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FROM SPACES OF POLYGONS TO SPACES OF POLYHEDRA FOLLOWING BAVARD, GHYS AND THURSTON

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ABSTRACT. After a work of W. P. Thurston, C. Bavard and É. Ghys built particular hyperbolic polyhedra from spaces of deformations of Euclidean polygons. We present this construction as a straightforward consequence of theory of mixed-volumes.

The gluing of these polyhedra can be isometrically embedded into complex hyperbolic cone-manifolds constructed by Thurston from spaces of deformations of Euclidean polyhedra. It is then possible to deduce the metric structure of the spaces of polygons embedded in complex hyperbolic orbifolds discovered by P. Deligne and G. D. Mostow.

In [Thu98] W.P. Thurston described a natural complex hyperbolic structure on the space of convex polytopes in the Euclidean 3-space with fixed cone-angles. Applying this construction to polygons, C. Bavard and É. Ghys pointed out in [BG92] that spaces of convex Euclidean polygons with fixed angles are isometric to particular hyperbolic polyhedra, called (truncated) orthoschemes. In Section 1 we get the Bavard–Ghys’ results using the theory of mixed-area (mixed-volume for polygons). By the way we get Proposition 1.6 which is new. The use of Alexandrov–Fenchel Theorem can appear artificial at this point (see the discussion after Theorem 1.1), but mixed-area theory lights the relations between convex polygons and hyperbolic orthoschemes *via* Napier cycles, see Subsection 1.1. Moreover it is very natural as mixed-area is the polar form of the quadratic form studied in [Thu98, BG92]. Above all it indicates a way to generalize Bavard–Ghys’ construction from spaces of polygons to spaces of polytopes of any dimension d . In the case of $d = 3$, the construction is related to the one of Thurston, but different. More explanations will be given in a forthcoming paper [FI]. Section 1 ends with a discussion about hyperbolic orthoschemes which are of Coxeter type, as it appears that the list given by Im Hoff in [IH90] is incomplete.

In Section 2 we glue some of these hyperbolic orthoschemes to get hyperbolic cone-manifolds. It can be seen as hyperbolization of space of configurations of weighted points on the circle. This has been done several times, especially in lower

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dimensions, but it seems that the link with orthoschemes was never clearly made. Proposition 2.3 is new as stated for all dimensions.

Section 3 describes a local parameterization of spaces of polyhedra, equivalent to the one in [Thu98], which points up a “complex mixed-area”. We outline next the remainder of the construction of [Thu98] which allows to find back in a simple way complex hyperbolic orbifolds listed by Mostow (our interest will be about a sublist first listed by Deligne and Mostow [DM86]).

Finally in Section 4 we check that spaces of polygons (isometrically) embed into spaces of polyhedra locally as real forms. This is a non-surprising and certainly well-known fact (see for example [KM95]), which is here checked with the parameterizations we defined. We then derive easily Theorem 4.2, which gives the metric structure of those sets of polygons in the Deligne–Mostow’s orbifolds (are they manifolds, orbifolds, or just cone-manifolds?).

Results presented above as news are news as far I know. References are given all along the text. The paper stays in primitive stage, and questions are asked.

Acknowledgments. I was brought into the subject by Jean–Marc Schlenker. The link with mixed-volume theory comes from discussions with Ivan Izestiev after he presented the content of [BI08]. The existence of the Coxeter polyhedron represented in Figure 4 was communicated to me by Anna Felikson and Pavel Tumarkin. I had fruitful discussions with them, as well as with Christophe Bavard and Ruth Kellerhals. A part of this work was done during my visits to the research group “Polyhedral surfaces” at TU Berlin, which I want to thank for hospitality.

1. SPACES OF POLYGONS AND HYPERBOLIC ORTHOSCHEMES

1.1. Basics about hyperbolic polyhedra, Napier cycles. We call the signature (N, Z, P) of a symmetric bilinear form (or a Hermitian form) the triple of its N negative eigenvalues, Z zero eigenvalues and P positive eigenvalues (with multiplicity). We denote by $\mathbb{R}^{n,1}$ the Minkowski space of dimension $(n + 1)$, that is \mathbb{R}^{n+1} endowed with the bilinear form of signature $(1, 0, n - 1)$

$$\langle x, y \rangle_1 = -x_0y_0 + x_1y_1 + \dots + x_ny_n.$$

A vector x of Minkowski space is said to be *positive* if $\langle x, x \rangle_1 > 0$ and *non-positive* otherwise. The hyperbolic space of dimension n is the following submanifold of $\mathbb{R}^{n,1}$ together with the induced metric

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n,1} | \langle x, x \rangle_1 = -1, x_0 > 0\}.$$

A convex polyhedron P of \mathbb{H}^n is the non-empty intersection of \mathbb{H}^n with a convex polyhedral cone of $\mathbb{R}^{n,1}$ with vertex the origin. If two facets (*i.e.* codimension 1 faces) of P intersect in \mathbb{H}^n , their outwards normals in $\mathbb{R}^{n,1}$ span a Riemannian plane and the angle between these two vectors is the exterior dihedral angle between the facets. The interior dihedral angle between the facets is π minus the exterior dihedral angle. In this paper, the *dihedral angle* is the interior dihedral angle.

Let us consider the central projection in $\mathbb{R}^{n,1}$ onto the hyperplane $\{x_0 = 1\}$. The image of the hyperbolic space under this projection is the interior of the unit ball of \mathbb{R}^n . It is endowed with the metric for which the projection is an isometry. In this model, known as the Klein projective model of the hyperbolic space, geodesics are straight lines. The unit sphere in this model is the boundary at infinity of the hyperbolic space. In this paper we call a *convex generalized polyhedron* of the hyperbolic space (the hyperbolic part of) a convex polytope of \mathbb{R}^n such that all its

edges meet the interior of the unit ball (a *polytope* is a compact polyhedron). A vertex lying outside the interior of the ball is called *hyperideal*. It is *ideal* if it lies on the unit sphere and *strictly hyperideal* otherwise. A vertex in the interior of the unit ball is called *finite*. A strictly hyperideal vertex corresponds to a positive vector of $\mathbb{R}^{n,1}$. The polyhedron is *truncated* if we cut it along the hyperplanes orthogonal to its strictly hyperideal vertices. We get a new hyperbolic polyhedron with new facets, one for each strictly hyperideal vertex v . Such a new facet has the property to be orthogonal to all the facets which had v as a vertex. A hyperbolic convex generalized polyhedron with only finite vertices is compact, and it is of finite volume if it has only finite and ideal vertices.

A particular important class of hyperbolic polyhedra (compact or finite volume) is the one of *Coxeter polyhedra*, who have all their dihedral angles equal to π/k for an integer $k > 1$. This implies in particular that the polyhedron is simple (that means that at each finite vertex come n facets). For more details about Coxeter polyhedra we refer to [Vin85, vin93]. Coxeter polyhedra are represented by *Coxeter diagrams*. Each facet is represented by a node. If two facets intersect orthogonally the nodes are not joined. If the two facets intersect with an angle π/k , $k > 2$, the nodes are joined by a line with a k above. If the facets intersect at infinity we put a ∞ on the line, and if the facets do not intersect the nodes are joined by a dashed line.

We now recall some definitions and results from [IH90]. A *Napier cycle* is a set of vectors $e_k \in \mathbb{R}^{n,1}$, $k \in \mathbb{Z}$ modulo $n + 3$, such that

- $\langle e_{k-1}, e_k \rangle < 0$ for all k ;
- $\langle e_k, e_j \rangle = 0$ for $2 \leq |j - k| \leq n + 1$.

Any $n + 1$ consecutive vectors of a Napier cycle form a basis of $\mathbb{R}^{n,1}$. There are three types of Napier cycles:

- *type 1*: two adjacent vectors are non-positive;
- *type 2*: one vector is non-positive;
- *type 3*: all vectors are positive.

Usually an orthoscheme is defined as a simplex of the hyperbolic space \mathbb{H}^n such that a facet intersects non-orthogonally at most two other facets. Here we extend the definition to generalized simplices with at most two hyperideal vertices. Hence there are three kinds of truncated orthoschemes:

- *ordinary orthoschemes* have no strictly hyperideal vertices. The (ordered) set of its outward normals is a Napier cycle of type 1. (For the Coxeter ones,) Their Coxeter diagram is a linear chain of length n .
- *simply-truncated orthoschemes* are truncated by their only strictly hyperideal vertex. The set of its outward normals is a Napier cycle of type 2. Their Coxeter diagram is a linear chain of length $n + 1$.
- *doubly-truncated orthoschemes* are truncated by their two strictly hyperideal vertices. The set of its outward normals is a Napier cycle of type 3. Their Coxeter diagram is a cycle of length $n + 2$.

By abuse of language we will call *orthoscheme* a polyhedron of one of the three types above.

Conversely, the set of hyperplanes orthogonal to the positive vectors of a Napier cycle of type 1 (resp. 2,3) forms an ordinary (resp. simply-truncated, doubly-truncated) orthoscheme. Note that if the Coxeter diagram of a Coxeter hyperbolic polyhedron is cyclic or linear, then it is a Coxeter orthoscheme.

1.2. Euclidean polygons and Napier cycles. Let P be a convex polygon of \mathbb{R}^2 with $n+3$ vertices such that the origin is contained in its interior. We call F_k the edges of P , labeled in the cyclic order, α_k is the exterior angle between F_{k-1} and F_k , and h_k is the distance of F_k to the origin. The angles α_k satisfies

$$(A) \quad 0 < \alpha_k < \pi, \quad \sum_{k=1}^{n+3} \alpha_k = 2\pi.$$

We call ℓ_k the length of F_k , and we have (see Figure 1)

$$(1) \quad \ell_k = \ell_k^r + \ell_k^l = \frac{h_{k-1} - h_k \cos(\alpha_k)}{\sin(\alpha_k)} + \frac{h_{k+1} - h_k \cos(\alpha_{k+1})}{\sin(\alpha_{k+1})}.$$

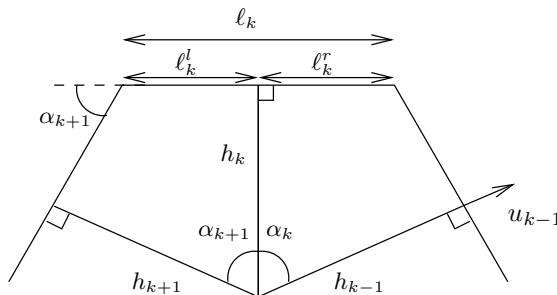


FIGURE 1. Notations for a convex polygon.

We identify the set of *heights* h_1, \dots, h_{n+3} (also called *support numbers*) with \mathbb{R}^{n+3} such that the set of outward unit normals u_1, \dots, u_{n+3} of the convex polygon P corresponds to the canonical basis. For a vector $P \in \mathbb{R}^{n+3}$, $h_k(P)$ is the k th coefficient of P for the basis u_1, \dots, u_{n+3} and $\ell_k(P)$ is defined by (1). We define the following bilinear form on \mathbb{R}^{n+3} :

$$m(P, Q) := -\frac{1}{2} \sum_{k=1}^{n+3} h_k(P) \ell_k(Q).$$

If P and Q are two convex polygons with outward unit normals u_1, \dots, u_{n+3} , $m(P, Q)$ is known as (minus) the mixed-area of P and Q , and $m(P, P)$ is minus the area of P , what is, up to the sign, the quadratic form used in [BG92] (here the minus sign stands only to get a more usual signature below). We get immediately that

$$(2) \quad m(u_k, u_k) = \frac{1}{2} \frac{\sin(\alpha_k + \alpha_{k+1})}{\sin(\alpha_k) \sin(\alpha_{k+1})},$$

that can be seen as (minus) the signed area of the triangle defined by u_k , see Figure 2. Moreover

$$(3) \quad m(u_k, u_j) = \begin{cases} 0 & \text{if } 2 \leq |j - k| \leq n + 1 \\ -\frac{1}{2} \frac{1}{\sin(\alpha_k)} & \text{if } j = k - 1 \\ -\frac{1}{2} \frac{1}{\sin(\alpha_{k+1})} & \text{if } j = k + 1 \end{cases}$$

It follows from (3) that m is symmetric, that is:

$$m(P, Q) = -\frac{1}{2} \sum_{k=1}^{n+3} h_k(Q) \ell_k(P).$$

This also comes from general properties of the mixed volume [Sch93, Ale05].

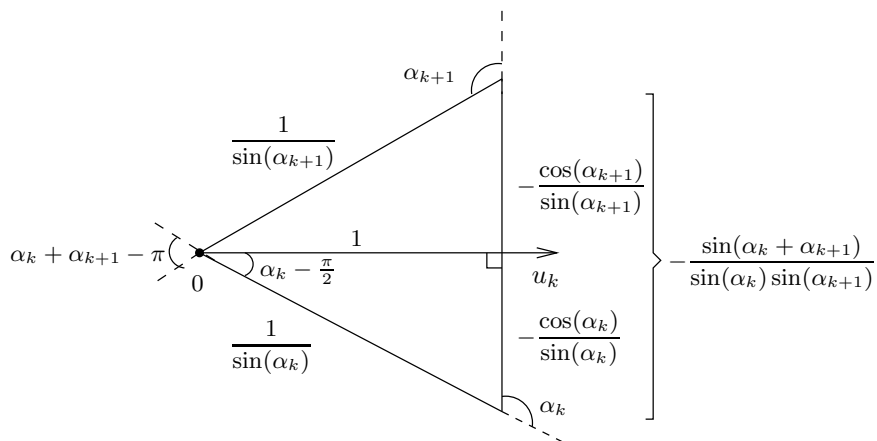


FIGURE 2. The geometric meaning of (2): it is (minus) the signed area of the triangle spanned by u_k .

Theorem 1.1. *The symmetric bilinear form m has signature $(1, 2, n)$.*

Theorem 1.1 is a straightforward adaptation of the analog result proved in [Thu98] dealing with convex polytopes of \mathbb{R}^3 (see Section 3). A wider statement (recalled in the proof of Proposition 1.6) is obtained in [BG92] with the same method. Theorem 1.1 is also proved with more efforts in [KNY99] in the case where all the α_k are equal. This statement is generalized to some cases of “convex generalized polygons” in [BI08, Lemma 3.15] (here “generalized” as another meaning than our).

Theorem 1.1 is also a particular case of classical results about mixed-volumes, even if it is far to be the simplest way to prove it. It appears in the Alexandrov’s proof of the so-called Alexandrov–Fenchel Theorem (or Alexandrov–Fenchel Inequality) for convex polytopes of \mathbb{R}^d (here $d = 2$) [Ale37, Sch93, Ale96]. Actually Theorem 1.1 can be derived from the Minkowski Inequality for convex polygons [Sch93, Note 1 p. 321], [Kla04]: if P and Q are convex polygons then

$$(4) \quad m(P, Q)^2 \geq m(P, P)m(Q, Q)$$

and equality occurs if and only if P and Q are homothetic. The way to go from Minkowski inequality to Theorem 1.1 is a part of Alexandrov's proof of Alexandrov–Fenchel Theorem. It is also done in a wider context in [Izm08, Appendix A]. Note that Minkowski Inequality (4) can be thought as the reversed Cauchy–Schwarz inequality in the case where the time-like vectors correspond to convex polygons. One interest to consider Bavard–Ghys' construction as a consequence of Alexandrov–Fenchel Theorem in the particular case when $d = 2$ rather than a consequence of Thurston's construction is that the former one generalizes immediately for any dimension d . This generalization will be the subject of [FI].

The heights of the sum of two polygons is the sum of the heights of the polygons, see *e.g.* [Ale96, Chapter IV]. It follows that a translation of a polygon P is the same as adding to the heights of P the heights of a point. Hence the kernel of m is made of heights spanning a point, as the area is invariant under translations (in the general case this is a step in the proof of Alexandrov–Fenchel Theorem, see [Ale96, Lemma III p.71], [Sch93, Proposition 3 p.329]). An example is given in Figure 3 for the case of triangles.

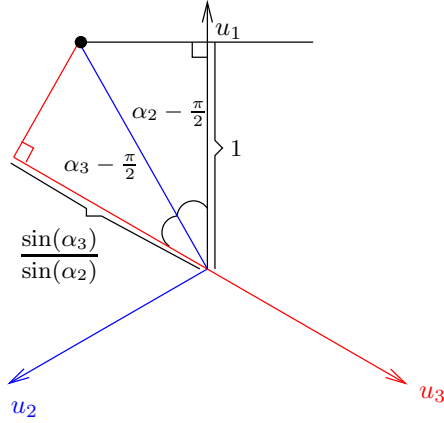


FIGURE 3. In the case of triangle with angles $(\alpha_1, \alpha_2, \alpha_3)$ the vector $\left(1, 0, -\frac{\sin(\alpha_3)}{\sin(\alpha_2)}\right)$ belongs to the kernel of m . The figure above checks it is heights of a point. Another vector spanning the kernel is $\left(0, 1, -\frac{\sin(\alpha_1)}{\sin(\alpha_2)}\right)$.

By Theorem 1.1 the quotient of \mathbb{R}^{n+3} by the kernel of m is isometric to the Minkowski space $\mathbb{R}^{n,1}$. We denote by Π the quotient map. By definition it follows from (3) and Theorem 1.1 that:

Corollary 1.2. *The set of vectors $(\Pi(u_1), \dots, \Pi(u_{n+3}))$ is a Napier cycle.*

Up to global isometries, this Napier cycle depends only on the angles α_k between the u_k . Let $(\alpha_1, \dots, \alpha_{n+3})$ be an ordered list of real numbers, up to cyclic permutations, satisfying (A). We denote by $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ the hyperbolic orthoscheme corresponding to the Napier cycle given by the Corollary above. We now check that, as announced in the title, $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is a space of convex polygons.

Lemma 1.3. *The orthoscheme $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is in bijection with the set of convex polygons with k th exterior angle equal to α_k , up to direct isometries and homotheties.*

From now we identify the space of polygons described in the lemma with $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$.

Proof. The interior of the hyperbolic polyhedron $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is the set of vectors P satisfying $m(P, u_k) < 0$ for all positive u_k , that is the set of P such that $\ell_k(P) > 0$ (it is not hard to see that this is true even if one or two u_k are non-positive). So P belongs to the set of convex polygons with the u_k as outward unit normals, which is the set of convex polygons with k th exterior angle equal to α_k , up to rotations as the u_k are arbitrarily placed in the plane. We saw that to quotient by the kernel of m is equivalent to consider the polygons up to translation. Finally $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is a subset of hyperbolic space, hence contains only polygons of unit area, that is the same as to consider set of polygons up to homotheties. \square

Proposition 1.4. *The hyperbolic orthoscheme $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ has the following properties.*

- i) *Its type is equal to 3 minus the number of k such that $\alpha_k + \alpha_{k+1} > \pi$.*
- ii) *It has non-obtuse dihedral angles, and if $\Pi(u_{k-1})$ and $\Pi(u_k)$ are space-like, the dihedral angle Θ between the corresponding facets is acute and verifies*

$$(5) \quad \cos^2(\Theta) = \frac{\sin(\alpha_{k-1}) \sin(\alpha_{k+1})}{\sin(\alpha_{k-1} + \alpha_k) \sin(\alpha_k + \alpha_{k+1})}.$$

- iii) *It is of finite volume and compact if and only if there does not exist k, k' such that $\alpha_k + \dots + \alpha_{k'} = \pi$.*

The last condition about compactness can be rephrased by saying that the polygons of $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ have no parallel edges.

Proof. First note that due to (2) the character of the vector $\Pi(u_k)$ is easily characterized:

- it is space-like if $\alpha_k + \alpha_{k+1} < \pi$,
- it is light-like if $\alpha_k + \alpha_{k+1} = \pi$,
- it is time-like if $\alpha_k + \alpha_{k+1} > \pi$.

and i) follows. The dihedral angles are either $\pi/2$ either minus their cosine is given by

$$(6) \quad \frac{m(u_{k-1}, u_k)}{\sqrt{m(u_{k-1}, u_{k-1})} \sqrt{m(u_k, u_k)}} = -\sqrt{\frac{\sin(\alpha_{k-1}) \sin(\alpha_{k+1})}{\sin(\alpha_{k-1} + \alpha_k) \sin(\alpha_k + \alpha_{k+1})}},$$

which is a real negative number if $\alpha_{k-1} + \alpha_k < \pi$ and $\alpha_k + \alpha_{k+1} < \pi$, that proves ii). Actually that dihedral angles are non-obtuse is always true for orthoschemes, as well as the fact that they are of finite volume [IH90, Proposition 2.1]. It remains to prove the assertion about the compactness. The polyhedron is not compact if and only if one of its face contains an ideal point. It can be the case if a vector u_k is light-like, that is if $\alpha_k + \alpha_{k+1} = \pi$. Suppose now u_k space-like. It is not hard to see that in the facet defined by u_k (see Figure 6), u_{k+1} is a light-like vector if and only if $\alpha_k + \alpha_{k+1} + \alpha_{k+2} = \pi$, and so on. \square

We easily check below that Equation (5) is another form of the main formula (7) of [BG92]. It also appear under the form (5) in [KNY99, MN00] in the cases $n = 2, 3$. We denote by U_k the line spanned by the vector u_k in \mathbb{R}^2 , and we define the cross-ratio as

$$[a, b, c, d] = \frac{d - a}{a - b} \frac{b - c}{c - d}.$$

Corollary 1.5. *Let Θ be as in ii) of Proposition 1.4. Then*

$$(7) \quad \tan^2(\Theta) = -[U_{k-1}, U_k, U_{k+1}, U_{k+2}].$$

Proof. This follows from (5) and from the well-known fact that

$$(8) \quad \frac{\sin(\alpha_{k-1}) \sin(\alpha_{k+1})}{\sin(\alpha_{k-1} + \alpha_k) \sin(\alpha_k + \alpha_{k+1})} = \frac{1}{1 - [U_{k-1}, U_k, U_{k+1}, U_{k+2}]}.$$

□

Note that in [BG92] the U_k are directions of the edges and not directions of the normals of the polygons, but the cross-ratio are the same.

We proved, following [BG92], that from any convex polygon one can construct a hyperbolic orthoscheme. It is possible to prove the converse. This point is absent from [BG92], but this reference contains the main point for the proof. Let us be more formal.

Let $A(n)$ be the space of all ordered lists $(\alpha_1, \dots, \alpha_{n+3})$ up to the action of the dihedral group \mathbb{D}_{n+3} , with α_k real numbers satisfying (A). Note that if $\sigma \in \mathbb{D}_{n+3}$ then $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ and $\mathcal{H}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n+3)})$ are isometric. We say that two elements $(\alpha_1, \dots, \alpha_{n+3})$ and $(\alpha'_1, \dots, \alpha'_{n+3})$ of $A(n)$ are equivalent if for a set of vectors of \mathbb{R}^2 $\square = \{u_1, \dots, u_{n+3}\}$ with α_k the angle between u_{k-1} and u_k there exists a set of vectors $\square' = \{u'_1, \dots, u'_{n+3}\}$ with α'_k the angle between u'_{k-1} and u'_k and a projective map φ of \mathbb{R}^2 fixing the origin and sending \square to \square' , i.e. $\varphi(u_k) = u'_k$ for all k . We denote by $\overline{A}(n)$ the quotient of $A(n)$ by this equivalence relation, and by $H(n)$ the space of orthoschemes of \mathbb{H}^n up to global isometries.

Proposition 1.6. *There is a bijection between $\overline{A}(n)$ and $H(n)$.*

Proof. Let $(\alpha_1, \dots, \alpha_{n+3}) \in A(n)$. These numbers define the bilinear form m which itself defines a Napier cycle. The orthoscheme is defined by the hyperplanes orthogonal to the positive vectors of the Napier cycle. Hence it is defined by the Gram matrix of the normalized positive vectors of the Napier cycle, whose coefficients are either 1 either given by (6), which is a projectively invariant formula (8). Hence there is a well-defined map from $\overline{A}(n)$ to $H(n)$.

Let $H \in H(n)$ and let (e_1, \dots, e_{n+3}) be a Napier cycle representing H . There are at most two vectors which are non-positive, and in this case they have to be consecutive. Up to change the labeling, we suppose that the potential non-positive vectors are e_{n+2} and e_{n+3} . We also assume that positive vectors are normalized. Suppose now that $(\alpha_1, \dots, \alpha_{n+3})$ and $(\alpha'_1, \dots, \alpha'_{n+3})$ in $A(n)$ both lead to H . Up to a projective transformation we can consider that $\alpha_1 = \alpha'_1$ and $\alpha_2 = \alpha'_2$. We denote by $s(\cdot, \cdot, \cdot)$ the function defined by the right side of (6). As $s(\alpha_1, \alpha_2, \alpha_3)$ and $s(\alpha'_1, \alpha'_2, \alpha'_3)$ are both equal to $m(e_1, e_2)$ it follows easily that $\alpha_3 = \alpha'_3$ (this is straightforward if one uses the cross-ratio). Next as $s(\alpha_2, \alpha_3, \alpha_4)$ and $s(\alpha'_2, \alpha'_3, \alpha'_4)$ are both equal to $m(e_2, e_3)$ it follows that $\alpha_4 = \alpha'_4$ and so on until $s(\alpha_n, \alpha_{n+1}, \alpha_{n+2}) = m(e_n, e_{n+1}) = s(\alpha_n, \alpha_{n+1}, \alpha'_{n+2})$ which gives $\alpha_{n+2} = \alpha'_{n+2}$.

The last equality $\alpha_{n+3} = \alpha'_{n+3}$ follows as the sum of the angles is equal to 2π . Hence the map from $\overline{A}(n)$ to $H(n)$ is injective.

For an arbitrary $H \in H(n)$, we can find back angles $(\alpha_1, \dots, \alpha_{n+3})$ in a similar way: α_1, α_2 and α_3 are chosen such that they are between 0 and π and such that $s(\alpha_1, \alpha_2, \alpha_3) = m(e_1, e_2)$ (it is clear that such a triple always exists). It is easy to see that the angle α_4 (supposed to be between 0 and π) is determined by $s(\alpha_2, \alpha_3, \alpha_4) = m(e_2, e_3)$ and so on until $s(\alpha_n, \alpha_{n+1}, \alpha_{n+2}) = m(e_n, e_{n+1})$ which determines $0 < \alpha_{n+2} < \pi$. Now we have to examine different cases depending on the character of e_{n+2} :

- if e_{n+2} is space-like, $0 < \alpha_{n+3} < \pi$ is given as above;
- if e_{n+2} is light-like, we define $\alpha_{n+3} = \pi - \alpha_{n+2}$. It is between 0 and π ;
- if e_{n+2} is time-like, we again define α_{n+3} by (6). The squared norm of e_{n+1} is positive and the one of e_{n+2} is negative. As they are elements of a Napier cycle, $m(e_{n+1}, e_{n+2})$ is negative, hence the right-hand side of (6) must belong to $i\mathbb{R}_-$. As $\sin(\alpha_{n+1})$ and $\sin(\alpha_{n+1} + \alpha_{n+2})$ are positive, as $\sin(\alpha_{n+2} + \alpha_{n+3})$ is negative (because e_{n+2} is time-like), then $\sin(\alpha_{n+3})$ must be positive, that means that $0 < \alpha_{n+3} < \pi$.

We are left with $n + 3$ angles between 0 and π . To each such set of angles the (signed) area gives a quadratic form. In our case this form has signature $(1, 2, n)$, because it is the Gram matrix of a Napier cycle. But it is known that the form has this signature only if the α_i sum up to 2π [BG92, Proposition p. 209]. Hence we constructed an element of $A(n)$. \square

As noted in the proof above [BG92] contains also a computation of the signature of the (signed) area form for spaces of non-convex polygons. In particular one can construct Euclidean and spherical polyhedra.

1.3. Coxeter orthoschemes. The aim of [IH90] (previously announced in [IH85]) is to find all the Coxeter orthoschemes. Some subfamilies were already known, especially in dimensions 2 and 3 (see [IH85, IH90] and [Vin75, Vin85] for more details). For these dimensions there exists infinite families of Coxeter orthoschemes. For dimension ≥ 4 Im Hof found a list of 75 Coxeter orthoschemes up to dimension 9 (he proved that they can't exist for higher dimension). In [BG92] the existence of Coxeter orthoschemes is checked, by showing for each one a list of real numbers representing the slopes of the lines parallel to the edges of a suitable convex polygon.

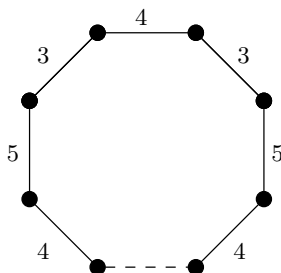


FIGURE 4. Tumarkin's polyhedron is a compact Coxeter orthoscheme of dimension 5 and type 3.

Figure 4 represents another example of Coxeter orthoscheme coming from [Tum07], that we will call Tumarkin's polyhedron. It is not hard to find a convex polygon of \mathbb{R}^2 whose space of angle-preserving deformations is isometric to Tumarkin's polyhedron. The following list gives the slopes of the lines containing the normals u_k :

$$\left(\sqrt{5}, -2, -1, 0, 1, \infty, -3, \frac{\sqrt{5}-3}{2} \right).$$

This list is a check of the existence of Tumarkin's polyhedron (dihedral angles can be easily computed with (7). The fact that all the slopes are different indicates that the polyhedron is compact). This polyhedron doesn't appear in [IH90] (hence not also in [BG92]). It seems that it has been forgotten by Im Hof, and then it is natural to ask

Question 1. *Is Im Hof list together with Tumarkin's polyhedron complete?*

There is at least two ways to answer the question: to check Im Hof's proof, that's involve a lot of hand computation, or to use a computer program. Effectively from results of [IH90] one can make a simple algorithm to find all Coxeter orthoschemes for each dimension from 4 to 9. I ran it in a common mathematic program and I got the Im Hof's list as well as Tumarkin's polyhedron. I didn't get more. But this does not prove that the list is now complete as it is possible that the program misses some values. One should find a more specialized program which asserts that computations are exact, but such investigation would lead too far from the aim of this paper.

We call a *rational angle* an angle of the form $q\pi$, $q \in \mathbb{Q}$. If H is an orthoscheme, we can't hope that there exists a set of rational angles $(\alpha_1, \dots, \alpha_n)$ such that $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is isometric to H . The question can be asked if H is a Coxeter orthoscheme, but I failed to find such set of angles for Tumarkin's polyhedron. Conversely two different sets of rational angles can lead to the same Coxeter orthoscheme, examples are shown in Figure 5.

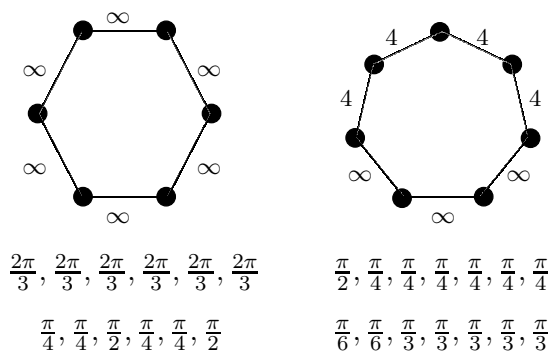


FIGURE 5. The Coxeter diagram on the left represents a Coxeter orthoscheme of dimension 3 and type 3 (the one shown in [Thu98, Figure 3]). The Coxeter diagram on the right represents a Coxeter orthoscheme of dimension 4 and type 3. Below each are two different lists of rational angles $(\alpha_1, \dots, \alpha_{n+3})$ such that $\mathcal{H}(\alpha_1, \dots, \alpha_{n+3})$ is isometric to the orthoscheme.

2. SPACES OF POLYGONS AND HYPERBOLIC CONE-MANIFOLDS

Let $\alpha_1, \dots, \alpha_{n+3}$ be $n+3$ real numbers satisfying (A). We denote by $R(\alpha_1, \dots, \alpha_{n+3})$ the set of all permutations of the set $\{\alpha_1, \dots, \alpha_{n+3}\}$ up to the action of the dihedral group. Hence $R(\alpha_1, \dots, \alpha_{n+3})$ has $(n-1)!/2$ elements. Each element $\sigma \in R(\alpha_1, \dots, \alpha_{n+3})$ will be identified with the orthoscheme $\mathcal{H}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n+3)})$. There is a natural way to glue all these polyhedra each other. Let α_k, α_j be such that $\alpha_k + \alpha_j < \pi$ (such pair always exists as the α_k satisfy (A)). It is easy to see that both orthoschemes $\mathcal{H}(\alpha_1, \dots, \alpha_k, \alpha_j, \dots, \alpha_n)$ and $\mathcal{H}(\alpha_1, \dots, \alpha_j, \alpha_k, \dots, \alpha_n)$ have a facet isometric to $\mathcal{H}(\alpha_1, \dots, \alpha_k + \alpha_j, \dots, \alpha_n)$, see Figure 6. We glue them isometrically along this facet. We do so for all facets of all orthoschemes in $R(\alpha_1, \dots, \alpha_{n+3})$. At the end we obtain a space which is by construction a hyperbolic cone-manifold of dimension n . We still denote by $R(\alpha_1, \dots, \alpha_{n+3})$ this space.

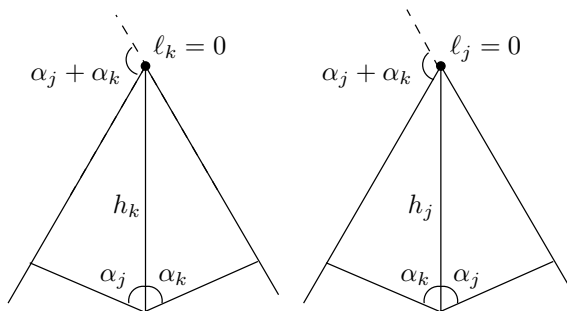


FIGURE 6. Let $\alpha_k + \alpha_j < \pi$. A facet of $\mathcal{H}(\alpha_1, \dots, \alpha_k, \alpha_j, \dots, \alpha_n)$ is defined by $m(u_k, P) = 0$, that is $\ell_k(P) = 0$ and this space is isometric to $\mathcal{H}(\alpha_1, \dots, \alpha_k + \alpha_j, \dots, \alpha_n)$. This last space is also isometric to the facet of $\mathcal{H}(\alpha_1, \dots, \alpha_j, \alpha_k, \dots, \alpha_n)$ defined by $m(u_j, P) = 0$.

Lemma 2.1. *The cone-manifold $R(\alpha_1, \dots, \alpha_{n+3})$ is connected. It has finite volume and it is compact if and only if there does not exist k, k' such that $\alpha_k + \dots + \alpha_{k'} = \pi$.*

Finiteness of the volume and description of the compactness are straightforward consequences of Proposition 1.4. To prove connectedness we will prove the following lemma, which implies the one above. We denote by $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ the double-covering of $R(\alpha_1, \dots, \alpha_{n+3})$ obtained by differentiating two lists of angles obtained one from the other by reversing the order.

Lemma 2.2. *The cone-manifold $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ is not connected if and only if there exists $\alpha_i, \alpha_j, \alpha_k$ ($i \neq j, j \neq k, k \neq i$) such that*

$$\alpha_i + \alpha_j > \pi, \alpha_j + \alpha_k > \pi, \alpha_k + \alpha_i > \pi.$$

In this case $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ has two connected components, which are identified by reversing the order of the angles.

Note that this last case can happen only if $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ contains orthoschemes of type 1.

Proof. To prove the lemma it suffices to know when there can't exist a sequence of glued polyhedra from

$$\mathcal{H}(\alpha_1, \dots, \alpha_i, \alpha_j, \dots, \alpha_{n+3})$$

to

$$\mathcal{H}(\alpha_1, \dots, \alpha_j, \alpha_i, \dots, \alpha_{n+3}).$$

Here are all the possible configurations.

- If all pairs α_i, α_j satisfy $\alpha_i + \alpha_j < \pi$ the polyhedra are glued along $\mathcal{H}(\alpha_1, \dots, \alpha_i + \alpha_j, \dots, \alpha_{n+3})$;
- If there is only one pair α_i, α_j such that $\alpha_i + \alpha_j > \pi$, we can glue

$$\mathcal{H}(\alpha_1, \dots, \alpha_i, \alpha_j, \alpha_{j+1}, \dots, \alpha_{n+3})$$

and

$$\mathcal{H}(\alpha_1, \dots, \alpha_i, \alpha_{j+1}, \alpha_j, \dots, \alpha_{n+3})$$

and so on, *i.e.* we can always permute α_j with the angle at its right, and as the list is up to cyclic order, at the end we arrive to $\mathcal{H}(\alpha_1, \dots, \alpha_j, \alpha_i, \dots, \alpha_{n+3})$, hence $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ is connected.

- By iterating the process described above, it is easy to show that $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ is still connected if there is $\alpha_i, \alpha_j, \alpha_k$ such that $\alpha_i + \alpha_j > \pi$, $\alpha_i + \alpha_k > \pi$ and $\alpha_j + \alpha_k < \pi$;
- In the same way it is easy to show that if $\alpha_i, \alpha_j, \alpha_k$ are as in the statement of the lemma,

$$\mathcal{H}(\alpha_1, \dots, \alpha_i, \alpha_j, \alpha_k, \dots, \alpha_{n+3})$$

and

$$\mathcal{H}(\alpha_1, \dots, \alpha_j, \alpha_i, \alpha_k, \dots, \alpha_{n+3})$$

can't be joined and hence $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ is not connected. Moreover every other polyhedron of $\tilde{R}(\alpha_1, \dots, \alpha_{n+3})$ can be joined to one of these two polyhedra above, and reversing the order of the angles is a bijection between these two components.

- As the sum of the α_k is equal to 2π there can't be a fourth angle α_l such that $\alpha_i, \alpha_j, \alpha_k$ are as in the statement of the lemma and $\alpha_l + \alpha_x > \pi$ for $x = i, j$ or k .

□

We know by the Poincaré Theorem [Thu98, Theorem 4.1] that $R(\alpha_1, \dots, \alpha_{n+3})$ is isometric to a hyperbolic orbifold if and only if the angle around each singular set of codimension 2 is of the form $2\pi/k$ with k integer > 0 . If these angles are 2π then the orbifold is a manifold. The singular sets of codimension 2 in $R(\alpha_1, \dots, \alpha_{n+3})$ correspond to codimension 2 faces of the polyhedra around which facets are glued. They are of two kinds:

- (1) if $\alpha_k + \alpha_{k+1} < \pi$ and $\alpha_j + \alpha_{j+1} < \pi$ with $2 \leq |j - k| \leq n + 1$, around the codimension 2 face isometric to

$$N := \mathcal{H}(\alpha_1, \dots, \alpha_k + \alpha_{k+1}, \dots, \alpha_j + \alpha_{j+1}, \dots, \alpha_{n+3})$$

are glued four orthoschemes, corresponding to the four ways of ordering (α_k, α_{k+1}) and (α_j, α_{j+1}) . As we know that the dihedral angle of each orthoscheme at such codimension 2 face is $\pi/2$, the total angle around N in $R(\alpha_1, \dots, \alpha_{n+3})$ is 2π . Hence metrically N is actually not a singular set.

- (2) if $\alpha_k + \alpha_{k+1} + \alpha_{k+2} < \pi$, around the codimension 2 face isometric to

$$S := \mathcal{H}(\alpha_1, \dots, \alpha_k + \alpha_{k+1} + \alpha_{k+2}, \dots, \alpha_{n+3})$$

are glued six orthoschemes corresponding to the six ways of ordering $(\alpha_k, \alpha_{k+1}, \alpha_{k+2})$.

Let us examine the angle θ around S . It is the sum of the six dihedral angles of the orthoschemes glued around it. Formula (5) giving the cosine of each dihedral angle is symmetric for two variables, hence θ is two times the sum of three different dihedral angles. Moreover

Proposition 2.3. *We have that $\cos(\theta/2)$ is equal to*

$$\frac{\sin(\alpha_1) \sin(\alpha_2) \sin(\alpha_3) - \sin(\alpha_1 + \alpha_2 + \alpha_3)(\sin(\alpha_1) \sin(\alpha_2) + \sin(\alpha_2) \sin(\alpha_3) + \sin(\alpha_3) \sin(\alpha_1))}{\sin(\alpha_1 + \alpha_2) \sin(\alpha_2 + \alpha_3) \sin(\alpha_3 + \alpha_1)}.$$

Proof. This formula is proved in [KNY99], hence we only outline the proof and sent the reader to this reference for more details about the computation. Note that in this reference the result is stated only for $n = 2, 3$ as they get (5) only for these dimensions. The idea is the following. We have to consider the sum of three dihedral angles, and as we know that they are acute, this sum is less than 2π . Hence the gluing of three orthoschemes having those dihedral angles can be isometrically embedded in the hyperbolic space (or at least a neighborhood of S). The gluing involves four facets glued around S , that gives four outward unit normals e_1, e_2, e_3, e_4 spanning a space-like plane in Minkowski space. Hence we can see these vectors as unit vectors in the Euclidean plane, and the problem is now reduced to find the angle between e_1 and e_4 knowing the angles between e_1 and e_2 , e_2 and e_3 and e_3 and e_4 (the exterior dihedral angles of the orthoschemes). \square

Here is a natural question.

Question 2. *For which $(\alpha_1, \dots, \alpha_{n+3})$ the cone-manifold $R(\alpha_1, \dots, \alpha_{n+3})$ is isometric to an orbifold?*

An intermediate step could be to know if there can exist such orbifolds for all dimension n . This is motivated by the fact that Coxeter orthoschemes and Mostow orbifolds (see Section 3) both don't exist for $n > 9$. Another analogy is that there exists (at least) 98 Coxeter orthoschemes (counting one for each infinite family in dimension ≤ 3) and 94 Mostow's orbifolds (but for dimension ≥ 4 there are at least 76 Coxeter orthoschemes and only 28 Mostow's orbifolds). But obviously there is no relation between the fact that $R(\alpha_1, \dots, \alpha_{n+3})$ is an orbifold and the fact that the orthoschemes constituting it are Coxeter. For example it is easy to check that $R(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ is a cone-manifold, made of Coxeter orthoschemes (some of them are isometric to the one of Figure 5 left).

There is an easy case when $R(\alpha_1, \dots, \alpha_{n+3})$ is a manifold: it is when there does not exist singular set of codimension 2, *i.e.* when the sum of each triple of angles is greater or equal to π . It is easy to check that this can be only for $n = 2, 3$. If $n = 3$ this implies that each triple of angles must sum up to π , that implies that all the angles are equal. Hence there is only one case, obtained by gluing 60 times the orthoscheme shown in Figure 5 left. For $n = 2$ there are infinitely many $R(\alpha_1, \dots, \alpha_5)$ which are manifolds, for examples the ones obtained by slightly deforming the angles in the list $(\frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5})$ (they are gluing of right-angled pentagons).

By running a simple computer program with the formula in the proposition above, I found only one triple of rational angles leading to a stratum of angle of

the form π/k (actually π):

$$(9) \quad \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{12} \right).$$

Once again this doesn't prove that it is the only possibility (among rational angles). It is easy to check that if $R(\alpha_1, \dots, \alpha_{n+3})$ is an orbifold with only (9) leading to a singular stratum then $n = 2$. Two examples of such orbifolds appear in Table 1, namely $R(\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ and $R(\frac{2\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4})$, and using (5) we note that the orthoschemes involved in the gluings are not of Coxeter type. This contrasts with Mostow's orbifolds, for which the angles have to be rational [DM86, 3.12].

It is proved in [KNY99, YNK02] that, if $n = 2$ or 3 , the metric structure of $R(\alpha_1, \dots, \alpha_{n+3})$ is uniquely determined by the angles $(\alpha_1, \dots, \alpha_{n+3})$, supposed it contains only orthoschemes of type 3. This contrasts with the fact that an orthoscheme is defined by an infinite number of lists of angles. The reason is that permutation of angles and projective transformations do not commute in general. This leads to the question:

Question 3. *Does $R(\alpha_1, \dots, \alpha_{n+3}) = R(\alpha'_1, \dots, \alpha'_{n+3})$ implies that $(\alpha_1, \dots, \alpha_{n+3}) = (\alpha'_1, \dots, \alpha'_{n+3})$ (up to the action of the dihedral group)?*

The study of deformation spaces of polygons with fixed angles appears in [Thu98] as a particular case of deformation spaces of polyhedra (see next section), it also appears as an exercise in [Thu97, Problem 2.3.12]. Detailed studies when $n = 5$ can be found in [KY93, Yos96, AY98] and in [AY99, MN00] when $n = 6$. Both cases are treated in [KNY99, YNK02]. Note that these references deal mainly with orthoschemes of type 3.

Due to the so-called Schwarz–Christoffel map $R(\alpha_1, \dots, \alpha_{n+3})$ can be thought as a real hyperbolization of configurations of points on the circle, depending on weights $(\alpha_1, \dots, \alpha_{n+3})$ [Thu98, KNY99, IP99, MN00]. For that reason this construction can be related to many other ones, see *e.g.* [KM95, IP99]. In [KM95] it is proved that spaces of polygons with fixed angles are homeomorphic to spaces of polygons with fixed edge lengths (so Lemma 2.2 is the analog of [KM95, Theorem 1, Lemma 6]). Natural metrics on moduli spaces of convex polygons in constant curvature Riemannian and Lorentzian spaces are also introduced in [Sch07].

3. SPACES OF POLYHEDRA

3.1. Configurations of points on the sphere. In [DM86] space of configurations of points on the sphere with suitable weights is endowed with a complex hyperbolic structure, depending on the weights. It is then possible to find a list of (compact or finite-volume) complex hyperbolic orbifolds. The list was enlarged in [Mos86] (some of them were known for a long, we refer to [DM86] for more details). Due to a generalization of the Schwarz–Christoffel map [Thu98, 8], space of configurations of weighted points on the sphere is homeomorphic to space of Euclidean cone-metrics on the sphere with prescribed cone-angles (see below) (this also comes from general theorems about determination of metrics on surfaces by curvatures [Tro86, Tro91]). In [Thu98] results of [DM86, Mos86] are recovered by studying such spaces of metrics on the sphere. The two constructions are outlined in parallel in [Koj01]. A bridge between the two constructions is done in a very clear way in [Tro07]. Moreover this last reference concerns also surfaces of higher genus. We won't review further the numerous works derived from [DM86] and [Thu98].

A *Euclidean metric with N cone singularities of positive curvature* on the sphere \mathbb{S}^2 is a flat metric on \mathbb{S}^2 minus N points x_1, \dots, x_N , $N \geq 3$, such that a neighborhood of each x_k is isometric to the neighborhood of the apex of a Euclidean cone of angle $0 < \theta_k < 2\pi$ (the curvature is $2\pi - \theta_k$). As we said such a metric is uniquely determined up to homotheties by the conformal class of the n punctured sphere and the numbers α_k (satisfying Gauss–Bonnet condition (A)). We denote by $C(\alpha_1, \dots, \alpha_{n+3})$ the set of Euclidean metrics on the sphere with $(n+3)$ cone singularities of positive curvature $2\alpha_i$, up to direct isometries and homotheties. Hence C is a connected topological manifold of dimension $2n$. We imply that the cone-singularities are labeled, that means that if x_k and x_j have same cone-angles, exchanging them lead to another metric (what we denote by $C(\alpha_1, \dots, \alpha_{n+3})$ is denoted by $P(2\alpha_1, \dots, 2\alpha_{n+3})$ in [Thu98]).

Due to the following famous theorem $C(\alpha_1, \dots, \alpha_{n+3})$ can also be defined as the space of convex polytopes of \mathbb{R}^3 with $n+3$ labeled vertices x_k such that the sum of the angles on the faces around x_k is $2\pi - 2\alpha_k$, up to Euclidean direct isometries and homotheties. In the following we will identify the metric and the polytope.

Theorem 3.1 (Alexandrov Theorem, [Ale42, Ale06]). *Let g be a Euclidean metric on the sphere with cone singularities of positive curvature. There exists a convex polytope P in \mathbb{R}^3 such that the induced metric on the boundary of P is isometric to g . Moreover P is unique up to ambient isometries.*

It is proved in [KM96] that spaces of configurations of points on the sphere are also homeomorphic to spaces of polygons in \mathbb{R}^3 with fixed edge lengths. Actually using a theorem of Minkowski it is not hard to see that such spaces of polygons are homeomorphic to spaces of convex polytopes of \mathbb{R}^3 with fixed face areas. In [KM95] they proved a duality between spaces of polygons with fixed angles and spaces of polygons with fixed edge lengths, so in dimension 3 this duality is expressed between spaces of polytopes with fixed cone-angles and spaces of polytopes with fixed face areas. The concerned theorem of Minkowski is proved for example in [Kla04]. In this reference it is proved simultaneously with Minkowski Inequality, which is the base result to prove Alexandrov–Fenchel Theorem. A quaternionic structure on spaces of polygons of \mathbb{R}^5 with fixed edge lengths is described in [FL04].

To emphasize the analogy with the case of polygons, we describe in the next subsection a local parameterization of $C(\alpha_1, \dots, \alpha_{n+3})$ close to the Thurston's one but different. The difference lies in the choice between two classical ways to unfold a polytope on the plane. The use of Alexandrov Theorem is never really necessary, but it simplifies some arguments and the construction uses elements coming from the original proof by A. D. Alexandrov.

3.2. Polytopes as complex polygons. A N -gon of the Euclidean plane is an ordered N -tuple of points (a_1, \dots, a_N) (the vertices) with line segments joining a_{k-1} to a_k (with $a_{N+1} = a_N$). Seeing the Euclidean plane as the complex plane, the set of N -gons is identified with \mathbb{C}^N . Let P be a convex polytope representing an element of $C(\alpha_1, \dots, \alpha_{n+3})$. We will describe a procedure which associates to P a $(2n+6)$ -gon $A(P)$.

Let us choose a point s on (the boundary of) P . The point s is a *source point*. We suppose that s is *generic*: it is not a vertex x_k and for each vertex x_k there is a unique shortest geodesic from s to x_k . If we cut P along these shortest geodesics, it can be unfolded into the plane as a $2(n+3)$ -gon: $(n+3)$ vertices are images of the

vertices x_k , which alternate with the $(n + 3)$ images of s . These last ones will be denoted by (s_1, \dots, s_{n+3}) . By an abuse of notation the image of the vertex x_k will be still denoted by letters x_k , but vertices of the $2(n + 3)$ -gon are labeled such that s_k is between x_{k-1} and x_k in the direct order. An example is shown in Figure 7.

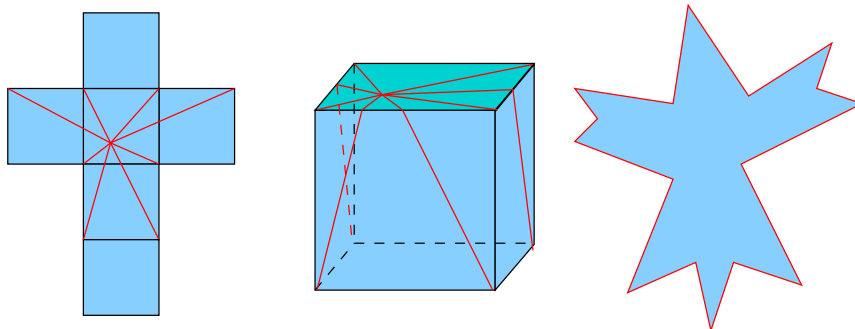


FIGURE 7. On the right is the Alexandrov unfolding of a cube. On the left is an edge unfolding, used to determine the shortest geodesics from the source point to the cone points.

This procedure is known as *Alexandrov unfolding* or as *star unfolding* (even if the resulting polygon is not necessarily star-shaped, hence we will avoid this last terminology), and seemly due to Alexandrov [Ale05, 4.1.2], [Ale06, VI,1]. Actually in those references the source point s is a vertex. With this restriction it is proved that the Alexandrov unfolding is non-overlapping in [Thu98, Proposition 7.1] (but in Figure 16 of [Thu98] the source point is generic). For our case with s a generic point the non-overlapping is proved in [AO92]. Hence $A(P)$ is a simple polygon. Alexandrov unfolding is used in [Web93] to parameterize spaces of cone-metrics on the sphere with four cone-singularities of positive curvature and one cone-singularity of negative curvature (*i.e.* the angle around the singularity is $> 2\pi$). Another way to unfold cone-metrics with five cone-points in the complex plane is described in [Par06].

By knowing only the (s_1, \dots, s_{n+3}) we can build back $A(P)$ and hence P , because the x_k are determined by

$$(10) \quad s_{k+1} - x_k = e^{i2\alpha_{k+1}}(s_k - x_k).$$

It follows that $A(P)$ is living in a complex vector space of (complex) dimension $(n + 3)$ what we identify with \mathbb{C}^{n+3} . On the space of $(2n + 6)$ -gons we define the following Hermitian form:

$$(11) \quad M(P, Q) = -\frac{1}{4i} \sum_{k=1}^{n+3} s_k(P) \overline{x_{k-1}(Q)} - x_{k-1}(P) \overline{s_k(Q)} + x_k(P) \overline{s_k(Q)} - s_k(P) \overline{x_k(Q)}$$

(recall that the signed area of the triangle $0ab$, $(a, b) \in \mathbb{C}^2$, is $\frac{1}{4i}(b\bar{a} - \bar{b}a)$). If P is an element of $C(\alpha_1, \dots, \alpha_{n+3})$, then $M(P, P)$ is (minus) the area of $A(P)$ (the face-area of P). Here is the analogous of Theorem 1.1.

Theorem 3.2. *The Hermitian form M has signature $(1, 2, n)$.*

This is proved by an induction on n . For $n = 0$, polytopes are doubled triangles and the result follows from Theorem 1.1 (see the proof of Lemma 4.1). This can also be seen directly by the facts that triangles have positive area and that area is invariant under translations.

For any n one can go back to $n - 1$ by the process called “cutting and gluing” and seemly due to Alexandrov [Ale06, Lemma 1, p 226], see also [Bus58, 17.5] and [Thu98, Proposition 3.3]. Cutting and gluing is as follows: if two cone points with curvatures $2\alpha_k$ and $2\alpha_j$ such that $\alpha_k + \alpha_j < \pi$ are sufficiently close (such points always exist), then we cut the geodesic joining the cone points. On the two resulting geodesics we can glue a cone of cone angle $2(\alpha_k + \alpha_j)$ at its apex such that the singularities at x_k and x_j disappear. The area of the old metric is then the area of the new metric less the area of the cone. In local coordinates this is simply translated as deleting a coordinate. The similar procedure for polygons instead of polytopes is the way used in [BG92] to prove Theorem 1.1.

This Hermitian form (11) can be considered as complex mixed-area. It is a classical one on the space of N -gons, see *e.g.* [FRS85] and references therein. In this reference the form is related to a larger family of Hermitian forms. Moreover polygons are considered as finite Fourier Series. It is possible to prove Alexandrov–Fenchel Theorem (actually Minkowski Inequality) for convex curves using Fourier Series, it is may be possible to compute the signature of M with this point of view. For more details we refer to [Gro96], especially Formula (4.3.3) and Remarks and References of 4.1.

To quotient \mathbb{C}^{n+3} with the kernel of M is the same as to consider the $(n+3)$ -gons up to translations. We identify this quotient with \mathbb{C}^{n+1} . Alexandrov unfolding is clearly an injective map from a neighborhood U of P in $C(\alpha_1, \dots, \alpha_{n+3})$ if the unfoldings are moreover considered up to rotations and homotheties. Hence U is mapped holomorphically onto a subset of the set of negative vectors of \mathbb{C}^{n+1} for a Hermitian form of signature $(1, n)$ up to complex conjugation: Alexandrov unfolding provides charts from $C(\alpha_1, \dots, \alpha_{n+3})$ to the complex hyperbolic space of (complex) dimension n $\mathbb{C}\mathbb{H}^n$ —we refer to [Eps87, Gol99] for details about complex hyperbolic space.

Let P be an element of $C(\alpha_1, \dots, \alpha_{n+3})$. Let T be a geodesic triangulation of P such that the cone-points are exactly the vertices T (we will call such triangulations *cone-triangulations*). The fact that cone-triangulations exist is obvious if we use Alexandrov Theorem (it suffices to triangulate the faces of the corresponding convex polytope), but precisely the existence of the triangulation is a step in the proof of this theorem, see *e.g.* [Bus58, p. 130], [Thu98, Proposition 3.1]. If we cut along some edges of T it is possible to unfold P onto the complex plane: this is an *edge unfolding* denoted by $E(P)$ (it is not necessarily non-overlapping). To each edge of T is associated a complex number, and $n + 1$ complex numbers suffice to recall P [Thu98, Proposition 3.2]: edge unfolding is another chart from $C(\alpha_1, \dots, \alpha_{n+3})$ to $\mathbb{C}\mathbb{H}^n$. We check that these two local coordinates are compatible.

Lemma 3.3. *Let U be a neighborhood of P in $C(\alpha_1, \dots, \alpha_{n+3})$ sufficiently small for A and E to be homeomorphisms on U . Then there is a linear isometry between $A(U)$ and $E(U)$.*

Proof. The Alexandrov unfolding $A(P)$ contains a simple polygon $SA(P)$ with vertices x_1, \dots, x_{n+3} (it is simple because $A(P)$ is). To go from $SA(P)$ to $A(P)$ one

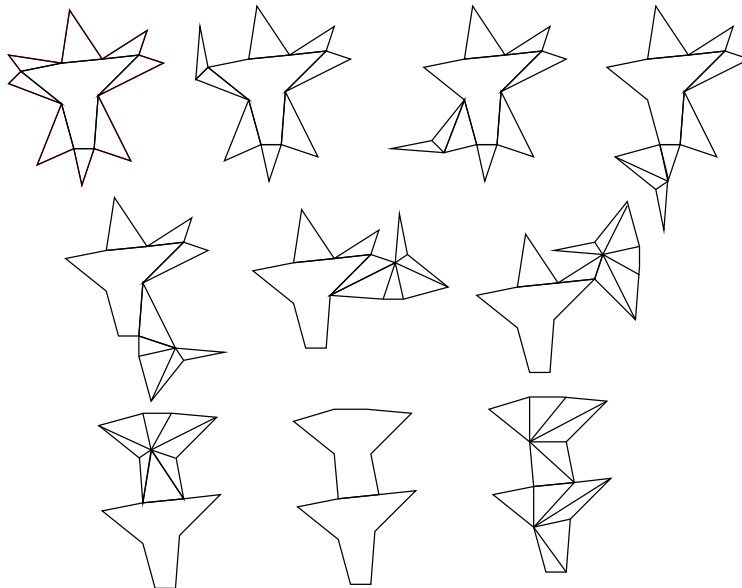


FIGURE 8. From Alexandrov unfolding to edge unfolding of the cube (example for the proof of Lemma 3.3).

has to add to each edge $x_{k-1}x_k$ of $SA(P)$ the triangle $T_k := x_{k-1}s_kx_k$. We will roll all these triangles around $SA(P)$. More precisely, we do on T_2 a rotation of angle $2\alpha_2$ and center x_2 . T_2 is now glued on the edge x_2s_3 . We rotate the union of T_2 and T_3 around x_3 by an angle α_3 , and so on. At the end all the T_k are glued around T_1 (all the s_k go to s_1), see Figure 8 for an example with the cube. The gluing of all the T_k around s_k gives a simple polygon (because angles around s sum up to 2π). Hence P is represented by the union (glued at the edge $x_{n+3}x_1$) of two simple polygons and each can be triangulated: we get an edge unfolding of a triangulation of P (note that there is no reason for the union to be simple). This procedure is linear in the s_k and preserves the area. \square

From now we summarize results of [Thu98]. Change of charts (for edge unfolding) is given by flips of edges of T (when two triangles form a quadrilateral, a *flip* of the diagonal consists of deleting it and choose the other diagonal). In terms of the complex coordinates, the change of charts is a linear map, and obviously an isometry. This gives a structure of complex hyperbolic manifold of complex dimension n on $C(\alpha_1, \dots, \alpha_{n+3})$. This manifold is not complete (as cone points can collide). We denote by $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ its metric completion. Then $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ has a structure of complex hyperbolic cone manifold (cone manifold structure for non constant curvature is less obvious to define as in the constant curvature case. We refer to [Thu98] for a precise definition).

If two cone-points x_k and x_j collide (that is possible only if $\alpha_k + \alpha_j < \pi$), this creates a singular stratum of (complex) codimension 1 in $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$. The main point is that the singular curvature around a stratum defined by the collision of two cone points with curvature $2\alpha_k, 2\alpha_j$ is $2\alpha_k + 2\alpha_j$ [Thu98, Proposition 3.5]. Hence it can be simply computed for which choices of $(\alpha_1, \dots, \alpha_{n+3})$ the cone-angles

around codimension 2 strata are of the form $2\pi/k$, that is, when $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ is a complex hyperbolic orbifold, due to the Poincaré Theorem [Thu98, Theorem 4.1]. There exists 36 of these orbifolds, which are listed in Table 1.

Moreover $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ has finite volume and it is compact if and only if there does not exist subset of $(\alpha_1, \dots, \alpha_{n+3})$ summing to π [Thu98, Proof of Theorem 0.2].

4. SPACES OF POLYGONS INTO SPACES OF POLYHEDRA

The set of convex polytopes contains *degenerated (convex) polytopes* which are obtained by doubling a convex polygon, that is by gluing isometrically along the edges a polygon together with its image by a reflection by a line. Such a reflection inverses the labeling of the angles, and hence there is a canonical injection f from $R(\alpha_1, \dots, \alpha_{n+3})$ to $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$. The metric structure on each space is given by the face-area, so it won't be surprising that f is an isometric map.

Lemma 4.1. *For the choice of an order on $(\alpha_1, \dots, \alpha_{n+3})$, the composition of f with the Alexandrov unfolding extends to an isometric linear map from $(\mathbb{R}^{n+1}, 2m)$ to (\mathbb{C}^{n+1}, M) .*

Proof. Let P be a convex polygon with exterior angles $(\alpha_1, \dots, \alpha_{n+3})$ and (u_1, \dots, u_{n+3}) its set of outward unit normals. We choose a point s in the interior of P such that s has a unique shortest geodesics to the vertices x_1, \dots, x_{n+3} of P . Without loss of generality, let us suppose that s is the origin of coordinates. We denote by s_k the image of s by a reflection by the edge $x_{k-1}x_k$. The polygon $s_1x_1s_2 \dots s_{n+3}x_{n+3}$ is an Alexandrov unfolding of the doubling of P , and then the linear extension of f is the following map from \mathbb{R}^{n+1} to \mathbb{C}^{n+1}

$$(h_1, \dots, h_{n+3}) \mapsto (2h_1u_1, \dots, 2h_{n+3}u_{n+3}).$$

It follows that $M(f(u_k), f(u_j)) = 0$ if $2 \leq |j - k| \leq n + 1$, and writing (10) as

$$x_k = \frac{1}{2i} \frac{1}{\sin(\alpha_{k+1})} (e^{i\alpha_{k+1}} s_k - e^{-i\alpha_{k+1}} s_{k+1})$$

we compute easily that

$$\begin{aligned} M(f(u_k), f(u_{k+1})) &= 4M(u_k, u_{k+1}) \\ &= -\frac{1}{i} \left(-\overline{u_k x_k(u_{k+1})} - x_k(u_k) \overline{u_{k+1}} \right) \\ &= \frac{1}{i} \left(\frac{1}{2i \sin(\alpha_{k+1})} + \frac{1}{2i \sin(\alpha_{k+1})} \right) \\ &= -\frac{1}{\sin(\alpha_{k+1})} \end{aligned}$$

and that

$$\begin{aligned} M(f(u_k), f(u_k)) &= 4M(u_k, u_k) \\ &= -\frac{1}{i} \left(\overline{u_k x_{k-1}(u_k)} - \overline{u_k} x_{k-1}(u_k) + \overline{u_k} x_k(u_k) - u_k \overline{x_k(u_k)} \right) \\ &= -\frac{1}{i} \left(\frac{1 \cos(\alpha_k)}{i \sin(\alpha_k)} + \frac{1 \cos(\alpha_{k+1})}{i \sin(\alpha_{k+1})} \right) \\ &= \frac{\sin(\alpha_k + \alpha_{k+1})}{\sin(\alpha_k) \sin(\alpha_{k+1})} \end{aligned}$$

that are two times m : the linear mapping is isometric. \square

It follows that on the image of \mathbb{R}^{n+1} in \mathbb{C}^{n+1} , the Hermitian form M has real values. Moreover this image has maximal real dimension, hence it is a real form of \mathbb{C}^{n+1} . To each real form is associated a unique real structure (= anti-linear involution) compatible with the Hermitian structure, whose fixed-points set is the real form. Here the real structure corresponds exactly to complex conjugation. We follow [Gol99] for the definitions and refer to it for more details.

This real structure on charts comes from a global isometric involution reversing orientation ρ on $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ that can be simply described as reflection of polytopes by a plane (in local charts, this is the same as making a reflection of the polygons by a line, that is the same as complex conjugation up to global direct isometries). As the vertices are labeled, the fixed-points set of ρ is exactly $R(\alpha_1, \dots, \alpha_{n+3})$. We get easily the metric structure of this set for the orbifolds of Deligne and Mostow.

Theorem 4.2. *Table 1 gives the metric structure of $R(\alpha_1, \dots, \alpha_{n+3})$ for Deligne–Mostow’s orbifolds.*

Proof. An easy case when $R(\alpha_1, \dots, \alpha_{n+3})$ is a manifold is when it has no singular set (of codimension 2). It occurs when for each triple $(\alpha_i, \alpha_j, \alpha_k)$ then $\alpha_i + \alpha_j + \alpha_k \geq \pi$. The two ones marked as orbifolds have only one singular stratum, which is represented by the triple $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{12})$ (see (9)).

From the discussion below Proposition 2.3 we know that the other examples are not manifolds neither orbifolds. We check this fact. If there exists three angles $\alpha_i, \alpha_j, \alpha_k$ such that the dihedral angle given by $(\alpha_i, \alpha_j, \alpha_k)$ and $(\alpha_j, \alpha_i, \alpha_k)$ is $\pi/4$, then the total angle around the singular set defined by these angles is at least four times $\pi/4$, plus something less than π (we know that these dihedral angles of orthoschemes are $< \pi/2$), hence it can’t be of the form $2\pi/k$. It is easy to see that such condition occurs for the triples $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6})$, $(\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{8})$, $(\frac{\pi}{10}, \frac{2\pi}{5}, \frac{2\pi}{5})$, $(\frac{\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12})$ and $(\frac{\pi}{18}, \frac{4\pi}{9}, \frac{4\pi}{9})$, which cover all the indicated cases in Table 1. \square

It is also possible to study spaces of cone-metrics without labeling the cone-points. In this case there are more orbifolds: if $\alpha_k = \alpha_j$ and if $\alpha_k + \alpha_j < \pi$, the angle around the stratum is half of the one obtained without labeling. Such orbifolds were founded in [Mos86, Mos88]. The complete list is achieved in [Thu98] (the list is known to be complete due to [DM86, 3.12] and [Fel97]). This list contains the list founded in [DM86] and listed in Table 1. In this case of non-labeling of the cone-points, the fixed-points set of ρ contains spaces of polygons and some polytopes obtained by doubling convex caps (which can be seen as convex isometric embeddings of Euclidean metrics with conical singularities on the closed disc). Answering the following question should be a step in the study of fixed-point set of ρ for Mostow’s orbifolds.

Question 4. *Is it possible to describe a (real) hyperbolic structure on the space of convex caps with fixed cone-angles?*

Many works have been done about real forms of complex hyperbolic orbifolds, with approaches different from the one here. We send the reader back to the

T	Angles	S	T	Angles	S
	Dimension 5		43	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M
3	$\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	45	$\frac{3\pi}{4}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	C
	Dimension 4		46	$\frac{5\pi}{8}, \frac{5\pi}{8}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
4	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	47	$\frac{\pi}{2}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	M
	Dimension 3		48	$\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}$	M
1	$\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M	49	$\frac{\pi}{10}, \frac{7\pi}{10}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$	C
5	$\frac{3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	57	$\frac{2\pi}{3}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}$	C
6	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	65	$\frac{7\pi}{12}, \frac{7\pi}{12}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}$	C
39	$\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	C	68	$\frac{5\pi}{6}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
44	$\frac{\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	C	69	$\frac{2\pi}{3}, \frac{7\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
66	$\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	70	$\frac{2\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}$	O
67	$\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	71	$\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$	O
	Dimension 2		72	$\frac{\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}$	M
2	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M	73	$\frac{7\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	M
7	$\frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C	74	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}$	M
8	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$	M	75	$\frac{\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}, \frac{5\pi}{12}$	M
9	$\frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$	M	78	$\frac{4\pi}{15}, \frac{8\pi}{15}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$	M
40	$\frac{5\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	C	79	$\frac{\pi}{18}, \frac{11\pi}{18}, \frac{4\pi}{9}, \frac{4\pi}{9}, \frac{4\pi}{9}$	C
41	$\frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{6}$	C	85	$\frac{7\pi}{10}, \frac{11\pi}{20}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$	C
42	$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$	M	89	$\frac{7\pi}{12}, \frac{7\pi}{24}, \frac{3\pi}{8}, \frac{3\pi}{8}, \frac{3\pi}{8}$	M

TABLE 1. The angles $(\alpha_1, \dots, \alpha_{n+3})$ are those for which $\overline{C}(\alpha_1, \dots, \alpha_{n+3})$ is a complex hyperbolic orbifold, given by the list in [DM86]. The column named **T** gives the number of the orbifold in Thurston's list [Thu98]. The column named **S** gives the structure of $R(\alpha_1, \dots, \alpha_{n+3})$: M means that it is a manifold, O that it is an orbifold and C that it is a cone-manifold.

following and the references therein for more details. For $n = 2$, see *e.g.* [AY98], for $n = 3$, [ACT07b, ACT06], for $n = 4$ [Yos01], [ACT07a] (announced in [ACT03]), for $n = 5$ [Chu07]. From fixed-points sets appearing in those works the following question arises:

Question 5. *Is it possible to describe a (real) hyperbolic structure on the space of centrally symmetric convex polytopes with fixed cone-angles?*

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