



**HAL**  
open science

## The odd-dimensional Goldberg Conjecture

Vestislav Apostolov, Tedi Draghici, Andrei Moroianu

► **To cite this version:**

Vestislav Apostolov, Tedi Draghici, Andrei Moroianu. The odd-dimensional Goldberg Conjecture. 2003. hal-00000563v1

**HAL Id: hal-00000563**

**<https://hal.science/hal-00000563v1>**

Preprint submitted on 26 Aug 2003 (v1), last revised 11 May 2004 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THE ODD-DIMENSIONAL GOLDBERG CONJECTURE

VESTISLAV APOSTOLOV, TEDI DRĂGHICI AND ANDREI MOROIANU

ABSTRACT. An odd-dimensional version of the Goldberg conjecture was formulated and proved in [2], by using an orbifold analogue of Sekigawa’s arguments in [4], and an approximation argument of K-contact structures with quasi-regular ones. We provide here another proof of this result.

2000 *Mathematics Subject Classification.* Primary 53C25, 53C26

## 1. INTRODUCTION

The celebrated Goldberg conjecture states that every compact almost Kähler Einstein manifold  $M$  is actually Kähler–Einstein. This conjecture was confirmed by Sekigawa [4] in the case when  $M$  has non-negative scalar curvature. The odd-dimensional analogues of Kähler manifolds are Sasakian manifolds, and those of almost Kähler manifolds are K-contact manifolds. In [2], Boyer and Galicki proved the following odd-dimensional analogue of Goldberg’s conjecture.

**Theorem 1.1.** [2] *Any compact Einstein K-contact manifold  $(M, g, \xi)$  is Sasakian.*

Their proof goes roughly as follows. First, an Einstein K-contact manifold has prescribed (positive) Einstein constant. If the K-contact structure is quasi-regular (i.e. the orbits of the Reeb vector field  $\xi$  are closed), then the quotient of  $M$  by the flow of  $\xi$  is an orbifold [5] which is Einstein with positive scalar curvature by the O’Neill formulas and one may apply Sekigawa’s proof to obtain the integrability. If  $\xi$  is non-regular, this quotient is no longer an orbifold (and even may not be a tractable topological space). To overcome this difficulty, the authors of [2] provide a beautiful argument to show that the Reeb vector field  $\xi$  can be approximate (in a suitable sense) by a sequence of quasi-regular Reeb vector fields  $\xi_i$  which define K-contact structures on a sequence of (no longer Einstein) metrics  $g_i$  approaching  $g$ . Then for the sequence of orbifolds that one obtains, one can use “approximative” Sekigawa formulas and eventually show that the K-contact structure is integrable.

The aim of this note is to give another proof of Theorem 1.1, by using the following simple observation. Instead of considering the quotient of  $M$  by the Reeb flow (which can well be irregular) we consider another almost Kähler manifold naturally associated to  $M$ , namely the cone over  $M$ . It is well-known that the cone is a smooth (although *non-compact*) almost Kähler manifold which is Kähler if and only if  $M$  is Sasakian. It therefore suffices to prove the integrability of the almost Kähler cone structure. A difficulty to apply directly the Sekigawa arguments in this situation is the non-compactness

---

The first author was supported in part by NSERC grant OGP0023879.

of the cone, which we overcome easily: we first apply a point-wise version of Sekigawa's formula on the cone manifold, and then integrate it on the level sets of the radial function (which are compact manifolds).

The use of this approach tempted us to extend the conjecture to the more general case of *contact metric structures*, when the metric is no longer bundle-like. Indeed, one could argue that the analogues of almost Kähler manifolds in odd dimensions are the contact metric structures, since they correspond to the level sets of the radial function of almost Kähler cone metrics. However, the analogue of the Goldberg conjecture in this setting, stating that *any compact Einstein contact metric manifold should be Sasakian-Einstein* turns out to be false, as we show by a counterexample in the last section.

## 2. PRELIMINARIES

Let  $(M, g)$  be a Riemannian manifold. We define the cone  $\bar{M} := M \times \mathbb{R}_+^*$  endowed with the metric  $\bar{g} = dr^2 + r^2g$ , and denote by  $\bar{\nabla}$  the covariant derivative of  $\bar{g}$ . It is well-known that the cone is a non-complete Riemannian manifold which can be completed at  $r = 0$  if and only if  $M$  is a round sphere.

Every vector field  $X$  on  $M$  induces in a canonical way a vector field  $(X, 0)$  on  $\bar{M}$ , which (with a slight abuse of notation) will still be denoted by  $X$ . Similarly, we denote by the same symbol the forms on  $M$  and their pull-backs to  $\bar{M}$  (with respect to the projection on the first factor). Let us denote by  $\partial_r$  the vector field  $\frac{\partial}{\partial r}$  on  $\bar{M}$ . The following formulas relate the covariant derivatives  $\nabla$  and  $\bar{\nabla}$ , and are immediate consequences of the definitions.

$$(1) \quad \bar{\nabla}_{\partial_r} \partial_r = 0; \quad \bar{\nabla}_X \partial_r = \bar{\nabla}_{\partial_r} X = \frac{1}{r} X; \quad \bar{\nabla}_X Y = \nabla_X Y - rg(X, Y) \partial_r.$$

Using this, we obtain for every vector  $X$  and a  $p$ -form  $\omega$  on  $M$

$$(2) \quad \bar{\nabla}_{\partial_r} \omega = -\frac{p}{r} \omega \quad \text{and} \quad \bar{\nabla}_X \omega = \nabla_X \omega - \frac{1}{r} dr \wedge X \lrcorner \omega,$$

$$(3) \quad \bar{\nabla}_{\partial_r} dr = 0 \quad \text{and} \quad \bar{\nabla}_X dr = rX^\flat.$$

The curvature tensors  $R$  and  $\bar{R}$  of  $M$  and  $\bar{M}$ , respectively, are related by

$$(4) \quad \bar{R}(\partial_r, \cdot) = 0 \quad \text{and} \quad \bar{R}(X, Y)Z = R(X, Y)Z + g(X, Z)Y - g(Y, Z)X.$$

**Definition 2.1.** A contact metric structure on a Riemannian manifold  $M$  is a unit length vector field  $\xi$  such that the 1-form  $\eta := \langle \xi, \cdot \rangle$  and the endomorphism  $\varphi$  associated to  $\frac{1}{2}d\eta$  are interrelated by

$$(5) \quad \varphi^2 = -1 + \eta \otimes \xi.$$

Since  $\varphi^2(\xi) = 0$ , we get  $|\varphi(\xi)|^2 = -\langle \xi, \varphi^2(\xi) \rangle = 0$ , so  $\varphi(\xi) = 0$ . In other words,  $\varphi$  defines a complex structure on the distribution orthogonal to  $\xi$ .

A contact metric structure  $(M, g, \xi, \varphi, \eta)$  is called K-contact if  $\xi$  is Killing. The K-contact structure  $(M, g, \xi, \varphi, \eta)$  is called Sasakian if

$$(6) \quad \nabla \cdot \nabla \xi = \xi \wedge \cdot.$$

Given a contact metric manifold  $(M, g, \xi, \varphi, \eta)$  we construct a 2-form  $\Omega$  on  $\bar{M}$ , defined by

$$(7) \quad \Omega = r dr \wedge \eta + \frac{r^2}{2} d\eta.$$

This 2-form is clearly compatible with  $\bar{g}$ , and therefore defines an almost complex structure  $J$  on  $\bar{M}$  by  $\Omega(\cdot, \cdot) = \bar{g}(J\cdot, \cdot)$ . Moreover,  $\Omega$  is obviously closed, meaning that  $(\bar{M}, J)$  is almost Kähler. It is well-known that  $\Omega$  is parallel (i.e.  $(\bar{M}, J)$  is Kähler) if and only if the contact structure  $\xi$  is Sasakian.

We close this section with the following

**Lemma 2.2.** (i) *The codifferentials on  $M$  and  $\bar{M}$  are related by*

$$(8) \quad \delta^{\bar{M}}(r^k \sigma) = r^{k-2} \delta^M \sigma, \quad \forall \sigma \in \Lambda^1 M.$$

(ii) *The Laplacians on  $M$  and  $\bar{M}$  are related by*

$$(9) \quad \Delta^{\bar{M}}(r^k f) = r^{k-2}(\Delta^M f - k(2n + k)f), \quad \forall f \in C^\infty(M).$$

*Proof.* (i) If  $(e_i)$  denotes a local orthonormal base on  $M$ , we have

$$\begin{aligned} \delta^{\bar{M}}(r^k \sigma) &= \sum_i \left( -\frac{e_i}{r} (r^k \sigma(\frac{e_i}{r})) + r^k \sigma(\bar{\nabla}_{\frac{e_i}{r}} \frac{e_i}{r}) \right) - \partial_r(r^k \sigma(\partial_r)) + r^k \sigma(\bar{\nabla}_{\partial_r} \partial_r) \\ &= \sum_i -r^{k-2} e_i(\sigma(e_i)) + r^{k-2} \sigma(\nabla_{e_i} e_i) - r \partial_r = r^{k-2} \delta^M \sigma. \end{aligned}$$

(ii) Similarly,

$$\begin{aligned} \Delta^{\bar{M}}(r^k f) &= \sum_i \left( -\frac{e_i}{r} \left( \frac{e_i}{r} (r^k f) \right) + \bar{\nabla}_{\frac{e_i}{r}} \frac{e_i}{r} (r^k f) \right) - \partial_r(\partial_r(r^k f)) \\ &= \sum_i \left( -r^{k-2} e_i(e_i(f)) + \frac{1}{r^2} (\nabla_{e_i} e_i - r \partial_r)(r^k f) \right) - k(k-1)r^{k-2} f \\ &= r^{k-2} \Delta^M f - k(2n+1)r^{k-2} f - k(k-1)r^{k-2} f \\ &= r^{k-2} (\Delta^M f - k(2n+k)f). \end{aligned}$$

□

### 3. PROOF OF THEOREM 1.1

Let  $(M^{2n+1}, g, \xi)$  be a compact K-contact Einstein manifold with Einstein constant  $\lambda$ . As  $\xi$  is Killing, we have  $\Delta \xi = 2\text{Ric}(\xi)$  so that taking the scalar product with  $\xi$  and integrating over  $M$  yields

$$\lambda \text{vol}(M) = \int_M \text{Ric}(\xi, \xi) d\mu = \frac{1}{2} \int_M \langle \Delta \xi, \xi \rangle d\mu = \frac{1}{2} \int_M |d\xi|^2 d\mu = \frac{1}{2} \int_M (4n) d\mu,$$

and, therefore,  $\lambda = 2n$ .

Consider now the cone  $\bar{M}$ , which is an almost Kähler manifold. We use the following Weitzenböck–type formula, taken from [1, Prop.2.1].

**Proposition 3.1.** *For any almost Kähler manifold  $(\bar{M}, \bar{g}, J, \Omega)$  with covariant derivative denoted by  $\bar{\nabla}$  and curvature tensor  $\bar{R}$ , the following point-wise relation holds:*

$$(10) \quad \begin{aligned} \Delta(s^* - s) = & -4\delta(J\delta^{\bar{\nabla}}(J\bar{\text{Ric}})) + 8\delta(\langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle) + 2|\bar{R}ic''|^2 \\ & - 8|\bar{R}''|^2 - |\bar{\nabla}^* \bar{\nabla} \Omega|^2 - |\phi|^2 + 4\langle \rho, \phi \rangle - 4\langle \rho, \bar{\nabla}^* \bar{\nabla} \Omega \rangle, \end{aligned}$$

where:  $s$  and  $s^*$  are respectively the scalar and  $*$ -scalar curvature,  $\phi(X, Y) = \langle \bar{\nabla}_{JX} \Omega, \bar{\nabla}_Y \Omega \rangle$ ,  $\bar{R}ic''$  is the  $J$ -anti-invariant part of the Ricci tensor  $\bar{\text{Ric}}$ ,  $\rho$  is the  $(1, 1)$ -form associated to the  $J$ -invariant part of  $\bar{\text{Ric}}$ ,  $\bar{\rho}^* := \bar{R}(\Omega)$  and  $\bar{R}''$  denotes a certain component of the curvature tensor.

In our situation, since  $M^{2n+1}$  is Einstein with constant  $2n$ , (4) shows that  $\bar{M}$  is Ricci-flat. So the formula above becomes

$$(11) \quad \Delta^{\bar{M}} s^* - 8\delta^{\bar{M}}(\langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle) = -8|\bar{R}''|^2 - |\bar{\nabla}^* \bar{\nabla} \Omega|^2 - |\phi|^2$$

We now use Lemma 2.2 in order to express the left-hand side of this equality in terms of the codifferential and Laplacian on  $M$ . From (4) we get  $\bar{\rho}^*(X, \partial_r) = 0$  and  $\bar{\rho}^*(X, Y) = \bar{g}(\bar{R}(\frac{e_i}{r}, J\frac{e_i}{r})X, Y) = \rho^*(X, Y)$ , for some 2-form  $\rho^*$  on  $M$ . Taking the scalar product with  $\Omega$  yields

$$(12) \quad s^* = \frac{1}{r^2} f$$

for some function  $f$  on  $M$ . Note that  $f$  is everywhere positive on  $M$  since  $s^* = s^* - s = |\bar{\nabla} \Omega|^2$  on  $\bar{M}$  (see [1], p. 777).

Now, from (2), (3) and (7) we get  $\bar{\nabla}_{\partial_r} \Omega = 0$  and  $\bar{\nabla}_X \Omega = r^2 \omega + r dr \wedge \tau_X$  for some 2-form  $\omega$  and 1-form  $\tau_X$  on  $M$ . Consequently, the 1-form  $\langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle$  on  $\bar{M}$  is easily seen to be of the form

$$(13) \quad \langle \bar{\rho}^*, \bar{\nabla} \cdot \Omega \rangle = \frac{1}{r^2} \alpha$$

for some 1-form  $\alpha$  on  $M$ . Using (12), (13) and Lemma 2.2, the equality (11) becomes

$$(14) \quad \frac{1}{r^4} (\Delta^M f + 2(2n - 2)f - 8\delta^M \alpha) = -8|\bar{R}''|^2 - |\bar{\nabla}^* \bar{\nabla} \Omega|^2 - |\phi|^2.$$

Integrating this last equation on each level set  $M_r := \{r = \text{constant}\}$  of  $\bar{M}$  yields

$$\int_{M_r} \frac{2(2n - 2)}{r^4} f + 8|\bar{R}''|^2 + |\bar{\nabla}^* \bar{\nabla} \Omega|^2 + |\phi|^2.$$

In particular, since  $f \geq 0$ ,  $\phi$  vanishes identically on  $\bar{M}$ , hence  $|\bar{\nabla}_X \Omega|^2 = -\phi(X, JX) = 0$  for every  $X$  on  $\bar{M}$ . Thus  $\bar{M}$  is Kähler, so  $M$  is Sasakian.

## 4. A COUNTEREXAMPLE TO FURTHER EXTENSION

As explained in the introduction, it was tempting to ask the following question, slightly more general than Theorem 1.1: *is every compact Einstein contact metric manifold Sasakian-Einstein?* We answer here this question in the negative by providing a counterexample.

**Proposition 4.1.** *The 1-form  $\eta := \cos t \, dx + \sin t \, dy$  defines a contact metric structure on the flat torus  $T^3$  (where  $t$ ,  $x$  and  $y$  are standard coordinates on  $T^3$  of periods  $2\pi$ ), which is not Sasakian.*

*Proof.* Let  $\xi$  be the vector field on  $T^3$ , which is the dual of  $\eta$  with respect to the flat metric  $g = dx^2 + dy^2 + dt^2$ . It is easy to check that (in dimension 3) the condition of being *metric contact* translates to  $\xi \lrcorner d\eta = 0$  and  $\eta \wedge d\eta = \text{vol}$ . Both equations are straightforward in our case. Finally,  $\eta$  does not define a Sasakian structure because its metric dual,  $\xi$ , is clearly not a Killing vector field.  $\square$

As a final remark, we note that constructing a counterexample *with positive scalar curvature* to the above question, amounts to finding eigenforms of degree 1 of the Laplace operator on the round sphere  $S^3$  with constant norm (other than the dual of the Killing vector fields defining the Hopf fibrations). We have not studied the question of existence of such forms thoroughly.

## REFERENCES

- [1] V. Apostolov, T. Drăghici, A. Moroianu, *A splitting theorem for Kähler manifolds whose Ricci tensors have constant eigenvalues*, Int. J. Math. **12** (2001), 769–789.
- [2] C. Boyer, K. Galicki, *Einstein manifolds and contact geometry*, Proc. Amer. Math. Soc. **129** (2001), 2419–2430.
- [3] S. Gallot, *Équations différentielles caractéristiques de la sphère*, Ann. Sci. Ec. Norm. Sup. Paris **12** (1979), 235–267.
- [4] K. Sekigawa, *On some compact Einstein almost Kähler manifolds*, J. Math. Soc. Japan **39** (1987), 677–684.
- [5] C. B. Thomas, *Almost regular contact manifolds*, J. Differential Geom. **11** (1976), 521–533.

VESTISLAV APOSTOLOV, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DU QUÉBEC, À MONTRÉAL,  
CASE POSTALE 8888, SUCCURSALE CENTRE-VILLE, MONTRÉAL (QUÉBEC), H3C 3P8, CANADA

*E-mail address:* apostolo@math.uqam.ca

TEDI DRĂGHICI, DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI  
FL 33199, USA

*E-mail address:* draghici@fiu.edu

ANDREI MOROIANU, CMAT, ÉCOLE POLYTECHNIQUE, UMR 7640 DU CNRS, 91128 PALAISEAU,  
FRANCE

*E-mail address:* am@math.polytechnique.fr