Universal Deformation Formulae for Three-Dimensional Solvable Lie groups

Pierre Bieliavsky, Philippe Bonneau, Yoshiaki Maeda

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Pierre Bieliavsky
Université Libre de Bruxelles, Belgium,
pbiel@ulb.ac.be
Philippe Bonneau
Université de Bourgogne, France,
bonneau@u-bourgogne.fr
Yoshiaki Maeda
Keio University, Japan,
maeda@math.keio.ac.jp

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Abstract

We apply methods from strict quantization of solvable symmetric spaces to obtain universal deformation formulae for actions of every three-dimensional solvable Lie group. We also study compatible co-products by generalizing the notion of smash product in the context of Hopf algebras. We investigate in particular the dressing action of the 'book' group on $SU(2)$.

1 Introduction

Let G be a group acting on a set M. Denote by $\tau : G \times M \to M : (g, x) \mapsto \tau_g(x)$ the (left) action and by $\alpha : G \times \mathrm{Fun}(M) \to \mathrm{Fun}(M)$ the corresponding action on the space of (complex valued) functions (or formal series) on M ($\alpha_g := \tau^{\star}_{g^{-1}}$). Assume that on a subspace $\mathbb{A} \subset \mathrm{Fun}(G)$, one has an associative $\mathbb{C}$-algebra product $\star^G_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that

(i) $\mathbb{A}$ is invariant under the (left) regular action of G on $\mathrm{Fun}(G)$,

(ii) the product $\star^G_{\mathbb{A}}$ is left-invariant as well i.e. for all $g \in G; a, b \in \mathbb{A}$, one has

$$(L^{\star}_g a) \star^G_{\mathbb{A}} (L^{\star}_g b) = L^{\star}_g (a \star^G_{\mathbb{A}} b).$$

Given a function on M, $u \in \mathrm{Fun}(M)$, and a point $x \in M$, one denotes by $\alpha^x(u) \in \mathrm{Fun}(G)$ the function on G defined as

$$\alpha^x(u)(g) := \alpha_g(u)(x).$$

Then one readily observes that the subspace $\mathbb{B} \subset \mathrm{Fun}(M)$ defined as

$$\mathbb{B} := \{u \in \mathrm{Fun}(M) \,|\, \forall x \in M : \alpha^x(u) \in \mathbb{A}\}$$

becomes an associative $\mathbb{C}$-algebra when endowed with the product $\star^M_{\mathbb{B}}$ given by

$$u \star^M_{\mathbb{B}} v(x) := (\alpha^x(u) \star^G_{\mathbb{A}} \alpha^x(v))(e) \qquad (1)$$

(e denotes the neutral element of G). Of course, all this can be defined for right actions as well.

Definition 1.1 *Such a pair* $(\mathbb{A}, \star^G_{\mathbb{A}})$ *is called a (left)* **universal deformation** *of* G*, while Formula (1) is called the associated* **universal deformation formula** *(briefly* **UDF***).*

In the present article, we will be concerned with the case where G is a Lie group. The function space $\mathbb{A}$ will be either
- a functional subspace (or a topological completion) of $C^\infty(G,\mathbb{C})$ containing the smooth compactly supported functions in which case we will talk about **strict deformation** (following Rieffel [Rie89]),
or,
- the space $\mathbb{A} = C^\infty(G)[[\hbar]]$ of formal power series with coefficients in the smooth functions on G in which case, we'll speak about **formal deformation**. In any case, we'll assume the product $\star^G_{\mathbb{A}}$ admits an asymptotic expansion of star-product type:

$$a \star^G_{\mathbb{A}} b \sim ab + \frac{\hbar}{2i}\mathbf{w}(\mathrm{d}u, \mathrm{d}v) + o(\hbar^2) \qquad (a, b \in C^\infty_c(G)),$$

where $\mathbf{w}$ denotes some (left-invariant) Poisson bivector on G[BFFLS]. In the strict cases considered here, the product will be defined by an integral three-point kernel $K \in C^\infty(G \times G \times G)$:

$$a \star^G_{\mathbb{A}} b(g) := \int_{G\times G} a(g_1)\, b(g_2) K(g, g_1, g_2) \mathrm{d}g_1\, \mathrm{d}g_2 \qquad (a, b \in \mathbb{A})$$

where $\mathrm{d}g$ denotes a normalized left-invariant Haar measure on G. Moreover, our kernels will be of **WKB type** [Wei94, Kar94] i.e.:

$$K = A\, e^{\frac{i}{\hbar}\Phi},$$

with A (the **amplitude**) and Φ (the **phase**) in $C^\infty(G \times G \times G, \mathbb{R})$ being invariant under the (diagonal) action by left-translations.
Note that in the case where the group G acts smoothly on a smooth manifold M by diffeomorphisms: $\tau : G \times M \to M : (g, x) \mapsto \tau_g(x)$, the first-order expansion term of $u \star^M_{\mathbb{B}} v, \quad u, v \in C^\infty(M)$ defines a Poisson structure $\mathbf{w}^M$ on M which can be expressed in terms of a basis $\{X_i\}$ of the Lie algebra $\mathfrak{g}$ of G as:

$$\mathbf{w}^M = [\mathbf{w}_e]^{ij}\, X^\star_i \wedge X^\star_j, \tag{2}$$

where $X^\star$ denotes the fundamental vector field on M associated to $X \in \mathfrak{g}$.

Strict deformation theory in the WKB context was initiated by Rieffel in [Rie93] in the cases where G is either Abelian or 1-step nilpotent. Rieffel's work has led to what is now called 'Rieffel's machinery'; producing a whole class of exciting non-commutative manifolds (in Connes sense) from the data of Abelian group actions on $C^\star$-algebras [CoLa01].
The study of formal UDF's for non-Abelian group actions in our context was initiated in [GiZh98] where the case of the group of affine transformations of the real line ('$ax + b$') was explicitly described.
In the strict (non-formal) setting, UDF's for Iwasawa subgroups of $SU(1, n)$ have been explicitly given in [BiMas01]. These were obtained by adapting a method developed by one of us in the symmetric space framework [Bie00]. In particular when $n = 1$ one obtains strict UDF's for the group '$ax + b$' (we recall this in the appendix).
In the present work, we provide in the Hopf algebraic context (strict and formal) UDF's for every solvable three-dimensional Lie group endowed with any left-invariant Poisson structure. The method used to obtain those is based on strict quantization of solvable symmetric spaces, on existing UDF's for '$ax + b$' and on a generalization of the classical definition of smash products in the context of Hopf algebras. As an application we analyze the particular case of the dressing action of the Poisson dual Lie group of $SU(2)$ when endowed with the Lu-Weinstein Poisson structure.

2 UDF's for three-dimensional solvable Lie groups

In this section G denotes a three-dimensional solvable Lie group with Lie algebra $\mathfrak{g}$.

Definition 2.1 *Let* $\mathbf{w}$ *be a left-invariant Poisson bivector field on* G*. The pair* $(G, \mathbf{w})$ *is called a* **pre-symplectic** *Lie group.*

The terminology 'symplectic' is after Lichnérowicz [LiMe88] who studied this type of structures.
One then observes

Proposition 2.1 *(i) The orthodual* $\mathfrak{s}$ *of the radical of* $\mathbf{w}_e$

$$\mathfrak{s} := (rad\ \mathbf{w}_e)^{\perp}$$

is a subalgebra of $\mathfrak{g}$.

(ii) Every two-dimensional subalgebra of $\mathfrak{g}$ *can be seen as the orthodual of the radical of a Poisson bivector.*

(iii) The symplectic leaves of $\mathbf{w}$ *are the left classes of the analytic subgroup* S *of* G *whose Lie algebra is* $\mathfrak{s}$.

We now fix a pair $(\mathfrak{g}, \mathbf{w}_e)$ to be such a pre-symplectic Lie algebra with associated symplectic Lie algebra $(\mathfrak{s}, \mathbf{w}_e|_{\mathfrak{s}^\star\times\mathfrak{s}^\star})$. We assume

$$\dim \mathfrak{s} = 2.$$

We denote by $\mathfrak{d}$ the first derivative of $\mathfrak{g}$:

$$\mathfrak{d} := [\mathfrak{g}, \mathfrak{g}].$$

Note that $\mathfrak{d}$ is an Abelian algebra.

2.1 Case 1: $\dim \mathfrak{d} = 2$

In this case $\mathfrak{g}$ can be realized as a split extension of Abelian algebras:

$$0 \to \mathfrak{d} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{a} \to 0,$$

with $\dim \mathfrak{a} = 1$. We denote by

$$\rho : \mathfrak{a} \to \mathtt{End}(\mathfrak{d})$$

the splitting homomorphism.

2.1.1 $\mathfrak{s} \cap \mathfrak{d} \neq \mathfrak{d}$

In this case one sets

$$\mathfrak{p} := \mathfrak{s} \cap \mathfrak{d},$$

and one may assume

$$\mathfrak{a} \subset \mathfrak{s}.$$

Note that $\mathfrak{s}$ is then isomorphic to the Lie algebra of the group '$ax + b$'.

2.1.1.1 The representation ρ is semisimple

In this case one has $\mathfrak{q} \subset \mathfrak{d}$ such that

$$\mathfrak{q} \oplus \mathfrak{p} = \mathfrak{d} \text{ and}$$
$$[\mathfrak{a}, \mathfrak{q}] \subset \mathfrak{q}.$$

Let Q and S denote the analytic subgroups of G whose Lie algebras are $\mathfrak{q}$ and $\mathfrak{s}$ respectively. Consider the mapping

$$Q \times S \to G : (q, s) \mapsto qs. \tag{3}$$

Assume this is a global diffeomorphism. Endow the symplectic Lie group S with a left-invariant deformation quantization $\star^T$ as constructed in [BiMas01] or [BiMae02] (see also the appendix of the present paper). One then readily verifies

Theorem 2.1 *Let* $(\mathbf{H}^S, \star^T)$ *be an associative algebra of functions on* S *such that* $\mathbf{H}^S \subset Fun(S)$ *is an invariant subspace w.r.t. the left regular representation of* S *on* $Fun(S)$ *and* $\star^T$ *is a* S*-left-invariant product on* $\mathbf{H}^S$. *Then the function space* $\mathbf{H} := \{u \in Fun(G) \mid \forall p \in Q : u(q, .) \in \mathbf{H}^S\}$ *is an invariant subspace of* $Fun(G)$ *w.r.t. the left regular representation of* G *and the formula*

$$u \star v(q, s) := (u(q, .) \star^T v(q, .))(s)$$

defines a G*-left-invariant associative product on* $\mathbf{H}$.

2.1.1.2 The representation ρ is not semisimple

Since it cannot be nilpotent one can find bases $\{A\}$ of $\mathfrak{a}$ and $\{X, Y\}$ of $\mathfrak{d}$ such that

$$\mathrm{span}\{X\} = \mathfrak{p}$$

and

$$\rho(A) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad \lambda \in \mathbb{R}_0.$$

Lemma 2.1 $\mathfrak{g}$ *can be realized as a subalgebra of the transvection algebra of a four-dimensional symplectic symmetric space* M.

Proof. The symplectic triple associated to the symplectic symmetric space (cf. [Bie95] or [Bie00] Definition 2.5) is $(\mathfrak{G}, \sigma, \Omega)$ where

$$\begin{aligned} \mathfrak{G} &= \mathfrak{K} \oplus \mathfrak{P} \\ \mathfrak{K} &= \mathrm{span}\{k_1, k_2\} \\ \mathfrak{P} &= \mathrm{span}\{e_1, f_1, e_2, f_2\}; \end{aligned}$$

with Lie algebra table given by

$$\begin{aligned} [f_1, k_1] &= e_1 \\ [f_1, k_2] &= e_2 + \lambda e_1 \\ [f_1, e_1] &= k_1 \\ [f_1, e_2] &= k_2 + \lambda k_1; \end{aligned}$$

f_2 being central. The symplectic structure on $\mathfrak{P}$ is given by

$$\Omega = \alpha e_1^* \wedge f_1^* + \beta e_2^* \wedge f_2^* \qquad (\alpha, \beta \neq 0).$$

The subalgebra

$$\mathrm{span}\{X := \frac{1}{2}(k_1 + e_1),\, Y := \frac{1}{2}(k_2 + e_2),\, A := f_1,\}$$

is then isomorphic to $\mathfrak{g}$. ∎

Note that the transvection subalgebra

$$\mathfrak{S} = \mathfrak{g} \oplus \mathbb{R} f_2$$

is transverse to the isotropy algebra $\mathfrak{K}$ in $\mathfrak{G}$. As a consequence the associated connected simply connected Lie group $\mathbf{S}$ acts acts transitively on the symmetric space M. Since $\dim M = \dim \mathbf{S} = 4$, one may identify M with the group manifold of $\mathbf{S}$. It turns out that the symplectic symmetric space M admits a strict deformation quantization which is transvection invariant [Bie00]. Therefore one has

Theorem 2.2 *The symplectic Lie group* $\mathbf{S}$ *admits a left-invariant strict deformation quantization. Since* G *is a group direct factor of* $\mathbf{S}$*, the deformed product on* $\mathbf{S}$ *restricts to* G *providing a UDF for* G.

2.1.2 $\mathfrak{s} = \mathfrak{d}$

In this case, a UDF is obtained by writing Moyal's formula where one replaces the partial derivatives ∂_i's by (commuting) left-invariant vector fields $\widetilde{X_i}$ associated with generators X_i of $\mathfrak{d}$: for $a, b \in C^\infty(G)[[\hbar]]$, one sets

$$a \star_\hbar b := a.\exp\left(\mathbf{w}_e|_{\mathfrak{d}}{}^{ij}\, \overleftarrow{\widetilde{X_i}} \wedge \overrightarrow{\widetilde{X_j}}\right).b$$

2.2 Case 2: $\dim \mathfrak{d} = 1$

In this case $\mathfrak{g}$ is isomorphic to either the Heisenberg algebra or to the direct sum of $\mathbb{R}$ with the Lie algebra of '$ax+b$'. The first case has been studied by Rieffel in [Rie89]. Let then

$$\mathfrak{g} = \mathfrak{L} \oplus \mathbb{R}Z$$

where $\mathfrak{L} = \mathrm{span}\{A, E\}$ with $[A, E] = E$. Up to automorphism one has either $\mathfrak{s} = \mathfrak{L}$, $\mathfrak{s} = \mathrm{span}\{E, Z\}$ or $\mathfrak{s} = \mathrm{span}\{A, Z\}$. The last ones are Abelian thus UDF's in these cases are obtained the same way as in Subsection 2.1.2. The first case reduces to an action of '$ax+b$' and has therefore been treated in [BiMas01] or [BiMae02].

3 Crossed, Smash and Co-products

Every algebra, coalgebra, bialgebra, Hopf algebra and vector space is taken over the field $k = \mathbb{R}$ or $\mathbb{C}$. For classical definitions and facts on these subjects, we refer to [Swe69], [Abe80] or more fundamentally to [MiMo65].
To calculate with a coproduct Δ, we use the Sweedler notation [Swe69]: $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$.

For the classical definitions of a B-module algebra, coalgebra and bialgebra we refer to [Abe80].

The literature on Hopf algebras contains a large collection of what we can call generically semi-direct products, or crossed products. Let us describe some of them. The simplest example of these crossed products is usually called the smash product (see [Swe68, Mol77]):

Definition 3.1 *Let B be a bialgebra and C a B-module algebra. The smash product $C\sharp B$ is the algebra constructed on the vector space $C \otimes B$ where the multiplication is defined as*

$$(f \otimes a) \overrightarrow{\star} (g \otimes b) = \sum_{(a)} f \ (a_{(1)} \rightharpoonup g) \otimes a_{(2)} b \ , \qquad f, g \in C \ , \quad a, b \in B. \tag{4}$$

Assuming B is cocommutative, we now introduce a generalization of the smash product.

Definition 3.2 *Let B be a cocommutative bialgebra and C a B-bimodule algebra (i.e. a B-module algebra for both, left and right, B-module structures). The* **L-R-smash product** *$C\natural B$ is the algebra constructed on the vector space $C \otimes B$ where the multiplication is defined by*

$$(f \otimes a) \star (g \otimes b) = \sum_{(a)(b)} (f \leftharpoonup b_{(1)})(a_{(1)} \rightharpoonup g) \otimes a_{(2)} b_{(2)} \ , \qquad f, g \in C \ , \quad a, b \in B. \tag{5}$$

Remark 3.1 The fact that the L-R-smash product $C\natural B$ is an associative algebra follows by an easy adaptation of the proof of the associativity for the smash product [Mol77].

In the same spirit, one has

Lemma 3.1 *If C is a B-bimodule bialgebra, the natural tensor product coalgebra structure on $C \otimes B$ defines a bialgebra structure to $C\natural B$.*
If C and B are Hopf algebras, $C\natural B$ is a Hopf algebra as well, defining the antipode by

$$\begin{aligned} J_\star(f \otimes a) &= \sum_{(a)} J_B(a_{(1)}) \rightharpoonup J_C(f) \leftharpoonup J_B(a_{(2)}) \otimes J_B(a_{(3)}) \qquad (6) \\ &= \sum_{(a)} (1_C \otimes J_B(a_{(1)})) \star (J_C(f) \otimes 1_B) \star (1_C \otimes J_B(a_{(2)})). \end{aligned}$$

Now by a careful computation, one proves

Proposition 3.1 *Let B be a cocommutative bialgebra, C be a B-bimodule algebra and $(C\natural B,\star)$ be their L-R-smash product.*
Let T be a linear automorphism of C (as vector space). We define:

(i) the product $\bullet^T$ on C by

$$f\bullet^T g=T^{-1}\left(T(f).T(g)\right);$$

(ii) the left and right B-module structures, $\overset{T}{\rightharpoonup}$ and $\overset{T}{\leftharpoonup}$, by

$$a\overset{T}{\rightharpoonup}f:=T^{-1}\left(a\rightharpoonup T(f)\right)\quad \text{and}\quad f\overset{T}{\leftharpoonup}a:=T^{-1}\left(T(f)\leftharpoonup a\right);$$

(iii) the product, $\star^T$, on $C\otimes B$ by

$$(f\otimes a)\star^T(g\otimes b)=\mathbf{T}^{-1}\left(\mathbf{T}(f\otimes a)\star\mathbf{T}(g\otimes b)\right)\qquad \text{where } \mathbf{T}:=T\otimes\mathtt{Id}.$$

Then $(C,\bullet^T)$ is a B-bimodule algebra for $\overset{T}{\rightharpoonup}$ and $\overset{T}{\leftharpoonup}$ and $\star^T$ is the L-R-smash product defined by these structures.
Moreover, if $(C,.,\Delta_C,J_C,\rightharpoonup,\leftharpoonup)$ is a Hopf algebra and a B-module bialgebra, then

$$C_T:=(C,\bullet^T,\Delta_C^T:=(T^{-1}\otimes T^{-1})\circ\Delta_C\circ T,J_C^T:=T^{-1}\circ J_C\circ T,\overset{T}{\rightharpoonup},\overset{T}{\leftharpoonup})$$

is also a Hopf algebra and a B-module bialgebra. Therefore, by Lemma 3.1,

$$(C_T\natural B,\star^T,\Delta^T=(23)\circ(\Delta_C^T\otimes\Delta_B),J_\star^T),$$

is a Hopf algebra, Δ^T being the natural tensor product coalgebra structure on $C_T\natural B$ (with $(23):C\otimes C\otimes B\otimes B\to C\otimes B\otimes C\otimes B$, $c_1\otimes c_2\otimes b_1\otimes b_2\mapsto c_1\otimes b_1\otimes c_2\otimes b_2$) and $J_\star^T$ being the antipode given on $C_T\natural B$ by Lemma 3.1. Also, one has

$$\Delta^T=(\mathbf{T}^{-1}\otimes\mathbf{T}^{-1})\circ(23)\circ(\Delta_C\otimes\Delta_B)\circ\mathbf{T}\quad\text{and}\quad J_\star^T=\mathbf{T}^{-1}\circ J_\star\circ\mathbf{T}\qquad\text{with } \mathbf{T}=T\otimes\mathtt{Id}.$$

4 Examples in deformation quantization

4.1 A construction on $T^\star(G)$

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $T^\star(G)$ its cotangent bundle. We denote by $\mathsf{U}\mathfrak{g}$, $\mathsf{T}\mathfrak{g}$ and $\mathsf{S}\mathfrak{g}$ respectively the enveloping, tensor and symmetric algebras of $\mathfrak{g}$. Let $\mathsf{Pol}(\mathfrak{g}^*)$ be the algebra of polynomial functions on $\mathfrak{g}^*$. We have the usual identifications:

$$\mathcal{C}^\infty(T^*G)\simeq\mathcal{C}^\infty(G\times\mathfrak{g}^*)\simeq\mathcal{C}^\infty(G)\hat{\otimes}\mathcal{C}^\infty(\mathfrak{g}^*)\supset\mathcal{C}^\infty(G)\otimes\mathsf{Pol}(\mathfrak{g}^*)\simeq\mathcal{C}^\infty(G)\otimes\mathsf{S}\mathfrak{g}.$$

First we deform $\mathsf{S}\mathfrak{g}$ via the "parametrized version" $\mathsf{U}_t\mathfrak{g}$ of $\mathsf{U}\mathfrak{g}$ defined by

$$\mathsf{U}_t\mathfrak{g}=\frac{\mathsf{T}\mathfrak{g}[[t]]}{<XY-YX-t[X,Y];X,Y\in\mathfrak{g}>}.$$

$\mathsf{U}_t\mathfrak{g}$ is naturally a Hopf algebra with $\Delta(X)=1\otimes X+X\otimes 1$, $\epsilon(X)=0$ and $S(X)=-X$ for $X\in\mathfrak{g}$. For $X\in\mathfrak{g}$, we denote by $\widetilde{X}$ (resp. $\overline{X}$) the left- (resp. right-) invariant vector field on G such that $\widetilde{X}_e=\overline{X}_e=X$. We consider the following $k[[t]]$-bilinear actions of $B=\mathsf{U}_t\mathfrak{g}$ on $C=\mathcal{C}^\infty(G)[[t]]$, for $f\in C$ and $\lambda\in[0,1]$:

(i) $(X\rightharpoonup f)(x)=t(\lambda-1)\ (\widetilde{X}.f)(x)$,

(ii) $(f\leftharpoonup X)(x)=t\lambda\ (\overline{X}.f)(x)$.

One then has

Lemma 4.1 *C is a B-bimodule algebra w.r.t. the above left and right actions (i) and (ii).*

Definition 4.1 *We denote by* $\star_\lambda$ *the star product on* $(\mathcal{C}^\infty(G) \otimes \mathsf{Pol}(\mathfrak{g}^*))\,[[t]]$ *given by the L-R-smash product on* $\mathcal{C}^\infty(G)[[t]] \otimes \mathsf{U}_t\mathfrak{g}$ *constructed from the bimodule structure of the preceding lemma.*

Proposition 4.1 *For* $G = \mathbb{R}^n$, $\star_{\frac{1}{2}}$ *is the Moyal star product (Weyl ordered),* $\star_0$ *is the standard ordered star product and* $\star_1$ *the anti-standard ordered one. In general* $\star_\lambda$ *yields the* λ*-ordered quantization, within the notation of M. Pflaum [Pfl99].*

Remark 4.1 In the general case, it would be interesting to compare our λ-ordered L-R smash product with classical constructions of star products on $T^\star(G)$, with Gutt's product as one example [Gut83].

4.2 Hopf structures

We have discussed (see Lemma 3.1) the possibility of having a Hopf structure on $C\natural B$. Let us consider the particular case of $\mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural\mathsf{U}_t\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural\mathsf{S}\mathbb{R}^n$ ($\mathbb{R}^n$ is commutative). $\mathsf{S}\mathbb{R}^n$ is endowed with its natural Hopf structure but we also need a Hopf structure on $\mathcal{C}^\infty(\mathbb{R}^n)[[t]] = \mathcal{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}[[t]]$. We will not use the usual one. Our alternative structure is defined as follows.

Definition 4.2 *We endow* $\mathbb{R}[[t]]$ *with the usual product, the co-product* $\Delta(P)(t_1, t_2) := P(t_1 + t_2)$*, the co-unit* $\epsilon(P) = P(0)$ *and the antipode* $J(t) = -t$*. We consider the Hopf algebra* $(\mathcal{C}^\infty(\mathbb{R}^n), ., \mathbf{1}, \Delta_C, \epsilon_C, J_C)$*, with pointwise multiplication, the unit* $\mathbf{1}$ *(the constant function of value* 1*), the coproduct* $\Delta_C(f)(x, y) = f(x+y)$*, the co-unit* $\epsilon(f) = f(0)$ *and the antipode* $J_C(f)(x) = f(-x)$*. The tensor product of these two Hopf algebras then yields a Hopf algebra denoted by*

$$(\mathcal{C}^\infty(\mathbb{R}^n)[[t]], ., \mathbf{1}, \Delta_t, \epsilon_t, J_t).$$

Note that Δ_t *and* J_t *are not linear in* t*. We then define, on the L-R smash* $\mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural\mathsf{S}\mathbb{R}^n$*,*

$$\Delta_\star := (23) \circ (\Delta_t \otimes \Delta_B), \quad \epsilon_\star := \epsilon_t \otimes \epsilon_B \quad \text{and} \quad J_\star \text{ as in Lemma 3.1 .}$$

Proposition 4.2 $(\mathcal{C}^\infty(\mathbb{R}^n)[[t]]\natural\mathsf{S}\mathbb{R}^n, \star_\lambda, \mathbf{1} \otimes 1, \Delta_\star, \epsilon_\star, J_\star)$ *is a Hopf algebra.*

Remark 4.2 The case $\lambda = \frac{1}{2}$ yields the usual Hopf structure on the enveloping algebra of the Heisenberg Lie algebra.

5 The 'Book' Algebra

Definition 5.1 *The* **book Lie algebra** *is the three-dimensional Lie algebra* $\mathfrak{g}$ *defined as the split extension of Abelian Lie algebras* $\mathfrak{a} := \mathbb{R}.A$ *and* $\mathfrak{d} := \mathbb{R}^2$*:*

$$0 \to \mathfrak{d} \to \mathfrak{g} \to \mathfrak{a} \to 0,$$

where the splitting homomorphism $\rho : \mathfrak{a} \to \mathtt{End}(\mathfrak{d})$ *is defined by*

$$\rho(A) := \mathtt{Id}_{\mathfrak{d}}.$$

The name comes from the fact that the regular co-adjoint orbits in $\mathfrak{g}^\star$ are open half planes sitting in $\mathfrak{g}^\star = \mathbb{R}^3$ ressembling the pages of an open book.
We are particularly interested in this example because the associated connected simply connected Lie group G turns out to be the solvable Lie group underlying the Poisson dual of $SU(2)$ when endowed with the Lu-Weinstein Lie-Poisson structure [LuWe90]. Explicitly, one has the following situation (see [LuRa91] or [Sle98]). Set $\mathfrak{K} := \mathfrak{su}(2)$ and consider the Cartan decomposition of $\mathfrak{K}^\mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$:

$$\mathfrak{K}^\mathbb{C} = \mathfrak{K} \oplus i\mathfrak{K}.$$

Fix a maximal Abelian subalgebra $\mathfrak{a}$ in $i\mathfrak{K}$, consider the corresponding root space decomposition

$$\mathfrak{K}^\mathbb{C} = \mathfrak{K}^+ \oplus \mathfrak{a}^\mathbb{C} \oplus \mathfrak{K}^-,$$

and set

$$\mathfrak{d} := \mathfrak{K}^+ \simeq \mathbb{C} \simeq \mathbb{R}^2.$$

The Iwasawa part $\mathfrak{a} \oplus \mathfrak{K}^+$ is then a realization of the book algebra $\mathfrak{g} = \mathfrak{a} \times \mathfrak{d}$. One denotes by

$$K^{\mathbb{C}} = KG$$

the Iwasawa decomposition of $K^{\mathbb{C}}$. For an element $\gamma \in K^{\mathbb{C}}$, we have the corresponding factorization:

$$\gamma = \gamma_K \gamma_G.$$

The dressing action

$$G \times K \xrightarrow{\tau} K \tag{7}$$

is then given by

$$\tau_g(k) := (gk)_K.$$

In the same way, denote by

$$\mathfrak{K}^{\mathbb{C}} \to \mathfrak{K} : A \mapsto A_{\mathfrak{K}}$$

the projection parallel to $\mathfrak{g}$. The infinitesimal dressing action of G on K is now given by

$$X^\star_k = -L_{k_{\star e}} \left[\left(\mathrm{Ad}(k^{-1})X\right)_{\mathfrak{K}} \right], \quad X \in \mathfrak{g}.$$

A choice of a pre-symplectic structure on G—or equivalently (up to a scalar) a choice of a two-dimensional Lie algebra $\mathfrak{s}$ in $\mathfrak{g}$— then yields, via the dressing action (7), a Poisson structure $\mathbf{w}^{\mathfrak{s}}$ on K (cf. (2)):

$$L_{k^{-1}_{\star_k}}(\mathbf{w}^{\mathfrak{s}}{}_k) := \left(\mathrm{Ad}(k^{-1})S_1\right)_{\mathfrak{K}} \wedge \left(\mathrm{Ad}(k^{-1})S_2\right)_{\mathfrak{K}}$$

where $\{S_1, S_2\}$ is a basis of $\mathfrak{s}$. Note that this Poisson structure is compatible with the Lu-Weinstein structure. For each choice of symplectic Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, the preceding section then produces deformations of $(K, \mathbf{w}^{\mathfrak{s}})$.

Appendix: UDF's for '$ax + b$'

Let $S =' ax + b'$ denote the two-dimensional solvable Lie group presented as follows. As a manifold, one has

$$S = \mathbb{R}^2 = \{(a, \ell)\},$$

and the multiplication law is given by

$$(a, \ell).(a', \ell') := (a + a',\, e^{-a'}\ell + \ell').$$

Observe that the symplectic form on S defined by

$$\omega := da \wedge dl$$

is left-invariant.
We now define two specific real-valued functions. First, the two-point function $A \in C^\infty(S \times S)$ given by

$$A(x_1, x_2) := \cosh(a_1 - a_2)$$

where $x_i = (a_i, \ell_i) \quad i = 1, 2$. Second, the three-point function $\Phi \in C^\infty(S \times S \times S)$ given by

$$\Phi(x_0, x_1, x_2) := \oint_{0,1,2} \sinh(a_0 - a_1)\ell_2$$

where $\oint$ stands for cyclic summation. One then has

Theorem 5.1 *[Bie00, BiMas01] There exists a family of functional subspaces*

$$\{\mathcal{H}\}_{\hbar\in\mathbb{R}} \subset \mathcal{C}^\infty(\mathcal{S})$$

such that

(i) for all $\hbar \in \mathbb{R}$,

$$C_c^\infty(S) \subset \mathcal{H}_\hbar;$$

(ii) For all $\hbar \in \mathbb{R}\backslash\{0\}$ and $u, v \in C_c^\infty(M)$, the formula:

$$u \star_\hbar v(x_0) := \int_{M\times M} u(x_1)\, v(x_2)\, A(x_1,x_2)\, e^{\frac{i}{\hbar}\Phi(x_0,x_1,x_2)}\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{8}$$

extends as an associative product on $\mathcal{H}_\hbar$ ($\mathrm{d}x$ denotes some normalization of the symplectic volume on (S,ω)). Moreover, (for suitable u, v and x_0) a stationary phase method yields a power series expansion of the form

$$u \star_\hbar v(x_0) \sim uv(x_0) + \frac{\hbar}{2i}\{u,v\}(x_0) + o(\hbar^2); \tag{9}$$

where $\{\ ,\ \}$ denotes the symplectic Poisson bracket on (S,ω).

(iii) The pair $(\mathcal{H}_\hbar, \star_\hbar)$ is a topological (pre) Hilbert algebra on which the group S acts on the left by automorphisms.

Now setting $\mathbb{R}^2 = \mathfrak{a} \times \mathfrak{l}$; $\quad a \in \mathfrak{a}, \ell \in \mathfrak{l}$, one gets the linear isomorphisms

$$\mathcal{C}^\infty(S) \simeq \mathcal{C}^\infty(\mathfrak{l})\hat{\otimes}\mathcal{C}^\infty(\mathfrak{a}) \simeq \mathcal{C}^\infty(\mathfrak{l})\hat{\otimes}\mathcal{C}^\infty(\mathfrak{l}^*) \supset \mathcal{C}^\infty(\mathfrak{l}) \otimes Pol(\mathfrak{l}^*) \simeq \mathcal{C}^\infty(\mathfrak{l}) \otimes \mathsf{U}\mathfrak{l}.$$

The quantization (8) on such a space turns out to be a L-R-smash product. Namely, one has

Proposition 5.1 *The formal version of the invariant quantization (8) is a L-R-smash product of the form $\star^T$ (cf. Proposition 3.1).*

Proof. Let $\mathcal{S}(\mathfrak{l})$ denote the Schwartz space on the vector space $\mathfrak{l}$. In [Bie00], one shows that the map

$$\begin{array}{ccc} \mathcal{S}(\mathfrak{l}) & \xrightarrow{T} & \mathcal{S}(\mathfrak{l}) \\ u & \mapsto & T(u) := F^{-1}\phi_\hbar^\star F(u), \end{array}$$

where

$$\phi_\hbar : \mathfrak{l}^* \to \mathfrak{l}^* : a \mapsto \frac{2}{\hbar}\sinh(\frac{\hbar}{2}a),$$

is a linear injection for all $\hbar \in \mathbb{R}$ (F denotes the partial Fourier transform w.r.t. the variable ℓ). An asymptotic expansion in a power series in $\hbar$ then yields a formal equivalence again denoted by T:

$$T := \mathtt{Id} + o(\hbar) : C^\infty(\mathfrak{l})[[\hbar]] \to C^\infty(\mathfrak{l})[[\hbar]].$$

Carrying the Moyal star product on $(\mathfrak{a} \times \mathfrak{l}, \omega)$ by $\mathbf{T} := T \otimes \mathtt{Id}$ yields a star product on S which coincides with the asymptotic expansion of the left-invariant star-product (8) [Bie00, BiMas01]. ∎

Subsection 4.2 then yields

Corollary 5.1 *The (formal) UDF (8) admits compatible co-product and antipode.*

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