

WONG-ROSAY THEOREM IN ALMOST COMPLEX MANIFOLDS

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ABSTRACT. We study the compactness of sequences of diffeomorphisms in almost complex manifolds in terms of the direct images of the standard integrable structure.

INTRODUCTION

The classical Wong-Rosay theorem states that every domain in the euclidean space \mathbb{C}^n or more generally in a complex manifold of dimension n , with an automorphism orbit accumulating at a \mathcal{C}^2 strictly pseudoconvex point, is biholomorphically equivalent to the unit ball in \mathbb{C}^n (see [4, 8, 9, 11]). The aim of this paper is to extend this theorem to strictly pseudoconvex domains in almost complex manifolds. Our main result can be considered as a compactness theorem for sequences of diffeomorphisms. This shows that the convergence of such sequences can be controlled in terms of the direct images of the standard complex structure. Our approach is based on the scaling method, introduced by S.Pinchuk [8] for the case of the integrable structure. In order to apply it in the almost complex case, we need substantial modifications. In particular we use lower estimates for the Kobayashi infinitesimal pseudometric on almost complex manifolds [5] and a priori estimates for J -holomorphic curves [10].

1. STATEMENT OF THE RESULTS

An almost complex manifold (M, J) is a smooth (\mathcal{C}^∞) real manifold equipped with an almost complex structure J , that is a \mathcal{C}^∞ -field of complex linear structures on the tangent bundle TM of M . Given two almost complex manifolds (M, J) and (M', J') and a smooth map f from M' to M we say that f is (J', J) -holomorphic if its differential $df : TM' \rightarrow TM$ satisfies $df \circ J' = J \circ df$ on TM . We denote by $\mathcal{O}_{(J', J)}(M', M)$ the set of (J', J) -holomorphic maps from M' to M and by $Diff_{(J', J)}(M', M)$ the set of (J', J) -holomorphic diffeomorphisms from M' to M . The set $Diff_{(J', J)}(M', M)$ is generically empty. However given a diffeomorphism f from M' to M and an almost complex structure J' on M' then $f \in Diff_{(J, J')}(M, M')$ for the almost complex structure $J = df \circ J' \circ df^{-1}$ naturally associated with f . If M and M' are two domains in \mathbb{C}^n , if $J = J' = J_{st}$, the usual standard structure on \mathbb{C}^n , and $f \in Diff_{(J_{st}, J_{st})}(M', M)$ we simply say that f is a biholomorphism from M' to M , or that M is biholomorphic to M' .

We consider the following situation :

- D is a bounded domain in \mathbb{C}^2 ,
- p is a point in a four dimensional almost complex manifold (M, J) , U is a relatively compact neighborhood of p in M , r is a \mathcal{C}^2 strictly J -plurisubharmonic function in a neighborhood of \bar{U} satisfying $r(p) = 0$ and $dr \neq 0$ on U ,
- $(r_\nu)_\nu$ is a sequence of \mathcal{C}^2 functions in a neighborhood of \bar{U} such that $\lim_{\nu \rightarrow \infty} \|r_\nu - r\|_{\mathcal{C}^2(\bar{U})} = 0$,
- for every ν , f^ν is a diffeomorphism from D to $G^\nu \subset M$ with $G^\nu \cap U = \{x \in U : r_\nu(x) < 0\}$
- for every ν , the almost complex structure $J_\nu := df^\nu \circ J_{st} \circ (df^\nu)^{-1}$ extends smoothly to \bar{U} .

Then we have :

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Theorem 1. *Assume that $\lim_{\nu \rightarrow \infty} \|J_\nu - J\|_{\mathcal{C}^2(\bar{U})} = 0$. If there is a point $x^0 \in D$ such that $\lim_{\nu \rightarrow \infty} f^\nu(x^0) = p$, then D is biholomorphic to the unit ball \mathbb{B}_2 in \mathbb{C}^2 .*

Theorem 1 implies the following compactness result :

Corollary 1. *Let D be a bounded domain in \mathbb{C}^2 , not biholomorphic to the unit ball \mathbb{B}_2 and let $G = \{x \in M : r(x) < 0\}$ be a relatively compact domain in (M, J) , where r is a \mathcal{C}^2 strictly J -plurisubharmonic function on M . Let $(f^\nu)_\nu$ be a sequence of diffeomorphisms from D to G , defined in a neighborhood of \bar{D} . If $\|J_\nu - J\|_{\mathcal{C}^2(\bar{D})} \rightarrow_{\nu \rightarrow \infty} 0$ then $(f^\nu)_\nu$ is compact.*

Finally we have an analogue of the Wong-Rosay theorem in almost complex manifolds :

Theorem 2. *Let $D = \{x \in M : r(x) < 0\}$ be a relatively compact domain in a four dimensional almost complex manifold (M, J) , where r is a \mathcal{C}^2 strictly plurisubharmonic function on M . Assume that the set $\text{Diff}_{(J, J_{st})}(D, \mathbb{B}_2)$ is empty. Then the set $\text{Diff}_{(J, J)}(D, D)$ is compact for the compact-open topology.*

2. PRELIMINARIES

The following Lemma shows that every almost complex structure J on an almost complex manifold may be represented locally at $p \in M$ as a small \mathcal{C}^2 deformation of the standard structure (see [5]).

Lemma 1. *For every $\lambda_0 > 0$ there exist a neighborhood U_0 of p and a coordinate diffeomorphism $z : U_0 \rightarrow \mathbb{B}_2$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$ and the direct image $\hat{J} = z_*(J)$ satisfies $\|\hat{J} - J_{st}\|_{\mathcal{C}^2(\mathbb{B}_2)} \leq \lambda_0$.*

Proof. Shrinking U if necessary there exists a diffeomorphism z from U onto \mathbb{B}_2 satisfying $z(p) = 0$ and $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$. For $\lambda > 0$ consider the dilation $d_\lambda : t \mapsto \lambda^{-1}t$ in \mathbb{C}^2 and the composition $z_\lambda = d_\lambda \circ z$. Then $\lim_{\lambda \rightarrow 0} \|(z_\lambda)_*(J) - J_{st}\|_{\mathcal{C}^2(\mathbb{B}_2)} = 0$. Setting $U = z_\lambda^{-1}(\mathbb{B}_2)$ for $\lambda > 0$ small enough, we obtain the desired statement. \square

In the sequel we will consider the diffeomorphism z given by Lemma 1, for sufficiently small λ_0 .

2.1. Plurisubharmonic functions. Let (M, J) be an almost complex manifold. We denote by TM the real tangent bundle of M and by $T_{\mathbb{C}}M$ its complexification. If $T^{(1,0)}M := \{X \in T_{\mathbb{C}}M : JX = iX\} = \{\zeta - iJ\zeta, \zeta \in TM\}$, $T^{(0,1)}M := \{X \in T_{\mathbb{C}}M : JX = -iX\} = \{\zeta + iJ\zeta, \zeta \in TM\}$ then $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. The set of complex forms of type $(1, 0)$ on M is defined by $T_{(1,0)}M = \{w \in T_{\mathbb{C}}^*M : w(X) = 0, \forall X \in T^{(0,1)}M\}$ and the set of complex forms of type $(0, 1)$ on M by $T_{(0,1)}M = \{w \in T_{\mathbb{C}}^*M : w(X) = 0, \forall X \in T^{(1,0)}M\}$. Then $T_{\mathbb{C}}^*M = T_{(1,0)}M \oplus T_{(0,1)}M$ and the operators ∂_J and $\bar{\partial}_J$ are defined on the space of smooth complex functions on M by $\partial_J u = du_{(1,0)} \in T_{(1,0)}M$ and $\bar{\partial}_J u = du_{(0,1)} \in T_{(0,1)}M$, for every complex smooth function u on M .

Let r be a \mathcal{C}^2 function on M . The Levi form of r is defined on TM by

$$\mathcal{L}^J(r)(X) := -d(J^*dr)(X, JX).$$

We recall that an upper semicontinuous function u on (M, J) is called J -plurisubharmonic on M if the composition $u \circ f$ is subharmonic on Δ for every $f \in \mathcal{O}_J(\Delta, M)$. Then we have the following characterization of J -plurisubharmonic functions (see [5]) :

Proposition 1. *Let u be a \mathcal{C}^2 real valued function on M . Then u is J -plurisubharmonic on M if and only if $\mathcal{L}^J(u)(X) \geq 0$ for every $X \in TM$.*

Hence, following [5], we say that a \mathcal{C}^2 real valued function u on M is *strictly J -plurisubharmonic* on M if $\mathcal{L}^J(u)$ is positive definite on TM .

We point out that the strict J -plurisubharmonicity is stable with respect to small \mathcal{C}^2 deformations of the almost complex structure and of the function.

2.2. Kobayashi-Royden pseudometric. We denote by Δ the unit disc in \mathbb{C} . Let (M, J) be an almost complex manifold. According to [7] for every $p \in M$ there is a neighborhood \mathcal{V} of 0 in $T_p M$ such that for every $v \in \mathcal{V}$ there exists $f \in \mathcal{O}_J(\Delta, M)$ satisfying $f(0) = p$, $df(0)(\partial/\partial x) = v$. We may therefore define the Kobayashi-Royden infinitesimal pseudometric $K_{(M, J)}$ and the integrated pseudodistance $d_{(M, J)}^K$ (the upper semicontinuity of $K_{(M, J)}$ on the tangent bundle TM of M is proved in [6]) :

Definition 1. (i) For $p \in M$ and $v \in T_p M$, $K_{(M, J)}(p, v)$ is the infimum of the set of positive α such that there exists a J -holomorphic disc $f : \Delta \rightarrow M$ satisfying $f(0) = p$ and $df(0)(\partial/\partial x) = v/\alpha$.

(ii) Let $p, q \in M$. Denote by $\Gamma_{p, q}$ the set of all \mathcal{C}^1 -paths $\gamma : [0, 1] \rightarrow M$ satisfying $\gamma(0) = p$, $\gamma(1) = q$. Then $d_{(M, J)}^K(p, q) = \inf_{\gamma \in \Gamma_{p, q}} \int_0^1 K_{(D, J)}(\gamma(t), \gamma'(t)) dt$.

As in the complex case, the integrated pseudodistance is decreasing under the action of (J', J) -holomorphic maps. Hence if $f \in \text{Diff}_{(J', J)}(M', M)$ then the inverse map f^{-1} is in $\text{Diff}_{(J, J')}(M, M')$ and we have for every $p', q' \in M'$:

$$(1) \quad d_{(M, J)}^K(f(p'), f(q')) = d_{(M', J')}^K(p', q').$$

The following results are proved in [5] :

Proposition A. (Localization principle) Let D be a domain in an almost complex manifold (M, J) , let $p \in \bar{D}$ and let U be a neighborhood of p in M (not necessarily contained in D). Let u be a \mathcal{C}^2 function on \bar{D} , negative and J -plurisubharmonic on D . We assume that $-L \leq u < 0$ on $D \cap U$ and that $u - c|z|^2$ is strictly J -plurisubharmonic on $D \cap U$, where c and L are positive constants and z is the diffeomorphism given by Lemma 1. Then there exist a positive constant s and a neighborhood $V \subset\subset U$ of p , depending on c and L only, such that for $q \in D \cap V$ and $v \in T_q M$ we have $K_{(D, J)}(q, v) \geq sK_{(D \cap U, J)}(q, v)$.

The next Proposition gives a lower bound on the Kobayashi-Royden infinitesimal pseudometric (more precise lower estimates are given in Theorem 1 of [5]).

Proposition B. Let D be a relatively compact domain in an almost complex manifold (M, J) . Let u be a \mathcal{C}^2 function on \bar{D} , satisfying $-L \leq u < 0$ on D and $u - c|z|^2$ is strictly J -plurisubharmonic on D , where c and L are positive constants and z is given by Lemma 1. Then there exist positive constants C and λ_0 , depending on c and L only, such that for every almost complex structure J' defined in a neighborhood of \bar{D} and such that $\|J' - J\|_{\mathcal{C}^2(\bar{D})} \leq \lambda_0$ we have : $K_{(D, J')}(p, v) \geq C\|v\|$ for every $p \in D$ and every $v \in T_p M$.

Finally we have the boundary behaviour of the Kobayashi pseudodistance in a strictly J -pseudoconvex domain.

Proposition C. Let D, D^ν be domains in an almost complex manifold (M, J) and let $p \in \partial D$. Assume that there is a neighborhood U of p such that $D \cap U = \{x \in U : r(x) < 0\}$, where r is a \mathcal{C}^2 strictly J -plurisubharmonic function in a neighborhood of \bar{U} . If $D^\nu \cap U = \{x \in U : r^\nu(x) < 0\}$, where r^ν is a sequence of \mathcal{C}^2 functions in a neighborhood of \bar{U} , converging to r in the $\mathcal{C}^2(\bar{U})$ convergence, then for every $q \in D \cap U$ there is ν_0 such that for $\nu \geq \nu_0$ we have : $\lim_{\substack{x \in D^\nu \\ x \rightarrow p}} d_{(D^\nu, J)}^K(x, q) = +\infty$.

3. PROOF OF THEOREM 1

In this Section we assume that the assumptions of Theorem 1 are satisfied.

3.1. Attraction property. The following Lemma is a direct application of Proposition C.

Lemma 2. For every $K \subset\subset D$ we have : $\lim_{\nu \rightarrow \infty} f^\nu(K) = p$.

Proof of Lemma 2. Let $K \subset\subset D$ be such that $x^0 \in K$. Since the function $x \mapsto d_D^K(x^0, x)$ is bounded from above by a constant C on K , it follows from the decreasing property of the Kobayashi pseudodistance that

$$(2) \quad d_{(G^\nu, J_\nu)}^K(f^\nu(x^0), f^\nu(x)) \leq C$$

for every ν and every $x \in K$. Moreover, r is strictly J -plurisubharmonic in a neighborhood of \bar{U} and the sequence $(r_\nu)_\nu$ converges to r in the $\mathcal{C}^2(\bar{U})$ convergence. It follows from Proposition C that for every $V \subset\subset U$, containing p , we have :

$$(3) \quad \lim_{\nu \rightarrow \infty} d_{(G^\nu, J_\nu)}^K(f^\nu(x^0), G^\nu \cap \partial V) = +\infty.$$

It follows from conditions (2) and (3) that $f^\nu(K) \subset V$ for every sufficiently large ν . This gives the statement. \square

According to [10] Corollary 3.1.2, there exist a neighborhood U of p in M and complex coordinates $z = (z_1, z_2) : U \rightarrow \mathbb{B}_2 \subset \mathbb{C}^2$, $z(p) = 0$ such that $z_*(J)(0) = J_{st}$ and moreover, a map $f : \Delta \rightarrow \mathbb{B}_2$ is J' := $z_*(J)$ -holomorphic if it satisfies the equations

$$(4) \quad \partial f_j / \partial \bar{\zeta} = A_j(f_1, f_2) \overline{(\partial f_j / \partial \zeta)}, \quad \text{for } j = 1, 2,$$

where $A_j(z) = O(|z|)$, $j = 1, 2$.

In order to obtain such coordinates, one can consider two transversal foliations of \mathbb{B} by J' -holomorphic curves (see [7]) and then take these curves into the lines $z_j = \text{const}$ by a local diffeomorphism. The direct image of the almost complex structure J under such a diffeomorphism has a diagonal matrix $J'(z_1, z_2) = (a_{jk}(z))_{jk}$ with $a_{12} = a_{21} = 0$ and $a_{jj} = i + \alpha_{jj}$ where $\alpha_{jj}(z) = \mathcal{O}(|z|)$ for $j = 1, 2$.

In what follows we omit the prime and denote this structure again by J . We also still denote by J_ν the structure $z_*(J_\nu)$ defined in \mathbb{B}_2 and we set $p^\nu := z(f^\nu(x^0))$. Hence we may assume that $U = \mathbb{B}_2$ and that $G^\nu \cap U = G^\nu \cap \mathbb{B}_2$ are domains in \mathbb{C}^2 , contained in \mathbb{B}_2 . Let $G := \{x \in \mathbb{B}_2 : r(x) < 0\}$. We may assume that the complex tangent space $T_0(\partial G) \cap J(0)T_0(\partial G) = T_0(\partial G) \cap iT_0(\partial G)$ is given by $\{z_2 = 0\}$. In particular, we have the following expansion for the defining function r of G on \mathbb{B}_2 : $r(z, \bar{z}) = 2\text{Re}(z_2) + 2\text{Re}K(z) + H(z, \bar{z}) + \mathcal{O}(|z|^3)$, where $K(z) = \sum \lambda_{k,l} z_k z_l$, $\lambda_{k,l} = \lambda_{l,k}$ and $H(z, \bar{z}) = \sum \lambda_{k,\bar{l}} z_k \bar{z}_l$, $\lambda_{k,\bar{l}} = \lambda_{l,\bar{k}}$.

Lemma 3. *The domain G is strictly J_{st} -pseudoconvex near the origin.*

Proof of Lemma 3. Consider a complex vector $v = (v_1, 0)$ tangent to ∂D at the origin. Let $f : \Delta \rightarrow \mathbb{C}^2$ be a J -holomorphic disc centered at the origin and tangent to v : $f(\zeta) = v\zeta + \mathcal{O}(|\zeta|^2)$. Since $A_2 = \mathcal{O}(|z|)$, it follows from the J -holomorphy equation (4) that $(f_2)_{\zeta\bar{\zeta}}(0) = 0$. This implies that $(r \circ f)_{\zeta\bar{\zeta}}(0) = H(v)$. Thus, the Levi form with respect to J coincides with the Levi form with respect to J_{st} on the complex tangent space of ∂G at the origin. \square

For sufficiently large ν let q^ν be the unique point on $\partial G^\nu \cap \mathbb{B}_2$ such that $|q^\nu - p^\nu| = \text{dist}(p^\nu, \partial G^\nu \cap \mathbb{B}_2)$. Since the sequence $(J_\nu)_\nu$ converges to J in the $\mathcal{C}^2(\bar{\mathbb{B}}_2)$ convergence, there exists a diffeomorphism A^ν defined on \mathbb{B}_2 such that $A^\nu(q^\nu) = 0$ and if $A^\nu(z) =: z^\nu = (z_1^\nu, z_2^\nu)$ then the expansion of r_ν in the z^ν -coordinates is given by : $r_\nu = 2\lambda_\nu^\nu \text{Re}(z_2^\nu) + 2\text{Re}(\sum_{k,l=1}^2 \lambda_{k,l}^\nu z_k^\nu z_l^\nu) + \sum_{k,l=1}^2 \lambda_{k,\bar{l}}^\nu z_k^\nu \bar{z}_l^\nu + \mathcal{O}(|z^\nu|^3)$, $\lambda_{k,l}^\nu = \lambda_{l,k}^\nu$, $\lambda_{k,\bar{l}}^\nu = \lambda_{l,\bar{k}}^\nu$, where the condition $\mathcal{O}(|z^\nu|^3)$ is uniform with respect to ν . Finally the diffeomorphism A^ν can be chosen such that $d(A^\nu) \circ J_\nu \circ d(A^\nu)^{-1}$ is represented by a diagonal matrix $(a_{jk}^\nu(z^\nu))_{jk}$ with $a_{12}^\nu = a_{21}^\nu = 0$ and $a_{jj}^\nu = i + \alpha_{jj}^\nu(z^\nu)$ where a_{jj}^ν converges to a_{jj} , with its first derivatives, uniformly on compact subsets of \mathbb{B}_2 . We note that A^ν converges to the identity map in any $\mathcal{C}^k(\bar{\mathbb{B}}_2)$ norm. Since r_ν converges to r in the $\mathcal{C}^2(\bar{\mathbb{B}}_2)$ norm by assumption, it follows that : $\lim_{\nu \rightarrow \infty} \lambda_2^\nu = 1$, $\lim_{\nu \rightarrow \infty} \lambda_{k,l}^\nu = \lambda_{k,l}$ and $\lim_{\nu \rightarrow \infty} \lambda_{k,\bar{l}}^\nu = \lambda_{k,\bar{l}}$ for $k, l = 1, 2$.

For every $\nu \geq 1$, let $\tau_\nu := \text{dist}(p^\nu, q^\nu)$. We define the dilation $\Lambda^\nu : A^\nu(\mathbb{B}_2) \rightarrow \mathbb{C}^2$ by $\Lambda^\nu(z^\nu) = (z_1^\nu / \sqrt{\tau_\nu}, z_2^\nu / \tau_\nu)$, in the z^ν -coordinates. Let \tilde{J}_ν be the almost complex structure

defined on $\Lambda^\nu \circ A^\nu(\mathbb{B}_2)$ by $\tilde{J}_\nu := \Lambda^\nu \circ dA^\nu \circ J_\nu \circ d(\Lambda^\nu)^{-1} \circ (\Lambda^\nu)^{-1}$. Since A^ν is $(J_\nu, dA^\nu \circ J_\nu \circ d(\Lambda^\nu)^{-1})$ -holomorphic, Λ^ν is (J_ν, \tilde{J}_ν) -holomorphic. Finally, we consider the domain

$$\tilde{G}^\nu := \Lambda^\nu \circ A^\nu(G^\nu \cap \mathbb{B}_2) = \{z \in \Lambda^\nu \circ A^\nu(\mathbb{B}_2) : r_\nu((\Lambda^\nu \circ A^\nu)^{-1}(z), \overline{(\Lambda^\nu \circ A^\nu)^{-1}(z)}) < 0\}.$$

The next Lemma gives the limit behaviour of the domains \tilde{G}^ν and of the almost complex structures \tilde{J}_ν .

Lemma 4. *The following conditions are satisfied :*

- (i) $\lim_{\nu \rightarrow \infty} \tilde{G}^\nu = \mathbb{G} = \{z \in \mathbb{C}^2 : \operatorname{Re}(z_2) + 2\operatorname{Re}(K(z_1, 0)) + H((z_1, 0), (\bar{z}_1, 0)) < 0\}$,
- (ii) $\lim_{\nu \rightarrow \infty} \tilde{J}_\nu = J_{st}$ uniformly on compact subsets of \mathbb{C}^2 .

Proof of Lemma 4. The convergence in statement (i) is the Hausdorff convergence on compact subsets. Condition (i) is a direct consequence of the convergence of $(r_\nu)_\nu$ to r and is similar to the usual complex case (see [8]).

Proof of (ii). For every ν the almost complex structure \tilde{J}_ν is represented by the diagonal matrix $(b_{jk}^\nu(z))_{jk}$ where $b_{jj}^\nu(z) = a_{jj}^\nu((\Lambda^\nu)^{-1}(z)) = i + \alpha_{jj}^\nu(\sqrt{\tau_\nu}z_1, \tau_\nu z_2) \xrightarrow{\nu \rightarrow \infty} i$, uniformly on compact subsets of \mathbb{C}^2 . \square

For every ν , let $F^\nu := \Lambda^\nu \circ A^\nu \circ f^\nu$. Then F^ν is a (J_{st}, \tilde{J}_ν) -holomorphic map from $(f^\nu)^{-1}(G^\nu \cap \mathbb{B}_2)$ to \tilde{G}^ν satisfying $F^\nu(x^0) = (0, -1)$ and we have :

Proposition 2. (i) *We may extract from $(F^\nu)_\nu$ a subsequence converging, uniformly on compact subsets of D , to a J_{st} -holomorphic map $F : D \rightarrow \mathbb{G}$,*

(ii) *We may extract from $((F^\nu)^{-1})_\nu$ a subsequence converging, uniformly on compact subsets of \mathbb{G} , to a J_{st} -holomorphic map $\tilde{F} : \mathbb{G} \rightarrow \tilde{D}$,*

(iii) *F is a biholomorphism from D to \mathbb{G} with $F^{-1} = \tilde{F}$.*

Statement (iii) is the conclusion of Theorem 1. Indeed it follows from Lemma 3 that $H((z_1, 0), (\bar{z}_1, 0)) = \alpha z_1 \bar{z}_1$ with $\alpha > 0$. Then the map $(z_1, z_2) \mapsto (\sqrt{\alpha}z_1, z_2 + H(z_1, 0))$ is a biholomorphism from \mathbb{G} to the unbounded representation of the unit ball $\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}(z_2) + |z_1|^2 < 0\}$.

Our proof of Proposition 2 is based on the method developed by F.Berteloot and G.Coeur e [3] and by F.Berteloot [2]. We first prove the following Lemma :

Lemma 5. *There exist $C_0 > 0$, $\delta_0 > 0$ and $r_0 > 0$ such that for every $0 < \delta < \delta_0$, for every $\nu \gg 1$ and for every J_ν -holomorphic disc $g^\nu : \Delta \rightarrow G^\nu$ we have :*

$$g^\nu(0) \in Q(0, \delta) \Rightarrow g^\nu(r_0\Delta) \subset Q(0, C_0\delta),$$

where $Q(0, \delta) := \{z \in \mathbb{C}^2 : |z_1| < \sqrt{\delta}, |z_2| < \delta\}$.

Proof of Lemma 5. Assume by contradiction that there exist $C_\nu \rightarrow \infty$, $\zeta_\nu \rightarrow 0$ in Δ and J_ν -holomorphic discs $g^\nu : \Delta \rightarrow G^\nu$ such that $g^\nu(0) \in Q(0, \delta_\nu)$ and $g^\nu(\zeta_\nu) \notin Q(0, C_\nu\delta_\nu)$. Let $d_\nu : z \mapsto (z_1/\sqrt{\delta_\nu}, z_2/\delta_\nu)$, let $h^\nu := d_\nu \circ g^\nu$ and let $J^\nu := d_\nu(J_\nu)$. We set $H^\nu := \{z \in d_\nu(U) : r_\nu \circ (d_\nu)^{-1}(z, \bar{z}) < 0\}$. It follows from Lemma 4 part (ii) that $\rho^\nu := r_\nu \circ d_\nu$ converges to $\rho := \operatorname{Re}(z_2) + |z_1|^2$, uniformly on compact subsets of \mathbb{C}^2 and J^ν converges to J_{st} , uniformly on compact subsets of \mathbb{C}^2 . There exists $A > 0$ such that the function $\rho + A\rho^2$ is strictly J_{st} -plurisubharmonic on $Q(0, 5)$. Hence for sufficiently large ν the function $\rho^\nu + A(\rho^\nu)^2$ is strictly J^ν -plurisubharmonic on $Q(0, 4)$. Moreover we can extend this function as a J^ν -plurisubharmonic function on H^ν . In particular it follows from Proposition A and Proposition B that there is a positive constant C such that $K_{(H^\nu, J^\nu)}(z, v) \geq C\|v\|$ for every $z \in H^\nu \cap Q(0, 3)$, $v \in \mathbb{C}^2$. Therefore, there exists a constant $C' > 0$ such that $\|dh_\nu(\zeta)\| \leq C'$ for any $\zeta \in (1/2)\Delta$ satisfying $h_\nu(\zeta) \in H^\nu \cap Q(0, 3)$. On the other hand, the sequence $|h_\nu(\zeta_\nu)|$ tends to $+\infty$. Denote by $[0, \zeta_\nu]$ the segment (in \mathbb{C}) joining the origin and ζ_ν and let $\zeta'_\nu \in [0, \zeta_\nu]$ be the point closest to the origin such that $h_\nu([0, \zeta'_\nu]) \subset H^\nu \cap \bar{Q}(0, 2)$ and $h_\nu(\zeta'_\nu) \in \partial Q(0, 2)$. Since $h_\nu(0) \in Q(0, 1)$, we have $|h_\nu(0) - h_\nu(\zeta'_\nu)| \geq C''$ for some constant $C'' > 0$. Let

$\zeta'_\nu = r_\nu e^{i\theta_\nu}$, $r_\nu \in]0, 1[$. Then

$$|h_\nu(0) - h_\nu(\zeta'_\nu)| \leq \int_0^{r_\nu} \|dh_\nu(te^{i\theta_\nu})\| dt \leq C' r_\nu \longrightarrow 0.$$

This contradiction proves Lemma 5.

We note that Lemma 5 is also satisfied replacing J_ν -holomorphic discs by (J_{st}, J_ν) -holomorphic maps. As a corollary we have the following

Lemma 6. *For any compact subset $K \subset D$ the sequence of norms $(\|F^\nu\|_{C^0(K)})_\nu$ is bounded.*

For the proof we can consider a covering of K by sufficiently small balls, similarly to [3], p.84. Indeed, consider a covering of K by the balls $p^j + r_0\mathbb{B}$, $j = 0, \dots, N$ where r_0 is given by Lemma 5 and $p^{j+1} \in p^j + r_0\mathbb{B}$ for any j . Since $\lim_{\nu \rightarrow \infty} (A^\nu \circ f^\nu)(x^0) = 0$, we obtain, for ν large enough, that $A^\nu \circ f^\nu(p^0 + r\mathbb{B}) \subset Q(0, 2C\tau_\nu)$, then $A^\nu \circ f^\nu(p^1 + r\mathbb{B}) \subset Q(0, 4C^2\tau_\nu)$. Continuing this process we obtain that $A^\nu \circ f^\nu(p^N + r\mathbb{B}) \subset Q(0, 2^N C^N \tau_\nu)$. Since $\tau_\nu \rightarrow 0$ and $F^\nu = \Lambda^\nu \circ A^\nu \circ f^\nu$ we obtain Lemma 6.

We prove now Proposition 2. *Part (i)*. Lemma 6 implies that the sequence $(F^\nu)_\nu$ is bounded (in the C^0 norm) on any compact subset K of D . Covering K by small bidiscs, consider two transversal foliations by holomorphic curves on every bidisc. Since the restriction of F^ν on every such curve is uniformly bounded in the C^0 -norm, it follows by the well-known elliptic estimates that it is bounded in C^l norm for every l (see [10]). Since the bounds are uniform with respect to curves, this implies that the sequence $(F^\nu)_\nu$ is bounded in every C^l -norm. So the family $(F^\nu)_\nu$ is relatively compact by Ascoli theorem. Let F be a cluster point of $(F^\nu)_\nu$. We still denote by $(F^\nu)_\nu$ an appropriate subsequence converging (uniformly on compact subsets of D) to F . Passing to the limit in the holomorphy condition $\tilde{J}_\nu \circ dF^\nu = dF^\nu \circ \tilde{J}_\nu$, we obtain that F is holomorphic with respect to J_{st} .

Part (ii). Let $0 < r < 1$, $B(0, r) = \{z \in \mathbb{C}^2 : \|z\| < r\}$ and let Φ be a biholomorphism from \mathbb{G} to \mathbb{B}_2 satisfying $\Phi((0, -1)) = (0, 0)$ (this is a biholomorphism for the standard structure J_{st} on \mathbb{G} and \mathbb{B}_2). According to Lemma 4 (ii) we have, for sufficiently large ν , the inclusion $\Phi^{-1}(B(0, r)) \subset \tilde{G}^\nu$. Since the sequence $(\tilde{J}_\nu)_\nu$ converges uniformly on compact subsets of \mathbb{G} to J_{st} (in C^2 norm) by Lemma 4 (iii), the sequence $(J'_\nu := d\Phi \circ \tilde{J}_\nu \circ d\Phi^{-1})_\nu$ converges to J_{st} uniformly on compact subsets of \mathbb{B}_2 (in C^2 norm). Fix $0 < r < 1$. Then there exists a positive constant C_r and $r < r' < 1$ such that for every $z \in B(0, r)$ and every unitary vector v in \mathbb{C}^2 there is a J_{st} -holomorphic disc $\varphi_{z,v} : \Delta \rightarrow \mathbb{B}_2$ satisfying $\varphi_{z,v}(0) = z$, $\varphi'(0) \in \mathcal{C}v$, $|\varphi'(0)| \geq C_r$ and $\varphi_{z,v}(\Delta(0, 1/2)) \subset B(0, r')$. According to Section 5.4a of [7] (study of the stability of J -holomorphic discs under small smooth deformations of a given almost complex structure J) there exists a positive constant C'_r and $r < r'' < 1$ such that for sufficiently large ν , for every $z \in B(0, r)$ and for every unitary vector $v \in \mathbb{C}^2$, there is a J'_ν -holomorphic disc $f_{J'_\nu, z, v} : \Delta \rightarrow B_2$, centered at z , satisfying

$$(5) \quad d(f_{J'_\nu, z, v})(0)(\partial/\partial x) = \alpha v \quad \text{with } \alpha \geq C'_r$$

and $f_{J'_\nu, z, v}(\Delta(0, 1/3)) \subset B(0, r'')$. Since for sufficiently large ν the J_{st} -holomorphic disc $\tilde{f}_{J'_\nu, z} := (\Phi \circ F^\nu)^{-1} \circ f_{J'_\nu, z}$ satisfies the inclusion $\tilde{f}_{J'_\nu, z}(\Delta) \subset D \subset \subset \mathbb{C}^2$, there exists a positive constant C' such that

$$(6) \quad |(\tilde{f}_{J'_\nu, z})'(0)| \leq C'$$

for sufficiently large ν .

It follows from conditions (5) and (6) that the first derivative of $(\Phi \circ F^\nu)^{-1}$ is bounded from below by a positive constant on $B(0, r)$, uniformly with respect to $\nu \gg 1$. By Ascoli Theorem we may extract from $((F^\nu)^{-1})_\nu$ a subsequence that converges uniformly on $B(0, r)$ to a holomorphic map $\tilde{F}_r : B(0, r) \rightarrow \tilde{D}$. In particular $\tilde{F}_{r'} = \tilde{F}_r$ on $B(0, r)$ for $r' > r$. This proves condition (ii).

Part (iii). We know that $F(x^0) = 0 \in \mathbb{G}$. Assume now that there is $x \in D$ such that $F(x) \in \partial\mathbb{G}$ and let γ be a C^1 -path in D such that $\gamma(0) = x^0$, $\gamma(1) = x$. We consider

$t_0 \leq 1$ such that $F(\gamma(0, t_0]) \subset \mathbb{G}$ and $F(\gamma(t_0)) \in \partial\mathbb{G}$. Since \mathbb{G} is complete hyperbolic (\mathbb{G} is biholomorphic to the unit ball in \mathbb{C}^2), we obtain that $\lim_{t \rightarrow t_0} d_{\mathbb{H}}^K(0, F(\gamma(t))) = \infty$. However, for every $t < 1$:

$$d_{\mathbb{G}}^K(0, F(\gamma(t))) \leq \sup_{s \in [0, 1]} d_D^K(x^0, \gamma(s)) < \infty$$

by the compactness of $\gamma([0, 1])$ in D . This is a contradiction, implying that $F(D) \subset \mathbb{G}$. We prove now that $\tilde{F}(\mathbb{G}) \subset D$. Since D is bounded in \mathbb{C}^2 , D is hyperbolic for the standard structure J_{st} . Let $r > 0$ and consider the Kobayashi ball $B_{(D, J_{st})}^K(x^0, r)$. Since f^ν is a biholomorphism from (D, J_{st}) to (G^ν, J_ν) , we have $f^\nu(B_{(D, J_{st})}^K(x^0, r)) = B_{(G^\nu, J_\nu)}(f^\nu(x^0), r)$ (see equality (1)). Since the sequence of points $(f^\nu(x^0))_\nu$ converges to p we obtain from Proposition C that $B_{(G^\nu, J_\nu)}(f^\nu(x^0), r) \subset\subset G^\nu \cap U$ for sufficiently large ν . In particular, since $(f^\nu)^{-1}$ is continuous on G^ν , we have : $B_{(D, J_{st})}^K(x^0, r) \subset\subset D$. This implies that D is complete hyperbolic. Assume that $q \in \mathbb{G}$ is such that $\tilde{F}(q) \in \partial D$. Then there exists $r > 0$ such that $q \in B_{(\mathbb{G}, J_{st})}^K(0, r)$. Let $\gamma : [0, 1] \rightarrow \mathbb{G}$ be a \mathcal{C}^1 -path such that $\gamma(0) = 0$, $\gamma(1) = q$ and $\gamma([0, 1]) \subset B_{(\mathbb{G}, J_{st})}^K(0, 2r)$. We define $t_0 \leq 1$ such that $\tilde{F}(\gamma([0, t_0])) \subset D$ and $\tilde{F}(\gamma(t_0)) \in \partial D$. By the complete hyperbolicity of D , we have : $\lim_{t \rightarrow t_0} d_{(D, J_{st})}^K(x^0, \tilde{F}(\gamma(t))) = \infty$, which contradicts the condition $d_{(D, J_{st})}^K(x^0, \tilde{F}(\gamma(t))) \leq d_{(\mathbb{G}, J_{st})}(0, \gamma t) \leq 2r$ for every $t < t_0$. Consequently : $\tilde{F}(\mathbb{G}) \subset D$. Now, since for every ν , $(F^\nu)^{-1} \circ F^\nu = id_D$ and $F^\nu \circ (F^\nu)^{-1} = id_{\mathbb{G}}$ we obtain, by taking the limit when $\nu \rightarrow \infty$, that $\tilde{F} \circ F = id_D$ and $F \circ \tilde{F} = id_{\mathbb{G}}$, meaning that F is a biholomorphism from D to \mathbb{G} . This proves Theorem 1. \square

Corollary 1 is a direct consequence of Theorem 1.

4. PROOF OF THEOREM 2

We explain now how to adapt the proof of Theorem 1 to obtain Theorem 2. Hence we assume in this Section that the assumptions of Theorem 2 are satisfied.

We have the following

Lemma 7. *For every $0 \in K \subset\subset \Delta$ we have : $\lim_{\nu \rightarrow \infty} (f^\nu \circ f)(K) = p$, uniformly with respect to $f \in \mathcal{O}_{(J_{st}, J)}(\Delta, D)$ such that $f(0) = x^0$.*

Proof of Lemma 7. The proof follows line by line the proof of Lemma 2, replacing f^ν by $f^\nu \circ f$ where $f \in \mathcal{O}_{(J_{st}, J)}(\Delta, D)$, $f(0) = x^0$. Indeed since the function $x \mapsto d_{(\Delta, J_{st})}^K(0, x)$ is bounded from above on K by a positive constant C , we have :

$$d_{(D, J)}^K(f^\nu \circ f(0), f^\nu \circ f(x)) \leq C$$

for every $\nu \geq 1$ and for every $x \in K$. Fix a neighborhood $V \subset\subset M$ of p . Since

$$\lim_{\nu \rightarrow \infty} d_{(D, J)}^K(f^\nu(x^0), D \cap \partial V) = \infty$$

we have $f^\nu \circ f(K) \subset V$ for sufficiently large ν , uniformly with respect to $f \in \mathcal{O}_{(J_{st}, J)}(\delta, D)$ such that $f(0) = x^0$. \square

Proof of Theorem 2. Assume by contradiction that there is a sequence $(f^\nu)_\nu$ in $Diff_{(J, J)}(D, D)$ and points $x^0 \in D$, $p \in \partial D$ such that $\lim_{\nu \rightarrow \infty} f^\nu(x^0) = p$. Lemma 7 implies that : $\lim_{\nu \rightarrow \infty} f^\nu(K) = p$, for every compact K in D .

Let U , z and A^ν defined as in page 5 and let Λ^ν is the dilation defined page 6. We consider $G^\nu := D$, $J_\nu := J$ and $\tilde{J}_\nu := \Lambda^\nu \circ d(A^\nu) \circ J \circ d(A^\nu)^{-1} \circ (\Lambda^\nu)^{-1}$ for every ν . We may assume as in the proof of Theorem 1 that $D \cap U$ is in \mathbb{C}^2 and we still denote by J the associated structure in \mathbb{C}^2 . If we set $\tilde{G}^\nu := \Lambda^\nu \circ A^\nu(D \cap U)$ then by Lemma 4 we have $\lim_{\nu \rightarrow \infty} \tilde{G}^\nu = \mathbb{G}$ and $\lim_{\nu \rightarrow \infty} \tilde{J}_\nu = J_{st}$, uniformly on compact subsets of \mathbb{C}^2 . If $F^\nu := \Lambda^\nu \circ A^\nu \circ f^\nu$ then $F^\nu \in Diff_{(J, \tilde{J}_\nu)}((f^\nu)^{-1}(D \cap U), \tilde{G}^\nu)$ and according to Proposition 2 the sequence $(F^\nu)_\nu$ converges after extraction to a (J, J_{st}) -holomorphic map F from D to \mathbb{G} .

We use the following quantitative version of Proposition 2.3.6 of [10] :

Proposition 3. *Let D' be a domain in \mathbb{C}^n . There is a positive constant δ_0 such that for every almost complex structure J' in a neighborhood of $\overline{D'}$ satisfying $\|J' - J_{st}\|_{C^2(\overline{D'})} \leq \delta_0$ we have*

$$(7) \quad \|f_{J'}\|_{C^1(K)} \leq C_{K,\delta_0} \|f_{J'}\|_{C^0(K)},$$

for every $f_{J'} \in \mathcal{O}_{(J_{st}, J')}(\Delta, D')$ and for every $K \subset\subset \Delta$, where C_{K,δ_0} is a positive constant depending only on K and δ_0 .

Let $U' \subset\subset U$ be a neighborhood of p such that

$$(8) \quad \|J - J_{st}\|_{C^2(\overline{U'})} \leq \delta_0.$$

Fix $0 < r < 1$, sufficiently close to 1. Since $\lim_{\nu \rightarrow \infty} \|\tilde{J}_\nu - J_{st}\|_{C^2(\overline{B(0,r)})} = 0$, it follows from [7] Section 4.5a that there is a covering \mathcal{R}_ν of $\overline{B(0,r)}$ by \tilde{J}_ν -holomorphic discs centered at the origin for sufficiently large ν , these discs being small deformations of the straight holomorphic discs in the ball. More precisely there exists $0 \in K_r \subset\subset \Delta$ and there exists a positive constant c_r such that for sufficiently large ν we have $\overline{B(0,r)} \subset \cup_{f \in \mathcal{R}_\nu} f(K_r)$ and

$$(9) \quad \inf_{f \in \mathcal{R}_\nu, \zeta \in K_r} \|df(\zeta)(\partial/\partial x)\| \geq c_r.$$

Moreover we may assume that $\Phi^{-1}(\overline{B(0,r)}) \subset G^\nu$. For $f \in \mathcal{R}_\nu$ consider the (J_{st}, J) -holomorphic map $g^\nu := (F^\nu)^{-1} \circ \Phi^{-1} \circ f$ from Δ to D . Since $g^\nu(0) = x^0$ it follows from Lemma 7 that there exists $\nu_0 \geq 1$ such that $f_{\nu_0} \circ g^\nu(K_r) \subset U'$, uniformly with respect to $\nu \gg 1$. It follows now from condition (8) and from Proposition 3 ($D \cap U' \subset \mathbb{C}^2$) that there exists a positive constant C_r such that $\|f_{\nu_0} \circ g^\nu\|_{C^1(K_r)} \leq C_r$ for sufficiently large ν , or equivalently that $\|g^\nu\|_{C^1(K_r)} \leq C'_r$ with $C'_r > 0$.

It follows from inequality (9) that $\|(F^\nu)^{-1} \circ \Phi^{-1}\|_{C^1(\overline{B(0,r)})} \leq C''_r$ for a positive constant C''_r and finally that $\|(F^\nu)^{-1}\|_{C^1(\Phi^{-1}(\overline{B(0,r)}))} \leq \tilde{C}_r$ for every sufficiently large ν , where \tilde{C}_r is a positive constant.

Since D is relatively compact in M , Ascoli Theorem implies that some subsequence of $((F^\nu)^{-1})_\nu$ converges to a (J_{st}, J) -holomorphic map \tilde{F} from \mathbb{G} to \overline{D} . The end of the proof of Theorem 2 follows line by line the proof of Proposition 2 (iii). Since $\tilde{F} \circ F = id_D$ and $F \circ \tilde{F} = id_{\mathbb{G}}$ we obtain that $F \in Diff_{(J, J_{st})}(D, \mathbb{G})$. This gives the contradiction. \square

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