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ESTIMATES OF THE KOBAYASHI METRIC ON ALMOST COMPLEX MANIFOLDS

HERVÉ GAUSSIER AND ALEXANDRE SUKHOV

ABSTRACT. We establish a lower estimate for the Kobayashi-Royden infinitesimal pseudometric on an almost complex manifold (M, J) admitting a bounded strictly plurisubharmonic function. We apply this result to study the boundary behaviour of the metric on a strictly pseudoconvex domain in M and to give a sufficient condition for the complete hyperbolicity of a domain in (M, J) . Finally we obtain the regularity up to the boundary of J -holomorphic discs attached to a totally real submanifold in M .

INTRODUCTION

The properties of the Kobayashi-Royden infinitesimal pseudometric on complex manifolds (with the standard integrable structure) have been intensively studied by many authors (see, for instance, [15] and references therein). On the contrary, its study on almost complex manifolds began quite recently (see [18], [9]) and many basic questions are still open.

The main goal of the present paper is to give a precise lower estimate for the Kobayashi-Royden pseudometric on a strictly pseudoconvex domain in an almost complex manifold. Such estimates are well known in the integrable case (see [11], [22]). However, the standard methods have no direct generalization to the case of an almost complex structure. For instance, the natural and usual technique consisting in osculating the boundary of a strictly pseudoconvex domain by balls cannot be used, since the Kobayashi-Royden metric of the unit ball with a non-standard almost complex structure cannot be computed explicitly. For this reason, we use another approach based on the construction of special classes of plurisubharmonic functions, developed essentially by N. Sibony [22] in the case of the standard complex structure (see also [10] by K. Diederich-J.E. Forneaess and [17] by N. Kerzman-J.P. Rosay). A corollary of our main results states that every point in an almost complex manifold has a complete hyperbolic neighborhood. Related results were obtained in [9] and [16].

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1. PRELIMINARIES

We will sometimes refer to results proved for complex manifolds and valid without substantial modification in the almost complex case. For coherence of exposition we will not present all of them in the Preliminaries. They will be stated with references in the different Sections.

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1.1. Almost complex manifolds. Let M be a smooth (everywhere in this paper this means C^∞) real $(2n)$ -dimensional manifold equipped with an almost complex structure J , that is a C^∞ -field of complex linear structures on the tangent bundle TM of M (we call (M, J) an *almost complex manifold*). For example, if $M = \mathbb{R}^{2n}$ we may consider *the standard complex structure* J_0 which associates to every point $p \in \mathbb{R}^{2n}$ the endomorphism of $T_p(M) = \mathbb{R}^{2n}$ given by the matrix

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n . Identifying \mathbb{R}^{2n} with \mathbb{C}^n by the isomorphism $(x, y) \mapsto z = x + iy$ for $x, y \in \mathbb{R}^n$ we see that the matrix J_0 corresponds to the multiplication by i . If D is a domain in \mathbb{R}^{2n} , then an almost complex structure J on D may be viewed as a matrix-valued map $J(x, y) = (a_{i,j}(x, y))_{1 \leq i, j \leq n}$ defined on D and satisfying $J^2 = -I_{2n}$, where $a_{i,j}$ is a smooth real valued function on D for every $1 \leq i, j \leq n$. In the general case of an almost complex manifold (M, J) the structure J admits such a representation in a local chart. Another possibility is to consider an almost complex structure as a tensor field of type $(1,1)$ on M . For instance, the standard complex structure J_0 on \mathbb{C}^n can be written in the form

$$J_0 = i \sum_{j=1}^n dz_j \otimes \frac{\partial}{\partial z_j} - d\bar{z}_j \otimes \frac{\partial}{\partial \bar{z}_j}.$$

This representation will be used in Lemma 3.

Let (M', J') and (M, J) be almost complex manifolds and let f be a smooth map from M' to M . We say that f is (J', J) -holomorphic if its differential $df : TM' \rightarrow TM$ satisfies

$$(1) \quad df \circ J' = J \circ df$$

on TM' . We denote by $\mathcal{O}_{(J', J)}(M', M)$ the set of (J', J) -holomorphic maps from M' to M . Everywhere in this paper Δ denotes the unit disc in \mathbb{C} . If $(M', J') = (\Delta, J_0)$, a (J_0, J) -holomorphic map is called a *J -holomorphic disc* and we denote by $\mathcal{O}_J(\Delta, M)$ the set of J -holomorphic discs in M . We note that generically, for an almost complex manifold (M', J') , equation (1) has no nontrivial solution, even when $(M, J) = (\Delta, J_0)$. On the contrary if $M' = \Delta$ then equation (1) represents a first order quasi-linear elliptic system which has nontrivial solutions (see [1]). An almost complex structure is called *integrable* if its Nijenhuis tensor vanishes; by the Newlander - Nirenberg theorem [13] this is equivalent to the existence of a local (J, J_0) -holomorphic diffeomorphism between M and an open subset of \mathbb{C}^n . If M is a real surface then (up to a biholomorphic equivalence) the only almost complex structure on M is the standard structure J_0 .

The general setting of the paper is the following. Let (M, J) be an almost complex manifold of real dimension $2n$ and let $p \in M$. We may assume that $J(p) = J_0$, the standard structure. Denote by \mathbb{B} the unit ball in \mathbb{C}^n . Then there is a neighborhood U of p and coordinates $(x_1, \dots, x_n, y_1, \dots, y_n) : U \rightarrow \mathbb{B} \subset \mathbb{R}^{2n}$, centered at p , satisfying $\partial/\partial y_j = J(p)(\partial/\partial x_j)$ for every $1 \leq j \leq n$. In general one can not find coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on U such that $\partial/\partial y_j = J(\partial/\partial x_j)$ on U , for every $1 \leq j \leq n$ (this would mean that U has a complex manifold structure). In particular, if we set $z_j = x_j + iy_j$ for $1 \leq j \leq n$, then the diffeomorphism $z := (z_1, \dots, z_n)$ between U and the unit ball $\mathbb{B} \subset \mathbb{C}^n$ is not (J, J_0) -holomorphic. However, z is a (J, \hat{J}) -holomorphic map where $\hat{J} = z^*(J) = (dz \circ J \circ dz^{-1})$ is the direct image of J under the map z . Let $|\bullet|$ be the usual Euclidean norm on \mathbb{C}^n . Then we define the distance between two points $p, q \in U$ by $dist(p, q) = |z(p) - z(q)|$. In a similar way, we define the length of

a vector $v \in T_p M$, where $p \in M$, by $\|v\| = |dz(p)v|$, where we identify $T_{z(p)}\mathbb{C}^n$ with \mathbb{C}^n .

An important special case of an almost complex manifold is a bounded domain D in \mathbb{C}^n equipped with an almost complex structure J , defined in a neighborhood of \bar{D} , and sufficiently close to the standard structure J_0 in the \mathcal{C}^2 norm on \bar{D} and every almost complex manifold may be represented locally in such a form. More precisely, we have the following Lemma.

Lemma 1. *Let (M, J) be an almost complex manifold. Then for every point $p \in M$ and every $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \rightarrow \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_0$ and the direct image $\hat{J} = z^*(J)$ satisfies $\|\hat{J} - J_0\|_{\mathcal{C}^2(\bar{\mathbb{B}})} \leq \lambda_0$.*

Proof. There exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto \mathbb{B} satisfying $z(p) = 0$ and $dz(p) \circ J(p) \circ dz^{-1}(0) = J_0$. For $\lambda > 0$ consider the dilation $d_\lambda : t \mapsto \lambda^{-1}t$ in \mathbb{C}^n and the composition $z_\lambda = d_\lambda \circ z$. Then $\lim_{\lambda \rightarrow 0} \|z_\lambda^*(J) - J_0\|_{\mathcal{C}^2(\bar{\mathbb{B}})} = 0$. Setting $U = z_\lambda^{-1}(\mathbb{B})$ for $\lambda > 0$ small enough, we obtain the desired statement. \square

Lemma 1 will be used throughout the paper. In particular, every almost complex structure J sufficiently close to the standard structure J_0 will be written $J = J_0 + \mathcal{O}(|z|)$. We also point out that in the case where (M, J) is a domain in \mathbb{R}^{2n} equipped with an almost complex structure J , the proof of Lemma 1 shows that for a fixed point p one may associate a neighborhood U of p and a family $J(\lambda) = J_\lambda$ of almost complex structures on U smoothly depending on a real parameter λ and such that $J(0) = J_0$ is the standard structure on \mathbb{C}^n (up to a linear conjugation) and $J(1) = J$. Consequently, by a small perturbation (or deformation) of the standard structure J_0 defined in a neighborhood of \bar{D} , where D is a domain in \mathbb{C}^n , we will mean a smooth one parameter family $(J_\lambda)_\lambda$ of almost complex structures defined in a neighborhood of \bar{D} , the real parameter λ belonging to a neighborhood of the origin, and satisfying : $\lim_{\lambda \rightarrow 0} \|J_\lambda - J_0\|_{\mathcal{C}^2(\bar{D})} = 0$.

1.2. Real submanifolds in an almost complex manifold. Let (M, J) be an almost complex manifold. We denote by TM the real tangent bundle of M and by $T_{\mathbb{C}}M$ its complexification. If :

$$T^{(1,0)}M := \{X \in T_{\mathbb{C}}M : JX = iX\} = \{\zeta - iJ\zeta, \zeta \in TM\},$$

$$T^{(0,1)}M := \{X \in T_{\mathbb{C}}M : JX = -iX\} = \{\zeta + iJ\zeta, \zeta \in TM\},$$

then $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. In particular every $X \in T_{\mathbb{C}}M$ has the unique decomposition $X = \zeta_1 + \zeta_2 - iJ(\zeta_1 - \zeta_2)$ where $\zeta_1, \zeta_2 \in TM$. Then we denote by \bar{X} the vector field $\bar{X} = \zeta_1 + \zeta_2 + iJ(\zeta_1 - \zeta_2)$ and we have the equivalence $X \in T^{(1,0)}M \Leftrightarrow \bar{X} \in T^{(0,1)}M$. Let T^*M denote the space of first-order differential forms (we call them one forms or forms of degree one) on M . By identifying $\mathbb{C} \otimes T^*M$ with $T_{\mathbb{C}}^*M := \text{Hom}(T_{\mathbb{C}}M, \mathbb{C})$ we may consider a complex one form on M as an element in $T_{\mathbb{C}}^*M$. In particular, if u is a smooth complex function on M , then $du \in T_{\mathbb{C}}^*M$. We define the set of complex forms of type $(1, 0)$ on M by :

$$T_{(1,0)}M = \{w \in T_{\mathbb{C}}^*M : w(X) = 0, \forall X \in T^{(0,1)}M\}$$

and the set of complex forms of type $(0, 1)$ on M by :

$$T_{(0,1)}M = \{w \in T_{\mathbb{C}}^*M : w(X) = 0, \forall X \in T^{(1,0)}M\}.$$

Then $T_{\mathbb{C}}^*M = T_{(1,0)}M \oplus T_{(0,1)}M$, meaning that every complex one form w on M may be uniquely decomposed as $w = w_{(1,0)} + w_{(0,1)}$, where $w_{(1,0)} \in T_{(1,0)}M$ and $w_{(0,1)} \in T_{(0,1)}M$. This allows to define the operators ∂_J and $\bar{\partial}_J$ on the space of

smooth functions defined on M : given a complex smooth function u on M , we set $\partial_J u = du_{(1,0)}$ and $\bar{\partial}_J u = du_{(0,1)}$. As usual, differential forms of any bidegree (p, q) on (M, J) can be defined by means of the exterior product.

Let Γ be a real smooth submanifold in M and let $p \in \Gamma$. We denote by $H^J(\Gamma)$ the J -holomorphic tangent bundle $T\Gamma \cap JT\Gamma$. We note that if Γ is a real hypersurface in M defined by $\Gamma = \{r = 0\}$ and $p \in \Gamma$ then by definition $H_p^J(\Gamma) = \{v \in T_p M : dr(p)(v) = dr(p)(J(p)v) = 0\}$. Considering $dr(p)$ as a complex linear form on $T_{\mathbb{C}}M$ we obtain that $H_p^J(\Gamma) = \{v \in T_p M : dr(p)(v - iJ(p)v) = 0\}$. However by definition of the operator $\bar{\partial}_J$ we have $\bar{\partial}_J(r)(p)(v - iJ(p)v) = 0$ for every $v \in T_p M$. It follows that $H_p^J(\Gamma) = \{v \in T_p M : \partial_J r(p)(v - iJ(p)v) = 0\}$. This identification will be used in the statement of Theorems 5 and 6.

We recall the notion of the Levi form of a hypersurface :

Definition 1. Let $\Gamma = \{r = 0\}$ be a smooth real hypersurface in M (r is any smooth defining function of Γ) and let $p \in \Gamma$.

(i) The Levi form of Γ at p is the map defined on $H_p^J(\Gamma)$ by $\mathcal{L}_\Gamma^J(X_p) = J^* dr[X, JX]_p$, where the vector field X is any section of the J -holomorphic tangent bundle $H^J\Gamma$ such that $X(p) = X_p$.

(ii) A real smooth hypersurface $\Gamma = \{r = 0\}$ in M is strictly J -pseudoconvex if its Levi form \mathcal{L}_Γ^J is positive definite on $H^J(\Gamma)$.

Remark 1. (i) the “strict J -pseudoconvexity” condition does not depend on the choice of a smooth defining function of Γ . Indeed if ρ is an other smooth defining function for Γ in a neighborhood of $p \in \Gamma$ then there exists a positive smooth function λ defined in a neighborhood of p such that $\rho = \lambda r$. In particular $(J^* dr)(p) = \lambda(p)(J^* d\rho)(p)$.

(ii) since the map $(r, J) \mapsto J^* dr$ is smooth the “strict J -pseudoconvexity” is stable under small perturbations of both the hypersurface and the almost complex structure.

Let $X \in TM$. It follows from the identity

$$(2) \quad d(J^* dr)(X, JX) = X(\langle J^* dr, JX \rangle) - JX(\langle J^* dr, X \rangle) - (J^* dr)[X, JX]$$

that

$$(J^* dr)[X, JX] = -d(J^* dr)(X, JX)$$

for every $X \in H^J\Gamma$, since $\langle dr, JX \rangle = \langle dr, JX \rangle = 0$ in that case. Hence we set

Definition 2. If r is a C^2 function on M then the Levi form of r is defined on TM by $\mathcal{L}^J(r)(X) := -d(J^* dr)(X, JX)$.

Let $p \in M$ and $v \in T_p M$. We will denote by $\mathcal{L}^J(r)(p)(v)$ the quantity $\mathcal{L}^J(r)(X)(p)$ where X is any section of TM such that $X(p) = v$. Obviously, the Levi form $\mathcal{L}^J(r)$ is determined by the form $-d(J^* dr)_{(1,1)}$ (where the $(1, 1)$ part of $-d(J^* dr)(X, JX)$ is taken with respect to J).

2. CONSTRUCTION OF J -PLURISUBHARMONIC FUNCTIONS WITH LOGARITHMIC SINGULARITIES

We start this Section with some general facts on J -plurisubharmonic functions.

2.1. Plurisubharmonic functions. We first recall the following definition :

Definition 3. An upper semicontinuous function u on (M, J) is called J -plurisubharmonic on M if the composition $u \circ f$ is subharmonic on Δ for every $f \in \mathcal{O}_J(\Delta, M)$.

If M is a domain in \mathbb{C}^n and $J = J_0$ then a J_0 -plurisubharmonic function is evidently a plurisubharmonic function in the usual sense. We recall that for a \mathcal{C}^2 function u defined on $U \subset \mathbb{C}^n$ we have $\mathcal{L}^{J_0}(u)(X) = -d(J_0^* du)(X, J_0 X) = 2i\partial\bar{\partial}u(X, J_0 X)$. Moreover u is plurisubharmonic on U if and only if $i\partial\bar{\partial}u \geq 0$ on U and u is strictly plurisubharmonic if and only if $i\partial\bar{\partial}u > 0$ on U . These notions may be carried to an almost complex manifold (M, J) and we have the following characterization of J -plurisubharmonic functions :

Proposition 1. *Let u be a \mathcal{C}^2 real valued function on M . Then u is J -plurisubharmonic on M if and only if $\mathcal{L}^J(u)(X) \geq 0$ for every $X \in TM$.*

This is a corollary of recent results of [8, 12].

Proof. Let $p \in M$ and let $X \in TM$. It follows from [19] that there exist $\alpha > 0$ and a J -holomorphic disc $f : \alpha\Delta \rightarrow M$ such that $f(0) = p$ and $df(0)(\partial/\partial x) = X(p)$. On the other hand, $\mathcal{L}^J(u)(p)(X) = -d(J^* du)(X, JX) = -d(J^* du)_{(1,1)}(X, JX)$. Then according to [12] (see also [8]) we have $f^* d(J^* du)_{(1,1)} = dJ_0^* d(u \circ f)$. This gives the equivalence by Definition 3. \square

Proposition 1 leads to the following definition :

Definition 4. *A \mathcal{C}^2 real valued function u on M is strictly J -plurisubharmonic on M if $\mathcal{L}^J(u)$ is positive definite on TM .*

Repeating the same argument as in the case of complex manifolds, one can show that if D is strictly J -pseudoconvex then there exists a neighborhood U of \bar{D} and a function ρ , J -plurisubharmonic on U and strictly J -plurisubharmonic in a neighborhood of ∂D , such that $D = \{\rho < 0\}$.

We have the following important example of a J -plurisubharmonic function on an almost complex manifold (M, J) :

Example 1. *For every point $p \in (M, J)$ there exists a neighborhood U of p and a diffeomorphism $z : U \rightarrow \mathbb{B}$ centered at p (ie $z(p) = 0$) such that the function $|z|^2$ is J -plurisubharmonic on U .*

Proof. Let $p \in M$, U_0 be a neighborhood of p and $z : U_0 \rightarrow \mathbb{B}$ be local complex coordinates centered at p , such that $dz \circ J(p) \circ dz^{-1} = J_0$ on \mathbb{B} . Consider the function $u(q) = |z(q)|^2$ on U_0 . For every $w, v \in \mathbb{C}^n$ we have $\mathcal{L}^{J_0}(u \circ z^{-1})(w)(v) = \|v\|^2$. Let $B(0, 1/2)$ be the ball centered at the origin with radius $1/2$ and let \mathcal{E} be the space of smooth almost complex structures defined in a neighborhood of $\overline{B(0, 1/2)}$. Since the function $(J', w) \mapsto \mathcal{L}^{J'}(u \circ z^{-1})(w)$ is continuous on $\mathcal{E} \times B(0, 1/2)$, there exist a neighborhood V of the origin and positive constants λ_0 and c such that $\mathcal{L}^{J'}(u \circ z^{-1})(q)(v) \geq c\|v\|^2$ for every $q \in V$ and for every almost complex structure J' satisfying $\|J' - J_0\|_{\mathcal{C}^2(\bar{V})} \leq \lambda_0$. Let U_1 be a neighborhood of p such that $\|z^*(J) - J_0\|_{\mathcal{C}^2(\overline{z(U_1)})} \leq \lambda_0$ and let $0 < r < 1$ be such that $B(0, r) \subset V$ and $U := z^{-1}(B(0, r)) \subset U_1$. Then we have the following estimate for every $q \in U$ and $v \in T_q M$: $\mathcal{L}^J(u)(q)(v) \geq c\|v\|^2$. Then $r^{-1}z$ is the desired diffeomorphism. \square

We also have the following

Lemma 2. *A function u of class \mathcal{C}^2 in a neighborhood of a point p of (M, J) is strictly J -plurisubharmonic if and only if one of the following equivalent conditions holds:*

(a) *there exists a neighborhood U of p with local complex coordinates $z : U \rightarrow \mathbb{B}$ centered at p , such that the function $u - c|z|^2$ is J -plurisubharmonic on U for some constant $c > 0$.*

(b) there exists a neighborhood U of p with local complex coordinates $z : U \rightarrow \mathbb{B}$ centered at p , such that the function $(u - c'|z|^2) \circ z^{-1}$ is J_0 -plurisubharmonic on \mathbb{B} for some constant $c' > 0$.

We point out that these two equivalent properties may be viewed as a definition of strict J -plurisubharmonicity in the case where the function u is just upper semicontinuous.

The function $\log |z|$ is J_0 -plurisubharmonic on \mathbb{C}^n and plays an important role in the pluripotential theory as the Green function for the complex Monge-Ampère operator on the unit ball. In particular, this function is crucially used in Sibony's method in order to localize and estimate the Kobayashi metric on a complex manifold. Unfortunately, after an arbitrarily small general almost complex perturbation of the standard structure this function is *not* plurisubharmonic with respect to a new structure (in any neighborhood of the origin), see for instance [8]. So in the remaining of the paper we will need the following statement communicated to the authors by E.Chirka :

Lemma 3. *Let p be a point in an almost complex manifold (M, J) . There exist a neighborhood U of p in M , a diffeomorphism $z : U \rightarrow \mathbb{B}$ centered at p and positive constants λ_0, A , such that the function $\log |z| + A|z|$ is J' -plurisubharmonic on U for every almost complex structure J' satisfying $\|J' - J\|_{C^2(\bar{U})} \leq \lambda_0$.*

Proof. The proof follows by a computation of the Levi form of Ψ_p . Let u be a C^2 function defined in a neighborhood U_0 of $p \in M$. Let us expand $\mathcal{L}^J(u)$, following [12], in the coordinates $z : U_0 \rightarrow \mathbb{B}$ centered at p , defined in Lemma 1, with the usual identification $\partial u / \partial z_j = (\partial(u \circ z^{-1}) / \partial z_j) \circ z$. Since $J^2 = -I$ and $\bar{J} = J$ then there exist complex constants $b_{k,l,m}, c_{k,l,m}$, defined for $1 \leq k, l, m \leq n$, and C^2 functions $f_{k,l}, g_{k,l}$, defined on U_0 for $1 \leq k, l \leq n$, bounded by $|z|^2$ on U_0 , such that :

$$\begin{aligned}
z^*(J) \circ z = J_0 &+ \sum_{1 \leq k, l \leq n} f_{k,l} dz_k \otimes \frac{\partial}{\partial z_l} \\
&+ \sum_{1 \leq k, l \leq n} \left(g_{k,l} + \sum_{1 \leq m \leq n} (b_{k,l,m} z_m + \bar{c}_{k,l,m} \bar{z}_m) \right) dz_k \otimes \frac{\partial}{\partial \bar{z}_l} \\
&+ \sum_{1 \leq k, l \leq n} \left(\bar{g}_{k,l} + \sum_{1 \leq m \leq n} (c_{k,l,m} z_m + \bar{b}_{k,l,m} \bar{z}_m) \right) d\bar{z}_k \otimes \frac{\partial}{\partial z_l} \\
&+ \sum_{1 \leq k, l \leq n} \bar{f}_{k,l} d\bar{z}_k \otimes \frac{\partial}{\partial \bar{z}_l}.
\end{aligned}$$

(see [12], we use the representation of an almost complex structure as a tensor field of type (1,1) on the unit ball \mathbb{B} with the coordinates z). It follows that

$$\begin{aligned}
J^* du &= i \sum_{j=1}^n \left(\frac{\partial u}{\partial z_j} dz_j - \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j \right) \\
&+ \sum_{1 \leq j, k, l \leq n} \left\{ (b_{jkl} z_l + \bar{c}_{jkl} \bar{z}_l) \frac{\partial u}{\partial \bar{z}_k} dz_j + (c_{jkl} z_l + \bar{b}_{jkl} \bar{z}_l) \frac{\partial u}{\partial z_k} d\bar{z}_j \right\} \\
&+ \sum_{j,k=1}^n \left(f_{j,k} \frac{\partial u}{\partial z_k} dz_j + \bar{f}_{j,k} \frac{\partial u}{\partial \bar{z}_k} d\bar{z}_j \right) \\
&+ \sum_{j,k=1}^n \left(g_{j,k} \frac{\partial u}{\partial \bar{z}_k} dz_j + \bar{g}_{j,k} \frac{\partial u}{\partial z_k} d\bar{z}_j \right).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
-d(J^* du)_{(1,1)} &= -2i \sum_{1 \leq j, k \leq n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \\
&+ \sum_{1 \leq j, k, l \leq n} \left(\bar{f}_{k,l} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_l} dz_j \wedge d\bar{z}_k - f_{k,l} \frac{\partial^2 u}{\partial z_l \partial \bar{z}_j} dz_k \wedge d\bar{z}_j \right) \\
&+ \sum_{1 \leq j, k, l \leq n} \frac{\partial^2 u}{\partial z_j \partial z_l} \left(\bar{g}_{k,l} + \sum_{1 \leq m \leq n} c_{k,l,m} z_m + \bar{b}_{k,l,m} \bar{z}_m \right) dz_j \wedge d\bar{z}_k \\
&+ \sum_{1 \leq j, k, l \leq n} \frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_l} \left(g_{k,l} + \sum_{1 \leq m \leq n} b_{k,l,m} z_m + \bar{c}_{j,l,m} \bar{z}_m \right) d\bar{z}_j \wedge dz_k \\
&+ \sum_{1 \leq j, k, l \leq n} \left(\frac{\partial \bar{f}_{k,l}}{\partial z_j} \frac{\partial u}{\partial \bar{z}_l} dz_j \wedge d\bar{z}_k + \frac{\partial f_{k,l}}{\partial \bar{z}_j} \frac{\partial u}{\partial z_l} d\bar{z}_j \wedge dz_k \right) \\
&+ \sum_{1 \leq j, k, l \leq n} \left(\frac{\partial \bar{g}_{k,l}}{\partial z_j} \frac{\partial u}{\partial z_l} dz_j \wedge d\bar{z}_k + \frac{\partial g_{k,l}}{\partial \bar{z}_j} \frac{\partial u}{\partial \bar{z}_l} d\bar{z}_j \wedge dz_k \right) \\
&+ \sum_{1 \leq j, k, l \leq n} \left(c_{k,l,j} \frac{\partial u}{\partial z_l} dz_j \wedge d\bar{z}_k + \bar{c}_{k,l,j} \frac{\partial u}{\partial \bar{z}_l} d\bar{z}_j \wedge dz_k \right).
\end{aligned}$$

Consider now the function $u = |z|$ on \mathbb{B} . We have the following estimates :

$$\begin{aligned}
\left| \frac{\partial |z|}{\partial z_j} \right| &= \frac{|\bar{z}_j|}{2|z|} \leq \frac{1}{2}, \\
\frac{\partial^2 |z|}{\partial z_j \partial \bar{z}_k} &= \frac{1}{2} \left(\frac{\delta_j^k}{|z|} - \frac{z_k \bar{z}_j}{2|z|^3} \right)
\end{aligned}$$

and

$$\left| \frac{\partial^2 |z|}{\partial z_j \partial z_k} \right| = \left| \frac{|\bar{z}_j \bar{z}_k|}{4|z|^3} \right| \leq \frac{1}{4|z|}.$$

Since $\mathcal{L}^{J_0}(u \circ z^{-1})(w)(v) \geq \|v\|^2/4|w|$ for every $w \in \mathbb{B} \setminus \{0\}$ and every $v \in \mathbb{C}^n$, it follows from the above expansion of $\mathcal{L}^J(u)$ that there exist a neighborhood U of p , $U \subset \subset U_0$, and a positive constant λ_0 such that $\mathcal{L}_u^{J'}(q)(v) \geq \|v\|^2/5|z(q)|$ for every $q \in U \setminus \{p\}$, every $v \in T_q M$ and every almost complex structure J' satisfying $\|J' - J\|_{\mathcal{C}^2(\bar{U})} \leq \lambda_0$. Moreover, we have the following estimates :

$$\left| \frac{\partial \log |z|}{\partial z_j} \right| = \left| \frac{\bar{z}_j}{2|z|^2} \right| \leq \frac{1}{2|z|},$$

$$\left| \frac{\partial^2 \log |z|}{\partial z_j \partial \bar{z}_k} \right| = \frac{1}{2} \left| \frac{\delta_j^k}{|z|^2} - \frac{z_k \bar{z}_j}{|z|^4} \right| \leq \frac{1}{|z|^2}$$

and

$$\left| \frac{\partial^2 \log |z|}{\partial z_j \partial z_k} \right| = \left| \frac{\bar{z}_j \bar{z}_k}{2|z|^4} \right| \leq \frac{1}{2|z|^2}.$$

Decreasing λ_0 if necessary, we obtain from the expansion of the Levi form that there exists a positive constant B such that $\mathcal{L}^{J'}(\log |z|)(q)(v) \geq -B\|v\|^2/|z(q)|$ for every $q \in U \setminus \{p\}$, every $v \in T_q M$ and every almost complex structure J' satisfying $\|J' - J\|_{\mathcal{C}^2(\bar{U})} \leq \lambda_0$. We may choose $A = 2B$ to get the result. \square

3. ESTIMATES OF THE KOBAYASHI-ROYDEN METRIC ON ALMOST COMPLEX MANIFOLDS

We start this Section with some general facts on the Kobayashi-Royden infinitesimal pseudometric.

3.1. Kobayashi hyperbolicity. Let (M, J) be an almost complex manifold. According to [19], for every $p \in M$ there exists a neighborhood \mathcal{V} of 0 in $T_p M$ such that for every $v \in \mathcal{V}$ there exists $f \in \mathcal{O}_J(\Delta, M)$ satisfying $f(0) = p$, $df(0)(\partial/\partial x) = v$. This allows to define the Kobayashi-Royden infinitesimal pseudometric $K_{(M, J)}$:

Definition 5. For $p \in M$ and $v \in T_p M$, $K_{(M, J)}(p, v)$ is the infimum of the set of positive α such that there exists a J -holomorphic disc $f : \Delta \rightarrow M$ such that $f(0) = p$ and $df(0)(\partial/\partial x) = v/\alpha$.

Most of the basic properties of the Kobayashi-Royden pseudometric in complex manifolds remains true in the almost complex case. In particular, since for every $f \in \mathcal{O}_{(J', J)}(M', M)$ and every $\varphi \in \mathcal{O}_J(\Delta, M')$ the composition $f \circ \varphi$ is in $\mathcal{O}_J(\Delta, M)$ we have :

Proposition 2. Let $f : (M', J') \rightarrow (M, J)$ be a (J', J) -holomorphic map. Then for every $p' \in M'$ and $v' \in T_{p'} M'$:

$$(3) \quad K_{(M, J)}(f(p'), df(p')(v')) \leq K_{(M', J')}(p', v').$$

For instance, if $M \subset M'$ and J' is an almost complex structure on M' then $K_{(M', J')}(p, v) \leq K_{(M, J)}(p, v)$ for every $p \in M$, $v \in T_p M$, the inclusion map from M to M' being (J', J) -holomorphic.

It is proved in [18] that $K_{(M, J)}$ is an upper semicontinuous function on the tangent bundle of M (this is a generalization of Royden's result [21] for the standard structure). We may therefore define the *integrated pseudodistance* $d_{(M, J)}^K$ on $M \times M$, generated by the Kobayashi-Royden infinitesimal pseudometric $K_{(M, J)}$, as follows :

Definition 6. Let $p, q \in M$. Denote by $\Gamma_{p,q}$ the set of all C^1 -paths $\gamma : [0, 1] \rightarrow M$ satisfying $\gamma(0) = p, \gamma(1) = q$. Then :

$$d_{(M,J)}^K(p, q) = \inf_{\gamma \in \Gamma_{p,q}} \int_0^1 K_{(D,J)}(\gamma(t), \gamma'(t)) dt.$$

As shown in [18] this pseudodistance coincides with the Kobayashi pseudodistance on (M, J) , defined (quite similarly to the case of a complex manifold) by :

$$Kob_{(M,J)}(p, q) = \inf_k \left\{ \sum_{j=1}^k \delta(w_{j-1}, w_j), w_j \in M, w_0 = p, w_k = q \right\},$$

where $\delta(w_{j-1}, w_j)$ denotes the infimum of the Poincaré distance between points ζ_{j-1} and ζ_j in Δ for which there is a J -holomorphic disc $f : \Delta \rightarrow M$, satisfying $f(\zeta_{j-1}) = w_{j-1}, f(\zeta_j) = w_j$.

If (M', J') and (M, J) are almost complex manifolds and $f : M' \rightarrow M$ is (J', J) -holomorphic, then Proposition 2 implies that for every $p', q' \in M'$ we have :

$$(4) \quad d_{(M,J)}^K(f(p'), f(q')) \leq d_{(M',J')}^K(p', q').$$

In particular, if M is a submanifold in M' and J' is an almost complex structure on M' then for every point $p, q \in M$ we have : $d_{(M',J')}^K(p, q) \leq d_{(M,J)}^K(p, q)$.

Similarly to the case of the integrable structure we have the following :

Definition 7. (i) An almost complex manifold (M, J) is called (Kobayashi) hyperbolic if $d_{(M,J)}^K$ is a distance.

(ii) Let $p \in M$. Then M is called locally hyperbolic at p if there exists a neighborhood U of p and a positive constant C such that for every $q \in U, v \in T_q M$: $K_{(M,J)}(q, v) \geq C\|v\|$.

It is well known (see [21]) that a complex manifold is hyperbolic if and only if it is locally hyperbolic at every point. This remains true in the almost complex case and we have the following quantitative version of this statement :

Lemma 4. Let (M, J) be an almost complex manifold, p be a point of $M, U \subset M$ be a neighborhood of p and $z : U \rightarrow \mathbb{B}^n$ local coordinates centered at p ($z(p) = 0$). Suppose that there exists a constant $N > 0$ such that $K_{(M,J)}(w, v) \geq N\|v\|$ for every $w \in U, v \in T_w M$. Then for every point $q \in z^{-1}(\mathbb{B})$ and every $\alpha \leq N$, the Kobayashi ball $B_{(M,J)}(q, \alpha)$ is contained in $\{w \in U : |z(w) - z(q)| < \alpha/N\}$.

We refer to Lemma 2.2 of [6] for its proof.

Lemma 4 implies the following proposition :

Proposition 3. If (M, J) is locally hyperbolic at every point then (M, J) is hyperbolic and $d_{(M,J)}^K$ induces the standard topology on M .

Definition 8. Let (M, J) be hyperbolic. Then (M, J) is called complete hyperbolic if the metric space $(M, d_{(M,J)}^K)$ is complete.

Since $K_{(M,J)}$ is upper semicontinuous by [18], it follows from [21] that :

Proposition 4. The metric space $(M, d_{(M,J)}^K)$ is complete if and only if the Kobayashi ball $B_{(M,J)}^K(p, r) := \{q \in M : d_{(M,J)}^K(p, q) < r\}$ is relatively compact in M for every $p \in M, r \geq 0$.

Let $f \in \mathcal{O}_{(J',J)}(M', M)$ be a diffeomorphism from M' to M . Then the inverse map $f^{-1} : M \rightarrow M'$ is in $\mathcal{O}_{(J,J')}(M, M')$ and we say that f is a (J', J) -biholomorphism from M' to M . We note, from the decreasing property of the Kobayashi-Royden infinitesimal pseudometric and of the Kobayashi pseudodistance, that for every $p', q' \in M'$ and every $v' \in T_{p'}M'$ we have the equalities $K_{(M,J)}(f(p'), df(p')(v')) = K_{(M',J')}(p', v')$ and $d_{(M,J)}^K(f(p'), f(q')) = d_{(M',J')}^K(p', q')$, since we may apply inequalities (3) and 4) to f and f^{-1} .

3.2. Local estimate of the Kobayashi-Royden infinitesimal pseudometric.

Let D be a bounded domain in \mathbb{C}^n . Recall that J_0 denotes the standard complex structure on \mathbb{C}^n . We start with a small perturbation of the standard structure and give local estimates of the Kobayashi-Royden infinitesimal pseudometric, following the method of N.Sibony [22] :

Theorem 1. *Let D be a bounded domain in \mathbb{C}^n . Assume that u is a \mathcal{C}^2 function on D such that $-L \leq u < 0$ and $u - c|z|^2$ is J_0 -plurisubharmonic on D , where c and L are positive constants. Then there are positive constants c' and λ_0 , depending on c and L only, such that for every almost complex structure J defined in a neighborhood of \bar{D} and such that $\|J - J_0\|_{\mathcal{C}^2(\bar{D})} \leq \lambda_0$ we have :*

$$K_{(D,J)}(p, v) \geq c' \frac{\|v\|}{|u(p)|^{1/2}},$$

for every $p \in D$, $v \in \mathbb{C}^n$.

Proof. Let θ be a smooth nondecreasing function on \mathbb{R}^+ such that $\theta(x) = x$ for $x \leq 1/3$ and $\theta(x) = 1$ for $x \geq 2/3$. According to Lemma 3 there exist positive constants λ_0 and A , such that the function $\log(|z - p|^2) + A|z - p|$ is J -plurisubharmonic on D for every $p \in D$ and for every $\|J - J_0\|_{\mathcal{C}^2(\bar{D})} < \lambda_0$. Let us fix $p \in D$ and consider a positive constant r that will be specified later. Then the function $\log(\theta(|z - p|/r^2)) + A|z - p|$ is J -plurisubharmonic on $D \cap B(p, r/\sqrt{3})$. Let $C = \sup_{1/3 \leq x \leq 1} (|\theta'/\theta|(x), (|\theta''\theta| + \theta'^2/\theta^2)(x))$. Since we have :

$$\begin{aligned} \frac{\partial \log(\theta(|z - p/r|^2))}{\partial z_j} &= \frac{\overline{(z_j - p_j)}\theta'(|z - p/r|^2)}{r^2\theta(|z - p/r|^2)}, \\ \frac{\partial^2 \log(\theta(|z - p/r|^2))}{\partial z_j \partial \bar{z}_k} &= \delta_j^k \frac{\theta'(|z - p/r|^2)}{r^2\theta(|z - p/r|^2)} \\ &\quad + \frac{\overline{(z_j - p_j)}(z_k - p_k)[\theta''(|z - p/r|^2)\theta(|z - p/r|^2) - (\theta'(|z - p/r|^2))^2]}{r^4(\theta(|z - p/r|^2))^2}, \end{aligned}$$

and

$$\frac{\partial^2 \log(\theta(|z - p/r|^2))}{\partial z_j \partial z_k} = \frac{\overline{(z_j - p_j)}\overline{(z_k - p_k)}[\theta''(|z - p/r|^2)\theta(|z - p/r|^2) - (\theta'(|z - p/r|^2))^2]}{r^4(\theta(|z - p/r|^2))^2},$$

we obtain the following estimates for $1/3 \leq |z - p/r|^2 \leq 1$:

$$\begin{aligned} \left| \frac{\partial \log(\theta(|z - p/r|^2))}{\partial z_j} \right| &\leq \frac{C}{r}, \\ \left| \frac{\partial^2 \log(\theta(|z - p/r|^2))}{\partial z_j \partial \bar{z}_k} \right| &\leq \frac{2C}{r^2} \end{aligned}$$

and

$$\left| \frac{\partial^2 \log(\theta(|z - p/r|^2))}{\partial z_j \partial \bar{z}_k} \right| \leq \frac{C}{r^2}.$$

Moreover

$$\left| \frac{\partial |z - p|}{\partial z_j} \right| \leq 1,$$

$$\left| \frac{\partial^2 |z - p|}{\partial z_j \partial \bar{z}_k} \right| \leq \frac{2}{|z - p|} \leq \frac{2\sqrt{3}}{r}$$

and

$$\left| \frac{\partial^2 |z - p|}{\partial z_j \partial z_k} \right| \leq \frac{1}{|z - p|} \leq \frac{\sqrt{3}}{r}$$

for $1/3 \leq |z - p/r|^2 \leq 1$.

Let $v(z) = |z - p|^2/r^2$. There exists a positive constant C_1 , independent of p and r , such that

$$\mathcal{L}^J(v)(z)(v) \geq (C_1/r^2)\|v\|^2$$

for $|z - p| \leq r$. So there exists a positive constant B , independent of p and r , such that the function

$$\log(\theta(|z - p/r|^2)) + A|z - p| + B|z - p/r|^2$$

is J -plurisubharmonic on

$$D \cap \{z \in \mathbb{C}^n : 1/3 \leq |z - p/r|^2 \leq 1\}.$$

Finally, decreasing λ_0 if necessary, we may assume that the function $|z - p|^2$ is J -plurisubharmonic on D . This implies that the function

$$\log(\theta(|z - p/r|^2)) + A|z - p| + B|z - p/r|^2$$

is J -plurisubharmonic on D for $\|J - J_0\|_{C^2(\bar{D})} \leq \lambda_0$, independently of p and r .

Consider now the function

$$\Psi_p(z) = \theta\left(\frac{|z - p|^2}{r^2}\right) \exp(A|z - p|) \exp(\tau u(z))$$

where $\tau = 1/|u(p)|$ and $r = (2B|u(p)|/c)^{1/2}$.

Since the function $\tau u - 2B|z - p/r|^2$ is J_0 -plurisubharmonic, we may assume, decreasing λ_0 if necessary, that the function $\tau u - B|z - p/r|^2$ is J -plurisubharmonic on D . Hence the function $\log(\Psi_p)$ is J -plurisubharmonic on D for every almost complex structure J satisfying $\|J - J_0\|_{C^2(\bar{D})} \leq \lambda_0$.

Let $v \in \mathbb{C}^n$ and let $f \in \mathcal{O}_J(\Delta, D)$ be such that $f(0) = p$ and $df(0)(\partial/\partial x) = v/\alpha$ where $\alpha > 0$. We have $f(\zeta) = p + df(0)(\zeta) + \mathcal{O}(|\zeta|^2)$. Setting $\zeta = \zeta_1 + i\zeta_2$ and using the J -holomorphy condition $df(0) \circ J_0 = J \circ df(0)$, we may write

$$df(0)(\zeta) = p + \zeta_1 df(0)(\partial/\partial x) + \zeta_2 J(df(0)(\partial/\partial x)).$$

Consider the function

$$\varphi(\zeta) = \Psi_p(f(\zeta))/|\zeta|^2$$

which is subharmonic on $\Delta \setminus \{0\}$. Since

$$\varphi(\zeta) = |f(\zeta) - p|^2/(r^2|\zeta|^2) \exp(\tau u(f(\zeta)))$$

and

$$|df(0)(\zeta)| \leq |\zeta|(\|I + J\| \|df(0)(\partial/\partial x)\|)$$

we obtain that $\limsup_{\zeta \rightarrow 0} \varphi(\zeta)$ is finite. Setting $\zeta_2 = 0$ we have :

$$\limsup_{\zeta \rightarrow 0} \varphi(\zeta) \geq \|v\|^2 \exp(2)/(r^2 \alpha^2).$$

Since there exists a positive constant C' , independent of p , such that $|z - p| \leq C'$ on D , we obtain, by applying the maximum principle to a subharmonic extension of ϕ on Δ , the inequality

$$\alpha \geq \sqrt{\frac{c}{2B \exp(1 + AC')}} \|v\|^2 / |u(p)|^{1/2}.$$

This completes the proof. \square

3.3. Localization of the Kobayashi-Royden pseudometric. In order to extend the previous result to an arbitrary almost complex manifold, we need to localize the Kobayashi-Royden metric. The next result is preparatory and will be used in the Localization principle.

Lemma 5. *Let $r < 1$ and let θ_r be a smooth nondecreasing function on \mathbb{R}^+ such that $\theta_r(s) = s$ for $s \leq r/3$ and $\theta_r(s) = 1$ for $s \geq 2r/3$. Let (M, J) be an almost complex manifold, and let p be a point of M . Then there exists a neighborhood U of p , positive constants $A = A(r)$, $B = B(r)$ and a diffeomorphism $z : U \rightarrow \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_0$ and the function $\log(\theta_r(|z|^2)) + \theta_r(A|z|) + B|z|^2$ is J -plurisubharmonic on U .*

Proof. Denote by w the standard coordinates in \mathbb{C}^n . It follows from Lemma 4 that there exist positive constants A and λ_0 such that the function $\log(|w|^2) + A|w|$ is J' -plurisubharmonic on \mathbb{B} for every almost complex structure J' , defined in a neighborhood of \mathbb{B} in \mathbb{C}^n and such that $\|J' - J_0\|_{\mathcal{C}^2(\mathbb{B})} \leq \lambda_0$. This means that the function

$$v(w) = \log(\theta_r(|w|^2)) + \theta_r(A|w|)$$

is J' -plurisubharmonic on $B(0, r') = \{w \in \mathbb{C}^n : |w| < r'\}$ for every such almost complex structure J' , where $r' = \inf(\sqrt{r/3}, r/3A)$. Decreasing λ_0 if necessary, we may assume that the function $|w|^2$ is strictly J' -plurisubharmonic on \mathbb{B} . Then, since v is smooth on $\mathbb{B} \setminus B(0, r')$, there exists a positive constant B such that the function $v + B|w|^2$ is J' -plurisubharmonic on \mathbb{B} for $\|J' - J_0\|_{\mathcal{C}^2(\mathbb{B})} \leq \lambda_0$. According to Lemma 1 there exists a neighborhood U of p and a diffeomorphism $z : U \rightarrow \mathbb{B}$ such that $\|z^*(J) - J_0\|_{\mathcal{C}^2(\mathbb{B})} \leq \lambda_0$. Then the function

$$v \circ z = \log(\theta_r(|z|^2)) + \theta_r(A|z|) + B|z|^2$$

is J -plurisubharmonic on U . \square

The following result allows to localize the Kobayashi-Royden pseudometric on an almost complex manifold. A similar statement was obtained by F. Berteloot [3] in the integrable case.

Theorem 2. *(Localization principle) Let D be a domain in an almost complex manifold (M, J) , let $p \in \bar{D}$, let U be a neighborhood of p in M (not necessarily contained in D) and let $z : U \rightarrow \mathbb{B}$ be the diffeomorphism given by Lemma 5. Let u be a \mathcal{C}^2 function on \bar{D} , negative and J -plurisubharmonic on D . We assume that $-L \leq u < 0$ on $D \cap U$ and that $u - c|z|^2$ is J -plurisubharmonic on $D \cap U$, where c and L are positive constants. Then there exist a positive constant s and a neighborhood $V \subset\subset U$ of p , depending on c and L only, such that for $q \in D \cap V$ and $v \in T_q M$ we have the following estimate:*

$$(5) \quad K_{(D, J)}(q, v) \geq s K_{(D \cap U, J)}(q, v).$$

Proof. Let $0 < r < 1$ be such that the set $V_1 := \{q \in U : |z(q)| \leq \sqrt{r}\}$ is relatively compact in U and let θ_r be a smooth nondecreasing function on \mathbb{R}^+ such that $\theta_r(s) = s$ for $s \leq r/3$ and $\theta_r(s) = 1$ for $s \geq 2r/3$. According to Lemma 5, there exist uniform positive constants A and B such that the function

$$\log(\theta_r(|z - z(q)|^2)) + \theta_r(A|z - z(q)|) + B|z|^2$$

is J -plurisubharmonic on U for every $q \in V$. By assumption the function $u - c|z|^2$ is J -plurisubharmonic on $D \cap U$. Set $\tau = 2B/c$ and define, for every point $q \in V$, the function Ψ_q by :

$$\begin{cases} \Psi_q(z) &= \theta_r(|z - z(q)|^2) \exp(\theta_r(A|z - z(q)|)) \exp(\tau u(z)) \text{ if } z \in D \cap U, \\ \Psi_q &= \exp(1 + \tau u) \text{ on } D \setminus U. \end{cases}$$

Then for every $0 < \varepsilon \leq B$, the function $\log(\Psi_q) - \varepsilon|z|^2$ is J -plurisubharmonic on $D \cap U$ and hence Ψ_q is J -plurisubharmonic on $D \cap U$. Since Ψ_q coincides with $\exp(\tau u)$ outside U , it is globally J -plurisubharmonic on D .

Let $f \in \mathcal{O}_J(\Delta, D)$ be such that $f(0) = q \in V_1$ and $(\partial f / \partial x)(0) = v/\alpha$ where $v \in T_q M$ and $\alpha > 0$. For ζ sufficiently close to 0 we have

$$f(\zeta) = q + df(0)(\zeta) + \mathcal{O}(|\zeta|^2).$$

Setting $\zeta = \zeta_1 + i\zeta_2$ and using the J -holomorphy condition $df(0) \circ J_0 = J \circ df(0)$, we may write :

$$df(0)(\zeta) = \zeta_1 df(0)(\partial/\partial x) + \zeta_2 J(df(0)(\partial/\partial x)).$$

Consider the function

$$\varphi(\zeta) = \Psi_q(f(\zeta))/|\zeta|^2$$

which is subharmonic on $\Delta \setminus \{0\}$. Since

$$\varphi(\zeta) = |f(\zeta) - q|^2 / |\zeta|^2 \exp(A|f(\zeta) - q|) \exp(\tau u(f(\zeta)))$$

for ζ close to 0 and

$$|df(0)(\zeta)| \leq |\zeta| (\|I + J\| \|df(0)(\partial/\partial x)\|)$$

we obtain that $\limsup_{\zeta \rightarrow 0} \varphi(\zeta)$ is finite. Moreover setting $\zeta_2 = 0$ we have

$$\limsup_{\zeta \rightarrow 0} \varphi(\zeta) \geq \|df(0)(\partial/\partial x)\|^2 \exp(-2B|u(q)|/c).$$

Applying the maximum principle to a subharmonic extension of φ on Δ we obtain the inequality

$$\|df(0)(\partial/\partial x)\|^2 \leq \exp(1 + 2B|u(q)|/c).$$

Hence, by definition of the Kobayashi-Royden infinitesimal pseudometric, we obtain for every $q \in D \cap V_1$, $v \in T_q M$:

$$K_{(D,J)}(q, v) \geq \left(\exp \left(-1 - 2B \frac{|u(q)|}{c} \right) \right)^{1/2} \|v\|.$$

Consider now the Kobayashi ball $B_{(D,J)}(q, \alpha) = \{w \in D : d_{(D,J)}^K(w, q) < \alpha\}$. It follows from Lemma 4 that there is a neighborhood V of p , relatively compact in V_1 and a positive constant $s < 1$, independent of q , such that for every $f \in \mathcal{O}_J(\Delta, D)$ satisfying $f(0) \in D \cap V$ we have $f(s\Delta) \subset D \cap U$. This gives the inequality (5). \square

The following result may be considered as a generalization of our estimates for the case of an almost complex manifold:

Theorem 3. *In the hypothesis of the previous theorem, there exists a neighborhood $U' \subset U$ of p depending on c and $\|J\|_{C^2(U)}$, a constant $c' > 0$ depending on c and L only such that we have the following estimate:*

$$(6) \quad K_{(D,J)}(q, v) \geq c' \frac{\|v\|}{|u(q)|^{1/2}},$$

for every neighborhood $V' \subset U'$ and every $q \in D \cap V'$ and $v \in T_q M$.

Proof. For every given $\lambda > 0$, there exists a neighborhood U' of p , the coordinate diffeomorphism $z' : U' \rightarrow \mathbb{B}$ such that the function $u \circ (z')^{-1}$ is J_0 -strictly plurisubharmonic on $z'(D) \cap \mathbb{B}$ and $\|(z')^*(J) - J_0\|_{C^2(\mathbb{B})} \leq \lambda$. It follows by the previous theorem that $K_{(D,J)}(q, v) \geq sK_{(D \cap U', J)}(q, v)$. Applying Theorem 1 to the direct image $(z')^*(J)$ on the domain $z'(D) \cap \mathbb{B}$ in \mathbb{C}^n , we obtain the desired statement. \square

Though this statement looks rather technical and uses local coordinates, it is quite general and useful for applications in such a form; in the last section it will be used to study of boundary behaviour of J -holomorphic curves near a totally real manifold. The above localization also leads to coordinate-free statements. For instance, we have the following

Theorem 4. *Let $D = \{\rho < 0\}$ be a relatively compact domain in an almost complex manifold (M, J) , where ρ is a C^2 defining function of D , strictly J -plurisubharmonic in a neighborhood of \bar{D} . Then there exist positive constants c and λ_0 such that for every almost complex structure J' defined in a neighborhood of \bar{D} and satisfying $\|J' - J\|_{C^2(\bar{D})} \leq \lambda_0$, for every $p \in D$ and every $v \in T_p M$ we have :*

$$(7) \quad K_{(D,J')}(p, v) \geq c \frac{\|v\|}{|\rho(p)|^{1/2}},$$

Proof. Let $p \in D$. For any given $\lambda > 0$, it follows from Lemma 1 that there exists a neighborhood U of p and a diffeomorphism $z : U \rightarrow \mathbb{B}$ such that $\|z^*(J) - J_0\|_{C^2(\mathbb{B})} \leq \lambda$ and such that the function $\rho \circ z^{-1}$ is strictly J_0 -plurisubharmonic on $z(D \cap U)$. Since the function ρ is strictly J -plurisubharmonic in a neighborhood of \bar{D} there is a positive constant c , independent of p , such that the function $\rho - c|z|^2$ is J -plurisubharmonic on $D \cap U$. Since the function ρ is negative on D , bounded from below on \bar{D} , it follows from Theorem 2 that there exists a positive constant s , independent of p , such that $K_{(D,J)}(p, v) \geq sK_{(D \cap U, J)}(p, v)$. Applying Theorem 1 to $(z(D \cap U), z^*(J))$ in \mathbb{C}^n , we obtain the desired statement. \square

3.4. Scaling and refined estimates. In order to prove the complete hyperbolicity of strictly pseudoconvex domains in an almost complex manifold we need a more precise estimate of the Kobayashi metric. To obtain this we apply the scaling method due to S.Pinchuk [20]. A similar idea was recently used by F.Berteloot [4] to give a short proof of Catlin's estimates [5] of the Kobayashi metric on domains of finite type in \mathbb{C}^2 .

Theorem 5. *Let $D = \{\rho < 0\}$ be a bounded domain in \mathbb{C}^n , where ρ is a C^2 defining function of D , strictly J_0 -plurisubharmonic in a neighborhood of \bar{D} . Then there exist positive constants c and λ_0 such that for every almost complex structure J defined in a neighborhood of \bar{D} and such that $\|J - J_0\|_{C^2(\bar{D})} \leq \lambda_0$ we have :*

$$K_{(D,J)}(p, v) \geq c \left[\frac{|\partial_J \rho(p)(v - iJ(p)v)|^2}{|\rho(p)|^2} + \frac{\|v\|^2}{|\rho(p)|} \right]^{1/2},$$

for every $p \in D$, $v \in \mathbb{C}^n$.

Proof. We note that according to Theorem 1 it is sufficient to prove the inequality near ∂D . Suppose by contradiction that there exists a sequence (p^ν) of points in D converging to a boundary point q , a sequence (v^ν) of unitary vectors and a sequence (J_ν) of almost complex structures defined in a neighborhood of \bar{D} , satisfying $\|J_\nu - J_0\|_{C^2(\bar{D})} \rightarrow_{\nu \rightarrow \infty} 0$, such that the quotient

$$(8) \quad K_{(D, J_\nu)}(p^\nu, v^\nu) \left[\frac{|\partial_{J_\nu} \rho(p^\nu)(v^\nu - iJ_\nu(p^\nu)v^\nu)|^2}{|\rho(p^\nu)|^2} + \frac{\|v^\nu\|^2}{|\rho(p^\nu)|} \right]^{-1/2}$$

tends to 0 as ν tends to ∞ . In view of the Localization principle, we may consider the Kobayashi metric on $D \cap U$ where U is a neighborhood of q . For sufficiently large ν denote by δ_ν the euclidean distance between p^ν and the boundary of D and by $q^\nu \in \partial D$ the unique point such that $|p^\nu - q^\nu| = \delta_\nu$. Without loss of generality we assume that $q = 0$, that $T_0(\partial D) = \{z \in \mathbb{C}^n : \operatorname{Re}(z_n) = 0\}$ and that $J_\nu(q^\nu) = J_0$ for every ν .

Consider a sequence of biholomorphic (for the standard structure) transformations T^ν in a neighborhood of the origin sending q^ν to 0 and such that the image $D^\nu := T^\nu(D)$ satisfies

$$T_0(\partial D^\nu) = \{z \in \mathbb{C}^n : \operatorname{Re}(z_n) = 0\}.$$

We point out that the sequence (T^ν) converges uniformly to the identity map since $q^\nu \rightarrow 0$ as $\nu \rightarrow \infty$ and hence that the sequence $((T^\nu)^{-1})_\nu$ is bounded. In particular, we may still denote by J_ν the direct image $(T^\nu)^*(J_\nu)$ and we may assume that every domain D^ν is defined in a neighborhood U of the origin by the function $\rho^\nu(z) = \operatorname{Re}(z_n) + |z|^2 + \mathcal{O}(|z|^3)$ and that the sequence $(\hat{p}^\nu = T^\nu(p^\nu) = (0', -\delta_\nu))_\nu$ is on the real inward normal to ∂D^ν at 0. Of course, the functions ρ^ν converge uniformly with all derivatives to the defining function ρ of D . Hence, restricting U if necessary, we have $D^\nu \cap U \subset \mathbb{H} \cap U$ where

$$\mathbb{H} := \{z \in \mathbb{C}^n : r(z) = \operatorname{Re}(z_n) + \frac{1}{2}|z'|^2 < 0\},$$

with $z' = (z_1, \dots, z_{n-1})$.

Consider the dilation Λ_ν defined on \mathbb{C}^n by $\Lambda_\nu(z) = (z'/(\delta_\nu)^{1/2}, z_n/\delta_\nu)$. If we set $J^\nu := \Lambda_\nu \circ J_\nu \circ (\Lambda_\nu)^{-1}$ then we have :

Lemma 6. $\lim_{\nu \rightarrow \infty} J^\nu = J_0$, uniformly on compact subsets of \mathbb{C}^n .

Proof. Let z^0 be a point in \mathbb{C}^n and let $v \in \mathbb{C}^n$. By definition, $J_\nu(z^0)(v) = (\Lambda^\nu \circ J_\nu((\Lambda^\nu)^{-1}(z^0))((\Lambda^\nu)^{-1}(v)))$. We may assume that the Taylor expansion of J_ν at the origin is given by $J_\nu = J_0 + \mathcal{O}(|z|)$ on U , uniformly with respect to ν . Hence :

$$\begin{aligned} J_\nu((\Lambda^\nu)^{-1}(z^0))((\Lambda^\nu)^{-1}(v)) &= J_0((\Lambda^\nu)^{-1}(v)) \\ &+ \mathcal{O}(|(\Lambda^\nu)^{-1}(z^0)| \|(\Lambda^\nu)^{-1}(v)\|). \end{aligned}$$

Consequently :

$$J^\nu(z^0)(v) = J_0(v) + \mathcal{O}(|(\Lambda^\nu)^{-1}(z^0)| \|v\|) \rightarrow_{\nu \rightarrow \infty} 0.$$

These estimates being uniform for z^0 in a compact of \mathbb{C}^n , this gives the result. \square

Consider now the Cayley transform

$$C_a : (z', z_n) \mapsto \left(\frac{z'}{z_n - 1}, \frac{z_n + 1}{z_n - 1} \right).$$

Then $C_a(\mathbb{H} \cap \Lambda_\nu(U)) = \mathbb{B}_n \cap C_a(\Lambda_\nu(U)) := B_n^\nu$. If we note $J'_\nu := dC_a \circ J^\nu \circ dC_a^{-1}$ then it follows from Lemma 6 that $\lim_{\nu \rightarrow \infty} J'_\nu = J_0$ uniformly on compact subsets of

$\bar{\mathbb{B}}_n \setminus \{(0', -1)\}$. Hence the function $|z|^2 - 1$, negative on B_n^ν , is uniformly strictly J'_ν -plurisubharmonic on B_n^ν for every sufficiently large ν , namely there exists a positive constant c , independent of ν , such that for every ν , every $q \in B_n^\nu$ and every $v \in \mathbb{C}^n$ we have $\mathcal{L}_{|z|^2}^{J'_\nu}(q)(v) \geq c\|v\|^2$. Then, according to Theorem 2, there exists a positive constant s , independent of ν , such that for sufficiently large ν (insuring that $B(0, 1/2) \subset B_n^\nu$), we have :

$$K_{(B_n^\nu, J'_\nu)}(0, v) \geq sK_{(B(0, 1/2), J'_\nu)}(0, v)$$

for every $v \in \mathbb{C}^n$. For sufficiently large ν , let f^ν be a J'_ν -holomorphic disc in $B(0, 1/2)$ such that $f^\nu(0) = 0$ and $df^\nu(0)(\partial/\partial x) = v/\alpha_\nu$, where $\alpha_\nu \geq K_{(B(0, 1/2), J'_\nu)}(0, v) - 1/\nu$. According to Proposition 2.3.6 of [23], the derivatives of the functions f^ν are uniformly bounded on compact subsets of Δ . It follows from Ascoli Theorem that we may extract from $(f^\nu)_\nu$ a subsequence, still called $(f^\nu)_\nu$, converging uniformly on compact subsets to a holomorphic disc f . In particular, the sequence $(df^\nu(0)(\partial/\partial x))_\nu$ converges to $df(0)(\partial/\partial x)$, implying that there exists a positive constant C' such that $\alpha_\nu \geq C'$ for sufficiently large ν . Hence, we have for sufficiently large ν :

$$K_{(B_n^\nu, J'_\nu)}(0, v) \geq \frac{sC'}{2}\|v\|$$

for every $v \in \mathbb{C}^n$.

Moreover for $v \in \mathbb{C}^n$ and for sufficiently large ν we have :

$$\begin{aligned} K_{(D^\nu \cap U, J_\nu)}(\hat{p}^\nu, v) &\geq K_{(\mathbb{H} \cap U, J_\nu)}(\hat{p}^\nu, v) = K_{(\mathbb{H} \cap \Lambda_\nu(U), J_\nu)}((0', -1), \Lambda(v)) \\ &= K_{(B_n^\nu, J'_\nu)}(0, d(Ca)((0', -1))(\Lambda_\nu(v))) \\ &\geq \frac{sC'}{2}\|d(Ca)((0', -1))(\Lambda_\nu(v))\|. \end{aligned}$$

This gives the inequality:

$$K_{(D^\nu \cap U, J_\nu)}(\hat{p}^\nu, v) \geq \frac{sC'}{4} \left(\frac{|v_1|^2}{\delta_\nu} + \dots + \frac{|v_{n-1}|^2}{\delta_\nu} + \frac{|v_n|^2}{\delta_\nu^2} \right)^{1/2}.$$

Since there exists a positive constant C_1 , independent of ν , such that $C_1\delta_\nu \leq |\rho(p^\nu)|$, we obtain that there is a positive constant C'' such that

$$K_{(D^\nu \cap U, J_\nu)}(\hat{p}^\nu, v) \geq C'' \left(\frac{\|v\|^2}{|\rho(p^\nu)|} + \frac{|v_n|^2}{|\rho(p^\nu)|^2} \right)^{1/2}.$$

However $|v_n|^2 = |dr(p^\nu)(v)|^2 + |dr(p^\nu)(J_\nu(q^\nu)(v))|^2$ and $J_\nu(q^\nu) = J_\nu(p^\nu) + \mathcal{O}(\delta_\nu)$. Then $|dr(p^\nu)(J_\nu(q^\nu)(v))|^2 = |dr(p^\nu)(J_\nu(p^\nu)(v))|^2 + \|v\|^2\mathcal{O}(\delta_\nu)$. Hence there exists a positive constant \tilde{C} such that

$$K_{(D^\nu \cap U, J_\nu)}(\hat{p}^\nu, v) \geq \tilde{C} \left(\frac{\|v\|^2}{|\rho(p^\nu)|} + \frac{|\partial J(p^\nu)(v - iJ_\nu(p^\nu)(v))|^2}{|\rho(p^\nu)|^2} \right)^{1/2},$$

contradicting the assumption on the quotient (8). This proves the desired estimate. \square

The following result may be considered as a generalization of Theorem 5 in an almost complex manifold:

Theorem 6. *Let $D = \{\rho < 0\}$ be a relatively compact domain in an almost complex manifold (M, J) , where ρ is a C^2 defining function of D , strictly J -plurisubharmonic in a neighborhood of \bar{D} . Then there exist positive constants c and λ_0 such that for*

every almost complex structure J' defined in a neighborhood of \bar{D} and such that $\|J' - J\|_{\mathcal{C}^2(\bar{D})} \leq \lambda_0$ we have :

$$(9) \quad K_{(D, J')}(p, v) \geq c \left[\frac{|\partial_J \rho(p)(v - iJ(p)v)|^2}{|\rho(p)|^2} + \frac{\|v\|^2}{|\rho(p)|} \right]^{1/2},$$

for every $p \in D$ and every $v \in T_p M$.

Proof. It is sufficient to prove the estimate near the boundary. Let $q \in \partial D$. For any given $\lambda > 0$, it follows from Lemma 1 that there exists a neighborhood U of q and a diffeomorphism $z : U \rightarrow \mathbb{B}$ such that $\|z^*(J) - J_0\|_{\mathcal{C}^2(\mathbb{B})} \leq \lambda$ and such that the function $\rho \circ z^{-1}$ is strictly J_0 -plurisubharmonic on $z(D \cap U)$. Since the function ρ is strictly J -plurisubharmonic in a neighborhood of \bar{D} there is a positive constant c such that the function $\rho - c|z|^2$ is J -plurisubharmonic on $D \cap U$. Since the function ρ is negative on D , bounded from below on \bar{D} , it follows from Theorem 2 that there exists a positive constant s such that $K_{(D, J)}(p, v) \geq sK_{(D \cap U, J)}(p, v)$ for any $p \in D \cap U$. Applying Theorem 5 to $(z(D \cap U), z^*(J))$ in \mathbb{C}^n , we obtain the desired statement. \square

4. APPLICATIONS

4.1. Hyperbolicity of almost complex manifolds. We begin with the following

Corollary 1. *Let $D = \{\rho < 0\}$ be a relatively compact domain in an almost complex manifold (M, J) , where ρ is a \mathcal{C}^2 defining function of D . Assume that ρ is strictly J -plurisubharmonic in a neighborhood of \bar{D} . Then D is hyperbolic.*

Proof. Let $p \in D$. It follows from Theorem 3 that (D, J) is hyperbolic at p . By Proposition 3, (D, J) is hyperbolic. \square

We note that Corollary 1 is no more true if ρ is not globally strictly J -plurisubharmonic on D , even for the standard structure. Consider for instance the pseudoconvex domain $D = \{z \in \mathbb{P}^2 : |P(z)|^2 < c\}$ where \mathbb{P}^2 is the complex projective space and $P = z_0 - z_1 - z_2$ in homogeneous coordinates. By a small perturbation of ∂D we can construct a strictly pseudoconvex domain D' containing the set $\{P = 0\}$. In particular D' is not hyperbolic.

Corollary 2. *Let $D = \{\rho < 0\}$ be a relatively compact domain in an almost complex manifold (M, J) where ρ is a \mathcal{C}^2 defining function of D , strictly J -plurisubharmonic in a neighborhood of \bar{D} . Then for every $p \in D$ and every $q \in \partial D$ we have :*

$$\lim_{\substack{x \in D \\ x \rightarrow q}} d_{(D, J)}^K(x, p) = +\infty.$$

Proof. Theorem 2 implies that there is a positive constant s and neighborhoods U and V of q , satisfying $V \subset\subset U$ and such that

$$K_{(D, J)}(x, v) \geq sK_{(D \cap U, J)}(x, v)$$

for every $x \in V$ and every $v \in T_x M$. Then Theorem 6 implies that we have :

$$K_{(D, J)}(x, v) \geq c \left[\frac{|\partial_J \rho(x)(v - iJ(x)v)|^2}{|\rho(x)|^2} + \frac{\|v\|^2}{|\rho(x)|} \right]^{1/2}$$

for a positive constant c independent of x and v . Now the conclusion follows by a standard integration argument since the Kobayashi distance contains a logarithmic term, see [15] for the details. \square

Corollary 3. *Let $D = \{\rho < 0\}$ be a relatively compact domain in an almost complex manifold (M, J) , where ρ is a \mathcal{C}^2 defining function of D , strictly J -plurisubharmonic in a neighborhood of \bar{D} . Then D is complete hyperbolic.*

Proof. We know by Corollary 1 that D is hyperbolic. In particular $d_{(D,J)}$ defines the usual topology on D . Then by Corollary 2 the Kobayashi balls are relatively compact in D . \square

As a particular case of Corollary 3 we have the following :

Corollary 4. *Let (M, J) be an almost complex manifold. Then every $p \in M$ has a complete hyperbolic neighborhood.*

Proof. Let $p \in M$. According to Example 1 there exist a neighborhood U of p and a diffeomorphism $z : U \rightarrow \mathbb{B}$, centered at p , such that the function $|z|^2$ is strictly J -plurisubharmonic on U . Hence the open set $\{q \in U : |z(q)| < 1/2\}$ is complete hyperbolic by Corollary 3. \square

4.2. Estimates for J -holomorphic discs attached to a totally real manifold.

We recall the following

Definition 9. *A real submanifold Γ in M is totally real if $H_p^J(\Gamma) = \{0\}$ for each $p \in \Gamma$.*

We note that this notion is stable under small perturbations of an almost complex structure. More precisely we have the following obvious statement :

Lemma 7. *Let Γ be a totally real submanifold of codimension $p \geq 1$ (resp. strictly pseudoconvex hypersurface) in \mathbb{C}^n . Then for every bounded open set U in \mathbb{C}^n there is a neighborhood \mathcal{U} of J_0 in $(\mathcal{C}^2(\bar{U}))$ such that $\Gamma \cap U$ is a totally real submanifold (resp. a strictly J -pseudoconvex hypersurface) in (\mathbb{C}^n, J) for every J in \mathcal{U} .*

J.C. Sikorav proved in [23] that the regularity of a J -holomorphic disc attached to a totally real submanifold W of an almost complex manifold (M, J) is related to the regularity of W (see Theorem 2.2.1). However the continuity of the disc up to the boundary is assumed. In what follows we obtain some regularity on the disc up to the boundary without this assumption. This application is a direct generalization of the results in [6] on the regularity of some J_0 -holomorphic discs. Using Theorem 3 and repeating without any modification the arguments of [6] we have the following :

Theorem 7. *Let r be a J -plurisubharmonic function in an almost complex manifold (M, J) and let $f \in \mathcal{O}_J(\Delta, M)$ such that $r \circ f \geq 0$ and $r \circ f(\zeta) \rightarrow 0$ as ζ tends to an open arc $\gamma \subset \partial\Delta$. Assume that there is $\zeta_0 \in \gamma$ such that the cluster set $C(f, \zeta_0)$ contains a point p such that r is strictly J -plurisubharmonic in a neighborhood of p . Then f extends to a Hölder $1/2$ -continuous mapping in a neighborhood of ζ_0 on $\Delta \cup \gamma$.*

We recall that $p \in C(f, \zeta_0)$ if and only if there is a sequence $(\alpha^\nu)_\nu$ of points in Δ such that $\lim_{\nu \rightarrow \infty} \alpha^\nu = \zeta_0$ and $p = \lim_{\nu \rightarrow \infty} f(\alpha^\nu)$.

Since every totally real \mathcal{C}^1 -submanifold in an almost complex manifold can be represented as the zero set of a certain \mathcal{C}^2 strictly plurisubharmonic nonnegative function ρ of class \mathcal{C}^2 in a neighborhood of M' (see [7]), we have as a direct consequence of Theorem 7 :

Theorem 8. *Let (M, J) be an almost complex manifold and let M' be a totally real \mathcal{C}^1 -submanifold of M . Let γ be an open arc in $\partial\Delta$ and let $f \in \mathcal{O}_J(\Delta, M)$ be such that the cluster set $C(f, \gamma)$ is contained in M' . Then f is Hölder $1/2$ -continuous on $\Delta \cup \gamma$.*

Finally it follows from Theorem 8 :

Corollary 5. *Let M' be a totally real C^1 -submanifold in an almost complex manifold (M, J) . If $f \in \mathcal{O}_J(\Delta, M)$ is such that $C(f, \partial\Delta)$ is contained in M' then the derivative of f is in $L^p(\Delta)$ for every $0 < p < 2$.*

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HERVÉ GAUSSIER	ALEXANDRE SUKHOV
C.M.I.	U.S.T.L.
39, RUE JOLIOT-CURIE,	CITÉ SCIENTIFIQUE
13453 MARSEILLE CEDEX 13	59655 VILLENEUVE D'ASCQ CEDEX
<i>E-mail address:</i> <code>gaussier@cmi.univ-mrs.fr</code> , <code>sukhov@agat.univ-lille1.fr</code>	