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# GENERIC BERNSTEIN-SATO POLYNOMIAL ON AN IRREDUCIBLE AFFINE SCHEME

ROUCHDI BAHLOUL

ABSTRACT. Given  $p$  polynomials with coefficients in a commutative unitary integral ring  $\mathcal{C}$  containing  $\mathbb{Q}$ , we define the notion of a generic Bernstein-Sato polynomial on an irreducible affine scheme  $V \subset \text{Spec}(\mathcal{C})$ . We prove the existence of such a non zero rational polynomial which covers and generalizes previous existing results by H. Biosca. When  $\mathcal{C}$  is the ring of an algebraic or analytic space, we deduce a stratification of the space of the parameters such that on each stratum, there is a non zero rational polynomial which is a Bernstein-Sato polynomial for any point of the stratum. This generalizes a result of A. Leykin obtained in the case  $p = 1$ .

## INTRODUCTION AND MAIN RESULTS

Fix  $n \geq 1$  and  $p \geq 1$  two integers and  $v \in \mathbb{N}^p$ . Let  $x = (x_1, \dots, x_n)$  and  $s = (s_1, \dots, s_p)$  be two systems of variables. Let  $\mathbf{k}$  be a field of characteristic <sup>1</sup> 0. Let  $\mathbf{A}_n(\mathbf{k})$  be the ring of differential operators with coefficients in  $\mathbf{k}[x] = \mathbf{k}[x_1, \dots, x_n]$  and  $\mathcal{D}$  (resp.  $\mathcal{O}$ ) be the sheaf of rings of differential operators (resp. analytic functions) on  $\mathbb{C}^n$  for which we denote by  $\mathcal{D}_{x_0}$  (resp.  $\mathcal{O}_{x_0}$ ) the fiber in  $x_0$ .

Let  $f = (f_1, \dots, f_p)$  be in  $\mathbf{k}[x]^p$  (resp.  $\mathcal{O}_{x_0}^p$ ) and consider the following functional identity:

$$b(s)f^s \in \mathbf{A}_n(\mathbf{k})[s] \cdot f^{s+v},$$

(resp.  $\mathcal{D}_{x_0}[s]$  instead of  $\mathbf{A}_n(\mathbf{k})[s]$ ) where  $f^{s+v} = f_1^{s_1+v_1} \dots f_p^{s_p+v_p}$ . This identity takes place in the free module generated by  $f^s$  over  $\mathbf{k}[x, \frac{1}{f_1 \dots f_p}, s]$  (resp.  $\mathcal{O}_{x_0}[\frac{1}{f_1 \dots f_p}, s]$ ).

The set of such  $b(s)$  is an ideal of  $\mathbf{k}[s]$  (resp.  $\mathbb{C}[s]$ ). This ideal is called the (global) Bernstein-Sato ideal of  $f$  (resp. local Bernstein-Sato ideal in  $x_0$ ) and we denote it by  $\mathcal{B}^v(f)$  (resp.  $\mathcal{B}_{x_0}^v(f)$ ). When  $p = 1$ , this ideal is principal and its monic generator is called the Bernstein polynomial associated with  $f$ . Historically, I.N. Bernstein [Be] introduced the (global) Bernstein polynomial and proved its existence (i.e. the fact that it is not zero). J.E. Björk [Bj] has given the proof in the analytic case. Let us cite also M. Kashiwara [K] who proved, moreover, the rationality of the roots of the local Bernstein polynomial. For  $p \geq 2$ , the algebraic case can be easily treated in the same way as for  $p = 1$ . For the analytic case, the proof of the non nullity of  $\mathcal{B}_{x_0}^v(f)$  is due to C. Sabbah ([S1] and [S2]). Let us also cite A. Gyoja [Gy] who proved that  $\mathcal{B}_{x_0}^v(f)$  contains a non zero *rational* polynomial. The absolute Bernstein-Sato polynomial naturally leads to the notion of a generic Bernstein-Sato polynomial which we shall explain in what follows.

Let  $\mathcal{C}$  be a unitary commutative integral ring with the following condition: For any prime ideal  $\mathcal{P} \subset \mathcal{C}$  and for any  $n \in \mathbb{N} \setminus \{0\}$ , we have:

$$n \in \mathcal{P} \Rightarrow 1 \in \mathcal{P}.$$

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<sup>1</sup>all the fields considered in this paper are of characteristic 0

This condition is equivalent to the fact that for any  $\mathcal{P} \subset \mathcal{C}$ , the fraction field of  $\mathcal{C}/\mathcal{P}$  is of characteristic 0. Note that this condition is satisfied if and only if there exists an injective ring morphism  $\mathbb{Q} \hookrightarrow \mathcal{C}$ .

We shall see  $\mathcal{C}$  as the ring of coefficients or parameters. Indeed, let  $f = (f_1, \dots, f_p)$  in  $\mathcal{C}[x]^p = \mathcal{C}[x_1, \dots, x_n]^p$ .

Let us denote by  $\mathbf{A}_n(\mathcal{C})$  the ring of differential operators with coefficients in  $\mathcal{C}[x]$ , that is the  $\mathcal{C}$ -algebra generated by  $x_i$  and  $\partial_{x_i}$  ( $i = 1, \dots, n$ ) where the only non trivial commutation relations are  $[\partial_{x_i}, x_i] = 1$  for  $i = 1, \dots, n$  (hence  $\mathcal{C}$  is in the center of  $\mathbf{A}_n(\mathcal{C})$ ).

We denote by  $\text{Spec}(\mathcal{C})$  (resp.  $\text{Specm}(\mathcal{C})$ ) the set of prime (resp. maximal) ideals of  $\mathcal{C}$  which is the spectrum of  $\mathcal{C}$  (resp. the maximal spectrum). For an ideal  $\mathcal{I} \subset \mathcal{C}$ , we denote by  $V(\mathcal{I}) = \{\mathcal{P} \in \text{Spec}(\mathcal{C}); \mathcal{P} \supset \mathcal{I}\}$  the affine scheme defined by  $\mathcal{I}$  and  $V_m(\mathcal{I}) = V(\mathcal{I}) \cap \text{Specm}(\mathcal{C})$ . Remark that we shall only work with the closed subsets of  $\text{Spec}(\mathcal{C})$  and forget the sheaf structure of a scheme.

We are going to introduce the notion of a generic Bernstein-Sato polynomial of  $f$  on an irreducible affine scheme  $V = V(\mathcal{Q}) \subset \text{Spec}(\mathcal{C})$  (that is when  $\mathcal{Q}$  is prime).

So let  $\mathcal{Q}$  be a prime ideal of  $\mathcal{C}$  and suppose that none of the  $f_j$ 's is in  $\mathcal{Q}[x]$ .

The main result of this article is the following.

**Theorem 1.** *There exists  $h \in \mathcal{C} \setminus \mathcal{Q}$  and  $b(s) \in \mathbb{Q}[s_1, \dots, s_p] \setminus 0$  such that*

$$h b(s) f^s \in \mathbf{A}_n(\mathcal{C})[s] f^{s+v} + \left( \mathcal{Q}[x, \frac{1}{f_1 \dots f_p}, s] \right) f^s.$$

Such a  $b(s)$  is called a (rational) generic Bernstein-Sato polynomial of  $f$  on  $V = V(\mathcal{Q})$  (see the notation and the remark below).

In the case where  $p = 1$ , the generic and relative (not introduced here) Bernstein polynomial has been studied by F. Geandier in [Ge] and by J. Briançon, F. Geandier and P. Maisonobe in [Br-Ge-M] in an analytic context (where  $f$  is an analytic function of  $x$ ). In [Bi] (see also [Bi2]), H. Biosca studied these notions with  $p \geq 1$  in the analytic and the algebraic context (that which we are concerned with) and proved that when

- $\mathcal{C} = \mathbb{C}[a_1, \dots, a_m]$  or
- $\mathcal{C} = \mathbb{C}\{a_1, \dots, a_m\}$  and

$\mathcal{Q} = (0)$  so that  $V$  is smooth and equal to  $\mathbb{C}^m$  or  $(\mathbb{C}^m, 0)$ , we have a generic Bernstein-Sato polynomial. It does not seem straightforward to adapt her proof to the case where  $\mathcal{Q} \neq (0)$  (i.e. when  $V$  is singular). Let us also say that she did not mention the fact that the polynomial she constructed is rational even though a detailed study of her proof shows that it is. As it appears, our main result covers and generalizes the previous existing results in this affine situation.

**Notation.** *Let  $\mathcal{P}$  be a prime ideal of  $\mathcal{C}$ . For  $c$  in  $\mathcal{C}$ , denote by  $[c]_{\mathcal{P}}$  the class of  $c$  in the quotient  $\mathcal{C}/\mathcal{P}$  and  $(c)_{\mathcal{P}} = \frac{[c]_{\mathcal{P}}}{1}$  this class viewed in the fraction field of  $\mathcal{C}/\mathcal{P}$ . We naturally extend these notations to  $\mathcal{C}[x]$ ,  $\mathbf{A}_n(\mathcal{C})$  and  $\mathcal{C}[x, \frac{1}{f_1 \dots f_p}, s]$ .*

**Remark.** *Using these notations, we can see that the polynomial  $b(s)$  of theorem 1 is a Bernstein-Sato polynomial of  $(f)_{\mathcal{P}}$  for any  $\mathcal{P} \in V(\mathcal{Q}) \setminus V(h)$ . This justifies the name of a generic Bernstein-Sato polynomial on  $V(\mathcal{Q})$ .*

As an application of theorem 1, we obtain some consequences :

**Corollary 2.** *Fix a positive integer  $d$  and a field  $\mathbf{k}$ .*

*For each  $j = 1, \dots, p$ , take  $f_j = \sum_{|\alpha| \leq d} a_{\alpha,j} x^\alpha$  with  $\alpha \in \mathbb{N}^n$  and  $a_{\alpha,j}$  an indeterminate.*

Take  $a = (a_{\alpha,j})$  for  $|\alpha| \leq d$  and  $j = 1, \dots, p$  such that we see  $f = (f_1, \dots, f_p)$  in  $\mathbf{k}[a][x]^p$ . Denote by  $m$  the number of the  $a_{\alpha,j}$ 's.

Then there exists a finite partition of  $\mathbf{k}^m = \cup W$  where each  $W$  is a locally closed subset of  $\mathbf{k}^m$  (i.e.  $W$  is a difference of two Zariski closed sets) such that for any  $W$ , there exists a polynomial  $b_W(s) \in \mathbb{Q}[s_1, \dots, s_p] \setminus 0$  such that for each  $a_0$  in  $W$ ,  $b_W(s)$  is in  $\mathcal{B}^v(f(a_0, x))$ .

**Remark.**

- This corollary generalizes to the case  $p \geq 2$  the main result of A. Leykin [L] and J. Briançon and Ph. Maisonobe [Br-Mai] in the case  $p = 1$ .
- There is another way to generalize these results: Given a well ordering  $<$  on  $\mathbb{N}^p$  compatible with sums, it is possible to prove the existence of a partition  $\mathbf{k}^m = \cup W$  into locally closed subsets with the following property: For any  $W$ , there exists a finite subset  $G_W \subset \mathbf{k}[a][x]$  such that for any  $a_0 \in W$ , the set  $G_W(a_0)$  is a  $<$ -Gröbner basis of the Bernstein-Sato ideal  $\mathcal{B}^v(f(a_0, x))$ , see [Br-Mai] and [Ba].

*Proof of Corollary 2.* We remark that we can give the same statement as in corollary 2 for any algebraic subset  $Y \subset \mathbf{k}^m$  as a space of parameters. The statement of corollary 2 will then follow from the proof of this more general statement, that we shall give by an induction on the dimension of  $Y$ . If  $\dim Y = 0$ , the result is trivial. Suppose  $\dim Y \geq 1$ . Write  $Y = V_m(Q_1) \cup \dots \cup V_m(Q_r)$  where the  $Q_i$ 's are prime ideals in  $\mathbf{k}^m$  (we identify the maximal ideals of  $\mathbf{k}[a]$  and the points of  $\mathbf{k}^m$ ). For each  $i$ , let  $h_i \in \mathbf{k}[a] \setminus Q_i$  and  $b_i(s) \in \mathbb{Q}[s] \setminus 0$  be the  $h$  and  $b(s)$  of theorem 1 applied to  $Q_i$ . Now, write

$$Y = \left( \bigcup_{i=1}^r V_m(Q_i) \setminus V_m(h_i) \right) \cup Y',$$

with  $Y' = \bigcup (V_m(Q_i) \cap V_m(h_i))$  for which  $\dim Y' < \dim Y$ . Apply the induction hypothesis to  $Y'$ . We obtain that  $Y$  is a union (not necessarily disjoint) of locally closed subsets  $V$  such that for each  $V$  there exists  $b_V(s) \in \mathbb{Q}[s] \setminus 0$  which is in  $\mathcal{B}^v(f(a_0, x))$  for any  $a_0 \in V$ . Let us show now how to obtain the announced partition. Let  $B$  be the set of the obtained polynomials  $b_V$ 's. Set  $B = \{b_1, \dots, b_e\}$ . For any  $i = 1, \dots, e$ , let  $E_i$  be the set of the  $V$ 's for which  $b_i = b_V$ . Put

- $W_1 = \bigcup_{V \in E_1} V$ ,
- $W_2 = \left( \bigcup_{V \in E_2} V \right) \setminus \left( \bigcup_{V \in E_1} V \right)$ ,
- $\vdots$
- $W_e = \left( \bigcup_{V \in E_e} V \right) \setminus \left( \bigcup_{V \in E_1 \cup \dots \cup E_{e-1}} V \right)$ .

Note that some of the  $W_i$ 's may be empty. The set  $\{(b_1, W_1), \dots, (b_e, W_e)\}$  gives a partition  $Y = \cup W_i$  in a way that  $b_i \in \mathcal{B}^v(f(a_0, x))$  for any  $a_0 \in W_i$ .  $\square$

**Corollary 3.** Take  $f_1(a, x), \dots, f_p(a, x) \in \mathcal{O}(U)[x]$  where  $\mathcal{O}(U)$  denotes the ring of holomorphic functions on a open subset  $U$  of  $\mathbb{C}^m$ .

Then there exists a finite partition of  $U = \cup W$  where each  $W$  is an (analytic) locally closed subset of  $U$  (i.e. each  $W$  is a difference of two analytic subsets of  $U$ ) such that for any  $W$ , there exists a rational non zero polynomial  $b(s)$  which belongs to  $\mathcal{B}^v(f(a_0, x))$  for any  $a_0 \in W$ .

**Remark.** *As it will appear in the proof, we have the same result if we replace  $\mathcal{O}(U)$  by  $\mathbb{C}\{a_1, \dots, a_m\}$  or  $\mathbf{k}[[a_1, \dots, a_m]]$  ( $\mathbf{k}$  being an arbitrary field).*

*Proof.* Let us write  $f_j(a, x) = \sum g_{\alpha,j}(a)x^\alpha$  where  $g_{\alpha,j} \in \mathcal{O}(U)$ . Let  $m$  be the number of the  $g_{\alpha,j}$ 's and let us introduce  $m$  new variables  $b_{\alpha,j}$ . Consider the (analytic) map  $\phi : U \ni a \mapsto (b_{\alpha,j} = g_{\alpha,j}(a))_{\alpha,j} \in \mathbb{C}^m$  where  $\mathbb{C}$  is a fixed arbitrary field. Now apply corollary 2 to this situation. Let  $\mathbf{k}^m = \cup W$  be the obtained partition and for any  $W$ , let  $b_W \in \mathbb{Q}[s]$  be the polynomial given in 2. Now apply  $\phi^{-1}$ . This gives a partition  $U = \cup \phi^{-1}(W)$ . Since  $\phi$  is analytic, the sets  $\phi^{-1}(W)$  are locally closed analytic subsets of  $U$ . It is then clear that for any  $W$  and  $a_0 \in \phi^{-1}(W)$ , we have  $b_W \in \mathcal{B}^v(f(a_0, x))$ .  $\square$

#### PROOF OF THE MAIN THEOREM

In order to prove theorem 1, we shall first prove the following.

**Theorem 4.** *Let  $\mathbf{k}$  be a field and  $f \in \mathbf{k}[x]^p$ . Then  $\mathcal{B}^v(f) \cap \mathbb{Q}[s]$  is not zero.*

Note that in [Br], the author proved (for  $p = 1$ ) that the global Bernstein polynomial has rational roots for any field  $\mathbf{k}$  of characteristic zero. The proof of 4 will use the following propositions.

**Proposition 5.** *Let  $\mathbf{K}$  be a subfield of a field  $\mathbf{L}$ . Suppose that  $f \in \mathbf{K}[x]^p$ . Let  $b(s) \in \mathbf{K}[s]$  be such that  $b(s)f^s \in \mathbf{A}_n(\mathbf{L})[s]f^{s+v}$ . Then*

$$b(s)f^s \in \mathbf{A}_n(\mathbf{K})[s]f^{s+v}.$$

*Proof.* The proof is inspired by [Br] in which the case  $p = 1$  is treated. As  $\mathbf{L}$  is a  $\mathbf{K}$ -vector space, let us take  $\{1\} \cup \{l_\gamma; \gamma \in \Gamma\}$  as a basis so that  $\mathbf{L}[x, s, \frac{1}{f_1 \dots f_p}]f^s$  is a free  $\mathbf{K}[x, s, \frac{1}{f_1 \dots f_p}]$ -module with  $\{f^s\} \cup \{l_\gamma f^s; \gamma \in \Gamma\}$  as a basis. Now let  $P$  be in  $\mathbf{A}_n(\mathbf{L})[s]$  such that  $b(s)f^s = Pf^{s+v}$ . We decompose  $P = P_0 + P'$  where  $P_0 \in \mathbf{A}_n(\mathbf{K})[s]$  and  $P'$  has its coefficients in  $\bigoplus_{\gamma \in \Gamma} \mathbf{K} \cdot l_\gamma$ . Now, we have:

$$b(s)f^s = P_0f^{s+v} + P'f^{s+v},$$

with  $b(s)f^s$  and  $P_0f^{s+v}$  in  $\mathbf{K}[x, s, \frac{1}{f_1 \dots f_p}]f^s$  and  $P'f^{s+v}$  in  $\bigoplus_{\gamma \in \Gamma} \mathbf{K}[x, s, \frac{1}{f_1 \dots f_p}]l_\gamma f^s$ . By identification, we obtain:

$$b(s)f^s = P_0f^{s+v}.$$

$\square$

**Proposition 6.** ([Br] and [Br-Mai]) *Given  $f \in \mathbb{C}[x]^p$ , we have :*

1. *The set  $\{\mathcal{B}_{x_0}^v(f); x_0 \in \mathbb{C}^n\}$  is finite.*
2.  *$\mathcal{B}^v(f)$  is the intersection of all the  $\mathcal{B}_{x_0}^v(f)$  where  $x_0 \in \mathbb{C}^n$ .*

*Proof of theorem 4.* We shall divide the proof into two steps:

(a) First, suppose that  $\mathbf{k} = \mathbb{C}$ . By [S1], [S2] and [Gy], as mentioned in the introduction, each  $\mathcal{B}_{x_0}^v(f)$  contains a non zero rational polynomial. By the previous proposition, we can take a finite product of these polynomials and obtain a rational polynomial in  $\mathcal{B}^v(f)$ .

(b) Now suppose that  $\mathbf{k}$  is arbitrary. Let  $c_1, \dots, c_N$  be all the coefficients that appear in the writing of the  $f_j$ 's and consider the field  $\mathbf{K} = \mathbb{Q}(c_1, \dots, c_N)$ . There exist  $e_1, \dots, e_N \in \mathbb{C}$  and an injective morphism of fields  $\phi : \mathbf{K} \rightarrow \mathbb{C}$  such that  $\phi(c_i) = e_i$  for any  $i$ . We

denote by the same symbol  $\phi$  the natural extension of  $\phi$  from  $\mathbf{K}[x]$  to  $\mathbb{C}[x]$  and from  $\mathbf{A}_n(\mathbf{K})[s]$  to  $\mathbf{A}_n(\mathbb{C})[s]$ . Now, consider in  $\mathbb{C}[s]$  the Bernstein-Sato ideal  $\mathcal{B}^v(\phi(f))$  (where  $\phi(f) = (\phi(f_1), \dots, \phi(f_p))$ ). Using the result of case (a), there exists  $b(s) \in \mathbb{Q}[s] \setminus 0$  that belongs to  $\mathcal{B}^v(\phi(f))$ . So we have a functional equation:

$$b(s)\phi(f)^s = P \cdot \phi(f)^{s+v},$$

where  $P \in \mathbf{A}_n(\mathbb{C})[s]$ . By proposition 5, we can suppose  $P \in \mathbf{A}_n(\phi(\mathbf{K}))[s]$ . Apply  $\phi^{-1}$  to this equation. Since  $b(s) \in \mathbb{Q}[s]$ ,  $\phi^{-1}(b(s)) = b(s)$ , thus we obtain:

$$b(s)f^s = \phi^{-1}(P) \cdot f^{s+v}.$$

In conclusion  $b(s)$  is in  $\mathcal{B}^v(f)$ . □

Now we dispose of a sufficient material to give the

*Proof of theorem 1.* By theorem 4, there exists a non zero rational polynomial  $b(s)$  in  $\mathcal{B}^v((f)_{\mathcal{Q}})$ . Hence, we have the following equation:

$$b(s) \left( \frac{[f]_{\mathcal{Q}}}{1} \right)^s = \frac{[U(s)]_{\mathcal{Q}}}{[h]_{\mathcal{Q}}} \cdot \left( \frac{[f]_{\mathcal{Q}}}{1} \right)^{s+v},$$

where  $U(s) \in \mathbf{A}_n(\mathbb{C})[s]$  and  $h \in \mathcal{C} \setminus \mathcal{Q}$ . It follows that:

$$h b(s) f^s - U(s) \cdot f^{s+v} \equiv 0 \pmod{\mathcal{Q}}$$

in  $\mathcal{C}[x, \frac{1}{f_1, \dots, f_p}, s] f^s$ . Since  $f_1 \cdots f_p \notin \mathcal{Q}[x]$  and  $\mathcal{Q}$  is prime, we obtain:

$$h b(s) f^s - U(s) \cdot f^{s+v} \in \mathcal{Q}[x, \frac{1}{f_1 \cdots f_p}, s] f^s.$$

□

This article is a more general and simplified version of some results of my thesis [Ba].

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