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THE ORCHARD MORPHISM

ROLAND BACHER

Abstract. We define and prove uniqueness of a natural homomorphism (called the Orchard morphism) from some groups associated naturally to a finite set E to the group $\mathcal{E}(E)$ of two-partitions of E representing equivalence relations having at most two classes on E.

As an application, we exhibit a natural equivalence relation on the set of points of generic finite configurations in $\mathbf{R}^d$.

1. Introduction

Let E be a set. The set $\mathcal{E}(E)$ of all equivalence relations on E into at most two equivalence classes can be endowed with a group structure, which we call the *group of two-partitions* since its elements represent partitions of E into at most two disjoint subsets. Elements of $\mathcal{E}(E)$ can be represented by functions $f : E \longrightarrow \{\pm 1\}$, well-defined up to multiplication by -1. Such a function endows E with the the equivalence relation given by the equivalence classes $f^{-1}(1)$ and $f^{-1}(-1)$.

We denote by

$$E^{(l)} = \{(x_1, \ldots, x_l) \in E^l \mid x_i \neq x_j \text{ for } 1 \leq i < j \leq l\}$$

the set of all sequences of length l without repetitions with values in E. Consider the multiplicative group $\{\pm 1\}^{E^{(l)}}$ formed by all functions $\varphi : E^{(l)} \longrightarrow \{\pm 1\}$. The symmetric group $\mathrm{Sym}(\{1, \ldots, l\})$ acts on such functions by permutations of the l arguments. The set $\mathcal{F}_+(E^{(l)})$ of all symmetric functions on which this action is trivial is a subgroup of $\{\pm 1\}^{E^{(l)}}$ while the set $\mathcal{F}_-(E^{(l)})$ of all antisymmetric functions on which $\mathrm{Sym}(\{1, \ldots, l\})$ acts by the the signature homomorphism is a free $\mathcal{F}_+(E^{(l)})-$set. Since the product of two antisymmetric functions is always symmetric, the union $\mathcal{F}_\pm(E^{(l)}) = \mathcal{F}_+(E^{(l)}) \cup \mathcal{F}_-(E^{(l)})$ is a subgroup of $\{\pm 1\}^{E^{(l)}}$.

Suppose now that the set E is finite. The main result of this paper is the existence of a non-trivial natural homomorphism from the finite group $\mathcal{F}_\pm(E^{(l)})$ to the finite group $\mathcal{E}(E)$ of two-partitions on E. Naturality means that this homomorphism is $\mathrm{Sym}(E)-$equivariant with respect to the obvious actions by automorphisms of $\mathrm{Sym}(E)$ on both groups $\mathcal{F}_\pm(E^{(l)})$ and $\mathcal{E}(E)$.

We call this homomorphism the *Orchard homomorphism*. The Orchard homomorphism is the unique natural homomorphism from $\mathcal{F}_\pm(E^{(l)})$ to $\mathcal{E}(E)$ which is non-trivial for $1 \leq l < \sharp(E)$. There exists however a natural

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non-trivial homomorphism $\mathcal{F}_{\pm}(E^{(2)}) \longrightarrow \mathcal{E}(E)$ distinct from the Orchard homomorphism (which is trivial in this case) if E has 2 elements.

The existence of such a homomorphism from the set of symmetric functions $\mathcal{F}_{+}(E^{(l)})$ to $\mathcal{E}(E)$ is not surprising. Its natural extension to the set $\mathcal{F}_{-}(E^{(l)})$ of antisymmetric functions is however not completely obvious.

Natural examples of antisymmetric functions $E^{(l)} \longrightarrow \{\pm 1\}$ where E is a finite set arise for instance by considering finite generic subsets of points in the oriented real affine space $\mathbf{R}^{l-1}$ where generic means that any set of $k \leq l$ points in E is affinely independent. One gets an antisymmetric function on $E^{(l)}$ by considering the orientation of simplices spanned by l linearly ordered points of E.

The case $l = 3$ may be illustrated as giving a natural rule to plant trees of two distinct species in an orchard: The Queen of Heart has randomly choosen n generic locations E in her future royal orchard and asks Alice to plant cherry- and plum-trees in a natural way. Alice assigns to each cyclically oriented triplet (a, b, c) of points in E the value 1 (respectively -1) if the points (a, b, c) define a positively (respectively negatively) oriented triangle (where Wonderland is supposed to be an oriented plane). This yields an antisymmetric function on $E^{(3)}$ whose image by the Orchard morphism is a two-partition prescribing the choice of the species (up to global permutation of all cherry- and plum-trees).

Figure 1 shows an example. The two-partition obtained by the Orchard morphism (given by Proposition 5.1 with $d = 2$) yields for the choosen nine positions three trees of one species and six trees of the remaining species. The paper [2] contains many more examples, some of which are monochromatic (all vertices belong to the same common equivalence class).

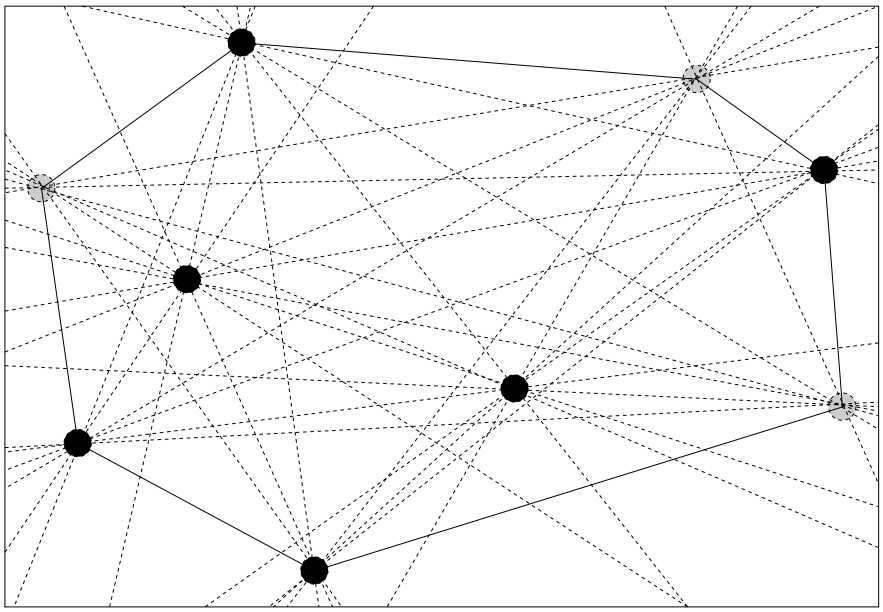

Figure 1: An orchard having 3 cherry- and 6 plumtrees in generic positions

Finally, the Orchard morphism exists also in the case where the finite set E is endowed with a natural fixpoint-free involution $\iota : E \longrightarrow E$ which can be thought of as a kind of orientation. We call such a set orientable.

Given an orientable set (E, ι) it is natural to consider only structures on E (equivalence relations, sets of functions etc.) which are invariant (perhaps up to a sign) under the involution ι. We define in this setting analogues of the groups $\mathcal{E}(E)$ and $\mathcal{F}_{\pm}(E^{(l)})$ considered above and construct the corresponding Orchard morphism.

2. Two-partitions

A *two-partition* is an unordered partition $\{A, B\}$ of a set $E = A \cup B$ into at most two disjoint subsets. Two-partitions are the same as equivalence relations having at most two classes. We will move freely between these two interpretations of two-partitions. The word "class" will often be used instead of "part of the two-partition" and a two-partition $\{A, B\}$ of E will generally be written as $A \cup B$ or $E = A \cup B$.

A two-partition $E = A \cup B$ can be given by a pair $\pm\alpha$ of opposite functions where

$$\alpha : E \longrightarrow \{\pm 1\}$$

is defined by $\alpha^{-1}(1) = A$ and $\alpha^{-1}(-1) = B$. The set $\mathcal{E}(E)$ of all such two-partitions is a vector space over the field $\mathbf{F}_2$ of two elements. Its dimension is $\sharp(E) - 1$ if E is a finite set. The pair ± 1 of constant functions represents the identity and the group law $(\pm\alpha)(\pm\beta)$ is the obvious product $\pm\alpha\beta$ of functions. Set-theoretically, the product $(A_1 \cup A_2)(B_1 \cup B_2)$ of 2 two-partitions on a set E is given by $E = C_1 \cup C_2$ where $C_1 = (A_1 \cap B_1) \cup (A_2 \cap B_2)$ and $C_2 = (A_1 \cap B_2) \cup (A_2 \cap B_1)$.

Consider a simple graph Γ (not necessarily finite) with vertices V and unoriented edges E. Its adjacency matrix A is the symmetric matrix with rows and columns indexed by elements of V. All its entries are zero except $A_{v,w} = 1$ where $v \neq w$ are adjacent vertices of Γ (i.e. $\{u, v\}$ is an edge of Γ).

Our main tool in what follows is the following trivial and probably well-known observation:

Lemma 2.1. *Let Γ be a simple graph with adjacency matrix A. Suppose that there exists a constant $\gamma \in \{0, 1\}$ such that*

$$A_{u,v} + A_{v,w} + A_{u,w} \equiv \gamma \pmod 2$$

for all triplets $(u, v, w) \in V^{(3)} \subset V^3$ of three distinct vertices.

Then either Γ or its complementary graph Γ^c (having adjacency matrix $A^c = J - I - A$ where J is the all one matrix and I the identity matrix) is a disjoint union of at most two complete graphs.

Proof. Up to replacing Γ by its complementary graph Γ^c we can suppose that $\gamma = 1$. This shows that given any three vertices of Γ, at least two of them are adjacent. The graph Γ consists thus of at most two connected components. If a connected component of Γ is not a complete graph, then this component contains two vertices u, v at distance 2 implying that $A_{u,v} = 0$. Since u and v are at distance 2 they share a common neighbour w for which we have $A_{u,w} = A_{v,w} = 1$. This yields a contradiction since $A_{u,v} + A_{u,w} + A_{v,w} = 2 \not\equiv \gamma \pmod 2$. $\square$

Given a set E we call a function $\sigma : E^{(2)} \longrightarrow \{\pm 1\}$ *symmetric* if $\sigma(x,y) = \sigma(y,x)$ for all $(x,y) \in E^{(2)}$.

Proposition 2.2. *Any symmetric function*

$$\sigma : E^{(2)} \longrightarrow \{\pm 1\}$$

with

$$\sigma(a,b)\sigma(b,c)\sigma(a,c) = \gamma \in \{\pm 1\}$$

independent of $(a,b,c) \in E^{(3)} = \{(a,b,c) \in E^3, \mid a \neq b \neq c \neq a\}$ *gives rise to a two-partition of* E.

Proof. Consider the simple graph Γ with vertices E and adjacency matrix having coefficients $A_{x,x} = 0$ and $A_{x,y} = \frac{\sigma(x,y)+1}{2},\ x \neq y$.

The graph Γ satisfies the assumptions of Lemma 2.1 and consists hence, up to a sign change of σ (which replaces Γ by its complementary graph), of at most two non-empty complete graphs. The connected components of Γ define a two-partition on E. □

Remark 2.3. *The two-partition described by Proposition 2.2 can be constructed as follows: set* $\gamma = \sigma(a,b)\sigma(b,c)\sigma(a,c)$ *for* $a \neq b \neq c \neq a$ *and choose an element* $x_0 \in E$. *Up to multiplication by* -1 *the function* $\alpha : E \longrightarrow \{\pm 1\}$ *defined by* $\alpha(x_0) = 1$ *and* $\alpha(x) = \gamma\sigma(x,x_0),\ x \neq x_0$ *is then independent of the choice of the element* x_0 *and the classes of the associated two-partition are given by* $\alpha^{-1}(1)$ *and* $\alpha^{-1}(-1)$.

3. Symmetric and antisymmetric functions

A function $\varphi : E^{(l)} \longrightarrow \{\pm 1\}$ (where E is a set) is $l-$*symmetric* or *symmetric* if

$$\varphi(\dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots) = \varphi(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_l)$$

for all $1 \leq i < l$ and $(x_1, \dots, x_l) \in E^{(l)}$. We denote by $\mathcal{F}_+(E^{(l)})$ the set of all symmetric functions from $E^{(l)}$ to $\{\pm 1\}$.

Similarly, such a function $\varphi : E^{(l)} \longrightarrow \{\pm 1\}$ is $l-$*antisymmetric* or *antisymmetric* if

$$\varphi(\dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots) = -\varphi(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_l)$$

for all $1 \leq i < l$ and $(x_1, \dots, x_l) \in E^{(l)}$. We denote by $\mathcal{F}_-(E^{(l)})$ the set of all antisymmetric functions from $E^{(l)}$ to $\{\pm 1\}$.

The set $\mathcal{F}_\pm(E^{(l)}) = \mathcal{F}_+(E^{(l)}) \cup \mathcal{F}_-(E^{(l)})$ of all symmetric or antisymmetric functions from $E^{(l)}$ to $\{\pm 1\}$ is a vector space over $\mathbf{F}_2$, of dimension $\binom{\sharp(E)}{l} + 1$ for $1 < l \leq \sharp(E)$ and E a finite set. The identity element is given by the symmetric constant function $E^{(l)} \longrightarrow \{1\}$ and the group-law is the usual product of functions.

We define the *signature homomorphisme* $\mathrm{sign} : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \{\pm 1\}$ by $\mathrm{sign}(\varphi) = 1$ if $\varphi \in \mathcal{F}_+(E^{(l)})$ is symmetric and $\mathrm{sign}(\varphi) = -1$ if $\varphi \in \mathcal{F}_-(E^{(l)})$ is antisymmetric. The set $\mathcal{F}_+(E^{(l)}) = \mathrm{sign}^{-1}(1)$ of all symmetric functions on $E^{(l)}$ is of course a subgroup of $\mathcal{F}_\pm(E^{(l)})$ and the set $\mathcal{F}_-(E^{(l)}) = \mathrm{sign}^{-1}(-1)$ of all antisymmetric functions on $E^{(l)}$ is a free $\mathcal{F}_+(E^{(l)})-$set.

4. The Orchard morphism

Given a finite set E, the aim of this section is to construct the *Orchard morphism*

$$\rho : \mathcal{F}_{\pm}(E^{(l)}) \longrightarrow \mathcal{E}(E) \ ,$$

a natural group homomorphism which factors through the quotient group $\mathcal{F}_{\pm}(E^{(l)})/\pm 1$ where ± 1 denote the obvious constant symmetric functions on $E^{(l)}$. Naturality means that ρ is equivariant with respect to the obvious actions of the symmetric group $\mathrm{Sym}(E)$ on $\mathcal{F}_{\pm}(E^{(l)})$ and $\mathcal{E}(E)$.

Given a totally ordered set X we denote by $\binom{X}{k}$ the set of all strictly increasing sequences of length k in X.

For an arbitrary set X, we define $\binom{X}{k}$ by choosing first an arbitrary total order relation on X.

Given a function $\varphi \in \mathcal{F}_{\pm}(E^{(l)})$ where E is finite, we define $\sigma_\varphi : E^{(2)} \longrightarrow \{\pm 1\}$ by setting

$$\sigma_\varphi(y,z) = \prod_{(x_1,\ldots,x_{l-1})\in\binom{E\setminus\{y,z\}}{l-1}} \varphi(x_1,\ldots,x_{l-1},y)\ \varphi(x_1,\ldots,x_{l-1},z)\ .$$

Proposition 4.1. *The function σ_φ is a well-defined symmetric function on $E^{(2)}$ such that*

$$\sigma_\varphi(a,b)\sigma_\varphi(b,c)\sigma_\varphi(a,c) = (sign(\varphi))^{\binom{\sharp(E)-3}{l-2}}$$

for all $(a,b,c) \in E^{(3)}$ where $sign(\varphi)$ is the signature homomorphism sending $l-$symmetric functions on $E^{(l)}$ to 1 and $l-$antisymmetric functions to -1.

Proof. Since every sequence $(x_1,\ldots,x_{l-1}) \in \binom{E\setminus\{y,z\}}{l-1}$ is involved twice in $\sigma_\varphi(y,z)$, the value of $\sigma_\varphi(y,z)$ is independent of the choice of a particular total order on $E\setminus\{y,z\}$. Symmetry ($\sigma_\varphi(y,z) = \sigma_\varphi(z,y)$) of σ_φ is obvious.

Consider now first an element $(x_1,\ldots,x_{l-1}) \in \binom{E\setminus\{a,b,c\}}{l-1}$. Such an element contributes always a factor 1 to the product $\sigma_\varphi(a,b)\sigma_\varphi(b,c)\sigma_\varphi(a,c)$. The product $\sigma_\varphi(a,b)\sigma_\varphi(b,c)\sigma_\varphi(a,c)$ is thus equal to the the product over all elements $(x_1,\ldots,x_{l-2}) \in \binom{E\setminus\{a,b,c\}}{l-2}$ of factors of the form

$$\begin{array}{l}\varphi(x_1,\ldots,x_{l-2},c,a)\ \varphi(x_1,\ldots,x_{l-2},c,b)\\ \varphi(x_1,\ldots,x_{l-2},a,b)\ \varphi(x_1,\ldots,x_{l-2},a,c)\\ \varphi(x_1,\ldots,x_{l-2},b,a)\ \varphi(x_1,\ldots,x_{l-2},b,c)\end{array}$$

and each of these $\binom{\sharp(E)-3}{l-2}$ factors yields a contribution of $\mathrm{sign}(\varphi)$. □

By Proposition 4.1 the function σ_φ satisfies the conditions of Proposition 2.2 and gives rise to a two-partition $\rho(\varphi) \in \mathcal{E}(E)$. We call the application $\rho : \mathcal{F}_{\pm}(E^{(l)}) \longrightarrow \mathcal{E}(E)$ defined in this way the *Orchard morphism.* Given an element $\varphi \in \mathcal{F}_{\pm}(E^{(l)})$ we call the two-partition $\rho(\varphi)$ the *Orchard-partition* of φ. The two-partition $\rho(\varphi)$ gives rise to the *Orchard-equivalence relation* partitioning the elements of E into at most two *Orchard classes.*

Remark 4.2. *(i) If $l = 1$, the two-partition $\rho(\varphi)$ on E given by the Orchard morphism is the obvious one with classes $\varphi^{-1}(1)$ and $\varphi^{-1}(-1)$.*

(ii) Consider a $2-$symmetric function $\varphi \in \mathcal{F}_{+}(E^{(2)})$ satisfying the condition of Proposition 2.2. By Proposition 2.2 it gives rise to a two-partition

on E. If E is finite, we get a second two-partition on E by considering the Orchard morphism $\rho(\varphi)$. An easy computation shows that these two-partitions coincide if $\sharp(E)$ is odd. If $\sharp(E)$ is even, the image of the Orchard morphism $\rho(\varphi) \in \mathcal{E}(E)$ is trivial for such a function φ.

Before stating the main result concerning the Orchard morphism, we recall the definition of equivariance: Let a group G act on two sets X and Y. An application $\psi : X \longrightarrow Y$ is $G-$*equivariant* if $\psi(g(x)) = g(\psi(x))$ for all $g \in G$ and $x \in X$. Given a set E, the symmetric group $\mathrm{Sym}(E)$ of all bijections of E acts in an obvious way on the groups $\mathcal{F}_\pm(E^{(l)})$ and $\mathcal{E}(E)$ and it is hence natural to study group homomorphisms from $\mathcal{F}_\pm(E^{(l)})$ to $\mathcal{E}(E)$ which are *natural*, i.e. $\mathrm{Sym}(E)-$equivariant.

Theorem 4.3. *For any finite set E and any natural integer $1 \leq l < \sharp(E)$, the Orchard morphism*

$$\rho : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \mathcal{E}(E)$$

is the unique natural group homomorphism which is non-trivial. Moreover, ρ factors through the quotient group $\mathcal{F}^l_\pm(E)/\{\pm 1\}$ (where, as always, ± 1 denote the constant symmetric functions on $E^{(l)}$).

Remark 4.4. *The Orchard morphism exists and is always trivial for $l = \sharp(E)$.*

For $n = l = 2$ there exists an "exotic" natural homomorphisme which is non-trivial: Defining $\rho' : \mathcal{F}_\pm(E^{(2)}) \longrightarrow \mathcal{E}(E)$ (where $E = \{1, 2\}$ has two elements) by $\rho'(\varphi) = (E = \{1, 2\})$ if $\varphi \in \mathcal{F}_+(E^{(2)})$ and $\rho'(\varphi) = (E = \{1\} \cup \{2\})$ if $\varphi \in \mathcal{F}_-(E^{(2)})$ we have a natural homomorphism distinct from the Orchard morphism (which is trivial in this case).

This failure is due the fact that both two-partitions on the set $E = \{1, 2\}$ are Sym(E)-invariant. However, for finite sets E having more than 2 elements, only the trivial two-partition is Sym$(E)-$invariant.

A *flip* is a symmetric function $f_X \in \mathcal{F}_+(E^{(l)})$ such that $f_X^{-1}(-1) \subset E^{(l)}$ consists (up to permutation of its elements) of a unique sequence $X = (x_1, \ldots, x_l) \in \binom{E}{l}$. We call the set $X = \{x_1, \ldots, x_l\}$ the *flipset* of the flip f_X.

The set $\{f_X\}_{X \in \binom{E}{l}}$ of all flips is obviously a basis of the subspace $\mathcal{F}_+(E^{(l)})$ of symmetric functions on $E^{(l)}$.

Lemma 4.5. *Given a flip $f_X \in \mathcal{F}_+(E^{(l)})$ and an arbitrary element $\varphi \in \mathcal{F}_\pm(E^{(l)})$ we have for $(a, b) \in E^{(2)}$*

$$\sigma_\varphi(a, b)\ \sigma_{(\varphi f_X)}(a, b) = -1$$

if and only if exactly one of the elements a, b belongs to X.

Proof. In the product defining $\sigma_\varphi(a, b)\sigma_{(\varphi f_X)}(a, b)$ the factor

$$\begin{aligned}
&\varphi(x_1, \ldots, x_{l-1}, a)\varphi(x_1, \ldots, x_{l-1}, b) \\
&\quad (\varphi\ f_X)(x_1, \ldots, x_{l-1}, a)(\varphi\ f_X)(x_1, \ldots, x_{l-1}, b) \\
&\quad\quad = f_X(x_1, \ldots, x_{l-1}, a)\ f_X(x_1, \ldots, x_{l-1}, b)
\end{aligned}$$

corresponding to $(x_1, \ldots, x_{l-1}) \in \binom{E\setminus\{a,b\}}{l-1}$ yields a contribution of 1 except if $X = \{x_1, \ldots, x_{l-1}, a\}$ or if $X = \{x_1, \ldots, x_{l-1}, b\}$. This happens at most once and only if exactly one of the elements a, b belongs to the set X. $\square$

Lemma 4.5 implies easily the following result.

Proposition 4.6. *(i) The classes of the two-partition $\rho(f_X)$ associated to a flip $f_X \in \mathcal{F}_+(E^{(l)})$ are given by X and $E \setminus X$.*

(ii) If two functions $\varphi, \psi = \varphi\ f_X \in \mathcal{F}_\pm(E^{(l)})$ differ by a flip then the corresponding equivalence relations $\rho(\varphi)$ and $\rho(\psi) = \rho(\varphi f_X)$ differ exactly on the subsets $X \times (E \setminus X)$ and $(E \setminus X) \times X$ of $E \times E$.

Proof of Theorem 4.3. Since the set of all flips generates $\mathcal{F}_+(E^{(l)})$ and since $\mathcal{F}_-(E^{(l)})$ is a free $\mathcal{F}_+(E^{(l)})-$set, the Orchard morphism ρ behaves well under composition by assertion (ii) of Proposition 4.6. Since the equivalence relation associated to a constant function $\pm 1 \in \mathcal{F}_+(E^{(l)})$ is obviously trivial, ρ defines a group homomorphism from the quotient group $\mathcal{F}_+(E^{(l)})/\{\pm 1\}$ into $\mathcal{E}(E)$.

Equivariance of ρ with respect to $\mathrm{Sym}(E)$ is obvious.

We have yet to show that every other natural ($\mathrm{Sym}(E)-$equivariant) homomorphism $\rho' : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \mathcal{E}(E)$ is either trivial or coincides with the Orchard morphism ρ.

A flip f_X is clearly invariant under the subgroup $\mathrm{Sym}(X) \times \mathrm{Sym}(E \setminus X) \subset \mathrm{Sym}(E)$. If $\sharp(E) > 2$, any two-partition invariant under $\mathrm{Sym}(X) \times \mathrm{Sym}(E \setminus X)$ of E is either trivial or equal to $X \cup (E \setminus X)$. This implies that we have either $\rho'(f_X) = 1$ or $\rho'(f_X) = \rho(f_X)$ for any $\mathrm{Sym}(E)-$equivariant homomorphism $\rho' : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \mathcal{E}(E)$. Since $\mathrm{Sym}(E)$ acts transitively on the set of all flips, the first case implies triviality of ρ' restricted to $\mathcal{F}_+(E^{(l)})$ while we have $\rho' = \rho$ for the restriction onto $\mathcal{F}_+(E^{(l)})$ in the second case. This conclusion holds also for $\sharp(E) = 2$ and $l = 1$ as can easily be checked.

If ρ' restricted to $\mathcal{F}_+(E^{(l)})$ is trivial, the identity $\mathcal{F}_\pm(E^{(l)}) = \varphi\ \mathcal{F}_+(E^{(l)})$ for any $\varphi \in \mathcal{F}_-(E^{(l)})$ shows that ρ' restricted to $\mathcal{F}_-(E^{(l)})$ is constant and hence trivial for $\sharp(E) > 2$ by $\mathrm{Sym}(E)-$equivariance. For $\sharp(E) = 2$ and $l = 2$, this conclusion fails as shown by the example of Remark 4.4.

We might hence suppose that $\rho' = \rho$ on $\mathcal{F}_+(E^{(l)})$. Choose an antisymmetric function $\varphi \in \mathcal{F}_-(E^{(l)})$. If $n = \sharp(E)$ is odd, choose a cyclic permutation $\beta \in \mathrm{Sym}(E)$ (of maximal length n) of E and consider

$$\tilde{\varphi}(x_1, \ldots, x_l) = \prod_{j=0}^{n-1} \varphi(\beta^j(x_1), \beta^j(x_2), \ldots, \beta^j(x_l))$$

where $\beta^0(x) = x$ and $\beta^j(x) = \beta(\beta^{j-1}(x))$ for $x \in E$. The function $\tilde{\varphi} : E^{(l)} \longrightarrow \{\pm 1\}$ is antisymmetric on $E^{(l)}$ and invariant under the cyclic subgroup generated by $\beta \in \mathrm{Sym}(E)$. The corresponding two-partition $\rho'(\tilde{\varphi})$ is also invariant under the cyclic permutation β and hence trivial since $n = \sharp(E)$ is odd. The equality $\mathcal{F}_-(E^{(l)}) = \tilde{\varphi}\mathcal{F}_+(E^{(l)})$ implies now the result.

Suppose now $n = \sharp(E)$ even. Choose an element $z \in E$ and a cyclic permutation β of all $(n-1)$ elements of $E \setminus \{z\}$. Setting

$$\tilde{\varphi}(x_1, \ldots, x_l) = \prod_{j=0}^{n-2} \varphi(\beta^j(x_1), \beta^j(x_2), \ldots, \beta^j(x_l))$$

for a fixed element $\varphi \in \mathcal{F}_-(E^{(l)})$ and reasoning as above we see that $\rho'(\tilde{\varphi}) \in \mathcal{E}(E)$ is either trivial or corresponds to the two-partition $\{z\} \cup (E \setminus \{z\})$. This implies that the same conclusion holds for $\rho'(\tilde{\varphi})\rho(\tilde{\varphi})$ and the identity $\mathcal{F}_-(E^{(l)}) = \tilde{\varphi}\mathcal{F}_+(E^{(l)})$ shows that the product $\rho'(\varphi)\rho(\varphi) \in \mathcal{E}(E)$ is constant for $\varphi \in \mathcal{F}_-(E^{(l)})$. By $\mathrm{Sym}(E)-$equivariance this is only possible if $n = 2$ (cf. Remark 4.4) or if $\rho'(\varphi)\rho(\varphi)$ is trivial which establishes the Theorem. □

4.1. **An easy characterisation of ρ restricted to $\mathcal{F}_+(E^{(l)})$.** In this subsection we give an elementary description of $\rho(\varphi)$ for $\varphi \in \mathcal{F}_+(E^{(l)})$ an $l-$symmetric function.

Given a finite set E and an $l-$symmetric function $\varphi \in \mathcal{F}_+(E^{(l)})$ we consider the function $\mu_\varphi : E \longrightarrow \{\pm 1\}$ defined by

$$\mu_\varphi(x) = \prod_{(x_1,\ldots,x_{l-1}) \in \binom{E\setminus\{x\}}{l-1}} \varphi(x_1, \ldots, x_{l-1}, x) \ .$$

Proposition 4.7. *The two classes of the Orchard relation $\rho(\varphi)$ are given by $\mu_\varphi^{-1}(1)$ and $\mu_\varphi^{-1}(-1)$.*

Proof. The result clearly holds for the two $l-$symmetric constant functions. The Proposition follows now from the fact that μ_φ and $\mu_{\varphi f_X}$ differ exactly on X for a flip $f_X \in \mathcal{F}_+(E^{(l)})$. □

Another proof can be given by remarking that μ_φ defines a non-trivial $\mathrm{Sym}(E)-$equivariant homomorphism into $\mathcal{E}(E)$ which must be the Orchard homomorphism by unicity.

Remark 4.8. *Setting*

$$\tilde{\mu}_\varphi(x) = \prod_{(x_1,\ldots,x_l) \in \binom{E\setminus\{x\}}{l}} \varphi(x_1, \ldots, x_l)$$

we have $\tilde{\mu}_\varphi = \mu_\varphi$, up to a sign given by

$$\prod_{(x_1,\ldots,x_l) \in \binom{E}{l}} \varphi(x_1, \ldots, x_l) \ .$$

5. Generic configurations of points in $\mathbf{R}^d$

A finite set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of n points in the oriented real affine space $\mathbf{R}^d$ is a *generic configuration* if any subset of at most $d+1$ points in $\mathcal{P}$ is affinely independent. Generic configurations of $n \leq d+1$ points in $\mathbf{R}^d$ are simply vertices of $(n-1)-$dimensional simplices. For $n \geq d+1$, genericity boils down to the fact that any set of $d+1$ points in $\mathcal{P}$ spans $\mathbf{R}^d$ affinely.

Two generic configurations $\mathcal{P}^1$ and $\mathcal{P}^2$ of $\mathbf{R}^d$ are *isomorphic* if there exists a bijection $\sigma : \mathcal{P}^1 \longrightarrow \mathcal{P}^2$ such that all pairs of corresponding $d-$dimensional simplices (with vertices $(P_{i_0}, \ldots, P_{i_d}) \subset \mathcal{P}^1$ and $(\sigma(P_{i_0}), \ldots, \sigma(P_{i_d})) \subset \mathcal{P}^2$)

have the same orientations (given for instance for the first simplex by the sign of the determinant of the $d\times d$ matrix with rows $P_{i_1}-P_{i_0},\dots,P_{i_d}-P_{i_0}$).

Two generic configurations $\mathcal{P}(-1)$ and $\mathcal{P}(+1)$ are *isotopic* if there exists a continuous path (with respect to the obvious topology on $\mathbf{R}^{dn}=\left(\mathbf{R}^d\right)^n$) of generic configurations $\mathcal{P}(t),\ t\in[-1,1]$, which joins them. Isotopic configurations are of course isomorphic. I ignore to what extend the converse holds.

Given a finite generic configuration $\mathcal{P}=\{P_1,\dots,P_n\}\subset\mathbf{R}^d$ we consider the $(d+1)-$antisymmetric function $\varphi:\mathcal{P}^{(d+1)}\longrightarrow\{\pm1\}$ defined by

$$\varphi(P_{i_0},\dots,P_{i_d})=1$$

if

$$\det(P_{i_1}-P_{i_0},P_{i_2}-P_{i_0},\dots,P_{i_d}-P_{i_0})>0$$

and $\varphi(P_{i_0},\dots,P_{i_d})=-1$ otherwise. The Orchard morphism $\rho(\varphi)\in\mathcal{E}(\mathcal{P})$ (extended to be trivial if $n\le d+1$) provides now a two-partition of the set $\mathcal{P}$.

The associated equivalence relation can be constructed geometrically as follows: Given two points $P,Q\in\mathbf{R}^d\setminus H$, call an affine hyperplane $H\subset\mathbf{R}^d$ *separating* if P,Q are not in the same connected component of $\mathbf{R}^d\setminus H$. For two points P,Q of a finite generic configuration $\mathcal{P}=\{P_1,\dots,P_n\}\subset\mathbf{R}^d$ we denote by $s(P,Q)$ the number of separating hyperplanes which are affinely spanned by d distinct elements in $\mathcal{P}\setminus\{P,Q\}$. The number $s(P,Q)$ depends obviously only of the isomorphism type of $\mathcal{P}$ and of $P,Q\in\mathcal{P}$.

Proposition 5.1. *The equivalence relation $\rho(\varphi)$ on a finite generic configuration $\mathcal{P}\subset\mathbf{R}^d$ is given by $P\sim Q$ if either $P=Q$ or if $s(P,Q)\equiv\binom{n-3}{d-1}$ (mod 2).*

Proof. Given two points $P,Q\in\mathcal{P}$ we have

$$\sigma(P,Q)=\prod_{(R_1,\dots,R_d)\in\binom{\mathcal{P}\setminus\{P,Q\}}{d}}\varphi(R_1,\dots,R_d,P)\varphi(R_1,\dots,R_d,Q)=(-1)^{\alpha(P,Q)}$$

where $\alpha(P,Q)$ denotes the number of subsets $(R_1,\dots,R_d)\in\mathcal{P}\setminus\{P,Q\}$ such that the two simplices with cyclically ordered vertices $(R_1,\dots,R_d,P)$ and $(R_1,\dots,R_d,Q)$ have opposite orientations. This happens if and only if the affine hyperplane containing the points $R_1,\dots,R_d$ separates P from Q. We have hence $\alpha(P,Q)=s(P,Q)$ and

$$\sigma_\varphi(P,Q)\sigma_\varphi(Q,R)\sigma_\varphi(P,R)=(-1)^{\binom{\sharp(\mathcal{P})-3}{(d+1)-2}}$$

and Proposition 5.1 follows from Remark 2.3. □

A *geometric flip* is a continuous path

$$t\longmapsto\mathcal{P}(t)=(P_1(t),\dots,P_n(t))\in\left(\mathbf{R}^d\right)^n,\ t\in[-1,1]$$

with $\mathcal{P}(t)=\{P_1(t),\dots,P_n(t)\}$ generic except for $t=0$ where there exists exactly one subset $\mathcal{F}(0)=(P_{i_0}(0),\dots,P_{i_d}(0))\subset\mathcal{P}(0)$, called the *flipset*, of $(d+1)$ points contained in an affine hyperplane spanned by any subset of d points in $\mathcal{F}(0)$. We require moreover that the simplices $(P_{i_0}(-1),\dots,P_{i_d}(-1))$ and $(P_{i_0}(1),\dots,P_{i_d}(1))$ carry opposite orientations. Geometrically this means

that a point $P_{i_j}(t)$ crosses the affine hyperplane spanned by $\mathcal{F}(t) \setminus \{P_{i_j}(t)\}$ for $t = 0$.

It is easy to see that two generic configurations $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbf{R}^d$ having n points can be related by a continuous path involving at most a finite number of geometric flips.

The next result follows directly from the fact that two configurations $\mathcal{P}(1)$ and $\mathcal{P}(-1)$ related by a geometric flip give rise to $(d+1)-$antisymmetric functions $\varphi_+, \varphi_- \in \mathcal{F}_-(\mathcal{P}^{(d+1)})$ which differ only by a flip:

Proposition 5.2. *Let $\mathcal{P}(-1), \mathcal{P}(+1) \subset \mathbf{R}^d$ be two generic configurations related by a flip with respect to a subset $\mathcal{F}(t)$ of $(d+1)$ points.*

(i) If two distinct points $P(t), Q(t)$ are either both contained in $\mathcal{F}(t)$ or both contained in its complement $\mathcal{P}(t) \setminus \mathcal{F}(t)$ then we have

$$P(-1) \sim Q(-1) \text{ if and only if } P(+1) \sim Q(+1) \ .$$

(ii) For $P(t) \in \mathcal{F}(t)$ and $Q(t) \notin \mathcal{F}(t)$ we have

$$P(-1) \sim Q(-1) \text{ if and only if } P(+1) \not\sim Q(+1) \ .$$

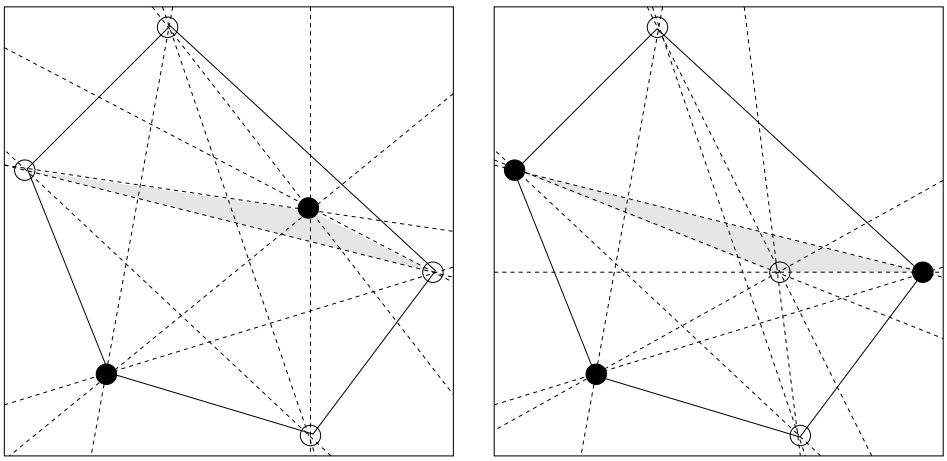

Figure 2: Two configurations of 6 points related by a geometric flip

Proposition 5.2 suggests also perhaps interesting problems concerning generic configurations: Call two generic configurations of n points in $\mathbf{R}^{2d+1}$ *Orchard-equivalent* if they are related by flips whose flipsets have always exactly $(d+1)$ points in each class.

More generally, flips are of different types according to the number of points of each class involved in the corresponding flipset. A very special type of flips are the *monochromatic* ones, defined as involving only vertices of one class in their flipset.

Understanding isomorphism classes of generic configurations up to flips subject to some restrictions (e.g. only monochromatic flips or configurations up to orchard-equivalence in odd dimensions) might be interesting.

We close this section by discussing two further examples.

Example. Consider a configuration $\mathcal{P} \subset \mathbf{S}^2 \subset \mathbf{R}^3$ consisting of n points contained in the Euclidean unit sphere $\mathbf{S}^2$ and which are generic as a a subset of $\mathbf{R}^3$ in the above sense, i.e. 4 distinct points of $\mathcal{P}$ are never contained in a common affine plane of $\mathbf{R}^3$. A stereographic projection $\pi : \mathbf{S}^3 \setminus \{N\} \longrightarrow \mathbf{R}^2$ with respect to a point $N \in \mathbf{S}^2 \setminus \mathcal{P}$ sends the set $\mathcal{P} \subset \mathbf{S}^2$ into a set

$\tilde{\mathcal{P}} = \pi(\mathcal{P}) \subset \mathbf{R}^2$ such that 4 points of $\tilde{\mathcal{P}}$ are never contained in a common Euclidean circle or line of $\mathbf{R}^2$. The Orchard relation on $\mathcal{P}$ can now be seen on $\tilde{\mathcal{P}}$ as follows: Given two distinct points $\tilde{P} \neq \tilde{Q} \in \tilde{\mathcal{P}}$ count the number $s(\tilde{P}, \tilde{Q})$ of circles or lines determined by 3 points in $\tilde{\mathcal{P}} \setminus \{\tilde{P}, \tilde{Q}\}$ which separate them. Two distinct points $P \neq Q \in \mathcal{P}$ are now Orchard-equivalent if and only if $s(\tilde{P}, \tilde{Q}) \equiv \binom{n-3}{2} \pmod 2$. This example can of course be generalised to finite generic configurations of points on the the $d-$dimensional unit sphere $\mathbf{S}^d \subset \mathbf{R}^{d+1}$ for $d \geq 2$.

Let $\mathcal{C}$ be a set of continuous real functions on $\mathbf{R}^k$. Suppose $\mathcal{C}$ is a $(d+1)-$dimensional vector space containing the constant functions. Call a set $\mathcal{P} \subset \mathbf{R}^k$ of n points $\mathcal{C}-$*generic* if for each subset $S = \{P_{i_1}, \ldots, P_{i_d})$ of d distinct points in $\mathcal{P}$ the set

$$I(S) = \{f \in \mathcal{C} \mid f(P_{i_j}) = 0,\ j = 0 \ldots, d\}$$

is a $1-$dimensional affine line and all $\binom{n}{d}$ affine lines in $\mathcal{C}$ of this form are distinct.

Given $P, Q \in \mathcal{P}$, call a set $S = \{P_{i_1}, \ldots, P_{i_d}\} \subset \mathcal{P} \setminus \{P, Q\}$ of d points as above $\mathcal{C}-$*separating* (or *separating* for short) if $f(P)f(Q) < 0$ for any $0 \neq f \in I(S)$ and denote by $s_{\mathcal{C}}(P, Q)$ the number of $\mathcal{C}-$separating subsets of $\mathcal{P}$.

Proposition 5.3. *The relation $P \sim_{\mathcal{C}} Q$ if either $P = Q$ or*

$$s_{\mathcal{C}}(P, Q) \equiv \binom{n-3}{d-1} \pmod 2$$

defines an equivalence relation having at most two classes on a set $\mathcal{P} = \{P_1, \ldots, P_n\} \subset \mathbf{R}^k$ of n points in $\mathbf{R}^k$ which are $\mathcal{C}-$generic.

Proof. Consider the linear map $V : \mathbf{R}^k \longrightarrow \mathbf{R}^d$ defined by $x \longmapsto V(x) = (b_1(x), \ldots, b_d(x))$ where $1, b_1, \ldots, b_d$ is a basis of the vector space $\mathcal{C}$. The image $V(\mathcal{P})$ of a $\mathcal{C}-$generic set $\mathcal{P} \subset \mathbf{R}^k$ is a generic subset of $\mathbf{R}^d$ and the relation defined by Proposition 5.3 coincides with the Orchard-relation described for instance by 5.1. □

Examples. (i) Considering the $(d+1)-$dimensional vector space of all affine functions in $\mathbf{R}^d$, Corollary 1.6 boils down to Theorem 1.1.

(ii) Consider the $6-$dimensional vector space $\mathcal{C}$ of all polynomial functions $\mathbf{R}^2 \longrightarrow \mathbf{R}$ of degree at most 2. A finite subset $\mathcal{P} \subset \mathbf{R}^2$ is $\mathcal{C}-$generic if and only if every subset of five points in $\mathcal{P}$ defines a unique conic and all these conics are distinct.

(iii) Consider the vector space $\mathcal{C}$ of all polynomials of degree $< d$ in x together with the polynomials $\lambda y,\ \lambda \in \mathbf{R}$. A subset $\mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ with $x_1 < x_2 < \ldots, < x_n$ is $\mathcal{C}-$generic if all $\binom{n}{d}$ interpolation polynomials in x defined by d points of $\mathcal{P}$ are distinct.

6. Orientable sets

In the following sections we consider a set E together with a fixpoint-free involution $\iota : E \longrightarrow E$. We call ι the *orientation-reversion* and the pair (E, ι) an *orientable set*. The aim of the following sections is to define the Orchard morphism for finite orientable sets. In this case, all groups

and homomorphisms are required to be also natural with respect to the involution ι.

Examples of orientable sets are for instance antipodal sets of points in $\mathbf{R}^d \setminus \{0\}$ or points of real Grassmannians endowed with orientations.

In the sequel we denote by $\pi : E \longrightarrow \overline{E} = E/\iota$ the quotient map $x \longrightarrow \overline{x} = \{x, \iota(x)\}$ onto the underlying (unoriented) quotient set. The set of all sections

$$\mathcal{S}(E, \iota) = \{s : \overline{E} \longrightarrow E | \pi \circ s(\overline{x}) = \overline{x} \ , \forall \overline{x} \in \overline{E}\}$$

is endowed with a free action of the group $(\pm 1)^{\overline{E}}$ of all functions $\overline{E} \longrightarrow \{\pm 1\}$ if we set $(\alpha s)(\overline{x}) = s(x)$ if $\alpha(\overline{x}) = 1$ and $(\alpha s)(\overline{x}) = \iota(s(x))$ otherwise where $\alpha : \overline{E} \longrightarrow \{\pm 1\}$ and $s \in \mathcal{S}(E, \iota)$. The quotient set $\mathcal{S}(E, \iota)/\{\pm 1\}$ corresponds to orientations defined up to global reversion (action of ι). We call an element of the quotient group $\mathcal{S}(E, \iota)/\{\pm 1\}$ a *semi-orientation* of the orientable set (E, ι).

Given an orientable set (E, ι), its automorphism group $\mathrm{Sym}(E, \iota)$ is the set of all $\iota-$equivariant permutations of E. Otherwise stated, a permutation $\pi : E \longrightarrow E$ belongs to $\mathrm{Sym}(E, \iota)$ if and only if $\pi(\iota(x)) = \iota(\pi(x))$ for all $x \in E$. As an abstract group, the group $\mathrm{Sym}(E, \iota)$ is easily seen to be isomorphic to the group of all isometries of the $e-$dimensional regular standard cube $[-1, 1]^e \subset \mathbf{R}^e$ where $2e = |E|$ is the cardinality of E. This group has $2^e\ e!$ elements and is the wreath product of $\mathrm{Sym}(\overline{E})$ with $\{\pm 1\}^e$. We have an obvious surjective homomorphisme $\mathrm{Sym}(E, \iota) \longrightarrow \mathrm{Sym}(\overline{E})$ with kernel $\{\pm 1\}^e$.

7. Two-sets of orientable sets

Given an orientable set (E, ι) it is natural to consider the set $\mathcal{E}(E, \iota)$ of all two-partitions of E which are invariant under ι. This set contains the subset $\mathcal{E}(E, \iota_+)$ consisting of all two-partitions factoring through π and inducing a two-partition on the quotient set $\overline{E}$. Otherwise stated, two elements x and $\iota(x)$ in an orbit under ι belong always to the same class. We call such a two-partition *even* since its classes are given $\alpha^{-1}(1)$ and $\alpha^{-1}(-1)$ where $\alpha : E \longrightarrow \{\pm 1\}$ is an even function with respect to the involution ι (it satisfies $\alpha(x) = \alpha(\iota(x))$ for all $x \in E$). Its complement $\mathcal{E}(E, \iota_-) = \mathcal{E}(E, \iota) \setminus \mathcal{E}(E, \iota_+)$, called the *odd* two-partitions, has equivalence classes defined as preimages of an odd function $\alpha : E \longrightarrow \{\pm 1\}$ satisfying $\alpha(x) = -\alpha(\iota(x))$ for all x. The set $\mathcal{E}(E, \iota_-)$ of all odd two-partitions on (E, ι) coincides with the set of semi-orientations of the orientable set (E, ι). Its elements are unordered pairs $\{s, \iota \circ s\}$ of complementary sections of the quotient map $\pi : E \longrightarrow \overline{E}$.

The set $\mathcal{E}(E, \iota) = \mathcal{E}(E, \iota_+) \cup \mathcal{E}(E, \iota_-)$ obtained by considering all even or odd two-partitions on the orientable set (E, ι) is a vector space (of dimension $\sharp(E)/2$ if E is finite) over $\mathbf{F}_2$. An element of $\mathcal{E}(E, \iota)$ is represented by $\pm\alpha$ where the function $\alpha : E \longrightarrow \{\pm 1\}$ is either even ($\alpha(\iota x) = \alpha(x)$ for all $x \in E$) or odd ($\alpha(\iota x) = -\alpha(x)$ for all $x \in E$) with respect to ι. The pair ± 1 of constant even functions represents the identity element and the group law is the usual product of (pairs of) functions. Given an element $\{\pm\alpha\} \in \mathcal{E}(E, \iota)$, we define a *parity homomorphism* $\mathcal{E}(E, \iota) \longrightarrow \{\pm 1\}$ by setting $\mathrm{parity}(\pm\alpha) = 1$ if $\{\pm\alpha\} \in \mathcal{E}(E, \iota_+)$ is a pair of even functions and $\mathrm{parity}(\pm\alpha) = -1$ if α is an odd function.

Given an orientable set (E,ι), we define the set $(E,\iota)^{(l)}$ as the set of all sequences $(x_1,\dots,x_l)\in E^l$ of length l such that $(\overline{x}_1,\dots,\overline{x}_l)\in\overline{E}^{(l)}$. Otherwise stated, such a sequence satisfies $\{x_i,\iota(x_i)\}\neq\{x_j,\iota(x_j)\}$ for $i\neq j$.

Proposition 7.1. *Any even (respectively odd) symmetric function*

$$\sigma:(E,\iota)^{(2)}\longrightarrow\{\pm1\}$$

with

$$\sigma(a,b)\sigma(b,c)\sigma(a,c)=\gamma\in\{\pm1\}$$

independent of $(a,b,c)\in(E,\iota)^{(3)}$ *gives rise to an even (respectively odd) two-partition on* (E,ι).

Proof. Results from Proposition 2.2 if σ is even.

For σ odd, choose a section $s:\overline{E}\longrightarrow E$ and define the two-partition in the obvious way on the section. This two-partition extends to a unique odd two-partition on (E,ι) which is independent of the choice of the section s. □

Remark 7.2. *The above equivalence relation can be constructed as follows: Choose a fixed base point* $x_0\in E$. *Set* $\alpha(x_0)=1$ *and* $\alpha(\iota(x_0))=$ *parity*(σ) *where parity*$(\sigma)=1$ *if* σ *is even and parity*$(\sigma)=-1$ *if* σ *is odd. For* $y\notin\{x_0,\iota(x_0)\}$ *we set* $\alpha(y)=\gamma\ \sigma(x_0,y)$ *with* $\gamma\in\{\pm1\}$ *as in Proposition 7.1.*

8. Symmetric and antisymmetric functions on orientable sets

Recall that $(E,\iota)^{(l)}$ denotes the set of all sequences $(x_1,\dots,x_l)\in E^l$ such that $(\overline{x_1},\dots,\overline{x_l})\in\overline{E}^{(l)}$.

One defines $l-$symmetric (respectively $l-$antisymmetric) functions on $(E,\iota)^{(l)}$ in the obvious way as the subset of functions which are invariant (respectively which change sign) under transposition of two arguments.

A symmetric or antisymmetric function $\varphi:(E,\iota)^{(l)}\longmapsto\{\pm1\}$ is *even* if

$$\varphi(x_1,x_2,\dots,x_l)=\varphi(\iota(x_1),x_2,\dots,x_l)=\varphi(x_1,\iota(x_2),x_3,\dots,x_l)=\dots\ .$$

We denote by $\mathcal{F}_\pm(E,\iota_+)^{(l)}$ the set of all even $l-$symmetric or $l-$antisymmetric functions. Notice that there exists an obvious bijection between $\mathcal{F}_\pm(E,\iota_+)^{(l)}$ and $\mathcal{F}_\pm(\overline{E}^{(l)})$.

Such a function is *odd* if

$$\varphi(x_1,x_2,\dots,x_l)=-\varphi(\iota(x_1),x_2,\dots,x_l)=-\varphi(x_1,\iota(x_2),x_3,\dots,x_l)=\dots\ .$$

The set of all odd symmetric or antisymmetric functions on $(E,\iota)^{(l)}$ will be denoted by $\mathcal{F}_\pm(E,\iota_-)^{(l)}$.

We denote by $\mathcal{F}_\pm(E,\iota)^{(l)}=\mathcal{F}_\pm(E,\iota_+)^{(l)}\cup\mathcal{F}_\pm(E,\iota_-)^{(l)}$ the set of all even or odd, $l-$symmetric or $l-$antisymmetric functions on the orientable set (E,ι). The set $\mathcal{F}_\pm(E,\iota)^{(l)}$ is of course a vector space (of dimension $\binom{\sharp(E)/2}{l}+2$ is E is finite) over $\mathbf{F}_2$. The set $\mathcal{F}_\pm(E,\iota_-)^{(l)}$ is a free $\mathcal{F}_\pm(E,\iota_+)^{(l)}-$set.

We define the signature and parity homomorphismes sign, parity $:\mathcal{F}_\pm(E,\iota)^{(l)}\longrightarrow\{\pm1\}$ by

$$\begin{array}{ll}\mathrm{sign}(\varphi)=1 \text{ if } \varphi\in\mathcal{F}_+(E,\iota)^{(l)}, & \mathrm{sign}(\varphi)=-1 \text{ if } \varphi\in\mathcal{F}_-(E,\iota)^{(l)}\ ,\\ \mathrm{parity}(\varphi)=1 \text{ if } \varphi\in\mathcal{F}_\pm(E,\iota_+)^{(l)}, & \mathrm{parity}(\varphi)=-1 \text{ if } \varphi\in\mathcal{F}_\pm(E,\iota_-)^{(l)}\ .\end{array}$$

9. The Orchard morphism for finite orientable sets

Given $\varphi \in \mathcal{F}_\pm(E,\iota)^{(l)}$ where (E,ι) is a finite orientable set, we define $\sigma_\varphi : (E,\iota)^{(2)} \longrightarrow \{\pm 1\}$ by setting

$$\sigma_\varphi(y,z) = \prod_{(\overline{x}_1,\ldots,\overline{x}_{l-1})\in\binom{\overline{E}\setminus\{\overline{y},\overline{z}\}}{l-1}} \varphi(x_1,\ldots,x_{l-1},y)\ \varphi(x_1,\ldots,x_{l-1},z)$$

where $x_1 = s(\overline{x}_1),\ldots,x_{l-1} = s(\overline{x}_{l-1})$ are obtained using an arbitrary section $s : \overline{E} \longrightarrow E$ of the quotient map $\pi : E \longrightarrow \overline{E} = E/\iota$.

Proposition 9.1. *Let $\varphi \in \mathcal{F}_\pm(E,\iota)^{(l)}$ be a function and define σ_φ as above.*

(i) The function σ_φ is well defined, symmetric and satisfies the identity

$$\sigma_\varphi(a,b)\sigma_\varphi(b,c)\sigma_\varphi(a,c) = (sign(\varphi))^{\binom{e-3}{l-2}}$$

for all $(a,b,c) \in (E,\iota)^{(3)}$ where $2e = |E| = 2\ |\overline{E}|$ is the cardinality of E.

(ii) If $\varphi \in \mathcal{F}_\pm(E,\iota_+)^{(l)}$ (i.e. φ even), then σ_φ is even.

(iii) If $\varphi \in \mathcal{F}_\pm(E,\iota_-)^{(l)}$ (i.e. φ odd), then σ_φ is even if $\binom{e-2}{l-1} \equiv 0 \pmod 2$ and odd otherwise.

Proof. Every element $(\overline{x}_1,\ldots,\overline{x}_{l-1}) \in \binom{\overline{E}\setminus\{\overline{y},\overline{z}\}}{l-1}$ is involved twice in $\sigma_\varphi(y,z)$ thus implying that the final value is independent of the choosen total order on $\overline{E}\setminus\{y,z\}$ and of the choosen section $s : \overline{E} \longrightarrow E$.

The definition of σ_φ is obviously symmetric with respect to its arguments.

The proof of the identity $\sigma_\varphi(a,b)\sigma_\varphi(b,c)\sigma_\varphi(a,c) = (\mathrm{sign}(\varphi))^{\binom{e-3}{l-2}}$ is exactly analogous to the corresponding proof in the non-orientable case.

Assertion (ii) is almost obvious since we have

$$\varphi(x_1,\ldots,x_{l-1},u) = \varphi(x_1,\ldots,x_{l-1},\iota(u))$$

for $u \in \{a,b\}$ and $\varphi \in \mathcal{F}_\pm(E^{(l)},\iota_+)$ even.

Assertion (iii) follows from the fact that

$$\varphi(x_1,\ldots,x_{l-1},u) = -\varphi(x_1,\ldots,x_{l-1},\iota(u))$$

for $u \in \{a,b\}$ and $\varphi \in \mathcal{F}_\pm(E^{(l)},\iota_-)$ odd and from the observation that the definition of $\sigma_\varphi(a,b)$ involves $\binom{e-2}{l-1}$ such factors. □

The Orchard morphism $\rho : \mathcal{F}_\pm(E^{(l)},\iota) \longrightarrow \mathcal{E}(E,\iota)$ associates to a function $\varphi \in \mathcal{F}_\pm(E^{(l)},\iota)$ the two-partition in $\mathcal{E}(E,\iota)$ associated to σ_φ by Proposition 7.1.

Theorem 9.2. *For $2l < \sharp(E)$ and $\sharp(E) \geq 6$, the oriented Orchard morphism is the unique $Sym(E,\iota)-$equivariant homomorphism from $\mathcal{F}_\pm(E,\iota)^{(l)}$ to $\mathcal{E}(E,\iota)$ which is non-trivial.*

Remark 9.3. *If (E,ι) is an orientable set containing 4 elements $\pm a, \pm b$ (with ι given by $\iota(a) = -a$ and $\iota(b) = -b$), there exist several non-trivial natural homomorphisms $\mathcal{F}_\pm(E,\iota)^{(l)} \longrightarrow \mathcal{E}(E,\iota)$ for $l = 1,2$.*

An example (distinct from the Orchard morphism) for $l = 1$ is given by $\rho'(\varphi) =$ trivial if $\varphi \in \mathcal{F}(E,\iota_+)^{(1)}$ and $\rho'(\varphi) = (E = \{\pm a\} \cup \{\pm b\})$ if $\varphi \in \mathcal{F}(E,\iota_-)^{(1)}$.

For $l = 2$, one can for instance extend the exotic homomorphism of the unoriented case (cf. Remark 4.2) in two ways by choosing an arbitrary

even two-partition as the image $\rho'(\varphi)$ for $\varphi \in \mathcal{F}_+(E,\iota_-)^{(2)}$. The image $\rho'(\psi)$ for $\psi \in \mathcal{F}_-(E,\iota_-)^{(2)}$ is then the unique remaining two-partition (i.e. $\rho'(\varphi)\rho'(\psi) = \rho'(\theta)$ with $\theta \in \mathcal{F}_-(E,\iota_+)^{(2)}$ the unique even non-trivial two-partition of (E,ι)).

Proof. The proof that ρ defines a homomorphism is as in the unoriented case.

The restriction of ρ to the subgroup $\mathcal{F}_\pm(E,\iota_+)^{(l)}$ consisting only of even functions coincides with the usual Orchard morphism $\mathcal{F}_\pm(\overline{E}) \longrightarrow \mathcal{E}(\overline{E})$ on $\overline{E}$ and the result holds for this restriction by Theorem 4.3.

It remains to show unicity of the restriction to $\mathcal{F}_\pm(E,\iota_-)$ of a natural homomorphisme ρ'. The identity $\mathcal{F}_\pm(E,\iota_-) = \varphi\mathcal{F}_\pm(E,\iota_+)$ for $\varphi \in \mathcal{F}_\pm(E,\iota_-)$ and $\mathrm{Sym}(E,\iota)-$equivariance show that such a homomorphism with trivial restriction to $\mathcal{F}_\pm(E,\iota_+)$ is trivial.

We can thus suppose that $\rho' = \rho$ on $\mathcal{F}_\pm(E,\iota_+)$. We denote by $e = \sharp(\overline{E})$ the halved cardinal of E.

Consider now a section $s : \overline{E} \longrightarrow E$ and the unique symmetric odd function $\varphi \in \mathcal{F}_+(E,\iota_-)$ defined by

$$\varphi(s(\overline{x}_{i_1}),\ldots,s(\overline{x}_{i_l})) = 1$$

for all $(\overline{x}_{i_1},\ldots,\overline{x}_{i_l}) \in \overline{E}^{(l)}$. $\mathrm{Sym}(E,\iota)-$equivariance of ρ' implies that $\rho'(\varphi)$ is either trivial or the semi-orientation associated to the section s. Choose now an element $\overline{x} \in \overline{E}$ and consider the corresponding function $\tilde{\varphi}$ associated as above to the section $\tilde{s}$ which coincides with s on $\overline{E} \setminus \{\overline{x}\}$ and sends $\overline{x}$ to $\iota(s(\overline{x}))$. The functions φ and $\tilde{\varphi}$ differ by the product of all $\binom{e-1}{l-1}$ flips with flipsets $\{\overline{x},\overline{y}_1,\ldots,\overline{y}_{l-1}\}$ where $(\overline{y}_1,\ldots,\overline{y}_{l-1}) \in \binom{\overline{E}\setminus\{\overline{x}\}}{l-1}$. An element $\overline{y} \in \overline{E} \setminus \{\overline{x}\}$ is involved in $\binom{e-2}{l-2}$ such flipsets and $\overline{x}$ is involved in $\binom{e-1}{l-1} = \binom{e-2}{l-2} + \binom{e-2}{l-1}$ such flipsets. This shows that $\rho'(\varphi) = \rho'(\tilde{\varphi})$ if $\binom{e-2}{l-1} \equiv 0 \pmod 2$ and $\mathrm{Sym}(E,\iota)-$equivariance forces $\rho'(\varphi)$ to be even. It coincides thus with the Orchard morphism.

If $\binom{e-2}{l-1} \equiv 1 \pmod 2$, the two-partitions $\rho'(\varphi)$ and $\rho'(\tilde{\varphi})$ differ exactly on $\pi^{-1}(\overline{x})$ and $\mathrm{Sym}(E,\iota)-$equivariance forces $\rho'(\varphi)$ to be the semi-orientation of $\mathcal{E}(E,\iota)$ associated to the section s. □

10. Geometric examples

In this section we discuss a few orientable sets arising from geometric configurations: finite generic antipodal configurations of points (or generic configurations of lines through the origin) in $\mathbf{R}^d$ and generic configurations of the real projective space $\mathbf{R}P^d$.

A finite *antipodal set* of $\mathbf{R}^d$ is a finite subset $\mathcal{P} \subset \mathbf{R}^d \setminus \{0\}$ invariant under the involution $x \longmapsto \iota(x) = -x$. We call such a set *generic* if the linear span of any subset $\{\pm x_1,\ldots,\pm x_k\} \subset \mathcal{P}$ is of dimension k for $k \leq d$. We get an element $\varphi \in \mathcal{F}_-(E,\iota)^{(d)}$ by considering the sign $\in \{\pm 1\}$ of

$$\det(x_1,\ldots,x_d)$$

for $(x_1,\ldots,x_d) \in (\mathcal{P},\iota)^{(d)}$ (where $\det(x_1,\ldots,x_d)$ denotes the non-zero determinant of the $d \times d$ matrix with rows $x_1,\ldots,x_d$).

Applying the oriented Orchard morphism ρ of the preceeding section to φ we get a two-partition $\rho(\varphi) \in \mathcal{E}(\mathcal{P}, \iota)$. Obviously, $\rho(\varphi)$ remains the same by rescaling each pair $\pm x \in \mathcal{P}$ by some strictly positive constant $\lambda_x \in \mathbf{R}_{>0}$.

We may rescale such an antipodal set in order to lie on the Euclidean sphere $\mathbf{S}^{d-1} = \{x \in \mathbf{R}^d \mid \quad \| x \| = 1\} \subset \mathbf{R}^d$. Similarly, we can interprete $\mathcal{P}$ as a set $\mathcal{L}$ of lines (defined by opposite pairs $\pm x \in \mathcal{P}$). The Orchard morphism $\rho(\varphi)$ endows then such a generic finite set of lines either with a two-partition (in the case where $\binom{\sharp(\mathcal{L})-2}{d-1} \equiv 0 \pmod 2$) or with a semi-orientation (if $\binom{\sharp(\mathcal{L})-2}{d-1} \equiv 1 \pmod 2$).

A finite subset $\mathcal{P} \subset \mathbf{R}P^d$ of the real projective space is generic if its completed preimage $\mathcal{L} = \overline{\pi^{-1}(\mathcal{P})} \subset \mathbf{R}^{d+1}$ is a finite set of generic lines in $\mathbf{R}^{d+1}$. If $\binom{\sharp(\mathcal{P})-2}{d} \equiv 0 \pmod 2$ we get a two-partition on such a set $\mathcal{P}$ by applying the Orchard morphism to $\mathcal{L}$.

In the case where the Orchard morphism endows $\mathcal{L}$ with a semi-orientation, we get also an interesting structure on $\mathcal{P}$ as follows:

Any pair $P, Q \in \mathcal{P}$ of distinct points defines two connected components on $L_{P,Q} \setminus \{P, Q\}$ where $L_{P,Q} \subset \mathbf{R}P^d$ denotes the projective line containing P and Q. One of these connected components is now selected by a semi-orientation on $\mathcal{L}$ by choosing the connected component of $L_{P,Q} \setminus \{P, Q\}$ whose preimage in $\mathbf{S}^d \subset \mathbf{R}^{d+1} = \pi^{-1}(\mathbf{R}P^d) \cup \{0\}$ joins elements of $\pi^{-1}(\mathcal{P})$ which are in the same class. We get in this way an immersion of the complete graph $K_{\mathcal{P}}$ with vertices $\mathcal{P}$ into the projective space $\mathcal{R}P^d$. It is straightforward to show that this immersion is homologically trivial: each cycle of $K_{\mathcal{P}}$ is immerged in a contractible way into $\mathbf{R}P^d$.

Remark 10.1. *A preliminary version of this paper (cf.* [1]*) contained also a section concerning simple arrangements of (pseudo)lines in the projective plane. The corresponding invariants (two-partitions and semi-orientations) are however not based on the Orchard-morphism but use only Proposition 7.1. They are thus not really related to the topic of this text and will be discussed elsewhere.*

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