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COMPUTATIONS OF BOTT-CHERN CLASSES ON $\mathbb{P}(E)$

CHRISTOPHE MOUROUGANE

Abstract. *We compute the Bott-Chern classes of the metric Euler sequence describing the relative tangent bundle of the variety $\mathbb{P}(E)$ of hyperplans of a holomorphic hermitian vector bundle (E,h) on a complex manifold. We give applications to the construction of the arithmetic characteristic classes of an arithmetic vector bundle $\overline{E}$ and to the computation of the height of $\mathbb{P}(\overline{E})$ with respect to the tautological quotient bundle $\mathcal{O}_{\overline{E}}(1)$.* [1]

Introduction

In the whole work, X will be a smooth complex analytic manifold of dimension n. For a vector bundle $F \to X$ on X we denote by $A^{p,q}(X,F)$ the space of smooth F-valued differential forms on X of type (p,q). On a holomorphic vector bundle $E \to X$ endowed with a hermitian metric h, there exists a unique connection $\nabla = \nabla_{E,h}$ compatible with both the holomorphic and the hermitian structures. It is called the Chern connection of (E,h). Its curvature $\frac{i}{2\pi}\nabla^2$ is multiplication by an endomorphism-valued 2-form that we will denote by $\Theta(E,h) \in A^{1,1}(X, Herm(E))$. The associated Chern forms are defined by

$$\det(I+tA) = \sum_{d=0}^{+\infty} t^d \det{}_d(A) \text{ and } c_d(E,h) := \det{}_d(\Theta(E,h)) \in A_{d,d}(X,\mathbb{C}).$$

The form $c_d(E,h)$ is closed and represents the Chern class $c_d(E)$ of E in $H^{2d}(X,\mathbb{R})$ through De Rham isomorphism. The Chern class polynomial $c_t(E) := \sum_{d=0}^{+\infty} t^d c_d(E)$ is multiplicative on exact sequences. Bott-Chern secondary classes are classes in

$$\widetilde{A^{d,d}}(X) := \frac{A^{d,d}(X,\mathbb{C})}{Imd' + Imd''}$$

which functorially represent the default of multiplicativity of the Chern form polynomial $c_t(E,h) := \sum_{d=0}^{+\infty} t^d c_d(E,h)$ on short exact sequences of hermitian vector bundles (see [Bi-G-S] theorem 1.29 and [G-S-2] theorem 1.2.2) :

- For any short exact sequence $(\mathcal{S}) = (0 \to S \to E \to Q \to 0)$, and any choice of metrics $h = (h_E, h_S, h_Q)$, $\widetilde{c_{d+1}}(\mathcal{S},h) \in \widetilde{A^{d,d}}(X)$,
$$c_t(E,h_E) - c_t(S,h_S)c_t(Q,h_Q) = -\frac{it}{2\pi} d'd'' \widetilde{c}_t(\mathcal{S},h)$$
where $\widetilde{c}_t(\mathcal{S},h) = \sum_{d=0}^{+\infty} t^d \widetilde{c_{d+1}}(\mathcal{S},h)$.
- If $(\mathcal{S},h)$ is metrically split, then $\widetilde{c}_t(\mathcal{S},h) = 0$.
- For any holomorphic map $f : X \to Y$ of complex analytic manifolds and any metric short exact sequence $(\mathcal{S},h)$ over Y, $\widetilde{c}_t(f^\star\mathcal{S}, f^\star h) = f^\star \widetilde{c}_t(\mathcal{S},h)$.

They were introduced by Bott and Chern in their study of the distribution of the values of holomorphic sections of hermitian vector bundles [B-C]. They were used by Donaldson for defining a functional on the space of hermitian metrics on

[1] Key words : Bott-Chern secondary classes, metric relative Euler sequence, arithmetic characteristic classes.
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a given holomorphic vector bundle in order to find Hermite-Einstein metrics [Do]. They were given an axiomatic definition and served as one kind of secondary objects (together with Green currents and analytic torsion forms) in the study by Bismut Gillet and Soulé of holomorphic determinant bundle [Bi-G-S]. They enter the very definition of arithmetic characteristic classes set by Gillet and Soulé [G-S-2].

Bott-Chern classes were computed in few cases, mainly when (E, h) is assumed to be flat. See the work of Gillet and Soulé in the the case of projective spaces [G-S-2] of Maillot in the case of Grassmanians [Ma] and of Tamvakis in the case of other flag manifolds [Ta]. In most of computed cases, Bott-Chern classes are made of closed forms.

We deal with a relative situation. Let $E \to X$ be a holomorphic vector bundle of rank r over X . Consider $\pi : \mathbb{P}(E) \to X$, the variety of hyperplans of E. The differential of the quotient map $E^\star - X \times \{0\} \to \mathbb{P}(E)$ over X gives rise to the relative Euler sequence (Σ) on $\mathbb{P}(E)$:

$$0 \to \mathcal{O}_E(-1) \to \pi^\star E^\star \to T \to 0$$

where T denotes the twisted relative tangent bundle $T_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(-1)$. The choice of a hermitian metric h on E enables to endow all the bundles in the sequence with a natural hermitian metric. Our aim is to compute Bott-Chern forms (i.e. representatives of Bott-Chern secondary classes) for this metric relative Euler sequence (Σ, h) in terms of the curvature form of (E, h). The case where X is a point is dealt with by Gillet and Soulé in [G-S-2].

We denote by α the curvature form $\Theta(\mathcal{O}_E(1), h)$ of $(\mathcal{O}_E(1), h)$. For $q \in \mathbb{N}$, we consider the forms $\Theta^q a^\star$ on $\mathbb{P}(E)$ given at the point $(x, [a^\star])$ of $\mathbb{P}(E)$, $(a^\star \in E^\star_x)$ by

$$\Theta^q a^\star = \frac{\langle \pi^\star \Theta(E^\star, h)^q a^\star, a^\star \rangle}{\|a^\star\|^2}.$$

The product is taken with respect to the wedge product in the form part and the composition in the endomorphism part. The generating function Θ_t for those forms is $\sum_{d=0}^{+\infty} (-t)^d (\Theta^d a^\star)$. We will also need the Harmonic numbers $\mathcal{H}_p = \sum_1^p \frac{1}{i}$ and their generating function $H(X) := \sum_{p=1}^{+\infty} \mathcal{H}_p X^p$. We prove

Main theorem. *The Bott-Chern secondary class polynomial $\widetilde{c_t}(\Sigma, h)$ for (Σ, h) is represented by the form*

$$-H\left(t\alpha\Theta_t + 1 - \Theta_t + t\Theta_t \frac{i}{2\pi} d'd'' \ln \Theta_t\right) \Theta_t \pi^\star c_t(E^\star, h).$$

This form is in general not closed. The first part of the proof consists in finding an expression in coordinates for $\widetilde{c_{d+1}}(\Sigma, h)$. Computations are simplified by the choice of a normal frame at a given point $(x_0, [a_0^\star])$ of $\mathbb{P}(E)$. The combinatorial analysis is rather intricate. The second part is devoted to identify the previously found expressions in terms of globally defined forms. Finally, the computation of the Bott-Chern forms generating polynomial enables to express the result in a concise way, for Bott-Chern forms follow a recursion formula.

Our second aim is to give some applications in the setting of Arakelov geometry. Let $\mathcal{X}$ be an arithmetic variety built on a scheme χ defined on the ring of integers of some number field. Let $\overline{E}$ be an arithmetic vector bundle on $\mathcal{X}$. We will write X for the smooth variety $\chi_{\mathbb{C}}(\mathbb{C})$ and (E, h) for the induced hermitian vector bundle on X. We give in section 8 an application of our computations to the construction of arithmetic Chern classes from arithmetic Segre classes. At algebraic level, the total

Chern class of $E^\star$ is the inverse of the total Segre class of E. This is also true at the level of forms (see section 7). But at the arithmetic level, some secondary purely complex analytic objects have to enter the definition of Segre classes in order to keep this inverse relation true in the arithmetic Chow group of the base scheme. This is due to the fact that $\overline{E}^\star \otimes \mathcal{O}_{\overline{E}}(1)$ having a nowhere vanishing section has nevertheless a non-zero top Chern form. Evaluating some fiber integrals of previously studied Bott-Chern forms, we give explicit expressions for the secondary classes involved. This answers a question raised by Bost and Soulé [Bo-S]. This in turn enables to compute the height of $\mathbb{P}(\overline{E})$ with respect to $\mathcal{O}_{\overline{E}}(1)$ in term of the top arithmetic Segre class and a secondary term from complex cohomology. The qualitative output is that complex characteristic classes of E are the only complex datas of (E, h) that accounts for the measure of the complexity of $\mathbb{P}(\overline{E})$ given by its height.

No wonder that such computations at least in the full relative case (see proposition 7) can be done for other characteristic classes than the Chern classes on other flag varieties of $\overline{E}$ to get expressions for their height.

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Contents

Introduction	1
1. Bott-Chern forms	3
2. Curvature computations	5
3. Computations in coordinates	6
4. Coordinates free expressions	10
5. On the Bott-Chern forms generating polynomial	13
6. Some special cases	13
6.1. On first parts of Bott-Chern forms	13
6.2. On first Bott-Chern forms	14
6.3. For curves and surfaces	14
7. On characteristic forms	14
7.1. From Segre forms to Chern forms	14
7.2. Computation of the class S	16
8. Some arithmetic applications	18
8.1. On arithmetic characteristic classes	18
8.2. On the height of $\mathbb{P}(\overline{E})$	20
References	22

1. Bott-Chern forms

General formulas for computing $\tilde{c}(\mathcal{S}, h)$ with induced and quotient metrics were given by Bott and Chern [B-C]. For sake of completeness, we outline their method.

The commutator of $End(E)$-valued forms α and β is defined to be $[\alpha, \beta] := \alpha\beta - (-1)^{\deg\alpha \deg\beta}\beta\alpha$.

Consider the sequence $0 \to S \xrightarrow{\iota} E \xrightarrow{p} Q \to O$ endowed with metric constructed from a hermitian metric h on E. Denote by ∇ (resp. ∇_S, ∇_Q) the Chern connection of (E, h) (resp. $(S, h_{|S})$, $(Q, h_{|Q})$). Consider the family of connections on E

$$\nabla_u := \nabla + (u-1)P_Q \nabla P_S$$

where $P_S = \iota\iota^\star$ (resp. $P_Q = p^\star p$) denotes the orthogonal projection of E onto $\iota(S)$ (resp. $\iota(S)^\perp$). The choice of a local holomorphic frame for E enables to express locally the connection ∇_u as $d + A_u$ for some matrix valued $(1,0)$-form

A_u. The curvature $\Theta(\nabla_u)$ of ∇_u is then given by the matrix valued $(1,1)$-form $\frac{i}{2\pi}(dA_u + A_u \wedge A_u)$. Recall that the connection ∇_u extends to $End(E)$-valued forms by $\nabla_u(\alpha) := [\nabla_u, \alpha]$. The formula for $\tilde{c}(\Sigma, h)$ relies on the following identities

$$\begin{aligned}
\nabla_u(\Theta(\nabla_u)) &= [\nabla_u, \Theta(\nabla_u)] = [\nabla_u, \nabla_u^2] = 0 \text{ Bianchi formula} \\
&= d(\Theta(\nabla_u)) + [A_u, \Theta(\nabla_u)] \\
\nabla_u(P_S) &= [\nabla_u, P_S] = u\frac{d}{du}\nabla_u \text{ got from } P_S\nabla' P_Q = P_Q \nabla'' P_S = 0 \\
&= d' P_S + [A_u, P_S].
\end{aligned}$$

That is

$$\begin{aligned}
d'\Theta(\nabla_u) = d\Theta(\nabla_u) &= -[A_u, \Theta(\nabla_u)] \\
d' P_S &= -[A_u, P_S] + u\frac{d}{du}\nabla_u.
\end{aligned}$$

Consider the polarization Det_k of $\det_k$, that is the symmetric k-linear form on $M_r(\mathbb{C})$ whose restriction on the small diagonal is $\det_k$. Note by $Det_k(A; B) = kDet_k(A, A, \cdots, A, B)$. The differential version of the $Gl(r, \mathbb{C})$-invariance of $\det_k$ shows that the contribution of the commutators $-[A_u, \cdot]$ vanishes. This leads to

$$d'\left(Det_k(\Theta(\nabla_u); P_S)\right) = uDet_k\left(\Theta(\nabla_u); \frac{d}{du}\nabla_u\right).$$

Now,

$$\begin{aligned}
\frac{i}{2\pi} d\left(\frac{d}{du}\nabla_u\right) &= \frac{i}{2\pi}\frac{d}{du}d(\nabla_u) = \frac{i}{2\pi}\frac{d}{du}d(A_u) = \frac{d}{du}\left(\Theta(\nabla_u) - \frac{i}{2\pi}A_u \wedge A_u\right) \\
&= -\frac{i}{2\pi}[A_u, \frac{d}{du}A_u] + \frac{d}{du}\left(\Theta(\nabla_u)\right).
\end{aligned}$$

The $Gl(r, \mathbb{C})$-invariance of $\det_i$ leads to

$$\begin{aligned}
-\frac{i}{2\pi} d'd'' Det_k(\Theta(\nabla_u); P_S) &= \frac{i}{2\pi} d\left(uDet_k\left(\Theta(\nabla_u); \frac{d}{du}\nabla_u\right)\right) \\
&= u\frac{d}{du}\det_k(\Theta(\nabla_u)). \qquad (1.1)
\end{aligned}$$

One then checks in a frame adapted to the $\mathcal{C}^\infty$ splitting $E \simeq S \oplus Q$ given by $\iota^\star \oplus p$, that

$$\Theta(\nabla_u) = \begin{vmatrix} (1-u)\Theta_S + u\iota^\star\Theta_E\iota & \iota^\star\Theta_E p^\star \\ up\Theta_E\iota & (1-u)\Theta_Q + up\Theta_E p^\star \end{vmatrix}$$

Hence integrating equation (1.1) between 0 and 1, we get

$$c_k(E, h) - c_k(S \oplus Q, \nabla_S \oplus \nabla_Q) = -\frac{i}{2\pi} d'd'' \int_0^1 \frac{\Phi_k(u) - \Phi_k(0)}{u} du \qquad (1.2)$$

where

$$\Phi_k(u) = Det_k(\Theta(\nabla_u); P_S) = \mathrm{coeff}_\lambda det_k(\Theta(\nabla_u) + \lambda P_S).$$

If moreover, the sub-bundle S is of rank 1, then

$$\Phi_{d+1}(u) = \det_d\left((1-u)\,\Theta_Q + up\Theta_E p^\star\right). \qquad (1.3)$$

2. Curvature computations

We will compute the curvature of T and of $\mathcal{O}_E(1)$ at a point $(x_0, [a_0^\star])$ of $\mathbb{P}(E)$ ($a_0^\star \in E^\star$) in a well-chosen frame.

We now recall the formula for the curvature of a quotient bundle. Consider the sequence $0 \to S \xrightarrow{\iota} E \xrightarrow{p} Q \to O$ endowed with metrics constructed from a hermitian metric on E. All the bundles we will consider, including the endomorphism bundles inherit a metric and a Chern connection from those of E. We will denote with a star the adjoint maps with respect to the given metric. Notice first that since $pp^\star = Id_Q$ and ∇_E is compatible with the metric, $p(\nabla_{Hom(Q,E)}p^\star) = 0$. Let s be a local smooth section of Q. We get first

$$\nabla_E(p^\star s) = (\nabla_{Hom(Q,E)}p^\star)s + p^\star\nabla_Q s = \iota\iota^\star(\nabla_{Hom(Q,E)}p^\star)s + p^\star\nabla_Q s$$

and then

$$p\nabla_E^2(p^\star s) = p(\nabla_{Hom(S,E)}\iota) \wedge \iota^\star(\nabla_{Hom(Q,E)}p^\star)s + \nabla_Q^2 s.$$

That is, taking the type into account, $(\nabla''_{Hom(S,E)}\iota = 0)$

$$\Theta_Q = p\Theta_E p^\star - \frac{i}{2\pi}p(\nabla'_{Hom(S,E)}\iota) \wedge \iota^\star(\nabla''_{Hom(Q,E)}p^\star).$$

We choose local coordinates $x_1, \cdots, x_n$ around x_0, and a frame $e_1^\star, \cdots, e_r^\star$ for $E^\star$ around x_0, normal at x_0 (i.e. $\langle e_i^\star, e_j^\star\rangle = \delta_{ij} - 2\pi\sum_{\lambda\mu} c_{\lambda\mu ji}x_\lambda\overline{x}_\mu + O(|x|^3)$) and such that $a_0^\star = e_1^\star(x_0)$. Notice that in this frame at x_0, $c_{\lambda\mu ij} = \overline{c_{\mu\lambda ji}}$, the connection $\nabla_{E^\star}$ is equal to d and the curvature $\Theta(E^\star, h)$ to $i\sum_{\lambda\mu jk} c_{\lambda\mu jk}dx_\lambda \wedge d\overline{x}_\mu(e_k^\star)^\star \otimes e_j^\star$. Recall the Euler sequence :

$$0 \to \mathcal{O}_E(-1) \xrightarrow{\iota} \pi^\star E^\star \xrightarrow{p} T \to 0.$$

On an appropriate open set around $(x, a_0^\star)$, the map $q : E^\star - X \times \{0\} \to \mathbb{P}(E)$ is given in coordinates by

$$q(x, \sum a_i e_i^\star) = (x, [a_1 : \cdots : a_r]) = (x, \frac{a_2}{a_1}, \cdots, \frac{a_r}{a_1}).$$

Hence the map p is given by

$$p\left(x, [\sum_1^r a_i e_i^\star], \sum_1^r b_j e_j^\star\right) = \left(x, [\sum a_i e_i^\star], \sum_{j\geq 2} \frac{b_j a_1 - b_1 a_j}{a_1^2}\frac{\partial}{\partial z_j}\right) \otimes (\sum a_i e_i^\star).$$

where for $2 \leq j \leq r$, $z_j := \frac{a_j}{a_1}$. Here and in the sequel we will also write $z_1 := \frac{a_1}{a_1} = 1$ and $dz_1 = 0$ for convenience. The adjoint of the map p is given by

$$p^\star\left(\frac{\partial}{\partial z_j} \otimes \frac{\sum a_i e_i^\star}{a_1}\right) = e_j^\star - \frac{\langle e_j^\star, \sum a_i e_i^\star\rangle}{\|\sum a_i e_i^\star\|^2}\sum a_i e_i^\star.$$

So, at the point $(x_0, [a_0^\star])$, the normality of the frame gives

$$\begin{aligned}\iota^\star(\nabla''p^\star)\frac{\partial}{\partial z_k} \otimes \frac{\sum a_i e_i^\star}{a_1} &= \iota^\star\nabla''\left(p^\star\frac{\partial}{\partial z_k} \otimes \frac{\sum a_i e_i^\star}{a_1}\right) - \iota^\star p^\star\nabla''\left(\frac{\partial}{\partial z_k} \otimes \frac{\sum a_i e_i^\star}{a_1}\right)\\ &= \iota^\star\nabla''\left(p^\star\frac{\partial}{\partial z_k} \otimes \frac{\sum a_i e_i^\star}{a_1}\right) = -d\overline{z}_k \otimes \frac{\sum a_i e_i^\star}{a_1}.\end{aligned}$$

Now,

$$p(\nabla'\iota)\left(\frac{\sum a_i e_i^\star}{a_1}\right) = p\nabla'\left(\iota\frac{\sum a_i e_i^\star}{a_1}\right) = \sum_2^r dz_j\frac{\partial}{\partial z_j} \otimes \frac{\sum a_i e_i^\star}{a_1}.$$

Summing up the previous computations we infer the formula for the curvature of $T = T_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(-1)$ at $(x_0, [a_0^\star])$

$$\Theta(T,h) = \sum_{2\leq j,k\leq r} (c_{jk} + \frac{i}{2\pi} dz_j \wedge d\overline{z}_k) \left(\frac{\partial'}{\partial z_k}\right)^\star \otimes \left(\frac{\partial'}{\partial z_j}\right) \tag{2.1}$$

where $\Theta(E^\star, h) = \sum_{1\leq j,k\leq r} c_{jk}(e_k^\star)^\star \otimes e_j^\star$, $c_{jk} := i\sum_{\lambda\mu} c_{\lambda\mu jk} dx_\lambda \wedge d\overline{x}_\mu$ $(c_{jk} = \overline{c_{kj}})$ and $\left(\frac{\partial'}{\partial z_j}\right)$ is $\left(\frac{\partial}{\partial z_j}\right) \otimes \frac{\sum a_i e_i^\star}{a_1}$.

For later use, we give the formula for the curvature of $(\mathcal{O}_E(1), h)$. Denote by Ω the positive form defined on the whole fiber of π over x_0 as the Fubini-Study metric of $(\mathbb{P}(E_{x_0}), h)$: in coordinates

$$\Omega := \frac{i}{2\pi} \frac{(1+|z|^2)\sum_{j\geq 2} dz_j \wedge d\overline{z}_j - \sum_{i,j\geq 2} \overline{z}_i z_j dz_i \wedge d\overline{z}_j}{(1+|z|^2)^2}.$$

Then,

$$\alpha := \Theta(\mathcal{O}_E(1), h) = \frac{i}{2\pi} d'd'' \log \|e_1^\star + \sum_{j\geq 2} z_j e_j^\star\|^2 = \Omega - \frac{\langle \pi^\star \Theta(E^\star, h) a^\star, a^\star\rangle}{\|a^\star\|^2}.$$

This equality is valid on the whole $\pi^{-1}(x_0)$. At the point $(x_0, [a_0^\star])$, it reduces to $\alpha = \frac{i}{2\pi}\sum_{j\geq 2} dz_j \wedge d\overline{z}_j - c_{11}$.

3. Computations in coordinates

We will compute the Bott-Chern forms of the metric Euler sequence at a point $(x_0, [a_0^\star])$ of $\mathbb{P}(E)$. The results will be given in terms of locally defined forms.

By the previous results (1.2), (1.3) and (2.1),

$$\Phi_{d+1}(u) = \det{}_d\Big(\Big| c_{jk} + (1-u)\frac{i}{2\pi} dz_j \wedge d\overline{z}_k \Big|_{2\leq j,k\leq r}\Big)$$

and

$$\widetilde{c_{d+1}} = \int_0^1 \frac{\Phi_{d+1}(u) - \Phi_{d+1}(0)}{u} du.$$

Our aim is to give an explicit coordinate free expression for the Bott-Chern forms $\widetilde{c_{d+1}}$.

For $d = 0$, $\Phi_1(u) = 1$ and $\widetilde{c_1} = 0$.
For $d = 1$,

$$\begin{aligned} \Phi_2(u) &= Trace\Big(\Big| c_{jk} + (1-u)\frac{i}{2\pi} dz_j \wedge d\overline{z}_k \Big|_{2\leq j,k\leq r}\Big) \\ &= c_1(\pi^\star E^\star) - c_{11} + (1-u)\Omega \end{aligned}$$

and $\widetilde{c_2} = -\Omega$.

We need some notations to go along the next computations. The set of positive integers will be denoted by $\mathbb{N}_\star$. For any positive integer r, we denote by $\mathbb{N}_r := \{1, 2, \cdots, r\}$ and Σ_r the group of permutations of $\mathbb{N}_r$. The notation $(a_1, a_2, \cdots, a_p)$ will be used for the cycle of length p, $a_1 \mapsto a_2 \mapsto \cdots \mapsto a_p \mapsto a_1$.

For a finite sequence B in $\mathbb{N}_\star^{(\mathbb{N}_\star)}$ of positive integers, we set $l(B) = c$ such that B belongs to $\mathbb{N}_\star^c$ (*its length*) and $|B| = \sum_{i=1}^c b_i$ (*its weight*).

A finite sequence P in $\mathbb{N}_\star^{(\mathbb{N}_\star)}$ of positive integers will be called a *partition* if it is non-decreasing. Its elements will then be called the parts. If $[1+\sum_{j=1}^i h_j, \sum_{j=1}^{i+1} h_j]$,

$(0 \leq i \leq q-1, |h(P)| = \sum_{j=1}^{q} h_j = l(P))$ are the biggest intervals where the parts are constant i.e. $p_1 = p_2 = \cdots = p_{h_1} < p_{h_1+1} = \cdots = p_{h_1+h_2} < p_{h_1+h_2+1} \cdots\cdots p_{h_1+h_2+\cdots h_{q-1}} < p_{h_1+h_2+\cdots h_{q-1}+1} = \cdots = p_{h_1+h_2+\cdots h_{q-1}+h_q} = p_{l(P)}$ then the sequence $h(P) = (h_i)_{1\leq i\leq q}$ will be called *the height* of the partition B. The symbol $h(B)!$ will denote $\prod_{i=1}^{q} h_i!$.

For a partition P in $\mathbb{N}_\star^{(\mathbb{N}_\star)}$ of positive integers, a permutation σ in Σ_r will be said to be of *type* P if $r = |P|$ and σ can be written as the product of $l(P)$ cycles C_i (with disjoint support) where C_i is of length p_i.

We define

$$c_i'(E^\star) \quad := \quad \det{}_i(|c_{jk}|_{2\leq j,k\leq r}).$$

For a cycle of length p, we will need ($p \in \mathbb{N}_\star, s \in \mathbb{N}_\star, s \leq p$, $Q \subset \mathbb{N}_p$)

$$\Omega_{p,s} := \frac{1}{p} \sum_{\substack{Q\subset\mathbb{N}_p \\ \sharp Q = s}} \Omega_{p,Q} \text{ with } \Omega_{p,Q} := \sum_{2\leq i_1, i_2, \cdots, i_p \leq r} \bigwedge_{a=1}^{p} C_{a,a+1}^{I,Q}$$

$$\text{where} \qquad \begin{cases} C_{a,a+1}^{I,Q} & = & \left(\frac{i}{2\pi}\right) dz_{i_a} \wedge d\overline{z}_{i_{a+1}} \text{ if } a \in Q \\ & = & c_{i_a i_{a+1}} \text{ otherwise} \end{cases}$$

where we have set $i_{p+1} := i_1$. The set Q denotes the locations of the $dz \wedge d\overline{z}$ terms. Hence, s is the total number of $dz \wedge d\overline{z}$ terms occurring in the cycle. Note that

$$\Omega_{p,1} = \frac{i}{2\pi} \sum_{2\leq i_1, i_2, \cdots, i_p \leq r} c_{i_1 i_2} c_{i_2 i_3} \cdots c_{i_{p-1} i_p} dz_{i_p} \wedge d\overline{z}_{i_1}.$$

We begin by computing the leading coefficient of our coming formula for Φ.

Lemma 1. *For fixed $d \in \mathbb{N}_\star$,*

$$\sum_{\substack{S,P\in\mathbb{N}_\star^{(\mathbb{N}_\star)} \\ S\leq P, |P|\leq d, |S|=d}} \frac{(-1)^{|P|+l(P)}}{l(P)!} \bigwedge_{i=1}^{l(P)} \Omega_{p_i,s_i} = \Omega^d.$$

Proof. First notice that $\Omega_{p,p} = \frac{1}{p}\Omega_{p,\mathbb{N}_p} = \frac{1}{p}(-1)^{p-1}\Omega^p$. Now,

$$\begin{aligned} \sum_{\substack{P,S\in\mathbb{N}_\star^{(\mathbb{N}_\star)} \\ S\leq P, |P|\leq d, |S|=d}} \frac{(-1)^{|P|+l(P)}}{l(P)!} \bigwedge_{i=1}^{l(P)} \Omega_{p_i,s_i} &= \sum_{\substack{P,S\in\mathbb{N}_\star^{(\mathbb{N}_\star)} \\ S=P, |P|=d}} \frac{(-1)^{|P|+l(P)}}{l(P)!} \bigwedge_{i=1}^{l(P)} \Omega_{p_i,p_i} \\ &= \sum_{\substack{P\in\mathbb{N}_\star^{(\mathbb{N}_\star)} \\ |P|=d}} \frac{1}{l(P)! \prod_{i=1}^{l(P)} p_i} \Omega^d = \Omega^d. \end{aligned}$$

The last equality is proven noticing that the map

$$\begin{array}{ccc} \Sigma_p & \to & \Sigma_p \\ \varphi & \mapsto & \big(\varphi(1), \varphi(2), \cdots, \varphi(p_1)\big)\big(\varphi(p_1+1), \varphi(p_1+2), \cdots, \varphi(p_2)\big)\cdots \end{array}$$

is surjective on the set of permutations of type P and is $\prod_{i=1}^{l(P)} p_i$-to one. Each permutation of type P is obtained by $l(P)!$ maps of this kind. □

We can now compute the function Φ involved in the expression for Bott-Chern forms. The first part of the proof of our main theorem will be done in three steps. In the following proposition, we first separate fiber differentials dz and base differentials arising in the curvature of $E^\star$

Proposition 1.

$$\Phi_{d+1}(u) = c'_d(E^\star) + \sum_{1\le s\le d}(1-u)^s \sum_{\substack{P,S\in\mathbb{N}_\star^{(\mathbb{N}_\star)}\\ l(P)=l(S),P\ge S\\ |P|\le d,|S|=s,(l(P)\le s)}} \frac{(-1)^{|P|+l(P)}}{l(P)!}c'_{d-|P|}(E^\star)\bigwedge_{m=1}^{l(P)}\Omega_{p_m,s_m}.$$

Proof. First remark that

$$\Phi_{d+1}(u) \;=\; \frac{1}{d!}\sum_{2\le i_1,i_2,\cdots,i_d\le r}\sum_{\sigma\in\Sigma_d}\varepsilon(\sigma)\bigwedge_{m=1}^{d}(c_{i_m i_{\sigma(m)}} + (1-u)\frac{i}{2\pi}dz_{i_m}\wedge d\overline{z}_{i_{\sigma(m)}}).$$

We first invert the summation over $\sigma\in\Sigma_d$ and the summation resulting from the development of the above wedge product. We then have to specify which $dz\wedge d\overline{z}$ terms we consider in the development. The part $c'_d(E^\star)$ is got when no $dz\wedge d\overline{z}$ terms are chosen. According to these, we write σ as product of cycles with disjoint supports, neglecting the precise expression of those cycles containing no $dz\wedge d\overline{z}$ terms. If for example we seek for the terms containing only dz_{i_1} and dz_{i_2} among the dz, we will only write explicitly the cycles in σ containing 1 and 2. The permutation σ is given by $P\in\mathbb{N}_\star^{(\mathbb{N}_\star)}$ with $|P|\le d$ and $\varphi:\coprod_{m=1}^{l(P)}\mathbb{N}_{p_m}^{(m)}\to\mathbb{N}_d$ injective and σ' permutation of $\mathbb{N}_d - im\varphi$ by

$$\sigma = \sigma'\prod_{m=1}^{l(P)}\left(\varphi(1^{(m)}),\varphi(2^{(m)}),\cdots,\varphi(p_m^{(m)})\right).$$

Notice that each σ is obtained $l(P)!\prod_{m=1}^{l(P)}p_m$ times. Hence,

$$\sum_{\sigma\in\Sigma_d} = \sum_P\frac{1}{l(P)!\prod_{m=1}^{l(P)}p_m}\sum_\varphi\sum_{\sigma'}.$$

Each $Q=\coprod_{m=1}^{l(P)}Q^{(m)}$, $Q^{(m)}\subset\mathbb{N}_{p_m}$ gives the summand ($\varepsilon(\sigma)=\varepsilon(\sigma')(-1)^{|P|+l(P)}$)

$$(-1)^{|P|+l(P)}\bigwedge_{\substack{\tau=1\\ \tau\notin im(\varphi)}}^{d}\varepsilon(\sigma')c_{i_\tau i_{\sigma'(\tau)}}\bigwedge_{m=1}^{l(P)}\bigwedge_{t=1}^{p_m}C^{I_\varphi,Q^{(m)}}_{t^{(m)},(t+1)^{(m)}}.$$

After commuting with $\sum_P\sum_\varphi\sum_Q$, rewrite $\displaystyle\sum_{2\le i_1,i_2,\cdots,i_d\le r}$ as $\displaystyle\sum_{\substack{2\le i_\tau\le r\\ \tau\notin im(\varphi)}}\sum_{m=1}^{l(P)}\sum_{\substack{2\le i_{\varphi(1^{(m)})}\le r\\ 2\le i_{\varphi(2^{(m)})}\le r\\ \vdots\\ 2\le i_{\varphi(p_m^{(m)})}\le r}}$.

We get the expression for $\Phi_{d+1}(u)$ as $c'_d(E^\star)$ plus

$$\frac{1}{d!}\sum_P\sum_\varphi\sum_Q\frac{(-1)^{|P|+l(P)}}{l(P)!\prod_{m=1}^{l(P)}p_m}(d-|P|)!c'_{d-|P|}(E^\star)\bigwedge_{m=1}^{l(P)}(1-u)^{\sharp Q^{(m)}}\Omega_{p_m,Q^{(m)}}.$$

The summand now does not depend on φ. Hence the summation over all injective $\varphi:\coprod_{m=1}^{l(P)}\mathbb{N}_{p_m}^{(m)}\to\mathbb{N}_d$ gives the factor $\dfrac{d!}{(d-|P|)!}$. We infer

$$\Phi_{d+1}(u) \;=\; c'_d(E^\star) + \sum_P\sum_Q\frac{(-1)^{|P|+l(P)}}{l(P)!\prod_{m=1}^{l(P)}p_m}c'_{d-|P|}(E^\star)\bigwedge_{m=1}^{l(P)}(1-u)^{\sharp Q^{(m)}}\Omega_{p_m,Q^{(m)}}. \tag{3.1}$$

Summing now over all Q with same size S gives

$$\begin{aligned}\Phi_{d+1}(u) &= c'_d(E^\star) + \sum_P \sum_S \frac{(-1)^{|P|+l(P)}}{l(P)!\prod_{m=1}^{l(P)} p_m} c'_{d-|P|}(E^\star) \bigwedge_{m=1}^{l(P)} (1-u)^{s_m} p_m \Omega_{p_m,s_m} \\ &= c'_d(E^\star) + \sum_P \sum_S \frac{(-1)^{|P|+l(P)}}{l(P)!} c'_{d-|P|}(E^\star)(1-u)^{|S|} \bigwedge_{m=1}^{l(P)} \Omega_{p_m,s_m}.\end{aligned}$$

□

We now simplify the expression of fiber differentials. Remark that when two $dz \wedge d\overline{z}$ terms are neighbors ($q_{t+1} = q_t + 1$),

$$\Omega_{p,\{q_1,q_2,\cdots,q_t,q_t+1,q_{t+2},\cdots,q_s\}} = -\Omega\Omega_{p-1,\{q_1,q_2,\cdots,q_{t-1},q_t,q_{t+2}-1,\cdots,q_s-1\}}$$

This remark will be improved in the following proposition.

Proposition 2. *For all $(p,s) \in \mathbb{N}^2$ with $s \leq p$,*

$$\Omega_{p,s} = \sum_{\substack{1\leq b_1\leq b_2\leq\cdots\leq b_s\leq p\\ |B|=p}} \frac{(-1)^{s-1}(s-1)!}{h(B)!}\Omega_{b_1,1}\Omega_{b_2,1}\cdots\Omega_{b_s,1}.$$

Proof. Recall that $\Omega_{p,s} := \frac{1}{p}\sum_{\substack{Q\subset\mathbb{N}_p\\ \sharp Q=s}} \sum_{2\leq i_1,i_2,\cdots,i_p\leq r} \bigwedge_{a=1}^p C_{a,a+1}^{I,Q}$. Gathering terms in the sum in the following way

$$\begin{aligned}&\big(d\overline{z}_{i_{q_1+1}} c_{i_{q_1+1}i_{q_1+2}} \quad \cdots \quad c_{i_{q_2-1}i_{q_2}} dz_{i_{q_2}}\big)\cdots\big(d\overline{z}_{i_{q_{s-1}+1}} c_{i_{q_{s-1}+1}i_{q_{s-1}+2}}\cdots c_{i_{q_s-1}i_{q_s}} dz_{i_{q_s}}\big)\\ &\qquad\qquad \big(-d\overline{z}_{i_{q_s+1}} c_{i_{q_s+1}i_{q_s+2}}\cdots c_{i_{p-1}i_p} c_{i_p i_1} c_{i_1 i_2}\cdots c_{i_{q_1-1}i_{q_1}} dz_{i_{q_1}}\big)\end{aligned}$$

we infer that each Q contributes to $\big(-\Omega_{q_2-q_1,1}\big)\big(-\Omega_{q_3-q_2,1}\big)\cdots\big(-\Omega_{q_s-q_{s-1},1}\big)\big(+\Omega_{p-q_s+q_1,1}\big)$. Set $b'_j := q_{j+1} - q_j$ for $1 \leq j \leq s-1$ and $b'_s := p - q_s + q_1$, we get a map ($\mathcal{P}_s(\mathbb{N}_p)$ denotes the set of subsets of $\mathbb{N}_p$ of cardinal s)

$$\begin{array}{ccccc}\mathcal{P}_s(\mathbb{N}_p) & \xrightarrow{\alpha} & (\mathbb{N}_\star)^s & \xrightarrow{\beta} & \{\text{ partitions of weight } p \text{ and length } s\}\\ Q & \mapsto & (b'_1, b'_2, \cdots, b'_s) & \mapsto & b_1 \leq b_2 \leq \cdots \leq b_s\end{array}$$

The fiber $\alpha^{-1}(B')$ has b'_s elements and the fiber $\beta^{-1}(B)$ $\frac{s!}{h(B)!}$. But

$$\sum_{B'\in\beta^{-1}(B)} b'_s = \frac{1}{s}\sum_{B'\in\beta^{-1}(B)} b'_1 + b'_2 + \cdots + b'_s = \frac{p}{s}\frac{s!}{h(B)!}.$$

Hence the composed map is of degree $\dfrac{p(s-1)!}{h(B)!}$ over B. □

We are now about the final step

Proposition 3. *For all $d \in \mathbb{N}$,*

$$\Phi_{d+1}(u) = c'_d(E^\star) + \sum_{1\leq s\leq p\leq d} \sum_{\substack{1\leq b_1\leq b_2\cdots\leq b_s\leq p\\ |B|=p}} (1-u)^s \frac{(-1)^{p+s}s!}{h(B)!} c'_{d-p}(E^\star)\Omega_{b_1,1}\Omega_{b_2,1}\cdots\Omega_{b_s,1}$$

Proof. Fix $s \leq p$ in $\mathbb{N}_\star$. The map which assigns to the product Q of $Q^{(m)} \in \mathcal{P}_{s_m}(\mathbb{N}_{p_m})$ the associated product of partitions $B^{(m)}$ is $\displaystyle\prod_{m=1}^{l(P)} \frac{p_m(s_m-1)!}{h(B^{(m)})!}$ to 1. The map which assigns to the latter product of partitions a partition B of weight p and of length s by concatenation and reordering is $\displaystyle\sum_{\substack{S\in(\mathbb{N}_\star)^{(\mathbb{N}_\star)}\\ |S|=s}} \frac{s!\prod_{m=1}^{l(S)} h(B^{(m)})!}{h(B)!\prod_{m=1}^{l(S)} s_m!}$ to 1. Hence,

the composed map is $\displaystyle\sum_{\substack{S\in(\mathbb{N}_\star)^{(\mathbb{N}_\star)}\\|S|=s}} \frac{s!\prod_{m=1}^{l(S)} p_m}{h(B)!\prod_{m=1}^{l(S)} s_m}$ to 1. Back to the formula (3.1), we obtain a new expression for $\Phi_{d+1}(u)$.

$$\begin{aligned}\Phi_{d+1}(u) &= c'_d(E^\star)+\sum_P\sum_Q \frac{(-1)^{|P|+l(P)}}{l(P)!\prod_{m=1}^{l(P)} p_m} c'_{d-|P|}(E^\star)\bigwedge_{m=1}^{l(P)}(1-u)^{\sharp Q^{(m)}}\Omega_{p_m,Q^{(m)}}\\ &= c'_d(E^\star)+\sum_{1\le s\le p\le d}\ \sum_{\substack{1\le b_1\le b_2\cdots\le b_s\le p\\|B|=p}}\ \sum_{\substack{S\in(\mathbb{N}_\star)^{(\mathbb{N}_\star)}\\|S|=s}} c'_{d-p}(E^\star)(1-u)^s\\ &\qquad \frac{s!\prod_{m=1}^{l(S)} p_m}{h(B)!\prod_{m=1}^{l(S)} s_m}\frac{(-1)^{p+l(S)}}{l(S)!\prod_{m=1}^{l(S)} p_m}(-1)^{s+l(S)}\Omega_{b_1,1}\Omega_{b_2,1}\cdots\Omega_{b_s,1}.\end{aligned}$$

We recall once more the identity $\displaystyle\sum_{\substack{S\in(\mathbb{N}_\star)^{(\mathbb{N}_\star)}\\|S|=s}} \frac{1}{l(S)!\prod_{m=1}^{l(S)} s_m}=1$ to end the proof. □

Hence, we get the Bott-Chern forms by integrating.

Theorem 1.

$$\widetilde{c_{d+1}} = -\sum_{1\le s\le p\le d}\ \sum_{\substack{1\le b_1\le b_2\cdots\le b_s\le p\\|B|=p}} \mathcal{H}_s\frac{(-1)^{p+s}s!}{h(B)!}c'_{d-p}(E^\star)\Omega_{b_1,1}\Omega_{b_2,1}\cdots\Omega_{b_s,1}.$$

Here $\mathcal{H}_s$ is the harmonic number $\displaystyle\sum_1^s\frac{1}{i}=\int_0^1\frac{1-(1-u)^s}{u}$. This formula reduces in the case of flat vector bundle to that of Gillet and Soulé (proposition 5.3 of [G-S-2]). Coordinates free expressions for $c'_d(E^\star)$ and $\Omega_{b,1}$ will be given in proposition 4 and 5. Notice that the relative degree of each summand is $2s$ and the degree in base variables is $2(d-p+|B|-s)=2(d-s)$. We can therefore restrict the range of s to $\max(d-dimX,1)\le s\le\min(r-1,d)$.

4. Coordinates free expressions

We now give coordinates free expressions of $c'_i(E^\star)$ and of the $\Omega_{p,1}$. We will express the results in terms of the following forms in $A^{q,q}(\mathbb{P}(E),\mathbb{C})$: for $q\in\mathbb{N}$, at the point $(x,[a^\star])\in\mathbb{P}(E)$,

$$\Theta^q a^\star := \frac{\langle\pi^\star\Theta(E^\star,h)^q a^\star, a^\star\rangle}{\|a^\star\|^2}.$$

Recall that $\Theta(E^\star,h)$ is in $A^{1,1}(X,End(E))$. The q-th power is taken with the wedge product in the form part and the composition in the endomorphism part. We will also need the form $\alpha=\Theta(\mathcal{O}_E(1),h)$.

Proposition 4.

$$c'_d(E^\star)=\sum_{\substack{p+m=d\\p,m\ge0}}(-1)^p(\Theta^p a^\star)\pi^\star c_m(E^\star,h)$$

where we have set $(\Theta^0 a^\star)=\pi^\star c_0(E^\star)=1$.

Proof. The expression for $\Theta^q a^\star$ in coordinates is

$$\frac{\sum_{J\in\{1,\cdots,r\}^{q+2}} \bigwedge_{m=1}^{q} c_{j_m j_{m+1}} a_{j_{q+1}} < e^\star_{j_1}, e^\star_{j_{q+2}} > \overline{a_{j_{q+2}}}}{\sum_{(j,j')\in\{1,\cdots,r\}^2} a_j < e^\star_j, e^\star_{j'} > \overline{a_{j'}}}$$

and is $\sum_{\substack{J\in\{1,\cdots,r\}^{q+1}\\ j_1=j_{q+1}=1}} \prod_{a=1}^{q} c_{j_a j_{a+1}}$ at the center x_0 of the normal frame for (E,h). Hence,

$$\begin{aligned} c'_1(E^\star) &= \pi^\star c_1(E^\star,h) - (\Theta^1 a^\star) \\ c'_2(E^\star) &= \pi^\star c_2(E^\star,h) - \left[c_{11} c'_1(E^\star) - \sum_{j\geq 2} c_{1j} c_{j1} \right] \\ &= \pi^\star c_2(E^\star,h) - (\Theta^1 a^\star)\pi^\star c_1(E^\star,h) + (\Theta^2 a^\star). \end{aligned}$$

In the same spirit of the proof of proposition 1, we write the permutations according to one of the positions of 1 (if any) : $\sigma = (\varphi(1),\varphi(2),\cdots,\varphi(p))\sigma'$ if $i_{\varphi(1)} = 1$.

$$\begin{aligned} &c_d(E^\star,h) = c'_d(E^\star) \\ &\quad + \frac{1}{d!} \sum_{\substack{1\leq i_1,i_2,\cdots,i_d\leq r\\ 1\in\{i_1,i_2,\cdots,i_d\}}} \sum_{p=1}^{d} \sum_{\substack{\varphi:\mathbb{N}_p\to\mathbb{N}_d\\ \text{injective}\\ i_{\varphi(1)}=1}} \sum_{\sigma'\in\Sigma_{d-p}} (-1)^{p+1} c_{1 i_{\varphi(2)}} c_{i_{\varphi(2)} i_{\varphi(3)}} \cdots c_{i_{\varphi(p)} 1} \bigwedge_{\tau\notin im\varphi} c_{i_\tau, i_{\sigma'(\tau)}}. \end{aligned}$$

Rewriting $\sum_{\substack{1\leq i_1,i_2,\cdots,i_d\leq r\\ 1\in\{i_1,i_2,\cdots,i_d\}}}$ as $\sum_{1\leq i_{\varphi(2)},\cdots,i_{\varphi(p)}\leq r} \sum_{\substack{1\leq i_\tau\leq r\\ \tau\notin im\varphi}}$, then replacing the sum $\sum_{\substack{\varphi:\mathbb{N}_p\to\mathbb{N}_d \text{injective},\\ i_{\varphi(1)}=1}}$ by product by factor $\dfrac{d!}{d-p)!}$ we infer

$$c_d(E^\star,h) = c'_d(E^\star) + \sum_{p=1}^{d} (-1)^{p+1} (\Theta^p a^\star) \pi^\star c_{d-p}(E^\star,h).$$

□

Before computing the $\Omega_{p,1}$, we need a lemma which relies on special properties of the normal frame.

Lemma 2. *In the center of a normal frame for an holomorphic Hermitian vector bundle (E,h), $d\Theta_E = 0$ and $id'd''\Theta_E = 0$.*

Proof. Call H the metric matrix and A the connection 1-form for the Chern connection of (E,h) in a frame normal at x_0. By the compatibility of the connection with the metric and the holomorphic structure, we infer that $d'H = AH$. Then $A(x_0) = 0$. On the whole chart, $\Theta_E = \frac{i}{2\pi}(dA + A\wedge A)$. So, $d\Theta_E = \Theta_E\wedge A - A\wedge\Theta_E$ (Bianchi identity). Hence, $d\Theta_E(x_0) = 0$. Now, at x_0

$$\begin{aligned} d'd''\Theta_E(x_0) = d'd\Theta_E(x_0) &= d'\Theta_E\wedge A + \Theta_E\wedge d'A - d'A\wedge\Theta_E + A\wedge d'\Theta_E \\ &= \Theta_E\wedge d'A - d'A\wedge\Theta_E \end{aligned}$$

From, $d'H = AH$ and $A(x_0) = 0$ we get $d'A(x_0) = 0$ (Notice that $\Theta_E(x_0) = dA(x_0)$ is of type $(1,1)$). □

Proposition 5. *For $d \geq 1$, at the point $(x_0, [a_0^\star])$,*

$$\begin{aligned}\Omega_{d+1,1} &= \frac{i}{2\pi} d'd''(\Theta^d a^\star) + (\Theta^{d+1} a^\star) + \alpha(\Theta^d a^\star) \\ &+ \frac{i}{2\pi} \sum_{m=2}^{d} \sum_{\substack{B \in \mathbb{N}_\star^{(\mathbb{N}_\star)} \\ l(B)=m, |B|=d}} (-1)^m (d''\Theta^{b_1} a^\star)(\Theta^{b_2} a^\star)(\Theta^{b_3} a^\star) \cdots (\Theta^{b_{m-1}} a^\star)(d'\Theta^{b_m} a^\star)\end{aligned}$$

Proof. From the definition of the normal frame we get at x_0, $d\|a^\star\|^2 = 0$. Hence, at $(x_0, [a_0^\star])$,

$$\frac{i}{2\pi} d'd''(\Theta^d a^\star) = \frac{i}{2\pi} d'd''\langle \pi^\star \Theta(E^\star, h)^d a^\star, a^\star\rangle - (\Theta^d a^\star)\alpha.$$

From lemma 2 and the relation $\frac{i}{2\pi} d'd''\langle e_i^\star, e_j^\star\rangle = -c_{ji}$ we infer using the expression of $\langle \pi^\star \Theta(E^\star, h)^d a^\star, a^\star\rangle$ in coordinates

$$\begin{aligned}&\frac{i}{2\pi} d'd''\langle \pi^\star \Theta(E^\star, h)^d a^\star, a^\star\rangle \\ &= \frac{i}{2\pi} \sum_{J \in \{1,\cdots,r\}^{d+1}} \bigwedge_{m=1}^{d} c_{j_m j_{m+1}} dz_{j_{d+1}} \wedge d\overline{z_{j_1}} + \sum_{\substack{J \in \{1,\cdots,r\}^{d+2} \\ j_{d+1}=j_{d+2}=1}} \bigwedge_{m=1}^{d} c_{j_m j_{m+1}} (-c_{j_{d+2} j_1}) \\ &= \frac{i}{2\pi} \sum_{J \in \{1,\cdots,r\}^{d+1}} \bigwedge_{m=1}^{d} c_{j_m j_{m+1}} dz_{j_{d+1}} \wedge d\overline{z_{j_1}} - (\Theta^{d+1} a^\star).\end{aligned}$$

Now, to recover $\Omega_{d+1,1}$ we have to remove the value 1 for the j's in the first sum. Gathering the terms according to the fact that they contain at least l $j's$ achieving the value 1 ($j_{d+1} \neq 1, j_1 \neq 1$), we get (notice that we will interchange $dz_{j_{d+1}}$ and $d\overline{z_{j_1}}$)

$$\begin{aligned}&\frac{i}{2\pi} \sum_{J \in \{1,\cdots,r\}^{d+1}} \left(\bigwedge_{m=1}^{d} c_{j_m j_{m+1}} \right) dz_{j_{d+1}} \wedge d\overline{z_{j_1}} \\ &= \Omega_{d+1,1} + \frac{i}{2\pi} \sum_{l=1}^{d-1} (-1)^l \sum_{\substack{1<q_1<q_2< \\ \cdots<q_l<d+1}} \sum_{J-Q \in \{1,\cdots,r\}^{d+1-l}} (d\overline{z}_{j_1} c_{j_1 j_2} c_{j_2 j_3} \cdots c_{j_{q_1-1} 1}) \\ &\quad (c_{1 j_{q_1+1}} \cdots c_{j_{q_2-1} 1})(c_{1 j_{q_2+1}} \cdots c_{j_{q_3-1} 1}) \cdots (c_{1 j_{q_{l-1}+1}} \cdots c_{j_{q_l-1} 1})(c_{1 j_{q_l+1}} \cdots c_{j_d j_{d+1}} dz_{j_{d+1}}) \\ &= \Omega_{d+1,1} + \frac{i}{2\pi} \sum_{l=1}^{d-1} (-1)^l \Big(\sum_{1 \leq j_1, j_2, \cdots, j_{q_1-1} \leq r} d\overline{z}_{j_1} c_{j_1 j_2} c_{j_2 j_3} \cdots c_{j_{q_1-1} 1} \Big) \\ &\quad (\Theta^{q_2-q_1} a^\star)(\Theta^{q_3-q_2} a^\star) \cdots (\Theta^{q_l-q_{l-1}} a^\star) \Big(\sum_{1 \leq j_{q_l+1}, \cdots, j_{d+1} \leq r} c_{1 j_{q_l+1}} \cdots c_{j_d j_{d+1}} dz_{j_{d+1}} \Big)\end{aligned}$$

Using once more the normality of the frame, we compute at the point $(x_0, [a_0])$

$$\begin{aligned}\sum_{1 \leq j_1, j_2, \cdots, j_{q_1-1} \leq r} d\overline{z}_{j_1} c_{j_1 j_2} c_{j_2 j_3} \cdots c_{j_{q_1-1} 1} &= d''(\Theta^{q_1-1} a^\star) \\ \sum_{1 \leq j_{q_l+1}, \cdots, j_{d+1} \leq r} c_{1 j_{q_l+1}} \cdots c_{j_d j_{d+1}} dz_{j_{d+1}} &= d'(\Theta^{d+1-q_l} a^\star).\end{aligned}$$

Now, the proof will be complete if we set $b_1 := q_1 - 1 \in \mathbb{N}_\star$, $b_j := q_j - q_{j-1}$ for j between 2 and l and $b_{l+1} := d + 1 - q_l \in \mathbb{N}_\star$ and $m = l + 1$. □

5. On the Bott-Chern forms generating polynomial

Bott-Chern forms fulfill a recursion formula which is easily expressed in terms of their generating polynomial. The latter has a concise expression.

Set

$$c_t(E^\star,h) := \sum_{d=0}^{+\infty} t^d c_d(E^\star,h)\ ;\ \widetilde{c_t}(\Sigma,h) := \sum_{d=0}^{+\infty} t^d \widetilde{c_{d+1}}(\Sigma,h)\ ;\ \Theta_t := \sum_{d=0}^{+\infty}(-t)^d(\Theta^d a^\star).$$

We will also need as auxiliary notations

$$c'_t(E^\star) := \sum_{d=0}^{+\infty} t^d c'_d(E^\star)\ ;\ \Omega_t := \sum_{d=1}^{+\infty}(-t)^d\Omega_{d,1}\ ;\ H(X) := \sum_{s=1}^{+\infty}\mathcal{H}_s X^s.$$

Theorem 1 can be rewritten as

$$t^d\widetilde{c_{d+1}} = -\sum_{1\le s\le d}\ \sum_{\substack{B\in(\mathbb{N}_\star)^s\\ e+\sum b_i=d}} \mathcal{H}_s(-1)^s t^e c'_e(E^\star)(-t)^{b_1}\Omega_{b_1,1}(-t)^{b_2}\Omega_{b_2,1}\cdots(-t)^{b_s}\Omega_{b_s,1}.$$

Summing over d, we infer

$$\widetilde{c_t}(\Sigma,h) = -\sum_{1\le s}\mathcal{H}_s(-1)^s c'_t(E^\star)(\Omega_t)^s = -H(-\Omega_t)c'_t(E^\star).$$

Proposition 4 can be rewritten as $c'_t(E^\star) = \Theta_t\pi^\star c_t(E^\star,h)$. As for proposition 5 notice that

$$\begin{aligned}\sum_{d=0}^{+\infty}(-t)^{d+1}\frac{i}{2\pi}\sum_{m=2}^{d}\sum_{\substack{B\in\mathbb{N}_\star{}^{(\mathbb{N}_\star)}\\ l(B)=m,|B|=d}}&(-1)^m(d''\Theta^{b_1}a^\star)(\Theta^{b_2}a^\star)(\Theta^{b_3}a^\star)\cdots(\Theta^{b_{m-1}}a^\star)(d'\Theta^{b_m}a^\star)\\ &= -t\frac{i}{2\pi}\Big(\sum_{b_1=1}^{+\infty}(-t)^{b_1}d''\Theta^{b_1}a^\star\Big)\Big(\sum_{b_m=1}^{+\infty}(-t)^{b_m}d'\Theta^{b_m}a^\star\Big)\sum_{m=2}^{+\infty}(1-\Theta_t)^{m-2}\\ &= -t\frac{i}{2\pi}\frac{d''\Theta_t d'\Theta_t}{\Theta_t}\end{aligned}$$

so that

$$-\Omega_t = t\left(\frac{i}{2\pi}d'd''\Theta_t - \frac{\Theta_t-1}{t} + \alpha\Theta_t + \frac{i}{2\pi}\frac{d''\Theta_t d'\Theta_t}{\Theta_t}\right).$$

We are led to our main theorem

Theorem 2. *Bott-Chern forms for the metric Euler sequence can be chosen to be*

$$\widetilde{c_t}(\Sigma,h) = -H\left(t\alpha\Theta_t + t\frac{1-\Theta_t}{t} + t\Theta_t\frac{i}{2\pi}d'd''\ln\Theta_t\right)\Theta_t\pi^\star c_t(E^\star,h).$$

6. Some special cases

6.1. **On first parts of Bott-Chern forms.** The high relative degree part of Bott-Chern forms is easy to compute with our formulas. Half of the relative degree will be indicated by an extra indices. Theorem 1 reads

$$\begin{aligned}(\widetilde{c_{d+1}})_d &= -\mathcal{H}_d\Omega^d\\ (\widetilde{c_{d+1}})_{d-1} &= -\mathcal{H}_{d-1}\left(c'_1(E^\star)\Omega^{d-1} - (d-1)\Omega_{2,1}\Omega^{d-2}\right)\\ (\widetilde{c_{d+1}})_{d-2} &= -\mathcal{H}_{d-2}\Big(c'_2(E^\star)\Omega^{d-2} - (d-2)c'_1(E^\star)\Omega_{2,1}\Omega^{d-2}\\ &\quad +(d-2)\Omega_{3,1}\Omega^{d-3} + \frac{(d-2)(d-3)}{2}(\Omega_{2,1})^2\Omega^{d-4}\Big).\end{aligned}$$

6.2. **On first Bott-Chern forms.** The propositions 4 and 5 give theorem 1 an intrinsic form. Alternatively, we may use theorem 2. Explicitly, we get up to d'-exact and d''-exact forms,

$$\begin{aligned}
\widetilde{c_1} &= 0\\
\widetilde{c_2} &= -\left[\mathcal{H}_1\alpha\right] - (\Theta^1 a^\star)\\
\widetilde{c_3} &= -\left[\mathcal{H}_2\alpha^2 + \pi^\star c_1(E^\star,h)\alpha\right] - \left[\alpha + \pi^\star c_1(E^\star,h)\right](\Theta^1 a^\star) - \left(\frac{1}{2}(\Theta^1 a^\star)^2 - (\Theta^2 a^\star)\right).
\end{aligned}$$

The terms in bracket are $d'd''$-closed.

6.3. **For curves and surfaces.** We now assume that the manifold X is a curve or a surface. Only the high relative degree part will occur in theorem 1. Now, make use of

$$\begin{aligned}
\Omega &= \alpha + (\Theta^1 a^\star)\\
\Omega_{2,1} &= \frac{i}{2\pi}d'd''(\Theta^1 a^\star) + (\Theta^2 a^\star) + \alpha(\Theta^1 a^\star)\\
\Omega_{3,1} &= \frac{i}{2\pi}d'd''(\Theta^2 a^\star) + (\Theta^3 a^\star) + \alpha(\Theta^2 a^\star) + \frac{i}{2\pi}d''(\Theta^1 a^\star)d'(\Theta^1 a^\star).
\end{aligned}$$

Hence we proved that up to d'-exact and d''-exact forms Bott-Chern forms are given on curves by

$$\widetilde{c_{d+1}} = -\left[\mathcal{H}_d\alpha^d + \mathcal{H}_{d-1}\pi^\star c_1(E^\star,h)\alpha^{d-1}\right] - \left[\alpha^{d-1}\right](\Theta^1 a^\star)$$

and on surfaces by

$$\begin{aligned}
\widetilde{c_{d+1}} = &-\left[\mathcal{H}_d\alpha^d + \mathcal{H}_{d-1}\pi^\star c_1(E^\star,h)\alpha^{d-1} + \mathcal{H}_{d-2}\pi^\star c_2(E^\star,h)\alpha^{d-2}\right]\\
&-\left[\alpha^{d-1} + \pi^\star c_1(E^\star,h)\alpha^{d-2} - (d-2)\alpha^{d-3}\frac{i}{2\pi}d'd''\Theta^1 a^\star\right](\Theta^1 a^\star)\\
&-\left[\alpha^{d-2}\right]\left(\frac{1}{2}(\Theta^1 a^\star)^2 - (\Theta^2 a^\star)\right).
\end{aligned}$$

7. On characteristic forms

7.1. **From Segre forms to Chern forms.** The geometric Segre forms are defined for $i \in \mathbb{N}$ by

$$s'_i(E,h) := \pi_\star(\alpha^{r-1+i}) \in A^{i,i}(X,\mathbb{C}).$$

Consider the metric Euler exact sequence (Σ,h) twisted by $\mathcal{O}_E(1)$

$$0 \to \mathcal{O}_{\mathbb{P}(E)} \to \pi^\star E^\star \otimes \mathcal{O}_E(1) \to T_{\mathbb{P}(E)/X} \to 0$$

and its top Bott-Chern form $\widetilde{c_r}(\Sigma(1),h)$. Notice that $c_t(\mathcal{O}_{\mathbb{P}(E)}) = 1$ for the induced metric on $\mathcal{O}_{\mathbb{P}(E)}$ is the flat one. Degree considerations lead to the relation

$$c_r(\pi^\star E^\star \otimes \mathcal{O}_E(1),h) = -\frac{i}{2\pi}d'd''\left(\widetilde{c_r}(\Sigma(1),h)\right).$$

which in cohomology reduces to Grothendieck defining relation for Chern classes of E. Now, for $m \in \mathbb{N}$ set $S_{m+1}(E,h) := \pi_\star\left(\alpha^m\widetilde{c_r}(\Sigma(1),h)\right) \in \widetilde{A^{m,m}}(X)$. Define the class $R_{m+1}(E,h) \in \widetilde{A^{m,m}}(X)$ by

$$\sum_{m=0}^{+\infty} t^m R_{m+1}(E,h) = \left(\sum_{m=0}^{+\infty}(-t)^m c_m(E,h)\right)^{-1}\left(\sum_{m=0}^{+\infty} t^m S_{m+1}(E,h)\right).$$

For $m \geq 1$, using the projection formula, we infer

$$\begin{aligned}
&\sum_{\substack{p+q=m\\ p,q\in\mathbb{N}}} c_p(E^\star,h)s'_q(E,h)\\
&= \pi_\star\left(\sum_{\substack{p+q=m\\ p,q\in\mathbb{N}}} c_p(\pi^\star E^\star,h)\alpha^{r-1+q}\right) = \pi_\star\left(\sum_{\substack{p+q=m\\ p,q\in\mathbb{Z}}} c_p(\pi^\star E^\star,h)\alpha^{r-1+q}\right)\\
&= \pi_\star\left(\alpha^{m-1}\sum_{\substack{p+s=r\\ p,s\in\mathbb{Z}}} c_p(\pi^\star E^\star,h)\alpha^{s}\right) = \pi_\star\left(\alpha^{m-1}c_r(\pi^\star E^\star\otimes\mathcal{O}_E(1),h)\right)\\
&= -\frac{i}{2\pi}d'd''\pi_\star\left(\alpha^{m-1}\widetilde{c_r}(\Sigma(1),h)\right).
\end{aligned}$$

From the previous equation, we infer that the Chern forms of (E,h) are related to the geometric Segre forms s' by

$$\left(\sum_{p=0}^{+\infty}(-t)^p c_p(E,h)\right)\left(\sum_{q=0}^{+\infty}t^q s'_q(E,h)\right) = 1-\frac{it}{2\pi}d'd''\sum_{m=0}^{+\infty}t^m S_{m+1}(E,h)$$

that is

(7.1)

$$\left(\sum_{m=0}^{+\infty}(-t)^m c_m(E,h)\right)^{-1} = 1+\sum_{m=1}^{+\infty}t^m\left(s'_m(E,h)+\frac{i}{2\pi}d'd''R_m(E,h)\right).$$

The striking fact is that despite the appearance of the non-closed forms $\Theta^q a^\star$ in the expression of $\widetilde{c_r}(\Sigma(1),h)$ the forms R are $d'd''$-closed.

Proposition 6. *The forms R are $d'd''$-closed or equivalently the Chern forms are related to the geometric Segre forms s' by*

$$\left(\sum_{m=0}^{+\infty}c_m(E^\star,h)\right)\left(\sum_{m=0}^{+\infty}t^m s'_m(E,h)\right) = 1$$

Proof. We first compute the Segre forms.

$$\begin{aligned}
s'_m(E,h) &= \pi_\star(\alpha^{r-1+m}) = \binom{r-1+m}{m}\pi_\star\left(\Omega^{r-1}(-\Theta^1 a^\star)^m\right)\\
&= \binom{r-1+m}{m}(-1)^m\int_{\mathbb{P}^{r-1}}\Omega^{r-1}\frac{(\sum c_{ij}a_j\overline{a_i})^m}{|a|^{2m}}\\
&= \binom{r-1+m}{m}(-1)^m\sum_{I,J\in(\mathbb{N}_r)^m}c_{i_1j_1}c_{i_2j_2}\cdots c_{i_mj_m}\int_{\mathbb{P}^{r-1}}\Omega^{r-1}\frac{a_{j_1}\overline{a_{i_1}}a_{j_2}\overline{a_{i_2}}\cdots a_{j_m}\overline{a_{i_m}}}{|a|^{2m}}
\end{aligned}$$

For parity reasons, the integral of non real terms vanishes. We may hence restrict to $I=J$ as sets with multiplicities. That is there exists a permutation $\sigma\in\Sigma_m$ such that for each λ, $j_\lambda = i_{\sigma(\lambda)}$. One term may be gotten from different permutations if some i_λ equals some i_μ for $\lambda\neq\mu$. As in the case of partitions, we define the height $h(I)$ of a sequence of numbers I to be the sequence of cardinals of subsets of identical entries. The term $a_{j_1}\overline{a_{i_1}}a_{j_2}\overline{a_{i_2}}\cdots a_{j_m}\overline{a_{i_m}}$ can be written as $a_{i_{\sigma(1)}}\overline{a_{i_1}}a_{i_{\sigma(2)}}\overline{a_{i_2}}\cdots a_{i_{\sigma(m)}}\overline{a_{i_m}}$ for $h(I)!$-different permutations. We will make use of the formula

$$\int_{\mathbb{P}^{r-1}}\frac{|a_1|^{2m_1}|a_2|^{2m_2}\cdots|a_r|^{2m_r}}{|a|^{2m}}\Omega^{r-1} = \frac{(r-1)!\prod m_i!}{(r-1+m)!}$$

where $m_i \in \mathbb{N}$ are such that $\sum m_i = m$. Back to our computations, we get

$$\begin{aligned} s'_m(E,h) &= \binom{r-1+m}{m}(-1)^m \sum_{I\in(\mathbb{N}_r)^m}\sum_{\sigma\in\Sigma_m} \\ &\quad c_{i_1 i_{\sigma(1)}} c_{i_2 i_{\sigma(2)}} \cdots c_{i_m i_{\sigma(m)}} \int_{\mathbb{P}^{r-1}} \frac{\Omega^{r-1}}{h(I)!}\frac{|a_{i_1}|^2|a_{i_2}|^2\cdots|a_{i_m}|^2}{|a|^{2m}} \\ &= \frac{(-1)^m}{m!}\sum_{I\in(\mathbb{N}_r)^m}\sum_{\sigma\in\Sigma_m} c_{i_1 i_{\sigma(1)}} c_{i_2 i_{\sigma(2)}} \cdots c_{i_m i_{\sigma(m)}}. \end{aligned}$$

For a finite sequence N in $\mathbb{N}^{(\mathbb{N}_\star)}$ of non negative integers, a permutation σ in Σ_m will be said to be of *shape* $1^{n_1}2^{n_2}\cdots m^{n_m}$, ($\sum pn_p = m$) if it can be written as the product of n_p cycles of length p with disjoint support. There are $\frac{m!}{\prod_p n_p! p^{n_p}}$ permutations of shape $1^{n_1}2^{n_2}\cdots m^{n_m}$ in Σ_m and each will give the same contribution in the sum after having computed $\sum_{I\in(\mathbb{N}_\star)^m}$. For $p\in\mathbb{N}$, consider the closed forms on X given by

$$\theta_p := Trace(\Theta(E,h)^{\otimes p}) = (-1)^p \sum_{I\in(\mathbb{N}_r)^p} c_{i_1 i_2} c_{i_2 i_3}\cdots c_{i_p i_1}.$$

We now consider the total Segre form.

$$\begin{aligned} \sum_{m\in\mathbb{N}} t^m s'_m(E,h) &= \sum_m t^m \frac{1}{m!}\sum_{\substack{N\in\mathbb{N}^{dimX}\\ \sum pn_p = m}} \prod_{p=1}^{dimX} \theta_p^{n_p}\frac{m!}{\prod_p n_p! p^{n_p}} \\ &= \sum_{N\in\mathbb{N}^{dimX}} \prod_{p=1}^{dimX}\left(\frac{t^p\theta_p}{p}\right)^{n_p}\frac{1}{n_p!} = \prod_{p=1}^{dimX}\exp\left(\frac{t^p\theta_p}{p}\right). \end{aligned}$$

Noticing that the signature of a permutation of shape $1^{n_1}2^{n_2}\cdots m^{n_m}$ is $(-1)^{\sum_p n_p(p+1)}$, we also get

$$\sum_{m\in\mathbb{N}} t^m c_m(E^\star,h) = \sum_{N\in\mathbb{N}^{dimX}} \prod_{p=1}^{dimX}\left(-\frac{t^p\theta_p}{p}\right)^{n_p}\frac{1}{n_p!} = \prod_{p=1}^{dimX}\exp\left(-\frac{t^p\theta_p}{p}\right).$$

□

7.2. **Computation of the class S.** We now aim at finding expressions for the class S in terms of the geometric Segre forms s'

$$s'_m(E,h) = \pi_\star(\alpha^{r-1+m}) = \binom{r-1+m}{r-1}\pi_\star\big((-\Theta^1 a^\star)^m\Omega^{r-1}\big).$$

We will need as auxiliary tools the forms on X defined by

$$s^b_c(E,h) = \pi_\star\big((-1)^b(\Theta^b a^\star)\alpha^{r-1+c}\big) = \binom{r-1+c}{r-1}\pi_\star\big((-1)^b(\Theta^b a^\star)(-\Theta^1 a^\star)^c\Omega^{r-1}\big).$$

Integral computations of $\pi_\star$ similar to the previous one lead to

$$s^b_c(E,h) = \frac{(-1)^{b+c}}{(r+c)c!}\sum_{\substack{I\in\mathbb{N}_r^{c+1}\\ K\in\mathbb{N}_r^{b-1}}}\sum_{\sigma\in\Sigma_{c+1}} c_{i_1 i_{\sigma(1)}} c_{i_2 i_{\sigma(2)}}\cdots c_{i_c i_{\sigma(c)}} c_{i_{c+1}k_2} c_{k_2 k_3}\cdots c_{k_b i_{\sigma(c+1)}}$$

The number of permutations in Σ_{c+1} of shape $1^{n_1}2^{n_2}\cdots(c+1)^{n_{c+1}}$ for which $c+1$ is in a cycle of length q is $\frac{c!}{q^{n_q-1}(n_q-1)!\prod_{p\neq q} p^{n_p} n_p!}$. The contribution of such a cycle

is $(-1)^{q+b-1}\theta_{q+b-1}$ for we plug in $\Theta^b a^\star$ instead of $\Theta^1 a^\star$. Forward,

$$s_c^b(E,h) = \frac{1}{(r+c)} \sum_{\substack{N\in\mathbb{N}^{dimX} \\ \sum pn_p=c+1}} \sum_{q/n_q\geq 1} \theta_{q+b-1}\left(\frac{\theta_q}{q}\right)^{n_q-1} \frac{1}{(n_q-1)!}\prod_{p\neq q}\left(\frac{\theta_p}{p}\right)^{n_p}\frac{1}{n_p!}$$

Changing the order of the summations over q and N and then summing over c,

$$\begin{aligned}\sum_{c\in\mathbb{N}} t^{b+c}(r+c)s_c^b(E,h) &= \sum_{q=1}^{dimX} t^{q+b-1}\theta_{q+b-1}\prod_{p=1}^{dimX}\exp\left(\frac{t^p\theta_p}{p}\right) \\ &= \left(\sum_{m\in\mathbb{N}} t^m s'_m(E,h)\right)\left(\sum_{q=b}^{dimX} t^q\theta_q\right)\end{aligned}$$

which shows in particular that the forms $s_c^b(E,h)$ are explicitly computable from the geometric Segre forms only and that they are closed.

Define $\mathcal{H}_a^b$ to be $\displaystyle\sum_{i=1}^{a}\mathcal{H}_i\binom{a}{i}\binom{b}{i}^{-1}$.

Theorem 3. *The secondary form S is given by*

$$S_{m+1}(E,h) = -\sum_{a+b+c=m}\mathcal{H}_{r-1-a-b}^{r-1+c}c_a(E^\star,h)s_c^b(E,h)$$

Proof. For degree reason, $S_m=0$ for $m>dimX+1$. It follows from the relation

$$c_p(E\otimes L,h_E\otimes h_L)=\sum_{i+j=p}\binom{r-j}{p-j}c_1(L,h_L)^i c_j(E,h_E)$$

for every holomorphic hermitian vector bundle (E,h_E) and every holomorphic hermitian line bundle (L,h_L) and from the transgression techniques of [G-S-2] (Theorem 1.2.2) that $\widetilde{c_r}(\Sigma(1),h)$ is equal to $\sum_{i+j=r}\alpha^i\widetilde{c_j}(\Sigma,h)=\sum_{i+j=r}\alpha^i\widetilde{c_j}$. Recall that for $j>r$, $\widetilde{c_j}=0$.

$$\begin{aligned}S_{m+1}(E,h) &= \pi_\star\big(\alpha^m\widetilde{c_r}\,(\Sigma(1),h)\,\big)=\sum_{i+j=r+m}\pi_\star\big(\alpha^i\widetilde{c_j}\big) \\ &= \sum_{i+c+j=r+m}\binom{i+c}{c}\pi_\star\left((\widetilde{c_j})_{r-1-i}\Omega^i(-\Theta^1a^\star)^c\right)\end{aligned}$$

where we used the relation $\alpha=\Omega-\Theta^1a^\star$ valid on the whole fiber $\pi^{-1}(x_0)$.

Terms of full relative degree are simpler to compute than the whole Bott-Chern forms. The proposition below is a weak form of theorem 1 whose full strength is hence not needed for arithmetic applications.

Proposition 7. *On the fiber of π over x_0,*

$$(\widetilde{c_{d+1}})_f\Omega^{r-1-f}=-\mathcal{H}_f\binom{r-1}{f}^{-1}\binom{r-1-d+f}{f}\sum_{\alpha+\beta=d-f}\pi^\star c_\alpha(E^\star)(-1)^\beta(\Theta^\beta a^\star)\Omega^{r-1}.$$

Proof. Starting from

$$\Phi_{d+1}(u) = \sum_{2\leq i_1<i_2<\cdots<i_d\leq r}\sum_{\sigma\in\Sigma_d}\varepsilon(\sigma)\bigwedge_{m=1}^{d}\left(c_{i_m i_{\sigma(m)}}+(1-u)\frac{i}{2\pi}dz_{i_m}\wedge d\overline{z}_{i_{\sigma(m)}}\right)$$

and

$$\Omega^{r-1-f}=(r-1-f)!\sum_{A=(a_1<a_2<\cdots<a_{r-1-f})}\bigwedge_{j=1}^{r-1-f}\frac{i}{2\pi}dz_{a_j}\wedge d\overline{z}_{a_j}$$

we infer (with $I = J \cup A^c$ and $\sigma = \sigma'\sigma''$)

$$
\begin{aligned}
&(\Phi_{d+1}(u))_f \Omega^{r-1-f} \\
&= (1-u)^f (r-1-f)! \sum_{|A|=r-1-f} \bigwedge \frac{i}{2\pi} dz_{a_j} \wedge d\overline{z}_{a_j} \\
&\qquad \sum_{\substack{2\le j_1<j_2<\cdots<j_{d-f}\le r \\ J\subset A}} \sum_{\substack{\sigma'\in\Sigma_{d-f} \\ \sigma''\in\Sigma_f}} \varepsilon(\sigma') \bigwedge_{1\le m\le d-f} c_{j_m j_{\sigma'(m)}} \varepsilon(\sigma'') \bigwedge_{\substack{k=1 \\ A^c=(b_1<b_2<\cdots<b_f)}}^{f} \frac{i}{2\pi} dz_{b_k} \wedge d\overline{z}_{b_{\sigma''(k)}} \\
&= (1-u)^f (r-1-f)! f! \frac{\Omega^{r-1}}{(r-1)!} \sum_A \sum_{\substack{2\le j_1<j_2<\cdots<j_{d-f}\le r \\ J\subset A}} \sum_{\sigma'\in\Sigma_{d-f}} \varepsilon(\sigma') \bigwedge_{1\le m\le d-f} c_{j_m j_{\sigma'(m)}} \\
&= (1-u)^f \frac{(r-1-f)!f!}{(r-1)!} \Omega^{r-1} \sum_{|J|=d-f} \sum_{\substack{|A|=r-1-f \\ A\supset J}} det(c'_{J,J}) \\
&= (1-u)^f \frac{(r-1-f)!f!}{(r-1)!} \Omega^{r-1} \binom{r-1-d+f}{f} c'_{d-f}(E^\star).
\end{aligned}
$$

This leads to

$$
(\widetilde{c_{d+1}})_f \Omega^{r-1-f} = -\mathcal{H}_f \binom{r-1}{f}^{-1} \binom{r-1-d+f}{f} c'_{d-f}(E^\star)\Omega^{r-1}
$$

at the point $(x_0, [a_0^\star])$. To get an expression valid on the whole fiber, we refer to proposition 4. □

Back to the computation of S_{m+1}, with $a+b = (j-1)-(r-1-i) = i+j-r$,

$$
\begin{aligned}
&S_{m+1}(E,h) \\
&= -\sum_{\substack{0\le i\le r-1 \\ a+b+c=m}} \mathcal{H}_{r-1-i} \binom{r-1}{i}^{-1} \binom{r-1-a-b}{r-1-i} \binom{i+c}{c} c_a(E^\star,h)\pi_\star\left((-1)^b \Theta^b a^\star (-\Theta^1 a^\star)^c \Omega^{r-1}\right) \\
&= -\sum_{a+b+c=m} \sum_{i=1}^{r-1-a-b} \mathcal{H}_i \frac{(r-1-a-b)!(r-1-i+c)!}{(r-1)!(r-1-i-a-b)!c!} c_a(E^\star,h)\pi_\star\left((-1)^b \Theta^b a^\star (-\Theta^1 a^\star)^c \Omega^{r-1}\right) \\
&= -\sum_{a+b+c=m} \sum_{i=1}^{r-1-a-b} \mathcal{H}_i \frac{(r-1-a-b)!(r-1+c-i)!}{(r-1-a-b-i)!(r-1+c)!} c_a(E^\star,h) s_c^b(E,h) \\
&= -\sum_{a+b+c=m} \sum_{i=1}^{r-1-a-b} \mathcal{H}_i \binom{r-1-a-b}{i} \binom{r-1+c}{i}^{-1} c_a(E^\star,h) s_c^b(E,h).
\end{aligned}
$$

ending the proof of theorem 3. □

8. Some arithmetic applications

8.1. **On arithmetic characteristic classes.** We complete the scheme proposed by Elkik ([El], see also [G-S-3]) for the construction of the arithmetic characteristic classes assuming known the construction of the arithmetic first Chern class and the push forward operation in arithmetic Chow groups.

Let K be a number field, $\mathcal{O}_K$ its ring of integers and $S := spec(\mathcal{O}_K)$. Consider an arithmetic variety $\mathcal{X}$ on S (i.e. a flat regular projective scheme χ over S together with the collection of schemes $\chi_{\mathbb{C}} = \coprod_{\sigma:K\hookrightarrow\mathbb{C}} \chi_\sigma$) and an arithmetic vector bundle $\overline{E}$ of rank r on it (i.e. an locally free sheaf of $\mathcal{O}_\chi$-modules together with the collection

of corresponding vector bundles on $\chi_{\mathbb{C}}(\mathbb{C})$ endowed with a hermitian metric). Write X for the smooth variety $\chi_{\mathbb{C}}(\mathbb{C})$ and (E,h) for the induced hermitian vector bundle on X. All the objects we will consider on X are required to be invariant under the conjugaison given by the real structure of $\mathcal{X}$. The notation $A^{p,p}(X)$ is now used for the subspace of conjugaison invariant forms. We will denote by $\mathcal{D}(X)$ the space of (conjugaison invariant) currents on X and by $\widetilde{\mathcal{D}}(X)$ its quotient by $Imd' + Imd''$.

We collect some basic facts we will need on arithmetic intersection theory (see [G-S-1] for properties of the arithmetic Chow groups and [G-S-2] for the construction of the arithmetic Chern classes). The arithmetic Chow groups are defined to be the quotient by the subgroup generated by arithmetic principal cycles and pairs of the form $(0, \partial u + \overline{\partial} v)$ of the free Abelian group on pairs (Z, g_Z) of an algebraic cycle $Z \subset \chi$ and a Green current $g_Z \in \widetilde{\mathcal{D}}(X)$ for $Z(\mathbb{C})$ defined by the requirement that $\delta_{Z(\mathbb{C})} + dd^c g_Z$ be the current associated with a smooth form in $A(X)$. The notation $\delta_{Z(\mathbb{C})}$ is used for the current of integration along $Z(\mathbb{C})$. There are natural maps

$$\begin{array}{rcccc} a & : & A^{p-1,p-1}(X) & \to & \widehat{CH}^p(\mathcal{X}) \\ & & \eta & \mapsto & [(0,\eta)] \end{array} \qquad \begin{array}{rcccc} \omega & : & \widehat{CH}^p(\mathcal{X}) & \to & A^{p,p}(X) \\ & & [(Z,g_Z)] & \mapsto & \delta_{Z(\mathbb{C})} + dd^c g_Z \end{array}$$

The push-forward map on arithmetic Chow groups is defined component-wise. Hence, $a\pi_\star = \widehat{\pi}_\star a$. The product in $\widehat{CH}^\star(\mathcal{X})$ is defined by the formula : $[(Y, g_Y)] \cdot [(Z, g_Z)] = [(Y \cap Z, g_Y \star g_Z)]$ where the star product is given by

$$g_Y \star g_Z := g_Y \wedge \delta_Z + \omega([(Y, g_Y)]) \wedge g_Z \in \widetilde{\mathcal{D}}(X).$$

Hence, for $r \in \widehat{CH}^\star(\mathcal{X})$, $r \cdot a(\eta) = a(\omega(r) \wedge \eta)$.

As for the arithmetic Chern classes, they have the following properties :

- For every arithmetic vector bundle $\overline{E}$ on $\mathcal{X}$, $\omega\big(\widehat{c}_t(\overline{E})\big) = c_t(E,h)$.
- For every exact sequence $(\overline{\mathcal{S}}) = (0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0)$ of arithmetic vector bundles on $\mathcal{X}$,

$$\widehat{c}_t(\overline{E}) - \widehat{c}_t(\overline{S})\widehat{c}_t(\overline{Q}) = -a(t\widetilde{c}_t(\mathcal{S}(\mathbb{C}), h)) \tag{8.1}$$

- For every arithmetic line bundle $\overline{L}$ and every arithmetic vector bundle $\overline{E}$ of rank r on $\mathcal{X}$,

$$\widehat{c}_r(\overline{E} \otimes \overline{L}) = \sum_{p+q=r} \widehat{c}_p(\overline{E})\widehat{c}_1(\overline{L})^q. \tag{8.2}$$

We now explain the construction of the arithmetic characteristic classes. Use first the usual hermitian theory on X to define in $A^{d,d}(X)$ the following characteristic forms

- the Segre forms $s'_d(E,h) := \pi_\star(\Theta(\mathcal{O}_E(1), h)^{r-1+d})$,
- the Chern forms $c_d(E,h) := trace(\Lambda^d \Theta(E,h))$
- and the forms $\theta_d := trace(\Theta(E,h)^{\otimes d})$.

Define the generalized Segre forms $s^b_c(E,h) \in A^{b+c,b+c}(X)$ by

$$\sum_{c=0}^{+\infty} t^{b+c}(r+c)s^b_c(E,h) = s'_t(E,h) \sum_{q=b}^{dimX} t^q \theta_q.$$

Consider the (secondary) characteristic forms $S_{m+1}(E,h)$ and $R_{m+1}(E,h)$ in $A^{m,m}(X)$ given by the relations

$$\begin{aligned} S_{m+1}(E,h) &= -\sum_{a+b+c=m} \mathcal{H}^{r-1+c}_{r-1-a-b} c_a(E^\star, h) s^b_c(E,h) \\ R_t(E,h) &= s'_t(E,h) S_t(E,h). \end{aligned}$$

where $S_t(E,h)=\sum_{m=0}^{+\infty}t^mS_{m+1}(E,h)$ and $R_t(E,h)=\sum_{m=0}^{+\infty}t^mR_{m+1}(E,h)$. Our previous computations ensure that the class of $S_{m+1}(E,h)$ in $\widetilde{A^{m,m}}(X)$ is

$$S_{m+1}(E,h)=\pi_\star(\widetilde{c}_r(\Sigma(1),h)\Theta(\mathcal{O}_E(1),h)^m). \tag{8.3}$$

Define the m-th geometric Segre class of $\overline{E}$ to be

$$\widehat{s'_m}(\overline{E}):=\widehat{\pi}_\star\left(\widehat{c}_1(\mathcal{O}_{\overline{E}}(1))^{r-1+m}\right)$$

in the arithmetic Chow group $\widehat{CH}(\mathcal{X})$ of $\mathcal{X}$. Its m-th arithmetic Segre class $\widehat{s_m}(E,h)$ is defined as the class in $\widehat{CH}(\mathcal{X})$ obtained by adding in the definition of the m-th geometric Segre class an extra term computed precisely thanks to the previous top Bott-Chern form :

$$\widehat{s_m}(\overline{E}):=\widehat{s'_m}(\overline{E})+a(R_m(E,h)).$$

Comparing with the arithmetic Chern class polynomial of $\overline{E^\star}$ as defined in [G-S-2] we get

Theorem 4. *In the arithmetic Chow group of $\mathcal{X}$*

$$\widehat{c}_t(\overline{E^\star})\cdot\widehat{s}_t(\overline{E})=1.$$

Proof. This is in fact a precise form of formula (7.1) in the arithmetic setting. For the twisted arithmetic Euler sequence $\overline{\Sigma}(1)$,

$$0\to\mathcal{O}_{\mathbb{P}(\overline{E})}\to\pi^\star\overline{E^\star}\otimes\mathcal{O}_{\overline{E}}(1)\to T_{\mathbb{P}(\overline{E})/\mathcal{X}}\to 0$$

formulas (8.2) and (8.1) read

$$\begin{aligned}\sum_{p+q=r}\widehat{c}_p(\pi^\star\overline{E^\star})\widehat{c}_1(\mathcal{O}_{\overline{E}}(1))^q &= \widehat{c}_r(\pi^\star\overline{E^\star}\otimes\mathcal{O}_{\overline{E}}(1))\\ &= \widehat{c}_1(\mathcal{O}_{\mathbb{P}(\overline{E})})\widehat{c_{r-1}}(T_{\mathbb{P}(\overline{E})/\mathcal{X}})-a(\widetilde{c}_r(\Sigma(1),h))\\ &= -a(\widetilde{c}_r(\Sigma(1),h))\end{aligned}$$

By push-forward $\widehat{\pi}_\star$ to the Chow group of $\mathcal{X}$, using the previously recalled facts and formula (8.3), this leads to

$$\widehat{c}_t(\overline{E^\star})\cdot\widehat{s'_t}(\overline{E})=1-a(tS_t(E,h))$$

that is

$$\widehat{c}_t(\overline{E^\star})\cdot\widehat{s}_t(\overline{E})=1-a(tS_t(E,h))+\widehat{c}_t(\overline{E^\star})\cdot a(tR_t(E,h)).$$

But

$$\widehat{c}_t(\overline{E^\star})\cdot a(R_t(E,h))=a(\omega(\widehat{c}_t(\overline{E^\star}))R_t(E,h))=a(c_t(E^\star,h)R_t(E,h))=a(S_t(E,h))$$

thanks to proposition 6. □

This provides an alternative definition of the arithmetic Chern classes and hence of all the arithmetic characteristic classes without using the splitting principle. This point of view is closer to that of Fulton [Fu].

8.2. **On the height of $\mathbb{P}(\overline{E})$.** For any vector bundle E of rank r on a smooth complex compact analytic manifold X of dimension n, we define the analytic height of $\mathbb{P}(E)\xrightarrow{\pi}X$ by

$$h_{\mathcal{O}_E(1)}(\mathbb{P}(E))=\int_{\mathbb{P}(E)}c_1(\mathcal{O}_E(1))^{n+r-1}.$$

It follows from Fubini theorem that

$$h_{\mathcal{O}_E(1)}(\mathbb{P}(E))=\int_X\pi_\star\left(c_1(\mathcal{O}_E(1))^{n+r-1}\right)=\int_X s'_n(E).$$

According to Hartshorne, the ampleness of $E \to X$ is defined to be the ampleness of $\mathcal{O}_E(1) \to \mathbb{P}(E)$ hence implies the positivity of the analytic height of $\mathbb{P}(E)$ from its very definition.

In the arithmetic setting, for an arithmetic vector bundle $\overline{E}$ of rank r over an arithmetic variety $\mathcal{X} \xrightarrow{f} S$ of relative dimension n over S, we define the arithmetic height of $\mathbb{P}(\overline{E}) \xrightarrow{\pi} \mathcal{X}$ with respect to $\mathcal{O}_{\overline{E}}(1)$ by

$$\widehat{h}_{\mathcal{O}_{\overline{E}}(1)}(\mathbb{P}(\overline{E})) = \widehat{deg}\widehat{f}_\star\widehat{\pi}_\star\left(\widehat{c_1}(\mathcal{O}_{\overline{E}}(1))^{dim\mathbb{P}(\overline{E})}\right)$$

where $\widehat{deg} : \widehat{CH}^1(S) \to \widehat{CH}^1(\mathbb{Z}) \to \mathbb{R}$ the last map sending $(0, \lambda)$ to $\lambda/2$ (see [Bo-G-S]). From the definition of the arithmetic Segre class, we infer

$$\widehat{h}_{\mathcal{O}_{\overline{E}}(1)}(\mathbb{P}(\overline{E})) = \widehat{deg}\widehat{f}_\star\widehat{s'}_{n+1} = \widehat{deg}\widehat{f}_\star\widehat{s}_{n+1} - \frac{1}{2}\int_X R_{n+1}.$$

The complex cohomological term $-\frac{1}{2}\int_X R_{n+1}$ is computed thanks to theorem 3. For example,

$$\begin{aligned} R_1 &= S_1 = -\sum_{i=1}^{r-1}\mathcal{H}_i s'_0(E,h) = -\sum_{i=1}^{r-1}\mathcal{H}_i. \\ R_2 &= s'_1 S_1 + S_2 = -\left(1+\frac{1}{r}\right)\left(\sum_{i=1}^{r-1}\mathcal{H}_i\right)s'_1(E,h) \end{aligned}$$

In rank 2,
$$\begin{aligned} S_{m+1}(E,h) &= \frac{1}{m+1}s'_m(E,h) \\ R_{m+1}(E,h) &= -\sum_{p+q=m}\frac{1}{q+1}s'_p(E,h)s'_q(E,h) \end{aligned}$$

In rank 3,
$$\begin{aligned} S_{m+1}(E,h) &= -\frac{1}{m+1}c_1(E^\star,h)s'_{m-1}(E,h) - \frac{3m+5}{(m+1)(m+2)}s'_m(E,h) \\ &= \frac{1}{m+1}\left(s'_1(E,h)s'_{m-1}(E,h) - s'_m(E,h)\right) - \frac{2m+3}{(m+1)(m+2)}s'_m(E,h). \end{aligned}$$

According to Zhang, a line bundle $\overline{L} \to \mathcal{X}$ is said to be ample if $L \to \chi$ is ample, $(L,h) \to X$ is semi-positive and for every large enough n there exists a $\mathbb{Z}$-basis of $\Gamma(\chi, L^{\otimes n})$ made of sections of sup norm less than 1 on X. This implies that the leading coefficient of the Hilbert function of $\overline{L}$ is positive ([Zh], lemma 5.3) which in turn implies the positivity of the arithmetic $dim\mathcal{X}$-fold intersection of $\widehat{c_1}(\overline{L})$ is positive ([G-S-3]). Hence, the ampleness of $\overline{E}$ that we define to be the ampleness of the associated line bundle $\mathcal{O}_{\overline{E}}(1) \to \mathbb{P}(\overline{E})$ implies the positivity of the arithmetic height of $\mathbb{P}(\overline{E})$.

Note however that it does not imply the positivity of the secondary term $-\frac{1}{2}\int_X R_{n+1}$ as it can be checked with Fulton-Lazarsfeld characterization of numerically positive polynomials on ample vector bundles [Fu-L] : the numerically positive polynomials in Chern classes are non-zero polynomials having non-negative coefficients in the basis of Schur polynomials in Chern polynomials. By Jacobi-Trudi formula, this basis is also the basis of Schur polynomials in Segre classes. For $r = 3$, $n = 3$ the third coefficient of $-\frac{1}{2}R_4$ in the degree 3 part of the basis of Schur polynomials in Segre classes $(s'_3, s'_1s'_2 - s'_3, {s'_3}^3 - 2s'_1s'_2 + s'_3)$ is $-\frac{1}{6}$.

References

[Bi-G-S] Bismut, J.-M. and Gillet, H. and Soulé, C., *Analytic torsion and holomorphic determinant bundles. I, II and III*, Comm. Math. Phys., 115, (1988), 1, 49–78, and 179–126 and 2, 301–351.

[Bo-G-S] Bost, J.-B. and Gillet, H. and Soulé, C., *Heights of projective varieties and positive Green forms*, J. Amer. Math. Soc., 7, (1994), 4, 903–1027.

[Bo-S] Bost, Jean Benoit and Soulé, Christophe, Conversation with Vincent Maillot, IHES 1993.

[B-C] Bott, Raoul and Chern, Shiing-Shen., *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, Acta Math., 114, (1965), 71–112.

[Do] Donaldson, S. K., *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. (3), 50, (1985), 1, 1– 26,

[El] Elkik, Renée, *Métriques sur les fibrés d'intersection*, Duke Math. J., 61, (1990), 1, 303–328.

[Fu] Fulton, William, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 2,Springer-Verlag,Berlin,1998.

[Fu-L] Fulton, William and Lazarsfeld, Robert, *Positive polynomials for ample vector bundles*, Ann. of Math. (2), Annals of Mathematics. Second Series, 118, (1983), 1, 35–60.

[G-S-1] Gillet, Henri and Soulé, Christophe, *Arithmetic intersection theory*, Publ. Math. Inst. Hautes Études Sci., 72, (1990), 93–174.

[G-S-2] Gillet, Henri and Soulé, Christophe, *Characteristic classes for algebraic vector bundles with Hermitian metric. I and II*, Ann. of Math. (2) 131 (1990) 1, 163–203 and 205–238.

[G-S-3] Gillet, Henri and Soulé, Christophe, *An arithmetic Riemann-Roch theorem* Invent. Math. 110 (1992) 473-543.

[Ma] Maillot, Vincent, *Un calcul de Schubert arithmétique*, Duke Math. J., 80 (1995) 1,195–221.

[Ta] Tamvakis, Harry, *Arithmetic intersection theory on flag varieties*, Math. Ann., 314, (1999), 4, 641–665.

[Zh] Zhang, Shouwu, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc., 8, (1995),1, 187–221.

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