  with  $\text{Ext}_{A^{\text{en}}}^r(A, U) \neq 0$ . The simple module  $U$  is isomorphic to a quotient of  $\text{Hom}_k(S, T)$  for  $S, T$  two simple  $A$ -modules. By assumption,  $\text{End}_A(S) \otimes_k \text{End}_A(T)^\circ$  is semi-simple, hence  $U$  is actually isomorphic to a direct summand of  $\text{Hom}_k(S, T)$ .

Then,

$$\text{Ext}_A^r(T, S) \xrightarrow{\sim} \text{Ext}_{A^{\text{en}}}^r(A, \text{Hom}_k(T, S)) \neq 0.$$

Hence,  $r \leq \text{gldim } A$ .

Now, given  $N$  an  $A$ -module,  $0 \rightarrow P^{-r} \otimes_A N \rightarrow \dots \rightarrow P^0 \otimes_A N \rightarrow N \rightarrow 0$  is a projective resolution of  $N$ , hence  $r \geq \text{gldim } A$ , so  $r = \text{gldim } A$ .  $\square$

**Remark 5.3.** Note that this Lemma doesn't hold if the residue fields of  $A$  are not separable extensions of  $k$ . Cf the case  $A = k'$  a purely inseparable extension of  $k$ .

Combining Lemmas 5.1 and 5.2, we get

**Proposition 5.4.** *Let  $A$  be a finite dimensional  $k$ -algebra or a commutative  $k$ -algebra essentially of finite type. Assume that given  $V$  a simple  $A$ -module, then  $Z(\text{End}_A(V))$  is a separable extension of  $k$ .*

*If  $A$  has finite global dimension, then  $D^b(A) = \langle \tilde{A} \rangle_{1+\text{gldim } A}$  and  $D^b(A\text{-mod}) = \langle A \rangle_{1+\text{gldim } A}$ . In particular,  $\dim D^b(A\text{-mod}) \leq \text{gldim } A$ .*

5.1.2. Following §3.2.2, we have the following result (cf [BoVdB, §3.4]).

**Proposition 5.5.** *Let  $X$  be a separated noetherian scheme over  $k$ . Assume there is a vector bundle  $\mathcal{L}$  on  $X$  and a resolution of the structure sheaf  $\mathcal{O}_\Delta$  of the diagonal in  $X \times X$*

$$0 \rightarrow \mathcal{F}^{-r} \rightarrow \dots \rightarrow \mathcal{F}^0 \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

*with  $\mathcal{F}^i \in \text{add}(\mathcal{L} \boxtimes \mathcal{L})$ .*

*Then,  $D^b(X\text{-qcoh}) = \langle \tilde{\mathcal{L}} \rangle_{1+r}$  and  $D^b(X\text{-coh}) = \langle \mathcal{L} \rangle_{1+r}$ .*

*Proof.* Let  $p_1, p_2 : X \times X \rightarrow X$  be the first and second projections. For  $C \in D^b(X\text{-qcoh})$ , we have  $C \simeq Rp_{1*}(\mathcal{O}_\Delta \otimes^{\mathbf{L}} p_2^* C)$ . It follows that  $C \in \langle \mathcal{L} \otimes_k R\Gamma(\mathcal{L} \otimes C) \rangle_{1+r}$ , hence  $C \in \langle \tilde{\mathcal{L}} \rangle_{1+r}$ . Since  $D^b(X\text{-qcoh})^c = D^b(X\text{-coh})$  (Corollary 4.35), the second assertion follows from Corollary 3.12.  $\square$

Note that the assumption of the Proposition forces  $X$  to be regular.

**Example 5.6.** Let  $X = \mathbf{P}_k^n$ . Let us recall results of Beilinson [Bei]. The object  $G = \mathcal{O} \oplus \cdots \oplus \mathcal{O}(n)$  is a classical generator for  $D^b(X\text{-coh})$ . We have  $\text{Ext}^i(G, G) = 0$  for  $i \neq 0$ . Let  $A = \text{End}(G)$ . We have  $D^b(X\text{-coh}) \simeq D^b(A\text{-mod})$ . We have  $\text{gldim } A = n$ , hence  $D^b(A\text{-mod}) = \langle A \rangle_{n+1}$  (Proposition 5.4), so  $D^b(\mathbf{P}^n\text{-coh}) = \langle \mathcal{O} \oplus \cdots \oplus \mathcal{O}(n) \rangle_{n+1}$ . Another way to see this is to use the resolution of the diagonal  $\Delta \subset X \times X$  :

$$0 \rightarrow \mathcal{O}(-n) \boxtimes \Omega^n(n) \rightarrow \cdots \rightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \rightarrow \mathcal{O} \boxtimes \mathcal{O} \rightarrow \mathcal{O}_\Delta \rightarrow 0.$$

By Proposition 5.36 below, it follows that  $\dim D^b(\mathbf{P}^n\text{-coh}) = n$ .

**Example 5.7.** In [Ka], Kapranov considers flag varieties (type  $A$ ) and smooth projective quadrics. For these varieties  $X$ , he constructs explicit bounded resolutions of the diagonal whose terms are direct sums of  $\mathcal{L} \boxtimes \mathcal{L}'$ , where  $\mathcal{L}$  and  $\mathcal{L}'$  are vector bundles. It turns out that these resolutions have exactly  $1 + \dim X$  terms (this is the smallest possible number). By Proposition 5.36, it follows that  $\dim D^b(X\text{-coh}) = \dim X$ .

Starting from a smooth projective variety  $X$ , there is a line bundle whose homogeneous coordinate ring is a Koszul algebra [Ba, Theorem 2]. Now, if the kernel of the  $r$ -th map of the resolution is a direct sum of sheaves of the form  $\mathcal{L} \boxtimes \mathcal{L}'$ , where  $\mathcal{L}, \mathcal{L}'$  are vector bundles, then  $\dim D^b(X\text{-coh}) \leq r$ . Note that this can work only if the class of  $\mathcal{O}_\Delta$  is in the image of the product map  $K_0(X) \times K_0(X) \rightarrow K_0(X \times X)$ . The case of flag varieties associated to reductive groups of type different from  $A_n$  would be interesting to study.

The following is our best result providing an upper bound for smooth schemes.

**Proposition 5.8.** *Let  $X$  be a smooth quasi-projective scheme over  $k$ . Let  $\mathcal{L}$  be an ample line bundle on  $X$ . Then, there is  $r \geq 0$  such that  $D^b(X\text{-qcoh}) = \langle \tilde{G} \rangle_{2 \dim X + 1}$  and  $D^b(X\text{-coh}) = \langle G \rangle_{2 \dim X + 1}$  where  $G = \mathcal{O} \oplus \mathcal{L}^{\otimes -1} \oplus \cdots \oplus \mathcal{L}^{\otimes -r}$ . In particular,  $\dim D^b(X\text{-coh}) \leq 2 \dim X$ .*

*Proof.* There is a resolution of the diagonal

$$\cdots \rightarrow C^{-i} \xrightarrow{d^{-i}} \cdots \rightarrow C^0 \xrightarrow{d^0} \mathcal{O}_\Delta \rightarrow 0$$

where  $C^i \in \text{add}(\{\mathcal{L}^{-j} \boxtimes \mathcal{L}^{-j}\}_{j \geq 0})$ . Denote by  $C$  the complex  $\cdots \rightarrow C^{-i} \xrightarrow{d^{-i}} \cdots \rightarrow C^0 \rightarrow 0$ . Let  $n = \dim X$ . Truncating, we get an exact sequence

$$0 \rightarrow C^{-2n-1} / \ker d^{-2n} \rightarrow C^{-2n} \rightarrow \cdots \rightarrow C^{-i} \xrightarrow{d^{-i}} \cdots \rightarrow C^0 \xrightarrow{d^0} \mathcal{O}_\Delta \rightarrow 0$$

Since  $X \times X$  is smooth of dimension  $2n$ , we have  $\text{Ext}^{2n+1}(\mathcal{O}_\Delta, C^{-2n-1} / \ker d^{-2n}) = 0$ . So, the distinguished triangle  $C^{-2n-1} / \ker d^{-2n} [2n] \rightarrow \sigma^{\geq -2n} C \rightarrow \mathcal{O}_\Delta \rightsquigarrow$  splits, *i.e.*,  $\mathcal{O}_\Delta$  is a direct summand of the complex  $\sigma^{\geq -2n} C$ . We conclude as in the proof of Proposition 5.5.  $\square$

**Remark 5.9.** We actually don't know any case of a smooth variety where we can prove that  $\dim D^b(X\text{-coh}) > \dim X$ . The first case to consider would be an elliptic curve.



5.1.3. For applications to finite dimensional algebras, we need to prove certain results for the derived category of differential modules. The theory of such derived categories mirrors that of the usual derived category of complexes of modules (forget the grading). We state here the constructions and results needed in this paper.

Let  $A$  be a  $k$ -algebra. A differential  $A$ -module is a  $(A \otimes_k k[\varepsilon]/(\varepsilon^2))$ -module. We view a differential  $A$ -module as a pair  $(M, d)$  where  $M$  is an  $A$ -module and  $d \in \text{End}_A(M)$  satisfying  $d^2 = 0$  is given by the action of  $\varepsilon$ . The homology of a differential  $A$ -module is the  $A$ -module  $\ker d / \text{im } d$ .

The category of differential  $A$ -modules has the structure of an exact category, where the exact sequences are those exact sequences of  $(A \otimes_k k[\varepsilon]/(\varepsilon^2))$ -modules that split by restriction to  $A$ . This is a Frobenius category and its associated stable category is called the homotopy category of differential  $A$ -modules.

A morphism of  $A$ -modules is a quasi-isomorphism if the induced map on homology is an isomorphism. We now define the derived category of differential  $A$ -modules, denoted by  $D\text{diff}(A)$ , as the localisation of the homotopy category of differential  $A$ -modules in the class of quasi-isomorphisms. These triangulated categories have a trivial shift functor.

We have a triangulated forgetful functor  $D(A) \rightarrow D\text{diff}(A)$ . Let  $X, Y$  be two  $A$ -modules and  $i \geq 0$ . Then, the canonical map  $\text{Ext}_A^i(X, Y) \xrightarrow{\sim} \text{Hom}_{D(A)}(X, Y[i]) \rightarrow \text{Hom}_{D\text{diff}(A)}(X, Y)$  is injective and we have an isomorphism  $\prod_{n \geq 0} \text{Ext}_A^n(X, Y) \xrightarrow{\sim} \text{Hom}_{D\text{diff}(A)}(X, Y)$ .

5.1.4.

**Lemma 5.10.** *Let  $A$  be a  $k$ -algebra. Let  $W$  be an  $A$ -module with  $\text{pdim } W \geq d$ . Then, there are  $A^{\text{en}}$ -modules  $M_0 = A, M_1, \dots, M_d$  which are projective as left and as right  $A$ -modules, elements  $\zeta_i \in \text{Ext}_{A^{\text{en}}}^1(M_i, M_{i+1})$  for  $0 \leq i \leq d-1$  such that  $(\zeta_{d-1} \cdots \zeta_0) \otimes_A 1_W$  is a non zero element of  $\text{Ext}_A^d(W, M_d \otimes_A W)$ .*

*Proof.* Let  $\dots \rightarrow C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} A \rightarrow 0$  be a projective resolution of the  $A^{\text{en}}$ -module  $A$ . Then,  $\dots \rightarrow C^{-2} \otimes_A W \rightarrow C^{-1} \otimes_A W \rightarrow C^0 \otimes_A W \rightarrow W \rightarrow 0$  is a projective resolution of  $W$ . Let  $\Omega^{-i}$  be the kernel of  $d^{i+1}$  for  $i \leq -1$  and  $\Omega^0 = A$ . Let  $\zeta_i \in \text{Ext}_{A^{\text{en}}}^1(\Omega^i, \Omega^{i+1})$  given by the exact sequence  $0 \rightarrow \Omega^{i+1} \rightarrow C^{-i} \xrightarrow{d^{-i}} \Omega^i \rightarrow 0$ .

Since  $\text{Ext}_A^d(W, -)$  is not zero, it follows that the exact sequence

$$0 \rightarrow \Omega^d \otimes_A W \rightarrow C^{-d+1} \otimes_A W \rightarrow \dots \rightarrow C^{-1} \otimes_A W \rightarrow C^0 \otimes_A W \rightarrow W \rightarrow 0$$

gives a non zero element  $\xi \in \text{Ext}_A^d(W, \Omega^d \otimes_A W)$ . This element is equal to  $(\zeta_{d-1} \cdots \zeta_0) \otimes_A 1_W$ .  $\square$

The following result is our main tool to produce lower bounds for the dimension.

**Lemma 5.11.** *Let  $A$  be a  $k$ -algebra. Let  $W$  be an  $A$ -module with  $\text{pdim } W \geq d$ . Let  $\mathcal{T}$  be  $D(A)$  or  $D\text{diff}(A)$ . Then,  $W \notin \langle \overline{A} \rangle_{\mathcal{T}, d}$ .*

*Proof.* Assume  $W \in \langle \overline{A} \rangle_{1+r}$  for some  $r \geq 0$ . Let  $W_{s-1} \rightarrow W_s \rightarrow V_s \rightsquigarrow$  be a family of distinguished triangles, for  $1 \leq s \leq r$ . We put  $V_0 = W_0$  and we assume  $V_s \in \langle \overline{A} \rangle$  for  $0 \leq s \leq r$  and  $W_r = W \oplus W'$  for some  $W'$ .

We use now Lemma 5.10. The element  $\zeta_i$  induces a natural transformation of functors  $M_i \otimes_A - \rightarrow M_{i+1}[1] \otimes_A -$  from  $D^b(A)$  to itself. Restricted to  $\langle \overline{A} \rangle$ , this transformation is zero. It follows from Lemma 4.9 that  $(\zeta_{d-1} \cdots \zeta_0) \otimes_A -$  vanishes on  $\langle \overline{A} \rangle_d$ . It follows that

$r \geq d$  (in case  $\mathcal{T}$  is the derived category of differential  $A$ -modules, note that the canonical map  $\text{Ext}_A^d(W, M_d \otimes_A W) \rightarrow \text{Hom}_{\mathcal{T}}(W, M_d \otimes_A W)$  is injective).  $\square$

We deduce the following crucial Proposition :

**Proposition 5.12.** *Let  $A$  be a commutative local noetherian  $k$ -algebra with maximal ideal  $\mathfrak{m}$ . Let  $\mathcal{T}$  be  $D(A)$  or  $D\text{diff}(A)$ . Then,  $A/\mathfrak{m} \notin \langle A \rangle_{\mathcal{T}, \text{Krulldim } A}$ .*

*Proof.* We know that  $\text{Krulldim } A \leq \text{gldim } A = \text{pdim}_A A/\mathfrak{m}$  (cf for example [Ma, Theorem 41]). The result follows now from Lemma 5.11.  $\square$

**Remark 5.13.** Let  $M, V \in D(A)$ . If  $V \in \overline{\langle M \rangle}_{D(A), i}$ , then  $F(V) \in \overline{\langle F(M) \rangle}_{D\text{diff}(A), i}$ , where  $F : D(A) \rightarrow D\text{diff}(A)$  is the forgetful functor.

From Lemma 5.11 and Propositions 5.23, we deduce

**Proposition 5.14.** *Let  $A$  be a noetherian  $k$ -algebra of global dimension  $d \in \mathbf{N} \cup \{\infty\}$ . Assume  $k$  is perfect. Then,  $d$  is the minimal integer  $i$  such that  $A\text{-perf} = \langle A \rangle_{i+1}$ .*

**Remark 5.15.** The dimension of  $D^b(A\text{-mod})$  can be strictly less than  $\text{gldim } A$  (this will be the case for example for a finite dimensional  $k$ -algebra  $A$  which is not hereditary but which is derived equivalent to a hereditary algebra). This cannot happen if  $A$  is a finitely generated commutative  $k$ -algebra, cf Proposition 5.36 below.

**Remark 5.16.** Let  $A = k[x]/(x^2)$  be the algebra of dual numbers. The indecomposable objects of  $D^b(A\text{-mod})$  are  $k[i]$  and  $L_n[i]$  for  $n \geq 1$  and  $i \in \mathbf{Z}$ , where  $L_n$  is the cone of a non-zero map  $k \rightarrow k[n]$ . It follows that  $D^b(A\text{-mod}) = \langle k \rangle_2$ , hence,  $\dim D^b(A\text{-mod}) = 1$  (cf Proposition 5.32 below).

Note that the dimension of the category of perfect complexes of  $A$ -modules is infinite by Proposition 5.23. Let us prove this directly. Given  $C$  a perfect complex of  $A$ -modules, there is an integer  $r$  such that  $\text{Ext}_{A^{\text{en}}}^i(A, A)$  acts as 0 on  $\langle C \rangle$  for  $i \geq r$ . On the other hand, given  $d$  an integer, then  $(\text{Ext}_{A^{\text{en}}}^1(A, A))^{rd}$  doesn't act by 0 on  $\text{Hom}_{D^b(A)}(L_{rd+1}, L_{rd+1}[rd])$  (note that  $L_{rd+1}$  is perfect). So,  $L_{rd+1} \notin \langle C \rangle_d$  by Lemma 4.9.

**Remark 5.17.** Let  $k$  be a field and  $A$  a finitely generated  $k$ -algebra. Can the dimension of  $D^b(A\text{-mod})$  be infinite ? We will show that the dimension is finite if  $A$  is finite dimensional (Proposition 5.32) or commutative and  $k$  is perfect (Theorem 5.38).

## 5.2. Finite global dimension.

5.2.1. We explain here a method of dévissage for derived categories of abelian categories with finite global dimension.

**Lemma 5.18.** *Let  $\mathcal{A}$  be an abelian category and  $C$  a complex of objects of  $\mathcal{A}$ . Assume  $H^1 C = \dots = H^i C = 0$  for some  $i \geq 0$ . Let  $0 \rightarrow \ker d^0 \xrightarrow{\alpha} L^0 \xrightarrow{f^0} \dots \xrightarrow{f^i} L^{i+1} \xrightarrow{\beta} C^{i+1}/\text{im } d^i \rightarrow 0$  be an exact sequence equivalent to  $0 \rightarrow \ker d^0 \rightarrow C^0 \rightarrow \dots \rightarrow C^{i+1} \rightarrow C^{i+1}/\text{im } d^i \rightarrow 0$  (i.e., giving the same element in  $\text{Ext}^{i+2}(C^{i+1}/\text{im } d^i, \ker d^0)$ ). Then,  $C$  is quasi-isomorphic to the complex*

$$\dots \rightarrow C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{a} L^0 \xrightarrow{f^0} \dots \xrightarrow{f^i} L^{i+1} \xrightarrow{b} C^{i+2} \xrightarrow{d^{i+2}} \dots$$

where  $a$  is the composite  $C^{-1} \xrightarrow{d^{-1}} \ker d^0 \xrightarrow{\alpha} L^0$  and  $b$  the composite  $L^{i+1} \xrightarrow{\beta} C^{i+1}/\text{im } d^i \xrightarrow{d^{i+1}} C^{i+2}$ .

*Proof.* It is enough to consider the case of an elementary equivalence between exact sequences. Let

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \ker d^0 & \longrightarrow & L^0 & \longrightarrow & \cdots & \longrightarrow & L^{i+1} & \longrightarrow & C^{i+1}/\operatorname{im} d^i & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \ker d^0 & \longrightarrow & C^0 & \longrightarrow & \cdots & \longrightarrow & C^{i+1} & \longrightarrow & C^{i+1}/\operatorname{im} d^i & \longrightarrow & 0 \end{array}$$

be a commutative diagram, with the rows being exact sequences. Then, there is a commutative diagram

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & L^0 & \longrightarrow & \cdots & \longrightarrow & L^{i+1} & \longrightarrow & C^{i+2} & \longrightarrow & \cdots \\ & & & & \parallel & & \searrow & & \downarrow & & \downarrow & & \parallel & & \\ & & & & & & \ker d^0 & & & & & & & & \\ & & & & & & \nearrow & & & & & & & & \\ \cdots & \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & C^0 & \longrightarrow & \cdots & \longrightarrow & C^{i+1} & \longrightarrow & C^{i+2} & \longrightarrow & \cdots \end{array}$$

This induces a morphism of complexes from the first row to the last row of the diagram and this is a quasi-isomorphism.  $\square$

**Lemma 5.19.** *Let  $\mathcal{A}$  be an abelian category with finite global dimension  $\leq n$ . Let  $C$  be a complex of objects of  $\mathcal{A}$ . Assume  $H^i C = 0$  if  $n \nmid i$ . Then,  $C$  is quasi-isomorphic to  $\bigoplus_i (H^{ni} C)[-ni]$ .*

*Proof.* Pick  $i \in \mathbf{Z}$ . The sequence  $0 \rightarrow \ker d^{ni} \rightarrow C^{ni} \rightarrow \cdots \rightarrow C^{n(i+1)} \rightarrow C^{n(i+1)}/\operatorname{im} d^{n(i+1)-1} \rightarrow 0$  is exact. It defines an element of  $\operatorname{Ext}_{\mathcal{A}}^{n+1}(C^{n(i+1)}/\operatorname{im} d^{n(i+1)-1}, \ker d^{ni})$ . This group is 0 by assumption, hence the exact sequence is equivalent to  $0 \rightarrow \ker d^{ni} \rightarrow \ker d^{ni} \xrightarrow{0} 0 \cdots 0 \xrightarrow{0} C^{n(i+1)}/\operatorname{im} d^{n(i+1)-1} \rightarrow C^{n(i+1)}/\operatorname{im} d^{n(i+1)-1} \rightarrow 0$ . Lemma 5.18 shows that  $C$  is quasi-isomorphic to a complex  $D$  with  $d_D^{ni} = \cdots = d_D^{n(i+1)-1} = 0$ . Now, there is a morphism of complexes  $(H^{n(i+1)} C)[-n(i+1)] \rightarrow D$  that induces an isomorphism on  $H^{n(i+1)}$ . So, for every  $i$ , there is a map  $\rho_i$  in  $D(\mathcal{A})$  from  $(H^{ni} C)[-ni]$  to  $C$  that induces an isomorphism on  $H^{ni}$ . Let  $\rho = \sum_i \rho_i : \bigoplus_i (H^{ni} C)[-ni] \rightarrow C$ . This is a quasi-isomorphism.  $\square$

**Proposition 5.20.** *Let  $\mathcal{A}$  be an abelian category with finite global dimension  $\leq n$  with  $n \geq 1$ . Let  $C$  be a complex of objects of  $\mathcal{A}$ . Then, there is a distinguished triangle in  $D(\mathcal{A})$*

$$\bigoplus_i D_i \rightarrow C \rightarrow \bigoplus_i E_i \rightsquigarrow$$

where  $D_i = \sigma^{\geq ni+1} \tau^{\leq n(i+1)-1} C$  is a complex with zero terms outside  $[ni+1, \dots, n(i+1)-1]$  and  $E_i$  is a complex concentrated in degree  $ni$ .

*Proof.* Let  $i \in \mathbf{Z}$ . Let  $f_i$  be the composition of the canonical maps  $\tau^{\leq n(i+1)-1} C \rightarrow C$  with the canonical map  $\sigma^{\geq ni+1} \tau^{\leq n(i+1)-1} C \rightarrow \tau^{\leq n(i+1)-1} C$ . Then,  $H^r(f_i)$  is an isomorphism for  $ni+2 \leq r \leq n(i+1)-1$  and is surjective for  $r = ni+1$ . Let  $D = \bigoplus_i \sigma^{\geq ni+1} \tau^{\leq n(i+1)-1} C$  and  $f = \sum_i f_i : D \rightarrow C$ . Let  $E$  be the cone of  $f$ . We have an exact sequence

$$\begin{aligned} \cdots \rightarrow H^{ni-2} D &\xrightarrow{\sim} H^{ni-2} C \rightarrow H^{ni-2} E \rightarrow H^{ni-1} D \xrightarrow{\sim} H^{ni-1} C \rightarrow H^{ni-1} E \rightarrow H^{ni} D \rightarrow \\ &\rightarrow H^{ni} C \rightarrow H^{ni} E \rightarrow H^{ni+1} D \rightarrow H^{ni+1} C \rightarrow H^{ni+1} E \rightarrow H^{ni+2} D \xrightarrow{\sim} H^{ni+2} C \rightarrow \cdots \end{aligned}$$

Since  $H^{ni}D = 0$  for all  $i$ , we deduce that  $H^rE = 0$  if  $n \not\equiv r$ . The Proposition follows now from Lemma 5.19.  $\square$

**Remark 5.21.** Note there is a dual version to Proposition 5.20 obtained by passing to the opposite category  $\mathcal{A}^\circ$ .

5.2.2.

**Proposition 5.22.** *Let  $A$  be a ring with finite global dimension. Then,  $D^b(A) = \langle \tilde{A} \rangle_{2+2\text{gldim } A}$ . If  $A$  is noetherian, then  $D^b(A\text{-mod}) = \langle A \rangle_{2+2\text{gldim } A}$  and  $\dim D^b(A\text{-mod}) \leq 1 + 2\text{gldim } A$ .*

*Proof.* Put  $n = \text{gldim } A$ . Let  $C \in D^b(A)$ . Up to quasi-isomorphism, we can assume  $C$  is a bounded complex of projective  $A$ -modules. We now use Proposition 5.20. An  $A$ -module  $M$  has a projective resolution of length  $n + 1$ , hence  $M \in \langle \tilde{A} \rangle_{n+1}$ . So,  $\bigoplus_i E_i \in \langle \tilde{A} \rangle_{n+1}$ . Similarly, we have  $\bigoplus_i D_i \in \langle \tilde{A} \rangle_{n+1}$ .

The second part of the Lemma follows from Corollary 3.12.  $\square$

The following characterization of regular algebras is due to Van den Bergh in the noetherian case.

**Proposition 5.23.** *Let  $A$  be a ring. Then, the following conditions are equivalent*

- (i)  $A$  is regular, i.e.,  $\text{gldim } A < \infty$
- (ii)  $K^b(A\text{-Proj}) \xrightarrow{\sim} D^b(A)$
- (iii)  $D(A)$  is regular.

*If  $A$  is noetherian, these conditions are equivalent to the following*

- (i') every finitely generated  $A$ -module has finite projective dimension
- (ii')  $D^b(A\text{-mod}) = A\text{-perf}$
- (iii')  $A\text{-perf}$  is strongly finitely generated.

*Proof.* The equivalence between the first two assertions is clear, since  $D^b(A)$  is classically generated by the  $L[i]$ , where  $L$  runs over the  $A$ -modules and  $i \in \mathbf{Z}$ .

If  $D(A)$  is regular, then every cohomologically locally presented object is in  $D(A)^f$  (Corollary 4.25). So, (iii)  $\Rightarrow$  (ii) follows from Proposition 4.31.

Finally, (i)  $\Rightarrow$  (iii) follows from Proposition 5.22.

The proof for the remaining assertions are similar.  $\square$

For finite dimensional or commutative algebras over a perfect field, we obtained in Proposition 5.4 the better bound  $\dim D^b(A\text{-mod}) \leq \text{gldim } A$ . We don't know whether such a bound holds under the assumption of Proposition 5.22.

The construction of Proposition 5.22 is not optimal when  $A$  is hereditary, since the  $D_i$  in Proposition 5.20 are then zero, i.e., every object of  $D^b(A)$  is isomorphic to a direct sums of complexes concentrated in one degree. We get then the following result.

**Proposition 5.24.** *Let  $A$  be a hereditary ring. Then,  $D^b(A) = \langle \tilde{A} \rangle_2$ .*

*Assume now  $A$  is noetherian. Then,  $D^b(A\text{-mod}) = \langle A \rangle_2$ . If there are infinitely many isomorphism classes of indecomposable finitely generated  $A$ -modules, then  $\dim D^b(A\text{-mod}) = 1$ .*

**Remark 5.25.** Proposition 5.24 generalizes easily to quasi-hereditary algebras. Let  $\mathcal{C}$  be a highest weight category over a field  $k$  with weight poset  $\Lambda$  (i.e., the category of finitely generated

modules over a quasi-hereditary algebra). Then, there is a decomposition  $D^b(\mathcal{C}) = \mathcal{I}_1 \diamond \cdots \diamond \mathcal{I}_d$  such that  $\mathcal{I}_i \simeq D^b(k^{n_i}\text{-mod})$  for some  $n_i$  and where  $d$  is the maximal  $i$  such that there is  $\lambda_1 < \cdots < \lambda_i \in \Lambda$  [CPS, Theorem 3.9]. It follows from Lemma 3.4 that  $\dim D^b(\mathcal{C}) < d$ .

**Remark 5.26.** It would be interesting to classify algebraic triangulated categories of dimension 1. Which differential graded / finite dimensional algebras can have such a derived category? This relates to work on quasi-tilted algebras.

5.2.3. The following Lemma is related to the non-commutative [BoVdB, Lemma 4.2.4].

**Lemma 5.27.** *Let  $X$  be a quasi-projective scheme over a field and  $\mathcal{L}$  an ample sheaf. Then, there are  $r, l \geq 0$  such that  $\overline{\text{add}}(\{\mathcal{L}^{\otimes i}\}_{i \in \mathbf{Z}}) \subset \langle \tilde{G} \rangle_l$  where  $G = \mathcal{L}^{\otimes -r} \oplus \mathcal{L}^{\otimes -r+1} \oplus \cdots \oplus \mathcal{L}^{\otimes r}$ . If  $X$  is regular, then we can take  $l = 1 + \dim X$ .*

*Proof.* Pick  $s > 0$  such that  $\mathcal{L}^{\otimes s}$  is very ample and let  $i : X \rightarrow \mathbf{P}^N$  be a corresponding immersion (i.e.,  $\mathcal{L}^{\otimes s} \simeq i^*\mathcal{O}(1)$ ). Beilinson's resolution of the diagonal (cf example 5.6) shows that for every  $i < 0$ , there is an exact sequence of vector bundles on  $\mathbf{P}^N$

$$0 \rightarrow \mathcal{O}(i) \rightarrow \mathcal{O} \otimes V_0 \rightarrow \mathcal{O}(1) \otimes V_1 \rightarrow \cdots \rightarrow \mathcal{O}(N) \otimes V_N \rightarrow 0$$

where  $V_0, \dots, V_N$  are finite dimensional vector spaces. By restriction to  $X$ , we obtain an exact sequence

$$0 \rightarrow \mathcal{L}^{\otimes si} \xrightarrow{f^{-1}} \mathcal{O} \otimes V_0 \xrightarrow{f^0} \mathcal{L}^{\otimes s} \otimes V_1 \xrightarrow{f^1} \cdots \xrightarrow{f^{N-1}} \mathcal{L}^{\otimes sN} \otimes V_N \rightarrow 0.$$

This shows the first part of the Lemma with  $l = N + 1$ .

Assume now  $X$  is regular of dimension  $d$ . Then,  $\text{Ext}^{d+1}(M, \mathcal{L}^{\otimes si}) = 0$ , where  $M = \text{coker } f^{d-1}$ . Consequently,  $\mathcal{L}^{\otimes si}$  is a direct summand of the complex

$$0 \rightarrow \mathcal{O} \otimes V_0 \xrightarrow{f^0} \mathcal{L}^{\otimes s} \otimes V_1 \xrightarrow{f^1} \cdots \xrightarrow{f^{d-1}} \mathcal{L}^{\otimes sd} \otimes V_d \rightarrow 0.$$

Dualizing, we see that, for  $i > 0$ , then  $\mathcal{L}^{\otimes si}$  is a direct summand of a complex

$$0 \rightarrow \mathcal{L}^{\otimes -sd} \otimes V_d \rightarrow \cdots \rightarrow \mathcal{L}^{\otimes -s} \otimes V_1 \rightarrow \mathcal{O} \otimes V_0 \rightarrow 0.$$

The Lemma follows.  $\square$

**Proposition 5.28.** *Let  $X$  be a regular quasi-projective scheme over a field and  $\mathcal{L}$  an ample sheaf. Then,  $D^b(X\text{-qcoh}) = \langle \tilde{G} \rangle_{2(1+\dim X)^2}$  and  $D^b(X\text{-coh}) = \langle G \rangle_{2(1+\dim X)^2}$  for some  $r > 0$ , where  $G = \mathcal{L}^{\otimes -r} \oplus \cdots \oplus \mathcal{L}^{\otimes r}$ . In particular,  $\dim D^b(X\text{-coh}) \leq 2(1 + \dim X)^2 - 1$ .*

*Proof.* By Lemma 5.27, there is  $r > 0$  such that  $\overline{\text{add}}(\{\mathcal{L}^{\otimes i}\}_{i \in \mathbf{Z}}) \subset \langle \tilde{G} \rangle_{1+\dim X}$  for all  $i$ , where  $G = \mathcal{L}^{\otimes -r} \oplus \cdots \oplus \mathcal{L}^{\otimes r}$ . Let  $C \in D^b(X\text{-qcoh})$ . Up to isomorphism, we can assume  $C$  is a bounded complex with terms in  $\overline{\text{add}}(\{\mathcal{L}^{\otimes i}\}_{i \in \mathbf{Z}})$ , because  $X$  is regular. Now, as in the proof of Proposition 5.22, we get  $C \in \langle \overline{\text{add}}(\{\mathcal{L}^{\otimes i}\}_{i \in \mathbf{Z}}) \rangle_{2+2\dim X}$ .  $\square$

In the case of a curve, we have a slightly better (though probably not optimal) result.

**Proposition 5.29.** *Let  $X$  be a regular quasi-projective curve over a field. Then,  $\dim D^b(X\text{-coh}) \leq 3$ .*

### 5.3. Nilpotent ideals.

**Lemma 5.30.** *Let  $A$  be a noetherian ring and  $I$  a nilpotent (two-sided) ideal of  $A$  with  $I^r = 0$ . Let  $M \in D^b((A/I)\text{-mod})$  such that  $D^b((A/I)\text{-mod}) = \langle M \rangle_n$ . Then,  $D^b(A\text{-mod}) = \langle M \rangle_{rn}$ .*

*In particular,  $\dim D^b(A\text{-mod}) \leq r(1 + \dim D^b((A/I)\text{-mod})) - 1$ .*

*Proof.* Let  $C$  be a bounded complex of finitely generated  $A$ -modules. We have a filtration  $0 = I^r C \subset I^{r-1} C \subset \dots \subset IC \subset C$  whose successive quotients are bounded complexes of finitely generated  $(A/I)$ -modules and the Lemma follows.  $\square$

We have a geometric version as well.

**Lemma 5.31.** *Let  $X$  be a separated noetherian scheme,  $\mathcal{I}$  a nilpotent ideal sheaf with  $\mathcal{I}^r = 0$  and  $i : Z \rightarrow X$  the corresponding closed immersion. Let  $M \in D^b(Z\text{-coh})$  such that  $D^b(Z\text{-coh}) = \langle M \rangle_n$ . Then,  $D^b(X\text{-coh}) = \langle i_* M \rangle_{rn}$ . Similarly, for  $M \in D^b(Z\text{-qcoh})$  such that  $D^b(Z\text{-qcoh}) = \langle \widetilde{M} \rangle_n$ , then  $D^b(X\text{-qcoh}) = \langle \widetilde{i_* M} \rangle_{rn}$ .*

*In particular,  $\dim D^b(X\text{-coh}) \leq r(1 + \dim D^b(Z\text{-coh})) - 1$ .*

For an artinian ring  $A$ , the Loewy length  $\text{ll}(A)$  of  $A$  is the smallest integer  $i$  such that  $J(A)^{i+1} = 0$ , where  $J(A)$  is the Jacobson radical of  $A$ .

From Lemma 5.30, we deduce

**Proposition 5.32.** *Let  $A$  be an artinian ring. Then,  $D^b(A\text{-mod}) = \langle A/J(A) \rangle_{\text{ll}(A)}$ . In particular,  $\dim D^b(A\text{-mod}) \leq \text{ll}(A) - 1$ .*

### 5.4. Finiteness for derived categories of coherent sheaves. Let $k$ be a field.

5.4.1. Let us recall some facts on derived categories of schemes.

Let  $X$  be a separated scheme of finite type over  $k$ .

Then,  $D(X\text{-qcoh})$  is generated by a compact object and  $D(X\text{-qcoh})^c = X\text{-perf}$  [BoVdB, Theorem 3.1.1]. Let  $U$  be an open subscheme of  $X$  and  $D_{X-U}(X\text{-qcoh})$  be the full subcategory of  $D(X\text{-qcoh})$  of complexes whose cohomology sheaves are supported on  $X - U$ . Then,  $D_{X-U}(X\text{-qcoh})$  is generated by an object of  $D_{X-U}(X\text{-qcoh}) \cap X\text{-perf}$  (the proof of [BoVdB, Theorem 3.1.1] applies to this case as well). In particular, Theorem 4.27 applies to  $\mathcal{I} = D_{X-U}(X\text{-qcoh})$  (this is the original localization Theorem of Thomason and Trobaugh [ThTr, Proposition 5.2.2]).

Via the exact sequence  $0 \rightarrow D_{X-U}^b(X\text{-coh}) \rightarrow D^b(X\text{-coh}) \rightarrow D^b(U\text{-coh}) \rightarrow 0$ , Lemma 3.3 gives

**Lemma 5.33.** *We have  $\dim D^b(U\text{-coh}) \leq \dim D^b(X\text{-coh})$ .*

**Proposition 5.34.** *Let  $X$  be a quasi-projective scheme over  $k$ . Then, the following assertions are equivalent*

- (i)  $X$  is regular
- (ii) every object of  $D^b(X\text{-qcoh})$  is isomorphic to a bounded complex of locally free sheaves
- (iii)  $D^b(X\text{-coh}) = X\text{-perf}$
- (iv)  $X\text{-perf}$  is strongly finitely generated
- (v)  $D(X\text{-qcoh})$  is regular.

*Proof.* It is clear that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i).

By Proposition 5.28, we have (i) $\Rightarrow$ (ii)–(v).

Assume (iv). Since  $X$ -perf is strongly finitely generated, it follows from Lemma 3.3 that  $U$ -perf is strongly finitely generated for any affine open  $U$  of  $X$  because the restriction functor  $X$ -perf  $\rightarrow U$ -perf is dense by Thomason-Thobaug’s localization Theorem 4.27. So,  $U$  is regular by Proposition 5.23, hence  $X$  is regular. So, (iv) $\Rightarrow$ (i).

Note that (v) $\Rightarrow$ (iv) has been discussed in §4.4.2.  $\square$

**Remark 5.35.** One shows more generally that for  $X$  a quasi-compact separated scheme, if  $D(X\text{-qcoh})$  is regular, then  $X$  is regular. We don’t know about the converse, *i.e.*, how to generalize Proposition 5.28.

5.4.2.

**Proposition 5.36.** *Let  $X$  be a separated scheme of finite type over  $k$ . Then,  $\dim D^b(X\text{-coh}) \geq \dim X$ .*

*Proof.* Thanks to Lemma 5.31, we can assume  $X$  is reduced. Let  $M \in D^b(X\text{-coh})$  such that  $D^b(X\text{-coh}) = \langle M \rangle_{r+1}$ .

Pick a closed point  $x$  of  $X$  with local ring  $\mathcal{O}_x$  of Krull dimension  $\dim X$  such that  $M_x \in \langle \mathcal{O}_X \rangle$  (given  $F$  a coherent sheaf over  $X$ , there is a dense open affine  $U$  such that  $F|_U$  is projective. Now, a complex with projective cohomology splits). Then,  $k_x \in \langle \mathcal{O}_x \rangle_{r+1}$ . It follows from Proposition 5.12 that  $r \geq \text{Krulldim } \mathcal{O}_x = \dim X$ .  $\square$

From Propositions 5.4 and 5.36, we deduce

**Theorem 5.37.** *Let  $X$  be a smooth affine scheme of finite type over  $k$ . Then,  $\dim D^b(X\text{-coh}) = \dim X$ .*

5.4.3. The following Theorem is due to Kontsevich, Bondal and Van den Bergh for  $X$  non singular [BoVdB, Theorem 3.1.4].

**Theorem 5.38.** *Let  $X$  be a separated scheme of finite type over a perfect field  $k$ . Then, there is  $E \in D^b(X\text{-coh})$  and  $d \in \mathbf{N}$  such that  $D^b(X\text{-qcoh}) = \langle E \rangle_d$  and  $D^b(X\text{-coh}) = \langle E \rangle_d$ .*

*In particular,  $\dim D^b(X\text{-coh}) < \infty$ .*

Let us explain how the Theorem will be proved. It is enough to consider the case where  $X$  is reduced. Then, the diagonal is a direct summand of a perfect complex up to a complex supported on  $Z \times X$ , where  $Z$  is a closed subscheme with smooth dense complement. We conclude by induction by applying the Theorem to  $Z$ .

Let us start with two Lemmas.

**Lemma 5.39.** *Let  $A$  and  $B$  be two finitely generated commutative  $k$ -algebras, where  $k$  is perfect.*

*Let  $M$  be a finitely generated  $(B \otimes A)$ -module and  $\cdots \rightarrow P^{-1} \xrightarrow{d^{-2}} P^0 \xrightarrow{d^{-1}} M \xrightarrow{d^0} 0$  be an exact complex with  $P^i$  finitely generated and projective.*

*If  $M$  is flat as an  $A$ -module and  $B$  is regular of dimension  $n$ , then  $\ker d^{-n}$  is a projective  $(B \otimes A)$ -module.*

*Proof.* Let  $i \geq 1$ ,  $\mathfrak{m}$  a maximal ideal of  $A$  and  $\mathfrak{n}$  a maximal ideal of  $B$ . We have

$$\text{Tor}_i^{B \otimes A}(\ker d^{-n}, B/\mathfrak{n} \otimes A/\mathfrak{m}) \simeq \text{Tor}_{n+i}^{B \otimes A}(M, B/\mathfrak{n} \otimes A/\mathfrak{m}) \simeq \text{Tor}_{n+i}^B(M \otimes_A A/\mathfrak{m}, B/\mathfrak{n}) = 0$$

since  $B$  is regular with dimension  $n$ . It follows that  $\ker d^{-n}$  is projective (cf [Ma, §18, Lemma 5]).  $\square$

**Lemma 5.40.** *Let  $X$  be a separated noetherian scheme and  $Z$  a closed subscheme of  $X$ , given by the ideal sheaf  $\mathcal{I}$  of  $\mathcal{O}_X$ . For  $n \geq 1$ , let  $Z_n$  be the closed subscheme of  $X$  with ideal sheaf  $\mathcal{I}^n$  and  $i_n : Z_n \rightarrow X$  the corresponding immersion.*

*Then, given  $C \in D_Z^b(X\text{-coh})$ , there is  $n \geq 1$  and  $C_n \in D^b(Z_n\text{-coh})$  such that  $C \simeq i_{n*}C_n$ .*

*Proof.* Let  $\mathcal{F}$  be a coherent sheaf on  $X$  supported by  $Z$ . Then,  $\mathcal{I}^n \mathcal{F} = 0$  for some  $n$  and it follows that  $\mathcal{F} \xrightarrow{\sim} i_{n*}(i_n^* \mathcal{F})$ . More generally, a bounded complex of coherent sheaves on  $X$  that are supported by  $Z$  is isomorphic to the image under  $i_{n*}$  of a bounded complex of coherent sheaves on  $Z_n$  for some  $n$ .

Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Let  $\mathcal{F}_Z$  be the subsheaf of  $\mathcal{F}$  of sections supported by  $Z$ . By Artin-Rees' Theorem [Ma, §11.C Theorem 15], there is an integer  $r$  such that  $(\mathcal{I}^m \mathcal{F}) \cap \mathcal{F}_Z = \mathcal{I}^{m-r}(\mathcal{I}^r \mathcal{F}_Z)$  for  $m \geq r$ . Since  $\mathcal{F}_Z$  is a coherent sheaf supported by  $Z$ , there is an integer  $d$  such that  $\mathcal{I}^d \mathcal{F}_Z = 0$ . So,  $(\mathcal{I}^{r+d} \mathcal{F}) \cap \mathcal{F}_Z = 0$ . It follows that the canonical map  $\mathcal{F}_Z \rightarrow \mathcal{F}/(\mathcal{I}^{r+d} \mathcal{F})$  is injective.

We prove now the Lemma by induction on the number of terms of  $C$  that are not supported by  $Z$ .

Let  $C = 0 \rightarrow C^r \xrightarrow{d^r} \dots \xrightarrow{d^{s-1}} C^s \rightarrow 0$  be a complex of coherent sheaves on  $X$  with cohomology supported by  $Z$  and take  $i$  minimal such that  $C^i$  is not supported by  $Z$ .

Since  $C^{i-1}$  and  $H^i(C)$  are supported by  $Z$ , it follows that  $\ker d^i$  is supported by  $Z$ . So, there is an integer  $n$  such that the canonical map  $\ker d^i \rightarrow C^i/(\mathcal{I}^n C^i)$  is injective. Let  $R$  be the subcomplex of  $C$  with non zero terms  $R^i = \mathcal{I}^n C^i$  and  $R^{i+1} = d^i(\mathcal{I}^n C^i)$  — a complex homotopy equivalent to 0. Let  $D = C/R$ . Then, the canonical map  $C \rightarrow D$  is a quasi-isomorphism. By induction,  $D$  is quasi-isomorphic to a complex of coherent sheaves on  $Z_n$  for some  $n$  and the Lemma follows.  $\square$

*Proof of the Theorem.* We have  $D^b(X\text{-qcoh})^c = D^b(X\text{-coh})$  (Corollary 4.35). So, the assertion about  $D^b(X\text{-coh})$  follows immediately from the one about  $D^b(X\text{-qcoh})$  by Corollary 3.12. By Lemma 5.31, it is enough to prove the Theorem for  $X$  reduced.

Assume  $X$  is reduced and let  $d$  be its dimension. We now prove the Theorem by induction on  $d$  (the case  $d = 0$  is trivial).

Let  $U$  be a smooth dense open subscheme of  $X$ . The structure sheaf  $\mathcal{O}_{\Delta U}$  of the diagonal  $\Delta U$  in  $U \times X$  is a perfect complex by Lemma 5.39. By Thomason and Trobaugh's localization Theorem (cf §5.4.1 and Theorem 4.27), there is a perfect complex  $C$  on  $X \times X$  and a morphism  $f : C \rightarrow \mathcal{O}_{\Delta X} \oplus \mathcal{O}_{\Delta X}[1]$  whose restriction to  $U \times X$  is an isomorphism. Let  $G$  be a compact generator for  $D(X\text{-qcoh})$ . Then,  $G \boxtimes G$  is a compact generator for  $D((X \times X)\text{-qcoh})$  [BoVdB, Lemma 3.4.1]. So, there is  $r$  such that  $C \in \langle G \boxtimes G \rangle_r$  by Theorem 4.20 (3).

Let  $D$  be the cone of  $f$ . Then,  $H^*(D)$  is supported by  $Z \times X$ , where  $Z = X - U$ . It follows that there is a closed subscheme  $Z'$  of  $X$  with underlying closed subspace  $Z$ , a bounded complex  $D'$  of coherent  $\mathcal{O}_{Z' \times X}$ -modules and an isomorphism  $(i \times 1)_* D' \xrightarrow{\sim} D$  in  $D^b((X \times X)\text{-coh})$ , where  $i : Z' \rightarrow X$  is the closed immersion (Lemma 5.40). By induction, there is  $M \in D^b(Z'\text{-coh})$  and an integer  $l$  such that  $D^b(Z'\text{-qcoh}) = \langle \widetilde{M} \rangle_l$ .



Let  $p_1$  and  $p_2$  be the first and second projections  $X \times X \rightarrow X$  and  $\pi : Z' \times X \rightarrow Z'$  be the first projection. Let  $\mathcal{F} \in D^b(X\text{-qcoh})$ . We have a distinguished triangle

$$Rp_{1*}(C \otimes^{\mathbf{L}} p_2^* \mathcal{F}) \rightarrow \mathcal{F} \oplus \mathcal{F}[1] \rightarrow Rp_{1*}(D \otimes^{\mathbf{L}} p_2^* \mathcal{F}) \xrightarrow{\sim} .$$

Since  $C$  is perfect, we have  $C \otimes^{\mathbf{L}} p_2^* \mathcal{F} \in D^b((X \otimes X)\text{-qcoh})$ , hence  $Rp_{1*}(C \otimes^{\mathbf{L}} p_2^* \mathcal{F})$  has bounded cohomology. It follows that  $Rp_{1*}(D \otimes^{\mathbf{L}} p_2^* \mathcal{F})$  has bounded cohomology as well. We have

$$Rp_{1*}(D \otimes^{\mathbf{L}} p_2^* \mathcal{F}) \simeq Rp_{1*}(i \times 1)_*(D' \otimes^{\mathbf{L}} \mathbf{L}(i \times 1)^* p_2^* \mathcal{F}) \simeq i_* R\pi_*(D' \otimes^{\mathbf{L}} \mathbf{L}(i \times 1)^* p_2^* \mathcal{F})$$

Note that  $R\pi_*(D' \otimes^{\mathbf{L}} \mathbf{L}(i \times 1)^* p_2^* \mathcal{F})$  is an element of  $D^b(Z'\text{-qcoh})$ . So,  $Rp_{1*}(D \otimes^{\mathbf{L}} p_2^* \mathcal{F}) \in \langle \widetilde{i_* M} \rangle_l$ .

We have  $(G \boxtimes G) \otimes^{\mathbf{L}} p_2^* \mathcal{F} \simeq G \boxtimes (G \otimes^{\mathbf{L}} \mathcal{F})$ , hence  $Rp_{1*}((G \boxtimes G) \otimes^{\mathbf{L}} p_2^* \mathcal{F}) \simeq G \otimes R\Gamma(G \otimes^{\mathbf{L}} \mathcal{F}) \in \langle \widetilde{G} \rangle$  (note this has bounded cohomology). So,  $Rp_{1*}(C \otimes^{\mathbf{L}} p_2^* \mathcal{F}) \in \langle \widetilde{G} \rangle_r$ .

Finally,  $\mathcal{F} \in \langle \widetilde{i_* M} \oplus G \rangle_{l+r}$  and we are done.  $\square$

**Remark 5.41.** Note that the proof works under the weaker assumption that  $X$  is a separated scheme of finite type over  $k$  and the residue fields at closed points are separable extensions of  $k$ .

We don't know how to bound the dimension of  $D^b(X\text{-coh})$  for singular  $X$ . When  $X$  is zero dimensional, then  $\dim D^b(X\text{-coh}) = 0$  if and only if  $X$  is smooth.

We don't know whether the inequality  $\dim D^b((X \times Y)\text{-coh}) \leq \dim D^b(X\text{-coh}) + \dim D^b(Y\text{-coh})$  holds for  $X, Y$  separated schemes of finite type over a perfect field.

Last but not least, we don't know a single case where  $X$  is smooth and  $\dim D^b(X\text{-coh}) > \dim X$ . For example, we don't know whether  $\dim D^b(X\text{-coh}) = 1$  or  $2$  for  $X$  an elliptic curve over an algebraically closed field.

5.4.4. Let  $X$  and  $Y$  be noetherian separated schemes.

Let  $f : X \rightarrow Y$  be a morphism such that  $Rf_*$  and  $Rf^*$  restrict to functors between  $D^b(X\text{-coh})$  and  $D^b(Y\text{-coh})$ .

If the adjoint map  $\text{id}_{D^b(Y\text{-coh})} \rightarrow Rf_* Rf^*$  is a split injection, then  $\dim D^b(Y\text{-coh}) \leq \dim D^b(X\text{-coh})$  by Lemma 3.3.

This applies in the following cases :

- $f : X \rightarrow Y = X/G$  is the quotient map by a finite group  $G$  acting on  $X$  and such that the order of  $G$  is invertible on  $X$ . So,  $\dim D^b((X/G)\text{-coh}) \leq \dim D^b(X\text{-coh})$ .
- $f : X \rightarrow Y$  is the blowup of the smooth variety  $Y$  along the smooth subvariety  $Z$ . Then,  $\dim D^b(Y\text{-coh}) \leq \dim D^b(X\text{-coh})$ .

## 6. APPLICATIONS TO FINITE DIMENSIONAL ALGEBRAS

### 6.1. Auslander's representation dimension.

6.1.1. Let  $\mathcal{A}$  be an abelian category.

**Definition 6.1.** *The (Auslander) representation dimension  $\text{repdim } \mathcal{A}$  is the smallest integer  $i \geq 2$  such that there is an object  $M \in \mathcal{A}$  with the property that given any  $L \in \mathcal{A}$ ,*

- (a) *there is  $\tilde{L} \in \mathcal{A}$  with  $L$  a direct summand of  $\tilde{L}$  and an exact sequence*

$$0 \rightarrow M^{-i+2} \rightarrow M^{-i+3} \rightarrow \dots \rightarrow M^0 \rightarrow \tilde{L} \rightarrow 0$$

with  $M^j \in \text{add}(M)$  such that the sequence

$$0 \rightarrow \text{Hom}(M, M^{-i+2}) \rightarrow \text{Hom}(M, M^{-i+3}) \rightarrow \cdots \rightarrow \text{Hom}(M, M^0) \rightarrow \text{Hom}(M, \tilde{L}) \rightarrow 0$$

is exact

(b) there is  $\tilde{L}' \in \mathcal{A}$  with  $L$  a direct summand of  $\tilde{L}$  and an exact sequence

$$0 \rightarrow \tilde{L}' \rightarrow M'^0 \rightarrow M'^1 \rightarrow \cdots \rightarrow M'^{i-2} \rightarrow 0$$

with  $M'^j \in \text{add}(M)$  such that the sequence

$$0 \rightarrow \text{Hom}(M'^{i-2}, M) \rightarrow \cdots \rightarrow \text{Hom}(M'^1, M) \rightarrow \text{Hom}(M'^0, M) \rightarrow \text{Hom}(\tilde{L}', M) \rightarrow 0$$

is exact.

An object  $M$  that realizes the minimal  $i$  is called an *Auslander generator*.

Note that either condition (a) or (b) implies that  $\text{gldim End}_{\mathcal{A}}(M) \leq i$  (cf e.g. [ErHoIySc, Lemma 2.1]).

Note that  $\text{repdim } \mathcal{A} = 2$  if and only if  $\mathcal{A}$  has only finitely many isomorphism classes of indecomposable objects. Note also that  $\text{repdim } \mathcal{A} = \text{repdim } \mathcal{A}^\circ$ .

6.1.2. Take  $\mathcal{A} = A\text{-mod}$ , where  $A$  is a finite dimensional algebra over a field. Then, we write  $\text{repdim } A$  for  $\text{repdim } A\text{-mod}$ .

Let  $M \in \mathcal{A}$  and  $i \geq 2$ . If  $M$  satisfies (a) of Definition 6.1, then it contains an projective generator as a direct summand (take  $L = A$ ). More generally, the following are equivalent

- $M$  satisfies (a) of Definition 6.1 and  $M$  contains an injective cogenerator as a direct summand
- $M$  satisfies (b) of Definition 6.1 and  $M$  contains an projective generator as a direct summand
- $M$  satisfies (a) and (b) of Definition 6.1

So, the definition of representation dimension given here coincides with Auslander's original definition (cf [Au, §III.3] and [ErHoIySc, Lemma 2.1]) when  $A$  is not semi-simple. When  $A$  is semi-simple, Auslander assigns the representation dimension 0 whereas we define it to be 2 here. Iyama has shown [Iy] that the representation dimension of a finite dimensional algebra is finite.

Various classes of algebras with representation dimension 3 have been found : algebras with radical square zero [Au, §III.5, Proposition p.56], hereditary algebras [Au, §III.5, Proposition p.58] and more generally stably hereditary algebras [Xi, Theorem 3.5], special biserial algebras [ErHoIySc], local algebras of quaternion type [Ho].

6.1.3. One can weaken the requirements in the definition of the representation dimension as follows :

**Definition 6.2.** *The weak (resp. left weak, resp. right weak) representation dimension of  $\mathcal{A}$ , denoted by  $\text{wrepdim } \mathcal{A}$  (resp.  $\text{lwrepdim}(\mathcal{A})$ , resp.  $\text{rwrepdim}(\mathcal{A})$ ) is the smallest integer  $i \geq 2$  such that there is an object  $M \in \mathcal{A}$  with the property that given any  $L \in \mathcal{A}$ , there is a bounded complex  $C = 0 \rightarrow C^r \rightarrow \cdots \rightarrow C^s \rightarrow 0$  of  $\text{add}(M)$  with*

- $L$  isomorphic to a direct summand of  $H^0(C)$
- $H^d(C) = 0$  for  $d \neq 0$  and
- $s - r \leq i - 2$  (resp. and  $C^d = 0$  for  $d > 0$ , resp. and  $C^d = 0$  for  $d < 0$ ).

Note that  $\text{lwrepdim}(\mathcal{A}) = \text{rwrepdim}(\mathcal{A}^\circ)$ ,  $\text{wrepdim } \mathcal{A} = \text{wrepdim } \mathcal{A}^\circ$ ,  
 $\inf\{\text{lwrepdim } \mathcal{A}, \text{rwrepdim } \mathcal{A}\} \geq \text{wrepdim } \mathcal{A}$  and  $\text{repdim } \mathcal{A} \geq \sup\{\text{lwrepdim } \mathcal{A}, \text{rwrepdim } \mathcal{A}\}$ .

In order to obtain lower bounds for the representation dimension of certain algebras, we will actually construct lower bounds for the weak representation dimension.

**Remark 6.3.** All the definitions given here for abelian categories make sense for exact categories.

6.1.4. We study here self-injective algebras with representation dimension 3.

Recently, various properties have been found for algebras of representation dimension 3 (cf for example [IgTo]). Here is a result in this direction concerning self-injective algebras.

Note that  $\text{repdim } A = 2$  if and only if  $\dim A\text{-stab} = 0$ . Consequently, if  $\text{repdim } A = 3$ , then  $\dim A\text{-stab} = 1$  (cf Proposition 6.9 below).

Given  $M$  an  $A$ -module, we denote by  $\Omega M$  the kernel of a surjective map from a projective cover of  $M$  to  $M$  and by  $\Omega^{-1}M$  the cokernel of an injective map from  $M$  to an injective hull of  $M$ .

**Lemma 6.4.** *Let  $A$  be a self-injective  $k$ -algebra and  $C = 0 \rightarrow C^0 \rightarrow C^1 \rightarrow 0$  an indecomposable complex of finitely generated  $A$ -modules with  $H^0(C) = 0$  and  $H^1(C) = S$  simple. Then,*

- $C^0$  and  $C^1$  have no non-zero projective direct summand
- or  $C^1$  is projective indecomposable and  $C^0 \simeq \Omega S$ .

*Proof.* If  $C^0$  has a non-zero projective summand  $L$ , then  $L$  is injective and the restriction of  $d = d_C^0$  to  $L$  is a split injection. In particular,  $C$  has a direct summand isomorphic to  $0 \rightarrow L \xrightarrow{\text{id}} L \rightarrow 0$ , which is impossible.

Assume now that  $C^1$  has a submodule  $N$  such that  $C^1/N$  is projective indecomposable. If  $N \not\subseteq \text{im } d$ , then there is  $P \subseteq \text{im } d$  such that  $C^1 = N \oplus P$ . So,  $C$  has a direct summand isomorphic to  $0 \rightarrow P \xrightarrow{\text{id}} P \rightarrow 0$ : this is impossible. So,  $C^0 \xrightarrow{\sim} \text{im } d = N \oplus N'$  with  $N' \simeq \Omega S$ . The indecomposability of  $C$  gives  $N = 0$ .  $\square$

**Lemma 6.5.** *Let  $A$  be a self-injective  $k$ -algebra with  $\text{repdim } A = 3$  and  $M$  an Auslander generator. Assume  $\Omega^{-1}M$  has no simple direct summand. Then, the number of simple  $A$ -modules (up to isomorphism) is less than or equal to the number of isomorphism classes of non projective indecomposable summands of  $M$ .*

*Proof.* Since  $\text{repdim } A = 3$ , for every simple  $A$ -module  $S$ , there is an exact sequence  $0 \rightarrow M_1 \rightarrow M_0 \rightarrow S \rightarrow 0$  with  $M_0$  and  $M_1$  in  $\text{add}(M)$ . By Lemma 6.4, we can assume that  $M_0$  and  $M_1$  have no non-zero projective direct summands. Then, we have  $[S] = [M_0] - [M_1]$  in  $K_0(A\text{-mod})$ . It follows that the non-projective indecomposable summands of  $M$  generate  $K_0(A\text{-mod})$ .  $\square$

Let  $A$  and  $B$  be two self-injective algebras. A stable equivalence of Morita type between  $A$  and  $B$  is the data of a finite dimensional  $(A, B)$ -bimodule  $X$ , projective as an  $A$ -module and as a right  $B$ -module, and of a finite dimensional  $(B, A)$ -bimodule  $Y$ , projective as a  $B$ -module and as a right  $A$ -module, such that

$$\begin{aligned} X \otimes_B Y &\simeq A \oplus \text{projective as } (A, A) \text{ - bimodules} \\ Y \otimes_A X &\simeq B \oplus \text{projective as } (B, B) \text{ - bimodules.} \end{aligned}$$

Stables equivalences of Morita type preserve the representation dimension [Xi, Theorem 4.1]:

**Proposition 6.6.** *Let  $A$  and  $B$  be two self-injective  $k$ -algebras and  $X$  be an  $(A, B)$ -bimodule inducing a stable equivalence between  $A$  and  $B$ .*

*Let  $M$  be an Auslander generator for  $B$ . Then,  $X \otimes_B M$  is an Auslander generator for  $A$ . In particular,  $\text{repdim } A = \text{repdim } B$ .*

*Proof.* Let  $Y$  be a  $(B, A)$ -bimodule inverse to  $X$ . Let  $V$  be an  $A$ -module. Then,  $X \otimes_B Y \otimes_A V \simeq V \oplus P$  with  $P$  projective. Starting with an exact sequence resolving  $L = Y \otimes_A V$  as in (a) or (b) of Definition 6.1, we get one for  $L = V$  by applying  $X \otimes_B -$ . Now, applying  $\text{Hom}_B(X \otimes_B M, -)$  to that new exact sequence gives the same result as applying  $\text{Hom}_A(Y \otimes_A X \otimes_B M, -)$  to the original exact sequence. Since  $Y \otimes_A X \otimes_B M \simeq M \oplus$  projective, we indeed get an exact sequence.  $\square$

The following Proposition gives a bound for the number of non-projective simple modules of a self-injective algebra which is stably equivalent (à la Morita) to a given self-injective algebra with representation dimension 3.

**Proposition 6.7.** *Let  $A$  be a self-injective  $k$ -algebra with  $\text{repdim } A = 3$  and  $M$  an Auslander generator. Let  $B$  be a self-injective  $k$ -algebra. Assume there is a stable equivalence of Morita type between  $A$  and  $B$ .*

*Then, the number of simple non-projective  $B$ -modules (up to isomorphism) is less than or equal to twice the number of isomorphism classes of indecomposable summands of  $M$ .*

*Proof.* Replacing  $B$  by a direct factor, one can assume  $B$  has no simple projective module.

Let  $X$  be a  $(B, A)$ -bimodule inducing a stable equivalence and  $N = X \otimes_A M$ . Then,  $N$  is an Auslander generator for  $B$  and  $\text{repdim } B = 3$  (Proposition 6.6). Let  $R$  be the subgroup of  $K_0(B\text{-mod})$  generated by the classes of the non-projective indecomposable summands of  $N$ . Note that the rank of  $R$  is at most the number of isomorphism classes of non-projective summands of  $M$ .

Let  $S$  be a simple  $B$ -module with  $[S] \notin R$ . There is an exact sequence  $0 \rightarrow N_1 \rightarrow N_0 \rightarrow S \rightarrow 0$  with  $N_0$  and  $N_1$  in  $\text{add}(N)$  and by Lemma 6.4,  $N_0$  is a projective cover of  $S$  and  $N_1 \simeq \Omega S$ . In particular,  $\Omega S$  is a direct summand of  $N$ .

So, the number of simple  $B$ -modules with  $[S] \notin R$  is at most the number of isomorphism classes of indecomposable non-projective summands of  $N$ .  $\square$

**Remark 6.8.** This Proposition, which was the starting point of this paper, led us to look for self-injective algebras with representation dimension greater than 3. This Proposition is related to the problem of the equality of the number of simple non projective modules for two stably equivalent algebras.

**6.2. Stable categories of self-injective algebras.** Let  $k$  be a field.

6.2.1. For  $A$  self-injective, we denote by  $A\text{-stab}$  the stable category of  $A$ . This is the quotient of the additive category  $A\text{-mod}$  by the additive subcategory  $A\text{-proj}$ . The canonical functor  $A\text{-mod} \rightarrow D^b(A\text{-mod})$  induces an equivalence  $A\text{-stab} \xrightarrow{\sim} D^b(A\text{-mod})/A\text{-perf}$  ([KeVo, Exemple 2.3] and [Ri, Theorem 2.1]). This provides  $A\text{-stab}$  with a structure of triangulated category. Recall that  $\text{ll}(A)$  denotes the Loewy length of  $A$  (cf §5.3).

**Proposition 6.9.** *Let  $A$  be a self-injective algebra. Then,*

$$\text{ll}(A) \geq \text{repdim } A \geq \text{wrepdim } A \geq 2 + \dim A\text{-stab}.$$

*Proof.* The first inequality is [Au, §III.5, Proposition p.55] (use  $M = A \oplus A/J(A) \oplus A/J(A)^2 \oplus \dots$ ).

The second inequality is trivial (cf §6.1.3). For the last inequality, note that if we have a bounded complex of  $A$ -modules  $C = 0 \rightarrow C^r \rightarrow \dots \rightarrow C^s \rightarrow 0$  with  $H^i(C) = 0$  for  $i \neq 0$  and  $L$  is a direct summand of  $H^0(C)$ , then  $L \in \langle C^s \rangle \diamond \dots \diamond \langle C^r \rangle$ .  $\square$

6.2.2. The following Theorem gives the first known examples of algebras with representation dimension  $> 3$ .

**Theorem 6.10.** *Let  $n \geq 1$  be an integer. Then,  $\dim \Lambda(k^n)\text{-stab} = \text{repdim } \Lambda(k^n) - 2 = n - 1$ .*

*Proof.* Put  $A = \Lambda(k^n)$  and  $B = k[x_1, \dots, x_n]$ .

Let us recall a version of Koszul duality [Ke, §10.5, Lemma “The ‘exterior’ case”]. We have an equivalence of triangulated categories  $R\text{Hom}^\bullet(k, -)$  between  $D^b(A\text{-mod})$  and  $\mathcal{T}$ , the subcategory of the derived category of differential graded  $B$ -modules classically generated by  $B$  ( $\mathcal{T}$  is also the subcategory of compact objects by Corollary 4.29). Note that this is a special case of §3.4, using the fact that  $R\text{End}^\bullet(k)$  is a dg algebra quasi-isomorphic to its cohomology algebra  $B$ . This equivalence sends  $A$  to  $k$ , so it induces an equivalence of triangulated categories between  $A\text{-stab}$  and  $\mathcal{T}/\mathcal{I}$ , where  $\mathcal{I}$  is the subcategory of  $\mathcal{T}$  classically generated by  $k$ . Denote by  $F : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{I}$  the quotient functor.

Let  $M \in \mathcal{T}$  such that  $\mathcal{T}/\mathcal{I} = \langle F(M) \rangle_{\mathcal{T}/\mathcal{I}, r+1}$ . Up to isomorphism, we can assume  $M$  is finitely generated and projective as a  $B$ -module. Let  $\mathcal{F}$  be the sheaf over  $\mathbf{P}^{n-1}$  corresponding to the graded  $B$ -module  $M$ . The differential on  $M$  gives a map  $d : \mathcal{F} \rightarrow \mathcal{F}(1)$ . Let  $\mathcal{G} = \ker d(1)/\text{im } d$ . Pick  $x$  a closed point of  $\mathbf{P}^{n-1}$  such that  $\mathcal{G}_x$  is a projective  $\mathcal{O}_x$ -module. Then, there is a projective  $\mathcal{O}_x$ -module  $R$  such that  $\ker d_x = \text{im } d_x \oplus R$ . We have an exact sequence  $0 \rightarrow R \rightarrow \mathcal{F}_x \rightarrow \mathcal{F}_x/R \rightarrow 0$  of differential  $\mathcal{O}_x$ -modules. Since  $\mathcal{F}_x/R$  is acyclic, it follows that  $R \rightarrow \mathcal{F}_x$  is an isomorphism in  $D\text{diff}(\mathcal{O}_x)$ , the derived category of differential  $\mathcal{O}_x$ -modules. Let  $I(x)$  be the prime ideal of  $B$  corresponding to the line  $x$  of  $\mathbf{A}^n$ . Note that the differential graded  $B$ -module  $B/I(x)$  (the differential is 0) is in  $\mathcal{T}$ . So,  $F(B/I(x)) \in \langle F(M) \rangle_{\mathcal{T}/\mathcal{I}, r+1}$ , hence  $k_x \in \langle \mathcal{F}_x \rangle_{D\text{diff}(\mathcal{O}_x), r+1}$ , hence  $k_x \in \langle \mathcal{O}_x \rangle_{D\text{diff}(\mathcal{O}_x), r+1}$ . By Proposition 5.12, we get  $r \geq n - 1$ . Hence,  $\dim A\text{-stab} \geq n - 1 = \text{ll}(A) - 2$ . Now, Proposition 6.9 gives the conclusion.  $\square$

6.2.3.

**Proposition 6.11.** *Let  $G$  be a finite group and  $B$  a block of  $kG$ . Let  $D$  be a defect group of  $B$ . Then,  $\dim B\text{-stab} = \dim(kD)\text{-stab}$ .*

*Proof.* Recall that a defect group  $D$  of  $B$  is a (smallest) subgroup such that the identity functor of  $B\text{-mod}$  is a direct summand of  $\text{Ind}_D^G \text{Res}_D^G$ . Since  $kD$  is a direct summand of  $B$  as a  $(kD, kD)$ -bimodule, the Proposition follows from Lemma 3.3.  $\square$

**Theorem 6.12.** *Let  $G$  be a finite group,  $B$  a block of  $kG$  over a field  $k$  of characteristic 2. Let  $D$  be a defect group of  $B$ . Then,  $\text{repdim } B \geq 2 + \dim B\text{-stab} > r$ , where  $r$  is the 2-rank of  $D$ .*

*Proof.* The first inequality is given by Proposition 6.9. By Proposition 6.11, it suffices to prove the Theorem for  $G = D$  and  $B = kD$ . Let  $P$  be an elementary abelian 2-subgroup of  $D$  with rank the 2-rank of  $D$ . Then,  $\dim kP\text{-stab} \leq \dim kD\text{-stab}$  by Lemma 3.3. Now,  $kP \simeq \Lambda(k^r)$  and the Theorem follows from Theorem 6.10.  $\square$

Let us recall a conjecture of D. Benson :

**Conjecture 6.13** (Benson). *Let  $G$  be a finite group,  $B$  a block of  $kG$  over a field  $k$  of characteristic  $p$ . Then,  $\text{ll}(B) > p\text{-rank}(D)$ .*

From Theorem 6.12 and Proposition 6.9, we deduce :

**Theorem 6.14.** *Benson's conjecture 6.13 holds for  $p = 2$ .*

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