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# Hamiltonian stationary tori in the complex projective plane

Frédéric Hélein, Pascal Romon

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## Abstract

We analyze here Hamiltonian stationary surfaces in the complex projective plane as (local) solutions to an integrable system, formulated as a zero curvature equation on a loop group. As an application, we show in details why such tori are finite type solutions, and eventually describe the simplest of them: the homogeneous ones.

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Keywords: Hamiltonian stationary Lagrangian surfaces, tori, loop groups, integrable systems, finite type solutions.

## 1 Introduction

Hamiltonian stationary Lagrangian surfaces are Lagrangian surfaces of a given four-dimensional manifold endowed with a symplectic and a Riemannian structure, which are critical points of the area functional with respect to a particular class of infinitesimal variations preserving the Lagrangian constraint. This class is the set of compactly supported Hamiltonian vector fields. The Euler–Lagrange equations of this variational problem are highly simplified when we assume that the ambient manifold  $\mathcal{N}$  is Kähler. In that case we can make sense of a *Lagrangian angle function*  $\beta$  along any simply-connected Lagrangian submanifold  $\Sigma \subset \mathcal{N}$  (uniquely defined up to the addition of a constant). And as shown in [ScWo] the mean curvature vector of the submanifold is then  $\vec{H} = J\nabla\beta$ , where  $J$  is the complex structure on  $\mathcal{N}$  and  $\nabla\beta$  is the gradient of  $\beta$  along  $\Sigma$ . (We will revisit in the following this relationship between  $\vec{H}$  and  $\beta$  in the special case where  $\mathcal{N} = \mathbb{C}P^2$ .) It turns out that  $\Sigma$  is Hamiltonian stationary if and only if  $\beta$  is a harmonic function on  $\Sigma$ .

A particular subclass of solutions occurs when  $\beta$  is constant: the Lagrangian submanifold is then simply a minimal one. In the case where  $\mathcal{N}$  is a Calabi–Aubin–Yau manifold, such submanifolds admit an alternative characterization

as *special Lagrangian*, a notion which has been extensively studied recently because of its connection with string theories and the mirror conjecture, see [SYZ].

An analytical theory of two-dimensional Hamiltonian stationary Lagrangian submanifolds was constructed by R. Schoen and J. Wolfson [ScWo], proving the existence and the partial regularity of minimizers. In contrast our results in the present paper rest on the fact that, for particular ambient manifolds  $\mathcal{N}$ , Hamiltonian stationary Lagrangian surfaces are solutions of an integrable system. This was discovered first in the case when  $\mathcal{N} = \mathbb{C}^2$  in [HR1] and [HR2]. In a subsequent paper [HR3] we proved that the same problem is also completely integrable if we replace  $\mathbb{C}^2$  by any two-dimensional Hermitian symmetric space. Among these symmetric spaces one very interesting example is  $\mathbb{C}P^2$ , because any simply-connected Lagrangian surface in  $\mathbb{C}P^2$  can be lifted into a Legendrian surface in  $S^5$ . Furthermore the cone in  $\mathbb{C}^3$  over this Legendrian surface is actually a singular Lagrangian three-dimensional submanifold in  $\mathbb{C}^3$ ; and the cone in  $\mathbb{C}^3$  is Hamiltonian stationary if and only if the surface in  $\mathbb{C}P^2$  is so.

A similar correspondence has been remarked and used in [J], [McI3] and [Has] in the case of minimal Lagrangian surfaces in  $\mathbb{C}P^2$  and allows these Authors to connect results on minimal Lagrangian surfaces in  $\mathbb{C}P^2$  [Sh] to minimal Legendrian surfaces in  $S^5$  [MM] and special minimal cones in  $\mathbb{C}^3$ .

Our aim in this paper is the following:

- to expound in details the correspondence between Hamiltonian stationary Lagrangian surfaces in  $\mathbb{C}P^2$  and Hamiltonian stationary Legendrian surfaces in  $S^5$  and a formulation using a family of curvature free connections of this integrable system. We revisit here the formulation given in [HR3], using *twisted loop groups*. Roughly speaking it rests on the identifications  $\mathbb{C}P^2 \simeq SU(3)/S(U(2) \times U(1))$  and  $(S^5, \text{contact structure}) = (U(3)/U(2) \times U(1), A_3^3 = 0)$ , where  $A_3^3$  is a component of the Maurer–Cartan form. We also show that this problem has an alternative formulation, analogous to the theory of K. Uhlenbeck [U] for harmonic maps into  $U(n)$ , using *based loop groups*.
- to define the notion of *finite type* Hamiltonian stationary Legendrian surfaces in  $S^5$ : we give here again two definitions, in terms of twisted loop groups (which is an analogue to the description of finite type harmonic maps into homogeneous manifolds according to [BP]) and in terms of based loop groups (an analogue to the description of finite type harmonic maps into Lie groups according to [BFPP]). We prove the equivalence between the two definitions because we actually need this result for the following. We believe that this fact should be well known to some specialists in the harmonic maps theory, but we did not find it in the literature.
- we prove that all Hamiltonian stationary Lagrangian tori in  $\mathbb{C}P^2$  (and hence Hamiltonian stationary Legendrian tori in  $S^5$ ) are of finite type.

This is the main result of this paper. Our proof focuses on the case of Hamiltonian stationary tori which are not minimal, since the minimal case has been studied by many authors ([BFPP], [Sh], [MM], [Has], [McI1],[McI2], [McI3], [J]). The method here is adapted from the similar result for harmonic maps into Lie groups in [BFPP]. However the strategy differs slightly: we use actually the two existing formulations of finite type solutions, using twisted or based loop groups. One crucial step indeed is the construction of a *formal Killing field*, starting from a given torus solution. This step can be slightly simplified here in the twisted loop groups formulation, because the semi-simple element we start with is then just constant. However proving that the formal Killing field is *adapted* requires more work in the twisted loop groups formulation (actually we were not able to do it directly) than in the based loop groups formulation; here we take advantage from the two formulations to avoid the difficulties and to conclude.

- lastly we give some examples of Hamiltonian stationary Legendrian tori in  $S^5$ : we construct a family of solutions which are equivariant in some sense under the action of the torus, that we call *homogeneous* Hamiltonian stationary tori. These are the simplest examples that one can build.

Let us add that the structure of the integrable system studied here fits in a classification of elliptic integrable systems proposed by C.L. Terng [T], as a 2nd  $(U(3), \sigma, \tau)$ -system<sup>1</sup>, where  $\sigma$  is an involution of  $U(3)$  such that its fixed set is  $U(3)^\sigma \simeq U(2) \times U(1)$  and  $U(3)/U(2)^\sigma \simeq \mathbb{C}P^2$  and  $\tau$  is a 4th order automorphism (actually  $\tau^2 = \sigma$ ) which encodes the symplectic structure on  $\mathbb{C}P^2$  or the Legendrian structure on  $S^5$ .

*Notations* — For any matrix  $M \in GL(n, \mathbb{C})$ , we denote by  $M^\dagger := {}^t\overline{M}$ .

## 2 Geometrical description of Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}P^n$

### 2.1 The space $\mathbb{C}P^n$

The complex projective space  $\mathbb{C}P^n$  can be identified with the quotient manifold  $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ . We denote by  $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}P^n$  the canonical projection (a.k.a. Hopf fibration): if  $z = (z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$ ,  $\pi(z) = [z]$  is the equivalence class of  $z$  modulo  $\mathbb{C}^*$  (the complex punctured line spanned by  $z$ ) and is also written in homogeneous coordinates  $[z^1 : \dots : z^{n+1}]$ .

*The tangent bundle* — If  $[z] \in \mathbb{C}P^n$  a tangent vector  $\xi \in T_{[z]}\mathbb{C}P^n$  can be

<sup>1</sup>we have here exchanged the notations  $\sigma$  and  $\tau$  with respect to [T] in order to be consistent with our notations in [HR3]

represented by a  $\mathbb{C}$ -linear map  $\ell : \mathbb{C}z \longrightarrow \mathbb{C}^{n+1}$ ,  $\lambda z \longmapsto \ell(\lambda z)$  such that

$$\forall \lambda \in \mathbb{C}, \quad d\pi_{\lambda z}(\ell(\lambda z)) = \xi. \quad (1)$$

Note that  $\ell$  is not unique and given some  $\xi \in T_{[z]}\mathbb{C}P^n$  if  $\ell$  and  $\tilde{\ell}$  satisfy condition (1) with  $\xi$ , then there exists  $k \in \mathbb{C}$  s.t.  $\tilde{\ell}(\lambda z) = \ell(\lambda z) + k\lambda z$ . However by using the standard Hermitian product  $\langle \cdot, \cdot \rangle_{\mathbb{C}^{n+1}}$  on  $\mathbb{C}^{n+1}$  we can select an unique  $\ell^0$  which satisfies (1) and such that

$$\langle \ell^0(z), z \rangle_{\mathbb{C}^{n+1}} = 0. \quad (2)$$

Indeed it suffices to start from any  $\ell$  satisfying (1) and to set  $\ell^0(z) = \ell(z) + kz$ . Then condition (2) holds if and only if  $k = -\frac{\langle \ell(z), z \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2}$ .

*The Hermitian metric* — We can define a Hermitian product on  $\mathbb{C}P^n$  as follows. If  $\xi_1, \xi_2 \in T_{[z]}\mathbb{C}P^n$  we consider the linear maps  $\ell_1^0$  and  $\ell_2^0$  satisfying (2) and such that  $d\pi_z(\ell_1^0(z)) = \xi_1$  and  $d\pi_z(\ell_2^0(z)) = \xi_2$ . Then the Hermitian product of  $\xi_1$  and  $\xi_2$  is

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n} := \frac{\langle \ell_1^0(z), \ell_2^0(z) \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2},$$

a definition which is obviously invariant by transformations  $z \longmapsto \lambda z$ . If we had started with linear mappings  $\ell_1, \ell_2$  which lift respectively  $\xi_1$  and  $\xi_2$  according to (1) but without the condition (2) we could recover  $\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n}$  by substitution of  $\ell_a^0(z) = \ell_a(z) - \frac{\langle \ell_a(z), z \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2} z$  (for  $a = 1, 2$ ) in the above definition. It gives

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n} = \frac{\langle \ell_1(z), \ell_2(z) \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2} - \frac{\langle \ell_1(z), z \rangle_{\mathbb{C}^{n+1}} \langle z, \ell_2(z) \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^4}. \quad (3)$$

Note that the Hermitian metric  $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n}$  on  $\mathbb{C}P^n$  provides us with an Euclidean metric  $\langle \cdot, \cdot \rangle_E$  and a symplectic form  $\omega$  through<sup>2</sup>  $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n} = \langle \cdot, \cdot \rangle_E - i\omega(\cdot, \cdot)$ .

*The horizontal distribution and the connection* — For each  $z \in \mathbb{C}^{n+1} \setminus \{0\}$  we let  $\mathcal{H}_z$  to be the complex  $n$ -subspace in  $\mathbb{C}^{n+1}$  which is Hermitian orthogonal to  $z$  (and hence to the fiber of  $\pi$ ). Note that the definition of the Hermitian metric on  $\mathbb{C}P^n$  is such that  $\frac{1}{|z|_{\mathbb{C}^{n+1}}} d\pi_z : \mathcal{H}_z \longrightarrow T_{[z]}\mathbb{C}P^n$  is an isometry between complex Hermitian spaces (and  $d\pi_z$  allows us also to orient  $\mathcal{H}_z$ ). We call the subbundle  $\mathcal{H} := \cup_{z \in \mathbb{C}^{n+1} \setminus \{0\}} \mathcal{H}_z$  of  $T\mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}$  the *horizontal distribution*. It defines in a natural way a connection  $\nabla^{Hopf} \simeq \nabla^H$  on the Hopf bundle  $\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}P^n$ : if  $f : \mathbb{C}P^n \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}$  is a section of this bundle its covariant derivative is

$$(\nabla_{\xi}^H f)_{[z]} := \frac{\langle df_{[z]}(\xi), f(z) \rangle_{\mathbb{C}^{n+1}}}{|f(z)|_{\mathbb{C}^{n+1}}^2} f(z).$$

<sup>2</sup>note the the sign convention may vary in the literature, e.g. some Authors use  $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n} = \langle \cdot, \cdot \rangle_E + i\omega(\cdot, \cdot)$ .

In order to compute the curvature of this connection, let us choose two vector fields  $\xi_1$  and  $\xi_2$  on  $\mathbb{C}P^n$  such that  $[\xi_1, \xi_2] = 0$ . Then using (3) one computes that

$$\nabla_{\xi_1}^H (\nabla_{\xi_2}^H f) - \nabla_{\xi_2}^H (\nabla_{\xi_1}^H f) = (\langle \xi_2, \xi_1 \rangle_{\mathbb{C}P^n} - \langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n}) f = 2i\omega(\xi_1, \xi_2)f.$$

Hence the curvature of this connection is proportional to the symplectic form  $\omega$ . It has the following consequence:

**PROPOSITION 2.1** *Let  $\Omega$  be a simply connected open subset of  $\mathbb{R}^n$  and  $u : \Omega \rightarrow \mathbb{C}P^n$  be a smooth Lagrangian immersion, i.e. such that  $u^*\omega = 0$ . Then there exists a lift*

$$\begin{array}{ccc} & & \mathbb{C}^{n+1} \\ & \nearrow \hat{u} & \downarrow \pi \\ \Omega & \xrightarrow{u} & \mathbb{C}P^n \end{array}$$

such that  $(u^*\nabla^H)\hat{u} = 0$  (where  $u^*\nabla^H$  is the pull-back by  $u$  of the connection  $\nabla^H$ ). Moreover this lift is unique modulo a multiplicative constant in  $\mathbb{C}^*$ , is isotropic (i.e. the pull-back by  $\hat{u}$  of the symplectic form on  $\mathbb{C}^{n+1}$  vanishes) and we can choose it in such a way that  $|\hat{u}|_{\mathbb{C}^{n+1}} = 1$  everywhere (i.e. such that  $\hat{u} : \Omega \rightarrow S^{2n+1}$ ).

*Proof* — The curvature of  $u^*\nabla^H$  is the pull-back by  $u$  of the curvature of  $\nabla^H$ , i.e.  $2iu^*\omega$ . But this 2-form vanishes because  $u$  is Lagrangian and thus  $u^*\nabla^H$  is flat. Hence we can find a parallel section  $\hat{u}$  (unique up to multiplication by a non-zero constant). The condition  $(u^*\nabla^H)\hat{u} = 0$  writes  $\langle d\hat{u}, \hat{u} \rangle_{\mathbb{C}^{n+1}} = 0$ , which implies  $d|\hat{u}|_{\mathbb{C}^{n+1}}^2 = \langle d\hat{u}, \hat{u} \rangle_{\mathbb{C}^{n+1}} + \langle \hat{u}, d\hat{u} \rangle_{\mathbb{C}^{n+1}} = 0$ . Hence  $|\hat{u}|_{\mathbb{R}^{2n+2}}$  is constant and in particular we can choose  $\hat{u}$  to take values in  $S^{2n+1}$ . ■

Observe that in Proposition 2.1 the tangent  $n$ -subspace along  $\hat{u}$  at  $(x, y)$  is a subspace of  $\mathcal{H}_{\hat{u}(x, y)}$ : we then say that  $\hat{u}$  is a *Legendrian* immersion.

*The Levi-Civita connection on  $\mathbb{C}P^n$*  — It can be expressed in terms of the trivial connection  $D$  of  $(\mathbb{C}^{n+1}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{2n+2}})$  as follows. Let  $U \subset \mathbb{C}P^n$  be an open subset such that we can define a smooth local section  $\sigma : U \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$  of the Hopf bundle. For any  $m \in U$  we denote by  $\hat{m} := \sigma(m)$ . Let  $X$  be a smooth tangent vector field on  $\mathbb{C}P^n$  defined on  $U$  and, choosing some point  $m \in U$ , let  $\xi \in T_m\mathbb{C}P^n$ . In order to define the covariant derivative of  $X$  at  $m$  along  $\xi$  we let  $\hat{\xi} := d\sigma_m(\xi) \in T_{\hat{m}}\mathbb{C}P^n$  and  $\hat{X} := \sigma_*X$ . Then we set

$$(\nabla_{\xi}X)(m) := d\pi_{\hat{m}} \left( (D_{\hat{\xi}}\hat{X})(m) - \left( \frac{\langle \hat{\xi}, \hat{m} \rangle_{\mathbb{C}^{n+1}}}{|\hat{m}|_{\mathbb{C}^{n+1}}^2} X(m) + \frac{\langle \hat{X}(m), \hat{m} \rangle_{\mathbb{C}^{n+1}}}{|\hat{m}|_{\mathbb{C}^{n+1}}^2} \xi \right) \right) \quad (4)$$

(Of course this definition does not depend on the choice of the local section  $\sigma$ .) One can then check easily that this connection respects the metric and is torsion-free and so it is the Levi-Civita connection on  $\mathbb{C}P^n$ .

## 2.2 The mean curvature vector of a Lagrangian submanifold in $\mathbb{C}P^n$

We consider here a simply connected Lagrangian submanifold  $\Sigma$  of  $\mathbb{C}P^n$  and denote by  $u : \Sigma \rightarrow \mathbb{C}P^n$  its immersion map. According to Proposition 2.1, we can construct a Legendrian lift  $\hat{u} : \Sigma \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ . We also consider a smooth orthonormal moving frame along  $u$ , i.e. sections  $E_1, \dots, E_n$  of  $u^*T\mathbb{C}P^n$  such that for all  $x \in \Sigma$ ,  $(E_1(x), \dots, E_n(x))$  is a direct orthonormal basis over  $\mathbb{R}$  of  $T_{u(x)}\Sigma$  and, for all  $a$ , we let  $e_a := \hat{u}_*E_a$ . Then for all  $x \in \Sigma$ ,  $(e_1(x), \dots, e_n(x))$  is a direct orthonormal basis over  $\mathbb{R}$  of  $T_{\hat{u}(x)}\hat{u}(\Sigma)$ . We next define the *mean curvature vector*  $\vec{H}$  of the immersion  $u$  which is the normal component of the trace of the second fundamental form. The main point here is that, since  $T_{u(x)}\Sigma$  is Lagrangian, it is mapped by the complex structure<sup>3</sup>  $J$  of  $T_{u(x)}\mathbb{C}P^n$  to the normal space  $(T_{u(x)}\mathbb{C}P^n)^\perp$ . And so  $(JE_1(x), \dots, JE_n(x))$  is an orthonormal basis of the normal subspace to  $T_{u(x)}\Sigma$  in  $T_{u(x)}\mathbb{C}P^n$ . Hence

$$\vec{H} := \frac{1}{n} \sum_{a,b=1}^n \langle \nabla_{E_a} E_a, JE_b \rangle_E JE_b. \quad (5)$$

Note that then, since  $\langle e_a, \hat{u} \rangle_{\mathbb{C}^{n+1}} = \langle e_b, \hat{u} \rangle_{\mathbb{C}^{n+1}} = 0$ , we have according to (4)

$$\nabla_{E_a} E_b = d\pi_{\hat{u}(x)}(D_{E_a} e_b) = d\pi_{\hat{u}(x)}(D_{e_a} e_b).$$

Hence

$$\vec{H} = \frac{1}{n} \sum_{a,b=1}^n \langle D_{e_a} e_a, ie_b \rangle_{\mathbb{R}^{2n+2}} JE_b. \quad (6)$$

*The complex volume  $n$ -form along  $\hat{u}$*  — Let us consider on  $\mathbb{C}^{n+1}$  the complex volume  $(n+1)$ -form  $\Theta := dz^1 \wedge \dots \wedge dz^{n+1}$ . Then we construct the complex volume  $n$ -form along  $\hat{u}$  to be the section  $\hat{\theta}$  of  $\hat{u}^*(\Lambda^n \mathcal{H}^* \otimes \mathbb{C})$  by

$$\hat{\theta}_x := (\hat{u}(x) \lrcorner \Theta)|_{\mathcal{H}_{\hat{u}(x)}}, \quad \forall x \in \Sigma,$$

where  $\lrcorner$  is the interior product. Lastly we define the complex volume  $n$ -form  $\theta$  along  $u$  to be the section of  $u^*(\Lambda^n T^*\mathbb{C}P^n \otimes \mathbb{C})$  defined by  $\theta := \hat{u}^*\hat{\theta}$ . Using  $\hat{\theta}$  we define the *Lagrangian angle function* along  $\hat{u}$  to be the function  $\beta_{\hat{u}} : \Sigma \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  such that

$$e^{i\beta_{\hat{u}}(x)} = \theta_x(E_1(x), \dots, E_n(x)) = \Theta(\hat{u}(x), e_1(x), \dots, e_n(x)), \quad \forall x \in \Sigma.$$

It is not difficult to check that this definition is independent of the choice of the framing  $(E_1, \dots, E_n)$ . However all this construction depends on the choice of the Legendrian lift  $\hat{u}$ : another choice  $\hat{v}$  would lead to another complex volume  $n$ -form along  $\hat{v}$ , and another Lagrangian angle function  $\beta_{\hat{v}}$ , however  $\beta_{\hat{u}} - \beta_{\hat{v}}$  is a constant in  $\mathbb{R}/2\pi\mathbb{Z}$ .

<sup>3</sup>the complex structure  $J$  is the image of the canonical complex structure  $i$  on  $\mathcal{H}_{\hat{u}(x)}$  by  $d\pi_{\hat{u}(x)}$ .

LEMMA 2.2 *The mean curvature vector of the Lagrangian surface  $\Sigma$  can be computed using the Lagrangian angle function  $\beta_{\widehat{u}}$  through the relation*

$$\vec{H} = \frac{1}{n} J \nabla \beta_{\widehat{u}}. \quad (7)$$

*Proof* — We set here  $\beta = \beta_{\widehat{u}}$ . Relation (7) is equivalent to proving that

$$\forall V \in T_x \mathbb{C}P^n, \quad n \langle \vec{H}, JV \rangle_E = \langle \nabla \beta, V \rangle_E. \quad (8)$$

The left hand side of (8) is, by using (6) and the fact that  $J$  is an isometry,  $\sum_{a,b=1}^n \langle D_{e_a} e_a, ie_b \rangle_{\mathbb{R}^{2n+2}} \langle E_b, V \rangle_E$ . So we are led to the following computation, which uses first the fact that  $\langle e_a, ie_b \rangle_{\mathbb{R}^{2n+2}} = 0$ , second the fact that  $i$  is a complex structure, and third the fact that the Lie bracket  $[e_a, e_b]$  is tangent to  $\Sigma$ :

$$\begin{aligned} \langle D_{e_a} e_a, ie_b \rangle_{\mathbb{R}^{2n+2}} &= -\langle e_a, iD_{e_a} e_b \rangle_{\mathbb{R}^{2n+2}} \\ &= \langle ie_a, D_{e_a} e_b \rangle_{\mathbb{R}^{2n+2}} \\ &= \langle ie_a, D_{e_b} e_a + [e_a, e_b] \rangle_{\mathbb{R}^{2n+2}} \\ &= \langle ie_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}. \end{aligned}$$

So we have  $n \langle \vec{H}, JV \rangle_E = \sum_{a,b=1}^n \langle ie_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} \langle E_b, V \rangle_E$ . Now the right hand side of (8) can be computed as follows.. We derivate both sides of the relation  $e^{i\beta} = \Theta_{\widehat{u}}(\widehat{u}, e_1, \dots, e_n)$  with respect to  $e_b$  and use the decomposition of  $D_{e_b} e_a$  in the  $\mathbb{R}$ -orthonormal basis  $(\widehat{u}, e_1, \dots, e_n, i\widehat{u}, ie_1, \dots, ie_n)$ :

$$\begin{aligned} ie^{i\beta} D_{e_b} \beta &= D_{e_b} e^{i\beta} \\ &= D_{e_b} (\Theta(\widehat{u}, e_1, \dots, e_n)) \\ &= \Theta(e_b, e_1, \dots, e_n) + \sum_{a=1}^n \Theta(\widehat{u}, e_1, \dots, D_{e_b} e_a, \dots, e_n) \\ &= \sum_{a=1}^n \Theta(\widehat{u}, e_1, \dots, e_a, \dots, e_n) (\langle e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} + i \langle ie_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}) \\ &= ie^{i\beta} \sum_{a=1}^n \langle ie_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}. \end{aligned}$$

Here we used in the last line the fact that  $|e_a|_{\mathbb{R}^{2n+2}}^2 = 1 \implies \langle e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} = 0$ . Hence we are left with  $D_{e_b} \beta = \sum_{a=1}^n \langle ie_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}$ , which implies that

$$\langle \nabla \beta, V \rangle_E = \sum_{b=1}^n D_{e_b} \beta \langle E_b, V \rangle_E = \sum_{a,b=1}^n \langle ie_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} \langle E_b, V \rangle_E.$$

So we have proved (8). ■

A *Hamiltonian stationary* Lagrangian submanifold  $\Sigma$  in  $\mathbb{C}P^n$  is a Lagrangian submanifold which is a critical point of the  $n$ -volume functional  $\mathcal{A}$  under first variations which are *Hamiltonian vector fields* with compact support. This means that for any smooth function with compact support  $h \in C_c^\infty(\mathbb{C}P^n, \mathbb{R})$ , we have

$$\delta \mathcal{A}_{\xi_h}(\Sigma) := \int_{\Sigma} \left\langle \vec{H}, \xi_h \right\rangle_E d\text{vol} = 0,$$



where  $\xi_h$  is the Hamiltonian vector field of  $h$ , i.e. satisfies  $\omega(\xi_h, \cdot) + dh = 0$  or  $\xi_h = J\nabla h$ . We also remark that if  $f \in \mathcal{C}_c^\infty(\Sigma, \mathbb{R})$ , then there exist smooth extensions with compact support  $h$  of  $f$ , i.e. functions  $h \in \mathcal{C}_c^\infty(\mathbb{C}P^n, \mathbb{R})$  such that  $h|_\Sigma = f$ , and moreover the normal component of  $(\xi_h)|_\Sigma$  does not depend on the choice of the extension  $h$  (it coincides actually with  $J\nabla f$ , where  $\nabla$  is here the gradient with respect to the induced metric on  $\Sigma$ ). So we deduce from Lemma 2.2 that actually  $\delta\mathcal{A}_{\xi_h}(\Sigma) = \frac{1}{n} \int_\Sigma \langle \nabla\beta, \nabla f \rangle_E d\text{vol}$ . This implies the following.

**COROLLARY 2.3** *Any Lagrangian submanifold  $\Sigma$  in  $\mathbb{C}P^n$  is Hamiltonian stationary if and only if  $\beta$  is a harmonic function on  $\Sigma$ , i.e.*

$$\Delta_\Sigma \beta = 0.$$

The theory above extends to non simply connected surfaces  $\Sigma$  with the following restrictions. Let  $\gamma$  be a homotopically non trivial loop. The Legendrian lift of  $\gamma$  needs not close, so that in general its endpoints  $p_1, p_2 \in S^{2n+1}$  are multiples of each other by a factor  $e^{i\theta}$ . The same holds for the Lagrangian angle:  $\beta(p_2) \equiv \beta(p_1) + (n+1)\theta \pmod{2\pi}$  (since the tangent plane is also shifted by the Decktransformation  $z \mapsto e^{i\theta}z$ ). In particular  $\beta$  is not always globally defined on surfaces in  $\mathbb{C}P^n$  with non trivial topology, unless the Legendrian lift is globally defined in  $S^{2n+1}/\mathbb{Z}_{n+1}$  (here  $\mathbb{Z}_{n+1}$  stands for the cubic roots of unity in  $SU(n+1)$ ).

### 2.3 Conformal Lagrangian immersions into $\mathbb{C}P^2$

We now set  $n = 2$ . We suppose that  $\Omega$  is a simply connected open subset of  $\mathbb{R}^2 \simeq \mathbb{C}$  and consider a conformal Lagrangian immersion  $u : \Omega \rightarrow \mathbb{C}P^2$ . This implies that we can find a function  $\rho : \Omega \rightarrow \mathbb{R}$  and two sections  $E_1$  and  $E_2$  of  $u^*T\mathbb{C}P^2$  such that  $\forall (x, y) \in \Omega$ ,  $(E_1(x, y), E_2(x, y))$  is an Euclidean orthogonal basis over  $\mathbb{R}$  of  $T_{u(x, y)}u(\Omega)$  and

$$du = e^\rho (E_1 dx + E_2 dy).$$

We observe that, due to the fact that  $u$  is Lagrangian,  $(E_1, E_2)$  is also a *Hermitian* basis over  $\mathbb{C}$  of  $T_{u(x, y)}\mathbb{C}P^2$ .

Let  $\Omega \xrightarrow{\hat{u}} S^5$  be a parallel lift of  $u$  as in Proposition 2.1 and  $(e_1, e_2)$  be the

$$\begin{array}{ccc} \Omega & \xrightarrow{\hat{u}} & S^5 \\ & \searrow u & \downarrow \pi \\ & & \mathbb{C}P^2 \end{array}$$

unique section of  $\hat{u}^*\mathcal{H} \times \hat{u}^*\mathcal{H}$  which lifts<sup>4</sup>  $(E_1, E_2)$ . Then we have

$$d\hat{u} = e^\rho (e_1 dx + e_2 dy). \tag{9}$$

---

<sup>4</sup>recall that the condition that  $v \in (\hat{u}^*\mathcal{H})_{(x, y)}$  means that  $v$  is in the horizontal subspace  $\mathcal{H}_{\hat{u}(x, y)}$

Note that  $\forall(x, y) \in \Omega$ ,  $(e_1(x, y), e_2(x, y))$  is a Hermitian basis of  $\mathcal{H}_{\hat{u}(x, y)}$ , which is Hermitian orthogonal to  $\hat{u}(x, y)$ . Hence  $\forall(x, y) \in \Omega$ ,  $(e_1(x, y), e_2(x, y), \hat{u}(x, y))$  is a Hermitian basis of  $\mathbb{C}^3$ . Thus this triplet can be identified with some  $\hat{F}(x, y) \in U(3)$ . We hence get the diagram

$$\begin{array}{ccc} & & U(3) \\ & \nearrow \hat{F} & \downarrow (\cdot \cdot *) \\ \Omega & \xrightarrow{\hat{u}} & S^5 \\ & \searrow u & \downarrow \pi \\ & & \mathbb{C}P^2 \end{array}$$

the mapping  $(e_1, e_2, e_3) \mapsto e_3$ .

We define the Maurer–Cartan form  $\hat{A}$  to be the 1-form on  $\Omega$  with coefficients in  $\mathfrak{u}(3)$  such that  $d\hat{F} = \hat{F} \cdot \hat{A}$ . Then we remark that the horizontality assumption  $\langle d\hat{u}, \hat{u} \rangle_{\mathbb{C}^3} = 0$  exactly means that

$$\hat{A}_3^3 = 0. \quad (10)$$

Moreover the Lagrangian angle function  $\beta_{\hat{u}}$  along  $\hat{u}$  can be computed by

$$e^{i\beta_{\hat{u}}} = \Theta(e_1, e_2, \hat{u}) = \det \hat{F}.$$

As in [HR1] we consider a larger class of framings of  $u$  as follows.

**DEFINITION 2.4** *A Legendrian framing of  $u$  along  $\hat{u}$  is a map  $F : \Omega \rightarrow U(3)$  such that*

- $(\cdot \cdot *) \circ F = \hat{u}$
- $\det F = e^{i\beta_{\hat{u}}}$ .

It is easily seen that the first condition is equivalent to the fact that there exists a smooth map  $G : \Omega \rightarrow U(3)$  (a gauge transformation) of the type

$$G(x, y) = \begin{pmatrix} g(x, y) & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } g : \Omega \rightarrow U(2)$$

such that

$$F(x, y) = \hat{F}(x, y) \cdot G^{-1}(x, y).$$

And then the second one is equivalent to say that  $g$  takes values in  $SU(2)$ .

## 2.4 A splitting of the Maurer–Cartan form of a Legendrian framing

Using (9) and (10) one obtains the following decomposition of  $\hat{A}$ :

$$\hat{A} = \hat{A}_{\mathfrak{u}(1)} + \hat{A}_{\mathfrak{su}(2)} + \hat{A}_{\mathbb{C}^2},$$

with the notations

$$\widehat{A}_{\mathfrak{u}(1)} = \begin{pmatrix} \widehat{\alpha}_{\mathfrak{u}(1)} & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{A}_{\mathfrak{su}(2)} = \begin{pmatrix} \widehat{\alpha}_{\mathfrak{su}(2)} & 0 \\ 0 & 0 \end{pmatrix},$$

and  $\widehat{A}_{\mathbb{C}^2} = e^\rho \begin{pmatrix} 0 & \epsilon dz + \bar{\epsilon} d\bar{z} \\ -{}^t(\epsilon dz + \bar{\epsilon} d\bar{z}) & 0 \end{pmatrix},$

where  $\widehat{\alpha}_{\mathfrak{u}(1)}$  is a 1-form on  $\Omega$  with coefficients in  $\mathfrak{u}(1) \simeq \mathbb{R} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ ,  $\widehat{\alpha}_{\mathfrak{su}(2)}$  a 1-form on  $\Omega$  with coefficients in  $\mathfrak{su}(2)$ ,  $\epsilon := \frac{1}{2} \begin{pmatrix} 1 & \\ & -i \end{pmatrix}$  and  $\bar{\epsilon} := \frac{1}{2} \begin{pmatrix} 1 & \\ & i \end{pmatrix}$  (so that  $\epsilon dz + \bar{\epsilon} d\bar{z} = \begin{pmatrix} dx \\ dy \end{pmatrix}$ ). Note that  $\det F = e^{i\beta_{\widehat{a}}}$  implies  $\widehat{\alpha}_{\mathfrak{u}(1)} = \frac{d\beta_{\widehat{a}}}{2} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ . Now we let  $F : \Omega \rightarrow U(3)$  be a Legendrian framing and  $A := F^{-1} \cdot dF$ . The relation  $F = \widehat{F} \cdot G^{-1}$  implies that  $A = G \cdot \widehat{A} \cdot G^{-1} - dG \cdot G^{-1}$ . Hence

$$A = A_{\mathfrak{u}(1)} + A_{\mathfrak{su}(2)} + A_{\mathbb{C}^2},$$

where, using the fact that  $\mathfrak{u}(1)$  commutes with  $\mathfrak{su}(2)$ ,

$$A_{\mathfrak{u}(1)} = \begin{pmatrix} \alpha_{\mathfrak{u}(1)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\alpha}_{\mathfrak{u}(1)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} i \frac{d\beta_{\widehat{a}}}{2} & 0 & 0 \\ 0 & i \frac{d\beta_{\widehat{a}}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{\mathfrak{su}(2)} = \begin{pmatrix} \alpha_{\mathfrak{su}(2)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g \cdot \widehat{\alpha}_{\mathfrak{su}(2)} \cdot g^{-1} - dg \cdot g^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A_{\mathbb{C}^2} = e^\rho \begin{pmatrix} 0 & g \cdot (\epsilon dz + \bar{\epsilon} d\bar{z}) \\ -(g \cdot (\epsilon dz + \bar{\epsilon} d\bar{z}))^\dagger & 0 \end{pmatrix}.$$

We can further split the last term  $A_{\mathbb{C}^2}$  along  $dz$  and  $d\bar{z}$  as  $A_{\mathbb{C}^2} = A'_{\mathbb{C}^2} + A''_{\mathbb{C}^2}$  where

$$A'_{\mathbb{C}^2} := e^\rho \begin{pmatrix} 0 & g \cdot \epsilon \\ -(g \cdot \bar{\epsilon})^\dagger & 0 \end{pmatrix} dz \quad \text{and} \quad A''_{\mathbb{C}^2} := e^\rho \begin{pmatrix} 0 & g \cdot \bar{\epsilon} \\ -(g \cdot \epsilon)^\dagger & 0 \end{pmatrix} d\bar{z}.$$

## 2.5 Interpretation in terms of an automorphism

As expounded in [HR1] and [HR3] the key point in order to exploit the structure of an integrable system is to observe that the splitting  $A = A_{\mathfrak{u}(1)} + A_{\mathfrak{su}(2)} + A'_{\mathbb{C}^2} + A''_{\mathbb{C}^2}$  corresponds to a decomposition along the eigenspaces of the following automorphism in  $\mathfrak{u}(3)^\mathbb{C}$ , the complexification<sup>5</sup> of  $\mathfrak{u}(3)$ . We let  $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

<sup>5</sup>we can define  $\mathfrak{u}(3)^\mathbb{C}$  as the set  $M(3, \mathbb{C})$  with its standard complex structure and with the conjugation mapping  $c : M \mapsto -M^\dagger$ ; clearly  $c$  is a Lie algebra automorphism, an involution and the set of fixed points of  $c$  is  $\mathfrak{u}(3)$ .

Similarly the complexification  $U(3)^\mathbb{C}$  is the set  $GL(3, \mathbb{C})$  with its standard complex structure and the conjugation map  $C : G \mapsto (G^\dagger)^{-1}$ .

and

$$\begin{aligned} \tau : \mathfrak{u}(3)^{\mathbb{C}} &\longrightarrow \mathfrak{u}(3)^{\mathbb{C}} \\ M &\longmapsto - \begin{pmatrix} -J & 0 \\ 0 & 1 \end{pmatrix} \cdot {}^t M \cdot \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

It is then straightforward that  $\tau$  is a Lie algebra automorphism, that  $\mathfrak{u}(3)$  is stable by  $\tau$  and that  $\tau^4 = Id$ . Hence we can diagonalize the action of  $\tau$  over  $\mathfrak{u}(3)^{\mathbb{C}}$  and in the following we denote by  $\mathfrak{u}(3)_a^{\mathbb{C}}$  the eigenspace of  $\tau$  for the eigenvalue  $i^a$ , for  $a = -1, 0, 1, 2$ . We first point out that the eigenspaces  $\mathfrak{u}(3)_0^{\mathbb{C}}$  and  $\mathfrak{u}(3)_2^{\mathbb{C}}$ , with eigenvalues 1 and  $-1$  respectively, are the complexifications of  $\mathfrak{u}(3)_0$  and  $\mathfrak{u}(3)_2$  respectively, where

$$\mathfrak{u}(3)_0 := \left\{ \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} / g \in \mathfrak{su}(2) \right\} \text{ and } \mathfrak{u}(3)_2 := \left\{ \begin{pmatrix} \lambda i & 0 & 0 \\ 0 & \lambda i & 0 \\ 0 & 0 & \mu i \end{pmatrix} / \lambda, \mu \in \mathbb{R} \right\}.$$

This can be obtained by first computing that

$$\tau \begin{pmatrix} A & X \\ -{}^t Y & d \end{pmatrix} = \begin{pmatrix} J {}^t A J & -J Y \\ -{}^t X J & -d \end{pmatrix}, \quad \forall A \in M(2, \mathbb{C}), \forall X, Y \in \mathbb{C}^2, \forall d \in \mathbb{C},$$

and by using the fact that  $\forall A \in \mathfrak{sl}(2, \mathbb{C}), J {}^t A J = A$ . Similarly the eigenspaces  $\mathfrak{u}(3)_1^{\mathbb{C}}$  and  $\mathfrak{u}(3)_{-1}^{\mathbb{C}}$ , with eigenvalues  $i$  and  $-i$  respectively, are found to be

$$\mathfrak{u}(3)_1^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & X \\ -{}^t Y & 0 \end{pmatrix} / X, Y \in \mathbb{C}^2, J Y = -i X \right\}$$

and

$$\mathfrak{u}(3)_{-1}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & X \\ -{}^t Y & 0 \end{pmatrix} / X, Y \in \mathbb{C}^2, J Y = i X \right\}.$$

Now we have the following

LEMMA 2.5 *The eigenspaces  $\mathfrak{u}(3)_1^{\mathbb{C}}$  and  $\mathfrak{u}(3)_{-1}^{\mathbb{C}}$  can be characterized by*

$$\mathfrak{u}(3)_1^{\mathbb{C}} = \left\{ \lambda \begin{pmatrix} 0 & h \cdot \bar{\epsilon} \\ -(h \cdot \epsilon)^\dagger & 0 \end{pmatrix} / \lambda \in [0, \infty), h \in SU(2) \right\}$$

and

$$\mathfrak{u}(3)_{-1}^{\mathbb{C}} = \left\{ \lambda \begin{pmatrix} 0 & h \cdot \epsilon \\ -(h \cdot \bar{\epsilon})^\dagger & 0 \end{pmatrix} / \lambda \in [0, \infty), h \in SU(2) \right\}.$$

*Proof* — This can be proved either by adapting the argument in section 2.4 of [HR1] or by a straightforward computation which exploits the fact that  $\forall h \in SU(2), \bar{h} J = J h, J \epsilon = i \epsilon$  and  $J \bar{\epsilon} = -i \bar{\epsilon}$ . ■

We conclude that if, using  $\mathfrak{u}(3)^{\mathbb{C}} = \mathfrak{u}(3)_{-1}^{\mathbb{C}} \oplus \mathfrak{u}(3)_0^{\mathbb{C}} \oplus \mathfrak{u}(3)_1^{\mathbb{C}} \oplus \mathfrak{u}(3)_2^{\mathbb{C}}$ , we decompose  $A$  as

$$A = A_{-1} + A_0 + A_1 + A_2,$$

where each  $A_a$  is a 1-form with coefficients in the  $\mathfrak{u}(3)_a^{\mathbb{C}}$ , then we recover the previous splitting by setting  $A_0 = A_{\mathfrak{su}(2)}$ ,  $A_2 = A_{\mathfrak{u}(1)}$ ,  $A_{-1} = A'_{\mathbb{C}^2}$  and  $A_1 = A''_{\mathbb{C}^2}$ . Note that the two last conditions actually reflect the Lagrangian constraint on  $u$ .

*Remark*— Note that by the automorphism property  $[\mathfrak{u}(3)_a, \mathfrak{u}(3)_b] \subset \mathfrak{u}(3)_{a+b \bmod 4}$ .

## 2.6 Legendrian framings of Hamiltonian stationary Lagrangian immersions

Given the Legendrian framing  $F$  of a conformal Lagrangian immersion  $u$  in  $\mathbb{C}P^2$ , we define the family of deformations  $A_\lambda$  of its Maurer–Cartan form  $A$ , for  $\lambda \in S^1 \subset \mathbb{C}^*$  by

$$A_\lambda := \lambda^{-2}A'_2 + \lambda^{-1}A_{-1} + A_0 + \lambda A_1 + \lambda^2 A''_2, \quad (11)$$

where  $A'_2 := A_2(\partial/\partial z)dz$  and  $A''_2 := A_2(\partial/\partial \bar{z})d\bar{z}$ . We then have the following:

**THEOREM 2.6** *Given a conformal Lagrangian immersion  $u : \Omega \rightarrow \mathbb{C}P^2$  and a Legendrian framing  $F$  of  $u$ , the Maurer–Cartan form of  $F$  satisfies*

$$A_{-1} = A'_{-1} = A'_{\mathbb{C}^2} \text{ and } A_1 = A''_1 = A'_{\mathbb{C}^2}. \quad (12)$$

*Furthermore  $u$  is Hamiltonian stationary if and only if, defining  $A_\lambda$  as in (11),*

$$dA_\lambda + A_\lambda \wedge A_\lambda = 0, \quad \forall \lambda \in S^1. \quad (13)$$

*Remark*— For  $\lambda = 1$ ,  $A_1 = A$  and equation (13) is a consequence of its definition  $A := F^{-1} \cdot dF$ .

*Proof*— See [HR1] and [HR3]. ■

We remark that all the conditions that have been collected about the components  $A_a$  can be encoded by the following twisting condition on  $A_\lambda$ :

$$\forall \lambda \in S^1, \quad \tau(A_\lambda) = A_{i\lambda}.$$

Thus we are led to define the following *twisted loop algebra*

$$\Lambda\mathfrak{u}(3)_\tau := \{S^1 \ni \lambda \mapsto \xi_\lambda \in \mathfrak{u}(3) / \forall \lambda \in S^1, \tau(\xi_\lambda) = \xi_{i\lambda}\},$$

and  $A_\lambda$  is a 1-form on  $\Omega$  with coefficients in  $\Lambda\mathfrak{u}(3)_\tau$ .

Actually  $\Lambda\mathfrak{u}(3)_\tau$  is the Lie algebra of the following *twisted loop group*

$$\Lambda U(3)_\tau := \{S^1 \ni \lambda \mapsto g_\lambda \in U(3) / \forall \lambda \in S^1, \tau(g_\lambda) = g_{i\lambda}\},$$

where the Lie algebra automorphism  $\tau : \mathfrak{u}(3)^{\mathbb{C}} \rightarrow \mathfrak{u}(3)^{\mathbb{C}}$  has been extended to the Lie group automorphism by

$$\begin{aligned} \tau : U(3)^{\mathbb{C}} &\longrightarrow U(3)^{\mathbb{C}} \\ M &\longmapsto \begin{pmatrix} -J & 0 \\ 0 & 1 \end{pmatrix} \cdot {}^t M^{-1} \cdot \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Now if we assume that  $\Omega$  is simply connected, then relation (13) allows us to integrate  $A_\lambda$ , i.e. to find a map  $F_\lambda : \Omega \rightarrow U(3)$  for any  $\lambda \in S^1$  such that  $dF_\lambda = F_\lambda \cdot A_\lambda$ . Moreover if we choose some base point  $z_0 \in \Omega$ , then by requiring further that  $F_\lambda(z_0) = Id$ ,  $F_\lambda$  is unique. A key observation is then that  $\tau(A_\lambda) = A_{i\lambda}$  implies  $\tau(F_\lambda) = F_{i\lambda}$ ,  $\forall \lambda \in S^1$ . Hence, a conformal Lagrangian immersion  $u : \Omega \rightarrow \mathbb{C}P^2$  is Hamiltonian stationary if and only if any Legendrian lift  $F$  of it can be deformed into a map  $F_\lambda : \Omega \rightarrow \Lambda U(3)_\tau$ , such that  $F_\lambda^{-1} \cdot dF_\lambda$  has the form (11). Summarizing this result with the observations in the previous section we have:

**THEOREM 2.7** *Given a simply connected domain  $\Omega \subset \mathbb{C}$  and a base point  $z_0 \in \Omega$ , the set of Hamiltonian stationary conformal Lagrangian immersions  $u : \Omega \rightarrow \mathbb{C}P^2$  such that  $u(z_0) = [0 : 0 : 1]$  is in bijection with the set of maps  $F_\lambda : \Omega \rightarrow \Lambda U(3)_\tau$ , such that  $F_\lambda(z_0) = Id$  and the Fourier decomposition of  $A_\lambda := F_\lambda^{-1} \cdot dF_\lambda$ ,  $A_\lambda = \sum_{k \in \mathbb{Z}} \widehat{A}_k \lambda^k$  satisfies*

$$\forall k \in \mathbb{Z}, \quad k \leq -3 \implies \widehat{A}_k = 0, \quad (14)$$

$$\widehat{A}_{-2} = a(z) dz \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where } a \in C^\infty(\Omega, \mathbb{C}), \quad (15)$$

$$\widehat{A}_{-1} = \widehat{A}_{-1}(\partial/\partial z) dz, \quad \text{i.e. } \widehat{A}_{-1}(\partial/\partial \bar{z}) = 0. \quad (16)$$

*Proof* — For any conformal Lagrangian Hamiltonian stationary immersion  $u$  the existence of  $F_\lambda$  and the properties (14), (15) and (16) are immediate consequences of Theorem 2.6. Conversely for any map  $F_\lambda$ , conditions (14), (15) and (16) and the reality condition  $\overline{A_\lambda} = A_\lambda$  imply that  $A_\lambda$  must satisfy (11). In particular we remark that condition (15) is a reformulation of (10). Thus by theorem 2.6 we deduce that  $F_1$  is the Legendrian lift of some Hamiltonian stationary conformal Lagrangian immersion. ■

*Remark* — From the analysis of the Maurer–Cartan of a Legendrian lift we know that actually the function  $a$  in (15) is  $\frac{1}{2}\partial\beta/\partial z$ , where  $\beta$  is the Lagrangian angle function. In particular since  $u$  is Hamiltonian stationary  $\beta$  is harmonic and hence  $a$  is holomorphic.

## 2.7 An alternative characterization

We introduce here another construction using based loop groups for characterizing Hamiltonian stationary Lagrangian conformal immersions. Consider

$$E_\lambda := F_\lambda \cdot F^{-1}.$$

We can observe that  $E_\lambda$  is a map with values in the *based loop group*

$$\Omega U(3) := \{S^1 \ni \lambda \mapsto g_\lambda \in U(3)/g_{\lambda=1} = 1\},$$

since  $F_{\lambda=1} = F$ . It is easy to check that  $\Omega U(3)$  is a loop group, the Lie algebra of which is

$$\Omega\mathfrak{u}(3) := \{S^1 \ni \lambda \longmapsto \xi_\lambda \in \mathfrak{u}(3) / \xi_{\lambda=1} = 0\}.$$

Note that the (formal) Fourier expansion of an element  $\xi_\lambda \in \Omega\mathfrak{u}(3)$  can be written  $\xi_\lambda = \sum_{k \in \mathbb{Z} \setminus \{0\}} \widehat{\xi}_k (\lambda^k - 1)$ .

The Maurer–Cartan form of  $E_\lambda$  is

$$\begin{aligned} \Gamma_\lambda &:= E_\lambda^{-1} \cdot dE_\lambda \\ &= F \cdot (F_\lambda^{-1} \cdot dF_\lambda - F \cdot dF) \cdot F^{-1} = F \cdot (A_\lambda - A) \cdot F^{-1} \\ &= (\lambda^{-2} - 1)\Gamma'_2 + (\lambda^{-1} - 1)\Gamma_{-1} + (\lambda - 1)\Gamma_1 + (\lambda^2 - 1)\Gamma''_2, \end{aligned}$$

where  $\Gamma'_2 := F \cdot A'_2 \cdot F^{-1}$ ,  $\Gamma_{-1} := F \cdot A_{-1} \cdot F^{-1}$ ,  $\Gamma_1 := F \cdot A_1 \cdot F^{-1}$  and  $\Gamma''_2 := F \cdot A''_2 \cdot F^{-1}$ . We can observe in particular that

$$\Gamma'_2 = ia\pi^\perp dz, \quad \text{where } \pi^\perp := F \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \cdot F^{-1}.$$

Note that  $\pi^\perp$  is the Hermitian orthogonal projection in  $\mathbb{C}^3$  onto the plane  $\widehat{u}^\perp$  (moreover  $\pi^\perp$  is actually independent of the lift  $\widehat{u}$  chosen for  $u$ ).

Lastly we point out the following equivariance property with respect to the automorphism  $\tau_u$  defined<sup>6</sup> by

$$\tau_u(M) = F \cdot \tau(F^{-1} \cdot M \cdot F) \cdot F^{-1}.$$

We have obviously  $\tau_u^4 = 1$ . Moreover, setting

$$\begin{aligned} \gamma_\lambda &:= \lambda^{-2}\Gamma'_2 + \lambda^{-1}\Gamma_{-1} + \lambda\Gamma_1 + \lambda^2\Gamma''_2 \\ &= F \cdot (\lambda^{-2}A'_2 + \lambda^{-1}A_{-1} + \lambda A_1 + \lambda^2 A''_2) \cdot F^{-1} \end{aligned}$$

and  $\gamma := \gamma_{\lambda=1} = F \cdot (A'_2 + A_{-1} + A_1 + A''_2) \cdot F^{-1}$ , so that  $\Gamma_\lambda = \gamma_\lambda - \gamma$ , we have

$$\tau_u(\gamma_\lambda) = \gamma_{i\lambda}.$$

### 3 Finite type solutions

In [HR3] we showed how Theorem 2.7 allows us to adapt the theory of J. Dorfmeister, F. Pedit and H.Y. Wu [DPW], in order to build a Weierstrass

<sup>6</sup>We can remark that the definition of  $\tau_u$  is independent from the choice of the Legendrian framing  $F$  of  $u$ , and depends only on  $u$ . This means that for any pair of Legendrian framings  $F$  and  $\widehat{F}$  such that  $\widehat{F} = F \cdot G$ , where  $G = \begin{pmatrix} g & \\ & 1 \end{pmatrix}$  and  $g : \Omega \rightarrow SU(2)$ , we have  $\widehat{F} \cdot \tau(\widehat{F}^{-1} \cdot M \cdot \widehat{F}) \cdot \widehat{F}^{-1} = F \cdot \tau(F^{-1} \cdot M \cdot F) \cdot F^{-1}$ . This can be checked by a computation using the fact that  $\begin{pmatrix} \pm J & \\ & 1 \end{pmatrix} \cdot G = \overline{G} \cdot \begin{pmatrix} \pm J & \\ & 1 \end{pmatrix}$ .

type representation theory of *all* conformal Lagrangian Hamiltonian stationary immersions, i.e. using holomorphic data. Here we want to exploit Theorem 2.7 in order to construct a particular class of examples of solutions: the *finite type* ones.

### 3.1 Definitions

We invite the Reader to consult [BFPP], [G] or [H] for more details. We first observe that  $U(3)_0 := \{g \in U(3)/\tau(g) = g\}$ , the fixed set of  $\tau$ , is a subgroup of  $U(3)$ , the Lie algebra of which is  $\mathfrak{u}(3)_0$  (same observation about  $U(3)_0^{\mathbb{C}}$ ). Actually  $U(3)_0$  is isomorphic to  $SU(2)$  so that we make the identifications  $U(3)_0 \simeq SU(2)$  and  $\mathfrak{u}(3)_0 \simeq \mathfrak{su}(2)$ . We will need an Iwasawa decomposition of  $SU(2)^{\mathbb{C}}$  for our purpose: it will be a pair  $(SU(2), \mathfrak{B})$  of *subgroups* of  $SU(2)^{\mathbb{C}}$ , such that  $\forall g \in SU(2)^{\mathbb{C}}, \exists!(f, b) \in SU(2) \times \mathfrak{B}$  with  $g = f \cdot b$ , a property that we summarize by writing  $SU(2)^{\mathbb{C}} = SU(2) \cdot \mathfrak{B}$ . Moreover  $\mathfrak{B}$  is a solvable Borel subgroup. We can choose for example

$$\mathfrak{B} := \left\{ \begin{pmatrix} T_1^1 & 0 \\ T_2^1 & T_2^2 \end{pmatrix} / T_1^1, T_2^2 \in (0, \infty), T_1^2 \in \mathbb{C}, T_1^1 T_2^2 = 1 \right\}.$$

We denote by  $\mathfrak{b}$  the Lie algebra of  $\mathfrak{B}$ . The Iwasawa decomposition  $SU(2)^{\mathbb{C}} = SU(2) \cdot \mathfrak{B}$  immediately implies the vector space decomposition  $\mathfrak{su}(2)^{\mathbb{C}} = \mathfrak{su}(2) \oplus \mathfrak{b}$ , which leads to the definition of the two projection mappings  $(\cdot)_{\mathfrak{su}} : \mathfrak{su}(2)^{\mathbb{C}} \rightarrow \mathfrak{su}(2)$  and  $(\cdot)_{\mathfrak{b}} : \mathfrak{su}(2)^{\mathbb{C}} \rightarrow \mathfrak{b}$  such that

$$\forall \xi \in \mathfrak{su}(2)^{\mathbb{C}}, \quad \xi = (\xi)_{\mathfrak{su}} + (\xi)_{\mathfrak{b}} \quad \text{with } (\xi)_{\mathfrak{su}} \in \mathfrak{su}(2) \text{ and } (\xi)_{\mathfrak{b}} \in \mathfrak{b}.$$

Then we define the following twisted loop algebras

$$\Lambda \mathfrak{u}(3)_{\tau}^{\mathbb{C}} := \{S^1 \ni \lambda \mapsto \xi_{\lambda} \in \mathfrak{u}(3)^{\mathbb{C}} / \forall \lambda \in S^1, \tau(\xi_{\lambda}) = \xi_{i\lambda}\},$$

$$\Lambda_{\mathfrak{b}}^+ \mathfrak{u}(3)_{\tau}^{\mathbb{C}} := \{[\lambda \mapsto \xi_{\lambda}] \in \Lambda \mathfrak{u}(3)_{\tau}^{\mathbb{C}} / \forall k \in \mathbb{Z}, k \leq -1 \implies \widehat{\xi}_k = 0 \text{ and } \widehat{\xi}_0 \in \mathfrak{b}\},$$

where we use the Fourier decomposition  $\xi_{\lambda} = \sum_{k \in \mathbb{Z}} \widehat{\xi}_k \lambda^k$ .

The decomposition  $\mathfrak{su}(2)^{\mathbb{C}} = \mathfrak{su}(2) \oplus \mathfrak{b}$  can be extended to loop algebras, i.e. to the splitting  $\Lambda \mathfrak{u}(3)_{\tau}^{\mathbb{C}} = \Lambda \mathfrak{u}(3)_{\tau} \oplus \Lambda_{\mathfrak{b}}^+ \mathfrak{u}(3)_{\tau}^{\mathbb{C}}$ . This can be checked by using the Fourier decomposition of an element  $\xi_{\lambda} \in \Lambda \mathfrak{u}(3)_{\tau}^{\mathbb{C}}$ :

$$\sum_{k \in \mathbb{Z}} \widehat{\xi}_k \lambda^k = \left( \sum_{k < 0} \widehat{\xi}_k \lambda^k + (\widehat{\xi}_0)_{\mathfrak{su}} - \sum_{k > 0} (\widehat{\xi}_{-k})^{\dagger} \lambda^k \right) + \left( (\widehat{\xi}_0)_{\mathfrak{b}} + \sum_{k > 0} \left( \widehat{\xi}_k + (\widehat{\xi}_{-k})^{\dagger} \right) \lambda^k \right).$$

We will denote the corresponding projection mappings by  $(\cdot)_{\Lambda \mathfrak{su}} : \Lambda \mathfrak{u}(3)_{\tau}^{\mathbb{C}} \rightarrow \Lambda \mathfrak{u}(3)_{\tau}$  and  $(\cdot)_{\Lambda_{\mathfrak{b}}^+} : \Lambda \mathfrak{u}(3)_{\tau}^{\mathbb{C}} \rightarrow \Lambda_{\mathfrak{b}}^+ \mathfrak{u}(3)_{\tau}^{\mathbb{C}}$ .



We also introduce the following finite dimensional subspaces of  $\Lambda\mathfrak{u}(3)_\tau$ : for any  $p \in \mathbb{N}$  we let

$$\Lambda^{2+4p}\mathfrak{u}(3)_\tau := \left\{ [\lambda \mapsto \xi_\lambda] \in \Lambda\mathfrak{u}(3)_\tau / \xi_\lambda = \sum_{k=-2-4p}^{2+4p} \widehat{\xi}_k \lambda^k \right\}.$$

We can now define a pair of vector fields  $X_1, X_2 : \Lambda^{2+4p}\mathfrak{u}(3)_\tau \longrightarrow \Lambda\mathfrak{u}(3)_\tau$  by

$$X_1(\xi_\lambda) := [\xi_\lambda, (\lambda^{4p}\xi_\lambda)_{\Lambda_{\mathfrak{su}}}], \quad X_2(\xi_\lambda) := [\xi_\lambda, (i\lambda^{4p}\xi_\lambda)_{\Lambda_{\mathfrak{su}}}], \quad (17)$$

Note that  $\lambda^{4p}\xi_\lambda$  belongs to  $\Lambda\mathfrak{u}(3)_\tau^{\mathbb{C}}$ , so that  $(\lambda^{4p}\xi_\lambda)_{\Lambda_{\mathfrak{su}}}$  is well defined.

LEMMA 3.1 *Let  $p \in \mathbb{N}$  and  $X_1$  and  $X_2$  defined by (17). Then*

- $\forall \xi_\lambda \in \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ ,  $X_1(\xi_\lambda), X_2(\xi_\lambda) \in T_{\xi_\lambda} \Lambda^{2+4p}\mathfrak{u}(3)_\tau \simeq \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ , so that  $X_1$  and  $X_2$  are tangent vector fields to  $\Lambda^{2+4p}\mathfrak{u}(3)_\tau$ .
- $|\xi_\lambda|^2$  is preserved by  $X_1$  and  $X_2$ . Hence the flow of these vector fields are defined for all time
- The Lie bracket of  $X_1$  and  $X_2$  vanishes:

$$[X_1, X_2] = 0. \quad (18)$$

*Proof* — This result follows by a straightforward adaptation of the analogous results for harmonic maps in [BFPP] (see e.g. [G] and [H]). Note that the proof of (18) rests upon the crucial property that  $\Lambda\mathfrak{u}(3)_\tau$  and  $\Lambda_{\mathfrak{b}}^+\mathfrak{u}(3)_\tau^{\mathbb{C}}$  are Lie algebras (see e.g. [BP], [H]). ■

This result allows us to integrate simultaneously  $X_1$  and  $X_2$ . So for any  $\xi_\lambda^0 \in \Lambda^{2+4p}\mathfrak{u}(3)_\tau$  there exists a unique map  $\xi_\lambda : \mathbb{R}^2 \longrightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$  such that  $\xi_\lambda(z_0) = \xi_\lambda^0$  and

$$\frac{\partial \xi_\lambda}{\partial x}(x, y) = X_1(\xi_\lambda(x, y)) \quad \text{and} \quad \frac{\partial \xi_\lambda}{\partial y}(x, y) = X_2(\xi_\lambda(x, y)). \quad (19)$$

Denoting by  $z = x + iy \in \mathbb{C}$ , the system (19) can be rewritten

$$\begin{aligned} d\xi_\lambda &= \left[ \xi_\lambda, (\lambda^{4p}\xi_\lambda)_{\Lambda_{\mathfrak{su}}} dx + (i\lambda^{4p}\xi_\lambda)_{\Lambda_{\mathfrak{su}}} dy \right] \\ &= \left[ \xi_\lambda, (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}} \right]. \end{aligned}$$

Let us denote by  $A_\lambda := (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$ . Since the system (19) is overdetermined,  $A_\lambda$  should satisfy a compatibility condition. Indeed one can check that

$$dA_\lambda + A_\lambda \wedge A_\lambda = 0. \quad (20)$$

This relation can be proved by a method similar to the proof of (18) (see [H]). It implies that there exists a map  $F_\lambda : \mathbb{C} \longrightarrow \Lambda U(3)_\tau$  such that

$$dF_\lambda = F_\lambda \cdot A_\lambda. \quad (21)$$

Now observe that  $\lambda^{4p}\xi_\lambda = \sum_{k=-2}^{8p+2} \hat{\xi}_{k-4p}\lambda^k$  implies

$$A_\lambda = \lambda^{-2}\hat{\xi}_{-4p-2}dz + \lambda^{-1}\hat{\xi}_{-4p-1}dz + \left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{su}} - \lambda \left(\hat{\xi}_{-4p-1}\right)^\dagger d\bar{z} - \lambda^2 \left(\hat{\xi}_{-4p-2}\right)^\dagger d\bar{z}.$$

We recall that  $\hat{\xi}_{-4p-2} \in \mathfrak{u}(3)_2^\mathbb{C}$  and so has the form  $\text{diag}(ia, ia, ib)$ . Moreover we have the following

LEMMA 3.2 *If  $\xi_\lambda \longrightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$  and  $A_\lambda := (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$  are solutions of  $d\xi_\lambda = [\xi_\lambda, A_\lambda]$ , then  $\hat{\xi}_{-4p-2}$  is constant.*

*Proof* — The relevant term in the Fourier expansion of  $d\xi_\lambda = [\xi_\lambda, A_\lambda]$  gives

$$\begin{aligned} d\hat{\xi}_{-4p-2} &= \left[ \hat{\xi}_{-4p-2}, \left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{su}} \right] + \left[ \hat{\xi}_{-4p-1}, \hat{\xi}_{-4p-1} \right] dz + \left[ \hat{\xi}_{-4p}, \hat{\xi}_{-4p-2} \right] dz \\ &= \left[ \left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{b}}, \hat{\xi}_{-4p-2} \right]. \end{aligned}$$

But since the coefficients of  $\left(\hat{\xi}_{-4p}dz\right)_{\mathfrak{b}}$  are in  $\mathfrak{u}(3)_0^\mathbb{C}$  and  $\hat{\xi}_{-4p-2}$  takes values in  $\mathfrak{u}(3)_2$  we deduce that  $d\hat{\xi}_{-4p-2} = 0$ , because  $\mathfrak{u}(3)_0^\mathbb{C}$  and  $\mathfrak{u}(3)_2^\mathbb{C}$  commute.  $\blacksquare$

We deduce from this result that if we choose the initial value  $\xi_\lambda^0$  of  $\xi_\lambda$  to be such that  $\hat{\xi}_{-4p-2}^0 = \text{diag}(ia, ia, 0)$  then  $\hat{\xi}_{-4p-2}$  is equal to that value for all  $(x, y)$ . So in this case the map  $F_\lambda$  obtained by integrating  $A_\lambda$  satisfies all the requirements of Theorem 2.7. It implies that  $F_\lambda$  represents a (conjugate family) of Hamiltonian stationary conformal Lagrangian immersion(s). The category of such  $F_\lambda$ 's are exactly characterized by the following definition.

DEFINITION 3.3 *Let  $F_\lambda$  be a family of Hamiltonian stationary conformal Lagrangian immersions and let  $A_\lambda := F_\lambda^{-1} \cdot dF_\lambda$ . Then  $F_\lambda$  is called a family of finite type solutions if and only if there exists  $p \in \mathbb{N}$  and a map  $\xi_\lambda : \mathbb{C} \longrightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$  such that  $\hat{\xi}_{-4p-2} = \text{diag}(ia, ia, 0)$ , for some constant  $a \in \mathbb{C}$ , and*

$$d\xi_\lambda = [\xi_\lambda, A_\lambda] \tag{22}$$

$$(\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}} = A_\lambda. \tag{23}$$

We also need the following definition in which we introduce an a priori weaker notion of finite type solution.

DEFINITION 3.4 *Let  $F_\lambda$  be a family of Hamiltonian stationary conformal Lagrangian immersions and let  $A_\lambda := F_\lambda^{-1} \cdot dF_\lambda$ . Then  $F_\lambda$  is called a family of quasi-finite type solutions if and only if it satisfies the same requirements as in definition 3.3 excepted that condition (23) is replaced by*

$$\exists B \in \Omega^1 \otimes \mathfrak{u}(3)_0^\mathbb{C}, \quad (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}} = A_\lambda + B. \tag{24}$$

We shall see in Section 3.3 that both definitions are actually equivalent.

### 3.2 An alternative description of quasi-finite type solutions

We may as well characterize such finite type solutions in terms of  $E_\lambda = F_\lambda \cdot F^{-1}$ . For that purpose we need to introduce the untwisted loop Lie algebra

$$\Lambda^+ \mathfrak{u}(3)^{\mathbb{C}} := \{S^1 \ni \lambda \mapsto \xi_\lambda \in \mathfrak{u}(3)^{\mathbb{C}} / \xi_\lambda = \sum_{k=0}^{\infty} \widehat{\xi}_k \lambda^k\}$$

and observe that any  $\xi_\lambda = \sum_{k=-\infty}^{\infty} \widehat{\xi}_k \lambda^k \in \Lambda \mathfrak{u}(3)^{\mathbb{C}}$  can be split as

$$\xi_\lambda = \left( \sum_{k=-\infty}^{-1} \widehat{\xi}_k (\lambda^k - 1) - (\widehat{\xi}_k)^\dagger (\lambda^{-k} - 1) \right) + \left( \sum_{k=0}^{\infty} \widehat{\xi}_k \lambda^k + \sum_{k=1}^{\infty} \widehat{\xi}_{-k} + (\widehat{\xi}_{-k})^\dagger (\lambda^k - 1) \right),$$

and hence  $\Lambda \mathfrak{u}(3)^{\mathbb{C}} = \Omega \mathfrak{u}(3) \oplus \Lambda^+ \mathfrak{u}(3)^{\mathbb{C}}$ . This defines a pair of projection mappings  $(\cdot)_\Omega : \Lambda \mathfrak{u}(3)^{\mathbb{C}} \rightarrow \Omega \mathfrak{u}(3)^{\mathbb{C}}$  and  $(\cdot)_{\Lambda^+} : \Lambda \mathfrak{u}(3)^{\mathbb{C}} \rightarrow \Lambda^+ \mathfrak{u}(3)^{\mathbb{C}}$ .

Now consider a family  $F_\lambda$  of quasi-finite type, let  $A_\lambda := F_\lambda^{-1} \cdot dF_\lambda$ ,  $A := F^{-1} \cdot dF$  (where  $F = F_{\lambda=1}$ ) and  $\xi_\lambda$  be a solution of (22). We let

$$\eta_\lambda := F \cdot \xi_\lambda \cdot F^{-1} = \sum_{k=-2-4p}^{2+4p} F \cdot \widehat{\xi}_k \cdot F^{-1} \lambda^k.$$

Then (22) implies by a straightforward computation that

$$\begin{aligned} d\eta_\lambda &= F \cdot (d\xi_\lambda + [A, \xi_\lambda]) \cdot F^{-1} \\ &= F \cdot ([\xi_\lambda, A_\lambda] - [\xi_\lambda, A]) \cdot F^{-1} = [\eta_\lambda, \Gamma_\lambda], \end{aligned}$$

where  $\Gamma_\lambda := E_\lambda^{-1} \cdot dE_\lambda$ . Now setting  $R_\lambda := \sum_{k=-4p}^{2+4p} F \cdot \widehat{\xi}_k \cdot F^{-1} \lambda^k$ , we have

$$\begin{aligned} (\lambda^{4p} \eta_\lambda dz)_\Omega &= \left( \lambda^{-2} F \cdot \widehat{\xi}_{-2-4p} \cdot F^{-1} dz + \lambda^{-1} F \cdot \widehat{\xi}_{-1-4p} \cdot F^{-1} dz + \lambda^{4p} R_\lambda dz \right)_\Omega \\ &= (\lambda^{-2} - 1) F \cdot \widehat{\xi}_{-2-4p} \cdot F^{-1} dz + (\lambda^{-1} - 1) F \cdot \widehat{\xi}_{-1-4p} \cdot F^{-1} dz \\ &\quad - (\lambda - 1) \left( F \cdot \widehat{\xi}_{-1-4p} \cdot F^{-1} \right)^\dagger d\bar{z} - (\lambda^2 - 1) \left( F \cdot \widehat{\xi}_{-2-4p} \cdot F^{-1} \right)^\dagger d\bar{z}. \end{aligned}$$

But relation (24) implies in particular that  $\widehat{\xi}_{-2-4p} = A'_2(\partial/\partial z)$  and  $\widehat{\xi}_{-1-4p} = A_{-1}(\partial/\partial z)$ . So we deduce that

$$(\lambda^{4p} \eta_\lambda dz)_\Omega = F \cdot (A_\lambda - A) \cdot F^{-1} = \Gamma_\lambda.$$

Hence  $E_\lambda$  can be constructed by solving a system analogous to (22), (23), i.e.

$$d\eta_\lambda + [\Gamma_\lambda, \eta_\lambda] = 0 \quad \text{and} \quad \Gamma_\lambda = (\lambda^{4p} \eta_\lambda dz)_\Omega. \quad (25)$$

Conversely a similar computation shows that a solution of (25) gives rise to a quasi-finite type family of solutions by an inverse transformation, but we shall

prove more in the next section.

Note that system (25) can also be interpreted as a pair of commuting ordinary differential equations in the finite dimensional space  $\Lambda^{2+4p}\mathfrak{u}(3) := \{S^1 \ni \lambda \mapsto \eta_\lambda \in \mathfrak{u}(3)/\eta_\lambda = \sum_{k=-2-4p}^{2+4p} \widehat{\eta}_k \lambda^k\}$ . It is the analogue of the definition of a finite type solution according to [BFPP].

### 3.3 Quasi-finite type solutions are actually finite type

We show here the following

**THEOREM 3.5** *For any family  $F_\lambda$  of Hamiltonian stationary Lagrangian conformal immersions of quasi-finite type, i.e. such that there exists  $\xi_\lambda : \Omega \rightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$  which satisfies (22) and (24), there exists a gauge transformation  $F_\lambda \mapsto F_\lambda^G := F_\lambda \cdot G$ , where  $G \in C^\infty(\Omega, U(3)_0)$ , such that  $F_\lambda^G$  is of finite type. More precisely, denoting by  $A_\lambda^G := G^{-1} \cdot A_\lambda \cdot G + G^{-1} \cdot dG$  and  $\xi_\lambda^G := G^{-1} \cdot \xi_\lambda \cdot G$ , then  $d\xi_\lambda^G + [A_\lambda^G, \xi_\lambda^G] = G^{-1} \cdot (d\xi_\lambda + [A_\lambda, \xi_\lambda]) \cdot G = 0$  and  $(\lambda^{4p} \xi_\lambda^G dz)_{\Lambda_{su}} = A_\lambda^G$ .*

*Proof* — We set  $E_\lambda := F_\lambda \cdot F^{-1}$ ,  $\Gamma_\lambda := E_\lambda^{-1} \cdot dE_\lambda$  and  $\eta_\lambda := F \cdot \xi_\lambda \cdot F^{-1}$  and will use the results of the previous section.

A constant in  $\Lambda^{2+4p}\mathfrak{u}(3)_\tau$  associated to the quasi-finite type family — First (25), which is a reformulation of (22), implies

$$d(E_\lambda \cdot \eta_\lambda \cdot E_\lambda^{-1}) = E_\lambda \cdot (d\eta_\lambda + [\Gamma_\lambda, \eta_\lambda]) \cdot E_\lambda^{-1} = 0.$$

Hence

$$\eta_\lambda^0 := E_\lambda \cdot \eta_\lambda \cdot E_\lambda^{-1}$$

is a constant in  $\Lambda\mathfrak{u}(3)$ . Moreover

$$\eta_\lambda^0 = E_\lambda(z_0) \cdot \eta_\lambda(z_0) \cdot E_\lambda^{-1}(z_0) = \eta_\lambda(z_0) = F(z_0) \cdot \xi_\lambda(z_0) \cdot F^{-1}(z_0) = \xi_\lambda(z_0),$$

which proves that  $\eta_\lambda^0 \in \Lambda^{2+4p}\mathfrak{u}(3)_\tau$ .

An auxiliary map into  $\Lambda^+U(3)^{\mathbb{C}}$  — We let

$$\Theta_\lambda := (\lambda^{4p} \eta_\lambda dz)_{\Lambda^+} = \lambda^{4p} \eta_\lambda dz - (\lambda^{4p} \eta_\lambda dz)_\Omega.$$

Then using (25) we have  $\Theta_\lambda = \lambda^{4p} \eta_\lambda dz - \Gamma_\lambda$  and so

$$d\Gamma_\lambda + \Gamma_\lambda \wedge \Gamma_\lambda + d\Theta_\lambda - \Theta_\lambda \wedge \Theta_\lambda = -\lambda^{4p} (d\eta_\lambda + [\Gamma_\lambda, \eta_\lambda]) \left( \frac{\partial}{\partial \bar{z}} \right) dz \wedge d\bar{z} = 0.$$

But since  $d\Gamma_\lambda + \Gamma_\lambda \wedge \Gamma_\lambda = 0$  this implies that  $d\Theta_\lambda - \Theta_\lambda \wedge \Theta_\lambda = 0$ . Hence  $\exists! V_\lambda : \Omega \rightarrow \Lambda^+U(3)^{\mathbb{C}}$  such that

$$dV_\lambda = \Theta_\lambda \cdot V_\lambda \quad \text{and} \quad V_\lambda(z_0) = 1.$$

Now, starting from  $\lambda^{4p}\eta_\lambda dz = \Gamma_\lambda + \Theta_\lambda$ , we deduce that

$$\begin{aligned}\lambda^{4p}\eta_\lambda^0 dz &= E_\lambda \cdot \Gamma_\lambda \cdot E_\lambda^{-1} + E_\lambda \cdot \Theta_\lambda \cdot E_\lambda^{-1} \\ &= dE_\lambda \cdot E_\lambda^{-1} + E_\lambda \cdot dV_\lambda \cdot V_\lambda^{-1} \cdot E_\lambda^{-1} \\ &= d(E_\lambda \cdot V_\lambda) (E_\lambda \cdot V_\lambda)^{-1},\end{aligned}$$

which can be integrated into the relation

$$E_\lambda \cdot V_\lambda = e^{\lambda^{4p}(z-z_0)\eta_\lambda^0}.$$

An Iwasawa decomposition of  $e^{\lambda^{4p}(z-z_0)\eta_\lambda^0}$  — The latter implies

$$e^{\lambda^{4p}(z-z_0)\eta_\lambda^0} = F_\lambda \cdot F^{-1} \cdot V_\lambda.$$

From this relation and the fact that  $\eta_\lambda^0$  and  $F_\lambda$  are twisted we deduce that  $W_\lambda := F^{-1} \cdot V_\lambda$  is twisted. It is also a map with values in  $\Lambda^+U(3)_\tau^{\mathbb{C}}$ . However it may not be in  $\Lambda_{\mathfrak{B}}^+U(3)_\tau^{\mathbb{C}}$  in general, because in the development

$$F^{-1} \cdot V_\lambda = \widehat{W}_0 + \sum_{k=1}^{\infty} \widehat{W}_k \lambda^k,$$

we are not sure that  $\widehat{W}_0$  takes values in  $\mathfrak{B}$ . But it takes values in  $U(3)_0^{\mathbb{C}}$ , so by using the Iwasawa decomposition  $U(3)_0^{\mathbb{C}} = U(3)_0 \cdot \mathfrak{B}$  we know that  $\exists! G \in U(3)_0$ ,  $\exists! \widehat{B}_0 \in \mathfrak{B}$ ,  $\widehat{W}_0 = G \cdot \widehat{B}_0$ . Hence

$$G^{-1} \cdot F^{-1} \cdot V_\lambda = \widehat{B}_0 + \sum_{k=1}^{\infty} G^{-1} \cdot \widehat{W}_k \lambda^k$$

takes values in  $\Lambda_{\mathfrak{B}}^+U(3)_\tau^{\mathbb{C}}$ . So the splitting

$$e^{\lambda^{4p}(z-z_0)\eta_\lambda^0} = (F_\lambda \cdot G) (G^{-1} \cdot F^{-1} \cdot V_\lambda)$$

exactly reproduces the Iwasawa decomposition  $\Lambda U(3)_\tau^{\mathbb{C}} = \Lambda U(3)_\tau \cdot \Lambda_{\mathfrak{B}}^+U(3)_\tau^{\mathbb{C}}$  proved in [DPW].

*Conclusion* — Let us denote by  $F_\lambda^G := F_\lambda \cdot G$ ,  $A_\lambda^G := (F_\lambda^G)^{-1} \cdot dF_\lambda^G = G^{-1} \cdot A_\lambda \cdot G + G^{-1} \cdot dG$  and  $B_\lambda^G := G^{-1} \cdot F^{-1} \cdot V_\lambda$  and let us introduce

$$\xi_\lambda^G := (F_\lambda^G)^{-1} \cdot \eta_\lambda^0 \cdot F_\lambda^G.$$

(These definitions imply immediately  $d\xi_\lambda^G + [A_\lambda^G, \xi_\lambda^G] = 0$ .) The first main observation is that the relation  $\eta_\lambda^0 = E_\lambda \cdot \eta_\lambda \cdot E_\lambda^{-1} = F_\lambda \cdot F^{-1} \cdot \eta_\lambda \cdot F \cdot F_\lambda^{-1} = F_\lambda \cdot \xi_\lambda \cdot F_\lambda^{-1}$  implies

$$\xi_\lambda^G = G^{-1} \cdot F_\lambda^{-1} \cdot \eta_\lambda^0 \cdot F_\lambda \cdot G = G^{-1} \cdot \xi_\lambda \cdot G. \quad (26)$$

Second, from the relation

$$\begin{aligned} \lambda^{4p} \eta_\lambda^0 dz &= d\left(e^{\lambda^{4p}(z-z_0)\eta_\lambda^0}\right) \cdot e^{-\lambda^{4p}(z-z_0)\eta_\lambda^0} \\ &= d\left(F_\lambda^G \cdot B_\lambda^G\right) \cdot \left(F_\lambda^G \cdot B_\lambda^G\right)^{-1} \\ &= dF_\lambda^G \cdot \left(F_\lambda^G\right)^{-1} + F_\lambda^G \cdot dB_\lambda^G \cdot \left(B_\lambda^G\right)^{-1} \cdot \left(F_\lambda^G\right)^{-1}, \end{aligned}$$

we deduce that

$$\lambda^{4p} \xi_\lambda^G dz = \left(F_\lambda^G\right)^{-1} \cdot \left(\lambda^{4p} \eta_\lambda^0 dz\right) \cdot F_\lambda^G = \left(F_\lambda^G\right)^{-1} \cdot dF_\lambda^G + dB_\lambda^G \cdot \left(B_\lambda^G\right)^{-1}.$$

Hence, since  $B_\lambda^G$  takes values in  $\Lambda_{\mathfrak{g}}^+ U(3)_\tau^{\mathbb{C}}$ ,

$$A_\lambda^G = \left(F_\lambda^G\right)^{-1} \cdot dF_\lambda^G = \left(\lambda^{4p} \xi_\lambda^G dz\right)_{\Lambda_{\mathfrak{su}}}. \quad (27)$$

And relations (26) and (27) lead to the conclusion.  $\blacksquare$

## 4 All Hamiltonian stationary Lagrangian tori are of finite type

The subject of this section is to prove the following:

**THEOREM 4.1** *Let  $u : \mathbb{C} \rightarrow \mathbb{C}P^2$  be a doubly periodic Hamiltonian stationary Lagrangian conformal immersion. Then  $u$  is of finite type.*

We will actually prove a slightly more general result, since we can replace the doubly periodicity assumption by the hypothesis that the Maurer–Cartan form of any *Legendrian framing* of  $u$  is doubly periodic. This result of course implies immediately that Hamiltonian stationary Lagrangian tori are of finite type, since they always can be covered conformally by the plane.

Note also that the study of Hamiltonian stationary Lagrangian tori splits into exactly two subcases: the *minimal* Lagrangian tori and the *non minimal* Hamiltonian stationary Lagrangian ones. The first case occurs when the Lagrangian angle function along any Legendrian lift is locally constant, the second one when this function is harmonic and non constant. In the case of minimal Lagrangian surfaces, Theorem 4.1 is a special case of the result in [BFPP], since in this case  $u$  is a harmonic map into  $\mathbb{C}P^2$ , as discussed in [McI1], [McI2], [McI3] and [J]. The non minimal case however is not covered by the theory in [BFPP] and is the subject of this section.

Let  $F : \mathbb{C} \rightarrow U(3)$  be a Legendrian framing of  $u$ ,  $A := F^{-1} \cdot dF$  its Maurer–Cartan form and  $A_\lambda$  the family of deformations of  $A$  as defined by (11). The first basic observation is that  $A_2 \left(\frac{\partial}{\partial z}\right)$  is holomorphic and doubly periodic on  $\mathbb{C}$ , hence constant. Thus two cases occur: either  $A_2 \left(\frac{\partial}{\partial z}\right) = 0$ , which corresponds

to the minimal case that we exclude here, or  $A_2 \left( \frac{\partial}{\partial z} \right)$  is a constant different from 0, the case that we consider next.

In order to show Theorem 4.1 we need to prove that there exists some  $p \in \mathbb{N}$  and a map  $\xi_\lambda : \mathbb{C} \rightarrow \Lambda^{2+4p}\mathfrak{u}(3)_\tau$  such that  $d\xi_\lambda = [\xi_\lambda, A_\lambda]$  and  $A_\lambda = (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$ . But thanks to Theorem 3.5 it will enough to prove that  $A_\lambda - (\lambda^{4p}\xi_\lambda dz)_{\Lambda_{\mathfrak{su}}}$  is a 1-form with coefficients in  $U(3)_0$ . Our proof here follows a strategy inspired from [BFPP]: a first step consists in building a formal series  $Y_\lambda = \sum_{k=-2}^{\infty} \widehat{Y}_k \lambda^k$  which is a solution of  $dY_\lambda = [Y_\lambda, A_\lambda]$ . Such a series is called a *formal Killing field*. We will also require  $Y_\lambda$  to be *quasi-adapted*, i.e. is such that

$$(Y_\lambda dz)_{\Lambda_{\mathfrak{su}}} = A_\lambda + B, \quad \text{where the coefficients of } B \text{ are in } \mathfrak{u}(3)_0. \quad (28)$$

This is achieved through a recursion procedure.

In a second step we will show that the coefficients of  $Y_\lambda$  form a countable collection of doubly periodic functions satisfying an elliptic PDE and hence, by using a compactness argument, we conclude that they are contained in a finite dimensional space. Then we deduce the existence of  $\xi_\lambda$  using linear algebra.

#### 4.1 Construction of an adapted formal Killing field

We first introduce some notations. We denote by

$$\pi_0^\perp := \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

and  $a := \frac{1}{2} \frac{\partial \beta}{\partial z}$  (here a constant different from 0). Then  $A'_2 = ia\pi_0^\perp dz$ . We will also set  $X := A_{-1} \left( \frac{\partial}{\partial z} \right)$  and  $C := A_0 \left( \frac{\partial}{\partial z} \right)$ , so that

$$A_\lambda = \lambda^{-2} ia\pi_0^\perp dz + \lambda^{-1} X dz + C dz - C^\dagger d\bar{z} - \lambda X^\dagger d\bar{z} + \lambda^2 i\bar{a}\pi_0^\perp d\bar{z}.$$

We also introduce the linear map  $\text{ad}\pi_0^\perp : \mathfrak{u}(3)^\mathbb{C} \rightarrow \mathfrak{u}(3)^\mathbb{C}$ , acting by  $\xi \mapsto [\pi_0^\perp, \xi]$ <sup>7</sup>. We observe that  $\pi_0^\perp$  commutes with the elements in  $\mathfrak{u}(3)_0^\mathbb{C}$  and  $\mathfrak{u}(3)_2^\mathbb{C}$ . Moreover  $\forall a, b \in \mathbb{C}$

$$\left[ \pi_0^\perp, \begin{pmatrix} & a \\ \mp ib & \pm ia \\ & b \end{pmatrix} \right] = \begin{pmatrix} & a \\ \pm ib & \mp ia \\ & b \end{pmatrix},$$

i.e.  $\text{ad}\pi_0^\perp$  maps  $\mathfrak{u}(3)_{\mp 1}^\mathbb{C}$  to  $\mathfrak{u}(3)_{\pm 1}^\mathbb{C}$ . From that we deduce that  $V := \text{Ker ad}\pi_0^\perp$  coincides with  $\mathfrak{u}(3)_0^\mathbb{C} \oplus \mathfrak{u}(3)_2^\mathbb{C}$  and  $V^\perp := \text{Im ad}\pi_0^\perp$  coincides with  $\mathfrak{u}(3)_{-1}^\mathbb{C} \oplus \mathfrak{u}(3)_1^\mathbb{C}$  (note that  $V^\perp$  is actually the orthogonal subspace to  $V$  in  $\mathfrak{u}(3)^\mathbb{C}$ ). In our construction we will use extensively the following properties:

<sup>7</sup>actually the map  $\xi \mapsto i[\pi_0^\perp, \xi]$  corresponds to the complex structure on the Legendrian distribution.

- the map  $\text{ad}\pi_0^\perp|_{V^\perp \rightarrow V^\perp}$  is a vector space isomorphism (it is actually a involution on  $V^\perp$ ),
- the inclusions  $VV \subset V$ ,  $VV^\perp \subset V^\perp$ ,  $V^\perp V \subset V^\perp$  and  $V^\perp V^\perp \subset V$ . These properties can be checked by a direct computation using the fact that matrices in  $V$  are diagonal by blocks and the matrices in  $V^\perp$  are off-diagonal by blocks. (The three first properties can also be deduced from the definition of  $V$  and  $V^\perp$  and the fact that  $\text{ad}$  is a derivation).

We look for a formal Killing field  $Y_\lambda$ , i.e. a solution of the equation

$$dY_\lambda = [Y_\lambda, A_\lambda], \quad (29)$$

of the form  $Y_\lambda = (1 + W_\lambda)^{-1} \lambda^{-2} i a \pi_0^\perp (1 + W_\lambda)$ , where  $W_\lambda = \sum_{k=0}^{\infty} \widehat{W}_k \lambda^k$  as in [BFPP]. In order to have a well-posed problem (and in particular to guarantee the existence of an unique solution of this type) we assume that  $W_\lambda$  takes values in  $V^\perp$ . We start by evaluating (29) along  $\partial/\partial z$ . It gives, after conjugation by  $1 + W_\lambda$ :

$$\begin{aligned} & \lambda^{-2} \frac{\partial a}{\partial z} \pi_0^\perp + \lambda^{-2} a [\pi_0^\perp, \\ & \frac{\partial W_\lambda}{\partial z} (1 + W_\lambda)^{-1} - (1 + W_\lambda) (\lambda^{-2} i a \pi_0^\perp + \lambda^{-1} X + C) (1 + W_\lambda)^{-1}] = 0 \end{aligned} \quad (30)$$

Here the fact that  $a$  is a constant leads to an immediate simplification, namely that the bracket in the left hand side of (30) is 0. Thus equation (30) implies that  $\frac{\partial W_\lambda}{\partial z} (1 + W_\lambda)^{-1} - (1 + W_\lambda) (\lambda^{-2} i a \pi_0^\perp + \lambda^{-1} X + C) (1 + W_\lambda)^{-1}$  lies in  $V$ , hence there exists a map  $\varphi_\lambda : \mathbb{C} \rightarrow V$  such that

$$\frac{\partial W_\lambda}{\partial z} (1 + W_\lambda)^{-1} - (1 + W_\lambda) (\lambda^{-2} i a \pi_0^\perp + \lambda^{-1} X + C) (1 + W_\lambda)^{-1} = \varphi_\lambda$$

or

$$\frac{\partial W_\lambda}{\partial z} - (1 + W_\lambda) (\lambda^{-2} i a \pi_0^\perp + \lambda^{-1} X + C) = \varphi_\lambda (1 + W_\lambda),$$

which can be projected according to the splitting  $V \oplus V^\perp$  as

$$\begin{cases} \lambda^{-2} i a \pi_0^\perp + \lambda^{-1} W_\lambda X + C = -\varphi_\lambda \in V \\ \frac{\partial W_\lambda}{\partial z} - \lambda^{-2} i a W_\lambda \pi_0^\perp - \lambda^{-1} X - W_\lambda C = \varphi_\lambda W_\lambda \in V^\perp. \end{cases}$$

Substituting  $\varphi_\lambda$ ,

$$\frac{\partial W_\lambda}{\partial z} - \lambda^{-2} i a W_\lambda \pi_0^\perp - \lambda^{-1} X - W_\lambda C + \lambda^{-2} i a \pi_0^\perp W_\lambda + \lambda^{-1} W_\lambda X W_\lambda + C W_\lambda = 0$$

or

$$[i a \pi_0^\perp, W_\lambda] + \lambda (W_\lambda X W_\lambda - X) + \lambda^2 [C, W_\lambda] + \lambda^2 \frac{\partial W_\lambda}{\partial z} = 0$$



or

$$ia \sum_{n \geq 0} [\pi_0^\perp, \widehat{W}_n] \lambda^n + \sum_{n \geq 1} \left( \sum_{k=0}^{n-1} \widehat{W}_k X \widehat{W}_{n-1-k} \right) \lambda^n - \lambda X + \sum_{n \geq 2} \left( [C, \widehat{W}_{n-2}] + \frac{\partial \widehat{W}_{n-2}}{\partial z} \right) \lambda^n = 0.$$

Hence

$$\begin{cases} n = 0, & ia[\pi_0^\perp, \widehat{W}_0] = 0 \\ n = 1, & ia[\pi_0^\perp, \widehat{W}_1] + \widehat{W}_0 X \widehat{W}_0 - X = 0 \\ n \geq 2, & ia[\pi_0^\perp, \widehat{W}_n] + \sum_{k=0}^{n-1} \widehat{W}_k X \widehat{W}_{n-1-k} + [C, \widehat{W}_{n-2}] + \frac{\partial \widehat{W}_{n-2}}{\partial z} = 0 \end{cases}$$

and thus

$$\begin{cases} \widehat{W}_0 = 0 \\ \widehat{W}_1 = -ia^{-1}[\pi_0^\perp, X] \\ \widehat{W}_n = ia^{-1} \left[ \pi_0^\perp, \sum_{k=0}^{n-1} \widehat{W}_k X \widehat{W}_{n-1-k} + [C, \widehat{W}_{n-2}] + \frac{\partial \widehat{W}_{n-2}}{\partial z} \right] \end{cases}$$

We observe that the formal Killing field is *quasi-adapted* in the sense that the two first coefficients are the right ones:

$$\begin{aligned} Y_\lambda &= ia\lambda^{-2}(1+W_\lambda)^{-1}\pi_0^\perp(1+W_\lambda) = ia\lambda^{-2} \left( \pi_0^\perp - \lambda[\widehat{W}_1, \pi_0^\perp] + \mathcal{O}(\lambda^2) \right) \\ &= \lambda^{-2}ia\pi_0^\perp + \lambda^{-1}X + \mathcal{O}(1). \end{aligned}$$

Another pleasant property is that this formal field is automatically twisted (as in the case of  $\mathbb{C}^2$ , see [HR1]). Indeed using the fact that  $\tau$  is an automorphism for the product of matrices as well as for the Lie bracket (and so  $[\mathfrak{u}(3)_a^{\mathbb{C}}, \mathfrak{u}(3)_b^{\mathbb{C}}] \subset \mathfrak{u}(3)_{a+b}^{\mathbb{C}}$  and  $\mathfrak{u}(3)_a^{\mathbb{C}}\mathfrak{u}(3)_b^{\mathbb{C}} \subset \mathfrak{u}(3)_{a+b}^{\mathbb{C}}$ ), we obtain that

$$\tau(Y_\lambda) = \tau(1+W_\lambda)^{-1}(-\lambda^2)ia\pi_0^\perp\tau(1+W_\lambda).$$

Thus it is enough to show that  $1+W_\lambda$  is twisted, i.e.  $\tau(W_\lambda) = W_{i\lambda}$ . In terms of the Fourier decomposition of  $W_\lambda$  this is equivalent to proving that  $\widehat{W}_n$  belongs to  $\mathfrak{u}(3)_n^{\mathbb{C}}$ . Let us prove it by recursion. We already know that  $\widehat{W}_1 = -ia^{-1}[\pi_0^\perp, X]$  is in  $\mathfrak{u}(3)_1^{\mathbb{C}}$ . Assume that the result is true up to  $n-1$ , then

$$\sum_{p=0}^{n-1} \widehat{W}_p X \widehat{W}_{n-1-p} + [C, \widehat{W}_{n-2}] + \frac{\partial \widehat{W}_{n-2}}{\partial z}$$

belongs to  $\mathfrak{u}(3)_{n-2}^{\mathbb{C}}$ . And since  $\pi_0^\perp \in \mathfrak{u}(3)_2^{\mathbb{C}}$ ,  $\widehat{W}_n$  is in  $\mathfrak{g}_n^{\mathbb{C}}$ .

We now prove that (29) is also true along  $\partial/\partial\bar{z}$ . We follow here the same kind of arguments as in [BFPP] slightly simplified<sup>8</sup>. We want to show that

$$\frac{\partial Y_\lambda}{\partial\bar{z}} + \left[ A_\lambda \left( \frac{\partial}{\partial\bar{z}} \right), Y_\lambda \right] = 0$$

and for that purpose we rather consider the conjugate of the left hand side  $\zeta_\lambda = (1 + W_\lambda) \left( \frac{\partial Y_\lambda}{\partial\bar{z}} + [A_\lambda \left( \frac{\partial}{\partial\bar{z}} \right), Y_\lambda] \right) (1 + W_\lambda)^{-1}$ . We then prove two facts

- $\zeta_\lambda$  takes its values in  $V^\perp$ : this follows from the identity

$$\zeta_\lambda = \left[ \lambda^{-2} i a \pi_0^\perp, \frac{\partial W_\lambda}{\partial\bar{z}} (1 + W_\lambda)^{-1} - (1 + W_\lambda) A_\lambda \left( \frac{\partial}{\partial\bar{z}} \right) (1 + W_\lambda)^{-1} \right].$$

Note that since  $\zeta_\lambda$  is twisted the fact that  $\zeta_\lambda \in V^\perp$  implies also that  $\zeta_\lambda$  is an odd function of  $\lambda$  and so that

$$\zeta_\lambda = \sum_{k=0}^{\infty} \widehat{\zeta}_{2k-1} \lambda^{2k-1}. \quad (31)$$

- the relation

$$\frac{\partial \zeta_\lambda}{\partial z} = [\varphi_\lambda, \zeta_\lambda]. \quad (32)$$

Indeed  $d + \text{ad}A_\lambda$  is a flat connection and in particular  $\frac{\partial}{\partial z} + \text{ad}A_\lambda \left( \frac{\partial}{\partial z} \right)$  commutes with  $\frac{\partial}{\partial\bar{z}} + A_\lambda \left( \frac{\partial}{\partial\bar{z}} \right)$ . Hence

$$\left( \frac{\partial}{\partial z} + \text{ad}A_\lambda \left( \frac{\partial}{\partial z} \right) \right) \left( \frac{\partial}{\partial\bar{z}} + \text{ad}A_\lambda \left( \frac{\partial}{\partial\bar{z}} \right) \right) Y_\lambda = 0$$

i.e.

$$\left( \frac{\partial}{\partial z} + \text{ad}A_\lambda \left( \frac{\partial}{\partial z} \right) \right) \left( (1 + W_\lambda)^{-1} \zeta_\lambda (1 + W_\lambda) \right) = 0$$

Thus (32) follows from a computation which uses  $\varphi_\lambda = \frac{\partial W_\lambda}{\partial z} (1 + W_\lambda)^{-1} - (1 + W_\lambda) A_\lambda \left( \frac{\partial}{\partial z} \right) (1 + W_\lambda)^{-1}$ .

Now assume by contradiction that  $\zeta_\lambda \neq 0$ : in view of (31) there exists an integer  $k \in \mathbb{N}$  such that  $\widehat{\zeta}_{2k-1} \neq 0$  and  $\widehat{\zeta}_{2k-3} = 0$ . By substituting the Fourier decompositions in (32) and observing that the Fourier series expansion of  $\varphi_\lambda$  starts by  $\lambda^{-2} i a \pi_0^\perp$ , we deduce that  $0 = \partial \widehat{\zeta}_{2k-3} / \partial z = [i a \pi_0^\perp, \widehat{\zeta}_{2k-1}]$ ; but  $\text{ad}\pi_0^\perp$  is invertible on  $V^\perp$  and hence  $\widehat{\zeta}_{2k-1} = 0$ . So we get a contradiction.

<sup>8</sup>essentially the simplifications occur because the semi-simple term  $B$  of [BFPP] is here  $i a \pi_0^\perp$  which is constant for  $d$ , so we do not need to introduce a flat connection.

## 4.2 Polynomial Killing fields

We now deduce the existence of a non-trivial polynomial Killing field.

A first easy consequence of the results of the previous section is that, for all  $n \in \mathbb{N}$  and for all polynomial of the form  $P(\lambda) = a_n \lambda^{-4n} + a_{n-1} \lambda^{-4(n-1)} + \dots + a_0$ , where  $a_0, a_1, \dots, a_n \in \mathbb{C}$  and  $a_n \neq 0$ , then  $Z_\lambda := P(\lambda)Y_\lambda$  is again formal Killing field. Moreover it is quasi-adapted (modulo the multiplicative factor  $a_n \lambda^{-4n}$ ), i.e. the lower degree terms are  $a_n \lambda^{-4n} (ia \lambda^{-2} \pi_0^\perp + \lambda^{-1} X + O(\lambda^0))$ . Let us consider

$$Z_{\leq} := \sum_{k=-2-4n}^0 \widehat{Z}_k \lambda^k, \quad \text{and} \quad Z_{>} := \sum_{k=1}^{\infty} \widehat{Z}_k \lambda^k,$$

so that  $Z_\lambda = Z_{\leq} + Z_{>}$ . We study

$$R_\lambda := dZ_{\leq} + [A_\lambda, Z_{\leq}]. \quad (33)$$

We first remark that  $R_\lambda$  is necessarily of the form  $R_\lambda = \sum_{k=-4-4n}^2 \widehat{R}_k \lambda^k$ . But because of  $dZ_\lambda + [A_\lambda, Z_\lambda] = 0$ , we also have

$$R_\lambda = -dZ_{>} - [A_\lambda, Z_{>}], \quad (34)$$

which implies  $R_\lambda = \sum_{k=-1}^{\infty} \widehat{R}_k \lambda^k$ . Hence finally

$$R_\lambda = \lambda^{-1} \widehat{R}_{-1} + \widehat{R}_0 + \lambda^1 \widehat{R}_1 + \lambda^2 \widehat{R}_2.$$

Each term  $\widehat{R}_k$  can be evaluated through two different ways: by using (33) or (34). From (33) we obtain

$$\begin{cases} \widehat{R}_{-1}(\partial_z) = \partial_z \widehat{Z}_{-1} + [A_{-1}(\partial_z), \widehat{Z}_0] + [A_0(\partial_z), \widehat{Z}_{-1}] \\ \widehat{R}_0(\partial_z) = \partial_z \widehat{Z}_0 + [A_0(\partial_z), \widehat{Z}_0] \\ \widehat{R}_1(\partial_z) = 0 \\ \widehat{R}_2(\partial_z) = 0 \end{cases} \quad (35)$$

and

$$\begin{cases} \widehat{R}_{-1}(\partial_{\bar{z}}) = \partial_{\bar{z}} \widehat{Z}_{-1} + [A_0(\partial_{\bar{z}}), \widehat{Z}_{-1}] + [A_1(\partial_{\bar{z}}), \widehat{Z}_{-2}] + [A_2''(\partial_{\bar{z}}), \widehat{Z}_{-3}] \\ \widehat{R}_0(\partial_{\bar{z}}) = \partial_{\bar{z}} \widehat{Z}_0 + [A_0(\partial_{\bar{z}}), \widehat{Z}_0] + [A_1(\partial_{\bar{z}}), \widehat{Z}_{-1}] + [A_2''(\partial_{\bar{z}}), \widehat{Z}_{-2}] \\ \widehat{R}_1(\partial_{\bar{z}}) = [A_1(\partial_{\bar{z}}), \widehat{Z}_0] + [A_2''(\partial_{\bar{z}}), \widehat{Z}_{-1}] \\ \widehat{R}_2(\partial_{\bar{z}}) = [A_2''(\partial_{\bar{z}}), \widehat{Z}_0]. \end{cases} \quad (36)$$

From (34) we get

$$\begin{cases} \widehat{R}_{-1}(\partial_z) = -[A_2'(\partial_z), \widehat{Z}_1] \\ \widehat{R}_0(\partial_z) = -[A_2'(\partial_z), \widehat{Z}_2] - [A_{-1}(\partial_z), \widehat{Z}_1] \\ \widehat{R}_1(\partial_z) = -\partial_z \widehat{Z}_1 - [A_2'(\partial_z), \widehat{Z}_3] - [A_{-1}(\partial_z), \widehat{Z}_2] - [A_0(\partial_z), \widehat{Z}_1] \\ \widehat{R}_2(\partial_z) = -\partial_z \widehat{Z}_2 - [A_2'(\partial_z), \widehat{Z}_4] - [A_{-1}(\partial_z), \widehat{Z}_3] - [A_0(\partial_z), \widehat{Z}_2] \end{cases} \quad (37)$$

and

$$\begin{cases} \widehat{R}_{-1}(\partial_{\bar{z}}) = 0 \\ \widehat{R}_0(\partial_{\bar{z}}) = 0 \\ \widehat{R}_1(\partial_{\bar{z}}) = -\partial_{\bar{z}}\widehat{Z}_1 - [A_0(\partial_{\bar{z}}), \widehat{Z}_1] \\ \widehat{R}_2(\partial_{\bar{z}}) = -\partial_{\bar{z}}\widehat{Z}_2 - [A_0(\partial_{\bar{z}}), \widehat{Z}_2] - [A_1(\partial_{\bar{z}}), \widehat{Z}_1]. \end{cases} \quad (38)$$

Thus in order to obtain an expression of  $R_\lambda$  which does depend only on  $\widehat{Z}_{-1}$  and  $\widehat{Z}_0$ , we exploit (35) and the two last equations in (36). But instead of using the two first equations of (36) we take the two first ones of (38). This gives us

$$R_\lambda(\partial_z) = \lambda^{-1} \left( \partial_z \widehat{Z}_{-1} + [A_{-1}(\partial_z), \widehat{Z}_0] + [A_0(\partial_z), \widehat{Z}_{-1}] \right) + \left( \partial_z \widehat{Z}_0 + [A_0(\partial_z), \widehat{Z}_0] \right), \quad (39)$$

$$R_\lambda(\partial_{\bar{z}}) = \lambda \left( [A_1(\partial_{\bar{z}}), \widehat{Z}_0] + [A_2''(\partial_{\bar{z}}), \widehat{Z}_{-1}] \right) + \lambda^2 [A_2''(\partial_{\bar{z}}), \widehat{Z}_0]. \quad (40)$$

These relations will imply that  $\widehat{Z}_{-1}$  and  $\widehat{Z}_0$  satisfy a second order elliptic equation. In order to prove that we need to establish another relation between  $R_\lambda(\partial_z)$  and  $R_\lambda(\partial_{\bar{z}})$ . For that purpose recall that  $dA_\lambda + A_\lambda \wedge A_\lambda = 0$ , which means that the connection  $d + \text{ad}A_\lambda$  has a vanishing curvature. In particular

$$0 = (d + \text{ad}A_\lambda) \circ (d + \text{ad}A_\lambda) Z_\leq = dR_\lambda + [A_\lambda \wedge R_\lambda].$$

This implies

$$\frac{\partial R_\lambda(\partial_z)}{\partial \bar{z}} - \frac{\partial R_\lambda(\partial_{\bar{z}})}{\partial z} = [A_\lambda(\partial_z), R_\lambda(\partial_{\bar{z}})] - [A_\lambda(\partial_{\bar{z}}), R_\lambda(\partial_z)]. \quad (41)$$

A substitution of (39) and (40) in (41) gives a system of linear elliptic equations on  $\widehat{Z}_{-1}$  and  $\widehat{Z}_0$ . Since the space of solutions to this system which are periodic is finite dimensional, it turns out that  $\widehat{Z}_{-1}$  and  $\widehat{Z}_0$  belong to a finite dimensional vector space. Hence relations (39) and (40) force  $R_\lambda(\partial_z)$  and  $R_\lambda(\partial_{\bar{z}})$  to stay in a finite dimensional vector space.

We can conclude: let us consider

$$\mathcal{R} := \{R_\lambda/R_\lambda(\partial_z), R_\lambda(\partial_{\bar{z}})\} \text{ are given by (39) and (40) and satisfy (41)}.$$

It is a complex finite dimensional vector space. Let us also denote by  $\mathcal{P}_n := \{P(\lambda) = a_n \lambda^{-4n} + a_{n-1} \lambda^{-4(n-1)} + \dots + a_0 / (a_0, \dots, a_n) \in \mathbb{C}^{n+1}\}$  and  $\mathcal{P}_\infty := \cup_{n \in \mathbb{N}} \mathcal{P}_n$ .

The linear map  $\mathcal{P}_\infty \ni P(\lambda) \mapsto dZ_\leq + [A_\lambda, Z_\leq]$ , where  $Z_\leq = (P(\lambda)Y_\lambda)_\leq$  takes values in  $\mathcal{R}$  and so has a finite rank, say  $n$ . Then since  $\dim_{\mathbb{C}} \mathcal{P}_n = n+1$ , the map  $\mathcal{P}_n \ni P(\lambda) \mapsto dZ_\leq + [A_\lambda, Z_\leq]$  has a non trivial kernel: let  $P(\lambda) = \sum_{k=0}^n a_k \lambda^{-4k}$  be a non trivial polynomial in this kernel. Let  $4p$  be the degree of  $P$  in  $\lambda^{-1}$ , i.e. such that  $P(\lambda) = \sum_{k=0}^p a_k \lambda^{-4k}$  and  $a_p \neq 0$ . Without loss of generality we can assume that  $a_p = 1$ . Then  $\xi_\lambda := (P(\lambda)Y_\lambda)_\leq - (P(\lambda)Y_\lambda)_\leq^\dagger$  is a solution of (22) and (24).  $\blacksquare$

## 5 Homogeneous tori in $\mathbb{C}P^2$

We describe here the simplest examples of Hamiltonian stationary Lagrangian tori in  $\mathbb{C}P^2$ : the *homogeneous* Hamiltonian stationary Lagrangian tori, i.e. immersions  $u$  of  $S^1 \times S^1$  into  $\mathbb{C}P^2$  such that  $u(x+t, y) = e^{tA}u(x, y)$  and  $u(x, y+t) = e^{tB}u(x, y)$  for some skew-Hermitian matrices  $A$  and  $B$ . Notice that  $A$  and  $B$  are only defined up to addition with a multiple of  $iId$ . The simplest example is the Clifford torus, namely the image by the Hopf map  $\pi$  of the product torus  $\{z = (z^1, z^2, z^3); |z^1| = |z^2| = |z^3| = 1/\sqrt{3}\}$ . This torus is minimal. The main result states that all homogeneous Hamiltonian stationary Lagrangian tori are similar to the Clifford torus.

**THEOREM 5.1** *Any homogeneous Hamiltonian stationary Lagrangian torus in  $\mathbb{C}P^2$  is the image by the Hopf map of some Cartesian product  $r_1S^1 \times r_2S^1 \times r_3S^1 = \{z = (z^1, z^2, z^3); |z^1| = r_1, |z^2| = r_2, |z^3| = r_3\}$  where  $r_1^2 + r_2^2 + r_3^2 = 1$ , up to  $U(2)$  congruence. Moreover, the torus is special Lagrangian if and only if  $r_1 = r_2 = r_3 = \sqrt{3}$ .*

*Proof* — Let us first see why  $\pi(T)$  is a Hamiltonian stationary Lagrangian torus in  $\mathbb{C}P^2$ , where  $T = r_1S^1 \times r_2S^1 \times r_3S^1$ . Indeed it suffices to show that  $\pi(T)$  admits a Legendrian preimage. Let

$$f(x, y) := \left( r_1 e^{i((1-r_1^2)x - r_2^2 y)}, r_2 e^{i(-r_1^2 x + (1-r_2^2)y)}, r_3 e^{i(-r_1^2 x - r_2^2 y)} \right).$$

Then the orbit under the Hopf action of the image of  $f$  is exactly the 3-torus  $T$  above and  $\pi \circ f$  is doubly periodic with periods  $(2\pi, 0)$  and  $(0, 2\pi)$ . Note that this immersion is not conformal but there exists an orthonormal Hermitian moving frame  $(e_1, e_2)$  such that  $\frac{\partial f}{\partial x} = r_1 \sqrt{1-r_1^2} e_1$  and  $\frac{\partial f}{\partial y} = \frac{r_2}{\sqrt{1-r_1^2}} (r_3 e_2 - r_1 r_2 e_1)$ .

And it is easy to check that  $f$  is Legendrian (and flat). Its Lagrangian angle function is

$$\beta(x, y) = x(1 - 3r_1^2) + y(1 - 3r_2^2) + \pi$$

and since the metric is flat,  $\beta$  is clearly harmonic, and constant if and only if  $r_1 = r_2 = r_3 = 1/\sqrt{3}$ . Notice that many of these tori do not lift up to  $S^5$  as Legendrian tori (they do not close up). Indeed the Maslov class is not always an integer: for the implicit homology basis,  $t \mapsto (2\pi t, 0)$  and  $t \mapsto (0, 2\pi t)$ , it is  $(1 - 3r_1^2, 1 - 3r_2^2)$ . However, if all  $r_i^2$  are rational, the torus in  $\mathbb{C}P^2$  possesses a Legendrian toric multiple cover.

Suppose now that  $u : S^1 \times S^1 \rightarrow \mathbb{C}P^2$  is a homogeneous Lagrangian immersion. According to our definition  $u$  has a lift  $\hat{u}$  such that  $\pi(\hat{u}(x+t, y)) = \pi(e^{tA}\hat{u}(x, y))$  and  $\pi(\hat{u}(x, y+t)) = \pi(e^{tB}\hat{u}(x, y))$ . In particular  $\pi(e^{xA}e^{yB}p) = \pi(e^{yB}e^{xA}p)$ , for any  $p \in S^5$  in the image. However the image is never contained in a complex subspace of  $\mathbb{C}^3$ , hence  $[A, B] \in i\mathbb{R}Id$ . Since  $[A, B]$  is traceless,  $A$  and  $B$  commute.

The obvious (non Legendrian) lift in  $S^5$  is  $(x, y) \mapsto e^{xA}e^{yB}p$  where now  $p = (p_1, p_2, p_3)$  is a fixed point mapped by the Hopf map  $\pi$  to  $u(0, 0)$ . A

Legendrian lift  $\hat{u}$  takes the following form:  $\hat{u}(x, y) = e^{i\theta(x, y)} e^{xA} e^{yB} p$  for some function  $\theta$ . The horizontality condition implies  $\langle (i \frac{\partial \theta}{\partial x} Id + A)p, p \rangle_{\mathbb{C}^3} = 0$  so that  $\frac{\partial \theta}{\partial x} = i \frac{\langle Ap, p \rangle_{\mathbb{C}^3}}{|p|^2}$  is a constant. The same holds in the  $y$  direction so we can define the lift  $\hat{u}(x, y) = e^{x\hat{A} + y\hat{B}} p$  where  $\hat{A} = A + i \frac{\partial \theta}{\partial x} Id$  and  $\hat{B} = B + i \frac{\partial \theta}{\partial y} Id$  are two commuting skew-symmetric matrices. (Notice that  $\hat{u}$  is only defined on the universal cover  $\mathbb{R}^2$ .) The base point  $p$  depends of course on the choice of origin and is only defined up to multiplication by a complex unit number. Nevertheless it plays an important role.

Consider now the metric induced by  $\hat{u}$ . Due to homogeneity, it is a constant metric on the  $(x, y)$ -plane. By doing a simple change in variables, we may as well assume that the metric is the standard plane metric, in other words the immersion is isometric. (Of course that will change the matrices  $\hat{A}$  and  $\hat{B}$ , but since they are replaced by some real linear combination of themselves, the properties mentioned above still hold.) Henceforth we suppose that  $\hat{u}$  is an isometric homogeneous Legendrian immersion of the plane.

Up to a unitary rotation in  $\mathbb{C}^3$  we may suppose that  $\hat{A}$  is diagonal, and write  $\hat{A} = i \text{diag}(a_1, a_2, a_3)$  with real coefficients  $a_1, a_2, a_3$ . We need then to consider three cases.

1. The only possible case:  $B = i \text{diag}(b_1, b_2, b_3)$  is diagonal. Then the surface lies inside the three torus  $T = |p_1|S^1 \times |p_2|S^1 \times |p_3|S^1$ . Necessarily it lifts  $\pi(T)$ . Isometry will constraint the coefficients to be as above.
2. One and only one of the off-diagonal coefficients of  $B$  is non zero. We can assume it is  $b_{12}$  up to permutation of the coordinates. Commutation forces  $a_1 = a_2$ , while  $a_3 \neq a_1$ , otherwise we would get a contradiction:  $A$  cannot be a multiple of  $iId$ . Let us first look at equations involving  $A$ . The immersion being isometric in  $S^5$ ,  $|p| = |Ap| = 1$

$$1 = |p_1|^2 + |p_2|^2 + |p_3|^2 = a_1^2(|p_1|^2 + |p_2|^2) + a_3^2|p_3|^2$$

but it is also Legendrian, so

$$\omega(Ap, p) = \langle iAp, p \rangle_{\mathbb{C}^3} = a_1(|p_1|^2 + |p_2|^2) + a_3|p_3|^2 = 0.$$

Hence

$$|p_1|^2 + |p_2|^2 = \frac{a_3}{a_3 - a_1}, \quad |p_3|^2 = -\frac{a_1}{a_3 - a_1} \text{ and } a_1 a_3 = -1$$

excluding thus  $a_1 = 0$ , and finally

$$|p_1|^2 + |p_2|^2 = \frac{1}{1 + a_1^2}, \quad |p_3|^2 = \frac{a_1^2}{1 + a_1^2}.$$

Take now into account the Legendrian constraints on  $B$ :

$$B = \begin{pmatrix} ib_1 & b_{12} & 0 \\ -\overline{b_{12}} & ib_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$$

$$0 = \langle Bp, p \rangle_{\mathbb{C}^3} = i \left( \sum_1^3 b_j |p_j|^2 + 2\text{Im}(b_{12}\overline{p_1}p_2) \right)$$

$$0 = \langle Bp, Ap \rangle_{\mathbb{C}^3} = \sum_1^3 a_j b_j |p_j|^2 + 2a_1 \text{Im}(b_{12}\overline{p_1}p_2).$$

Uniting both, we deduce

$$a_1 b_1 |p_1|^2 + a_1 b_2 |p_2|^2 + a_1 b_3 |p_3|^2 = a_1 b_1 |p_1|^2 + a_1 b_2 |p_2|^2 + a_3 b_3 |p_3|^2.$$

Since  $|p_3| \neq 0$  and  $a_1 \neq a_3$ ,  $b_3$  vanishes. The remaining equations are:

$$|p_1|^2 + |p_2|^2 = \frac{1}{1 + a_1^2}$$

$$b_1 |p_1|^2 + b_2 |p_2|^2 + 2\text{Im}(b_{12}\overline{p_1}p_2) = 0 \quad (42)$$

$$(b_1^2 + |b_{12}|^2)|p_1|^2 + (b_2^2 + |b_{12}|^2)|p_2|^2 + 2(b_1 + b_2)\text{Im}(b_{12}\overline{p_1}p_2) = 1. \quad (43)$$

Finally we infer a contradiction: indeed equation (42) amounts to the existence of an isotropic vector  $(p_1, p_2)$  for the skew-hermitian matrix  $\begin{pmatrix} ib_1 & b_{12} \\ -\overline{b_{12}} & ib_2 \end{pmatrix}$ , and that requires its determinant  $|b_{12}|^2 - b_1 b_2$  to vanish. Plugging this into (43), we obtain

$$(b_1 + b_2)(b_1 |p_1|^2 + b_2 |p_2|^2 + 2\text{Im}(b_{12}\overline{p_1}p_2)) = 1,$$

obviously contradicting (42).

3. If at least two off-diagonal coefficients of  $B$  are non-zero, then  $A = ia_1 Id$ . But that contradicts  $\langle Ap, p \rangle = 0$ . So that case is also excluded.

Notice that in the language of integrable systems, homogeneous tori correspond to vacuum solutions and are of finite type for  $p = 0$ . ■

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