A remark on renormalized volume and Euler characteristic for ACHE 4-manifolds

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A REMARK ON RENORMALIZED VOLUME AND EULER CHARACTERISTIC FOR ACHE 4-MANIFOLDS

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Abstract. This note computes the "renormalized volume" and a renormalized Gauss-Bonnet-Chern formula for asymptotically complex hyperbolic Einstein (the so-called ACHE) 4-manifolds.

1. Introduction.

Asymptotically symmetric Einstein metrics exhibit many interesting phenomena [3, 9]. They were especially studied in the asymptotically *real hyperbolic* (or AHE) case, which enjoys fruitful relationships with physics through the ADS-CFT correspondence. They are also a useful tool in the study of conformal geometry in establishing links between the conformal geometry of a compact $(n-1)$-dimensional manifold (usually called the *boundary at infinity*) and a complete Einstein n-dimensional manifold (the AHE manifold). In this setting, an intriguing invariant, called *renormalized volume*, has been defined by C. R. Graham [8], after works by physicists such as Henningson and Skenderis [10].

In even dimensions n, the renormalized volume is an invariant of the Einstein metric only. If $n = 4$, its role in the formula for the Euler characteristic of the Einstein manifold has been moreover pointed out by M. T. Anderson [1], with applications in the study of the moduli space of Einstein asymptotically real hyperbolic metrics [2]. This formula is called "the renormalized Gauss-Bonnet-Chern formula": although the Einstein manifold is non-compact, all divergent terms in the integrals of the formula are shown to cancel, whereas renormalized volume appears as a finite limit contribution.

In odd dimensions n, the renormalized volume is not an invariant of the Einstein metric only but rather depends on a choice of a representative metric on the boundary at infinity in the conformal class. This makes it no less interesting, as it gives rise to the so-called *conformal anomaly* phenomenon: the difference between the renormalized volumes of two different choices of metric singles out a local differential operator on the boundary with nice properties [8].

The goal of this short note is to point out analogous results in the case of Einstein asymptotically *complex* hyperbolic (or ACHE) manifolds of dimension 4, where the boundary at infinity is now a strictly pseudoconvex 3-manifold, with the hope that such an object would be interesting for the study of 3-dimensional CR geometry.

Unfortunately, the situation is less pleasant than in the real case, as the renormalized volume is never an invariant of the the complete Einstein metric and always

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depends on the choice of some contact form (or in the usual language of CR geometry: a *pseudo-hermitian structure*) compatible with the CR-structure at infinity. This situation is reminiscent from that of Einstein asymptotically *real* hyperbolic manifolds of odd-dimensions (*i.e.* boundary at infinity of *even* dimension), this should come as no surprise as it is well-known that CR geometry enjoys lots of analogies with even-dimensional conformal geometry.

However, the fact that in the ACHE context the bulk Einstein manifold is even-dimensional brings some nice features. It turns out that adding some well-chosen local quantity at infinity can yield an invariant of the Einstein metric only. As expected, a renormalized Gauss-Bonnet characteristic formula can be obtained as well.

Our main results then read as follows (further notations and definitions are given in the next section):

1.1. **Theorem.** *Let (M, g) be a 4-dimensional Einstein asymptotically complex hyperbolic* (ACHE) *manifold, with boundary at infinity a compact strictly pseudoconvex CR 3-dimensional manifold X with contact distribution H and almost-complex structure J. Then there exists for any choice of compatible contact form η on X an invariant V of the pair (g, η) called* renormalized volume *of g relative to η.*

1.2. **Theorem.** *Under the same assumptions, if moreover R and τ are the curvature and torsion of the Webster-Tanaka connection on (X, H, J, η), then*

$$\mathcal{V} = \frac{3}{2}\, V - \int_X \left(\frac{R^2}{16} - \frac{5}{2}|\tau|^2 \right)$$

is an invariant of the metric g only, and

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(|W^g|^2 - \frac{1}{24} (\mathrm{Scal}^g)^2 \right) + \frac{1}{4\pi^2}\, \mathcal{V}.$$

As the model case of the complex hyperbolic plane shows, the appearance of the integral factor on the boundary seems unavoidable; see section 2 for further details. This shows than, rather than giving rise to a global invariant, the renormalized volume gives birth to a *conformal anomaly*, *i.e.* a formula relating the renormalized volume for some choice of pseudo-hermitian structure at infinity to its expression for some other choice at infinity, through a *local* differential expression. Namely, if we let $V(\eta)$ be the renormalized volume for a choice of contact form at infinity,

1.3. **Corollary.** *For each contact form η, there is a differential operator P_η on X such that, for any function f on X which never vanishes,*

$$V(f\eta) - V(\eta) \;=\; \int_X P_\eta(f)\, \eta \wedge d\eta,$$

In the *real hyperbolic* case [8], the conformal anomaly is given by differential operators with nice invariance properties. Our result in the *complex hyperbolic case* strongly suggests that it should be interesting to study the operator arising from the variation of

$$T(\eta) \;=\; \frac{R^2}{16} - \frac{5}{2}|\tau|^2$$

under deformations of the contact form in the same contact structure. Explicit derivations of the variations of Tanaka-Webster curvature R and torsion τ are given as an Appendix to this note; further study will be deferred to a future work.

Note moreover that N. Seshadri has given in [12] another version of the renormalized volume that covers all dimensions but for Kähler-Einstein metrics only rather than ACHE.

2. Definitions and notations.

Let (X^3, H, J_0) be a strictly pseudo-convex 3-dimensional CR manifold, *i.e.* a contact manifold with contact distribution H and almost complex structure J_0 on H. If η in any choice of compatible contact form, an associated metric γ may be defined on H by $\gamma = d\eta(\cdot, J_0\cdot)$. One gets from it a Reeb field ξ and a (Tanaka-Webster) connection ∇ whose torsion in the direction of ξ is $\tau = \nabla^{TW}_\xi \cdot - [\xi, \cdot]$.

Let M be a 4-manifold such that the complement of some compact set is diffeomorphic to $[r_0, +\infty[\times X$. We consider first the metric $g_0 = dr^2 + \mathrm{e}^{2r}\,\eta^2 + \mathrm{e}^r\,\gamma$ on $]r_0, +\infty[\times X$ and let C^∞_δ be the space of smooth functions on M such that $\mathrm{e}^{\delta r}\,\nabla^k f$ is bounded for any k. Any metric g on M such that $g - (dr^2 + \mathrm{e}^{2r}\,\eta^2 + \mathrm{e}^r\,\gamma)$ belongs to C^∞_δ for some $\delta > 0$ will be called an *asymptotically complex hyperbolic* metric. Moreover, (M, g) is said to be ACHE if g is an Einstein metric.

A lot of such metrics arise on pseudoconvex domains in $\mathbb{C}^2$ (and are Kähler-Einstein in this case [6]) whereas another important family was constructed by O. Biquard in [3]. The Biquard metrics are especially interesting in the case the boundary at infinity X is endowed with a non-embeddable CR structure, as they provide a substitute for the non-existing Kähler-Einstein metric.

In [4], the author and O. Biquard carefully studied the asymptotic behaviour of ACHE metrics, and precise asymptotic expansions were obtained. In all that follows, we consider an ACHE metric g on a neighbourhood of infinity $]r_0, +\infty[\times X$ in M. If a contact form η is given, there exists a canonical Tanaka-Webster connection ∇ on X. For any tensor field D on X, $D^{d...}_{a,bc...}$ $(a, b, c, d, ... \in \{1, \bar 1\})$ will denote the components of D (and subsequent Tanaka-Webster derivatives, separated by a comma from the original components) in a local orthonormal coframe $(\theta^1, \theta^{\bar 1})$, *i.e.* such that $d\eta = i\theta^1 \wedge \theta^{\bar 1}$. For instance we shall use expressions such as $\tau^a_{b,c...}$ for the (derivatives of the) torsion τ of ∇ and $R_{,ab...}$ for its curvature. We also denote $\vartheta^0 := \mathrm{e}^{-r}\,dr + i\eta$, and $\vartheta^1 := \theta^1 - \phi_\lrcorner\theta^1$. Last, in any power series expansion $\sum \phi_k(x)\,\mathrm{e}\,kr$, the k-the term ϕ_k (seen as a function on X) will be called *formally determined* if it can be computed with the knowledge of a finite jet of the CR structure at $x \in X$ only. The most interesting feature of ACHE metrics (and Kähler-Einstein metrics as well) is that they are not entirely formally determined. The results in [4] are summarized in the three following statements:

2.1. **Theorem** ([4]). *There exists on $]r_0, +\infty[\times X$ an integrable complex structure J given by a (not necessarily convergent) formal series, entirely determined formally from data at infinity. The first terms in its expansion is given by*

$$J = J_0 - 2\,\mathrm{e}^{-r}\,\tau + \mathrm{e}^{-2r}(2|\tau|^2 - J_0\nabla_\xi\tau) + o(\mathrm{e}^{-2r}),$$

or equivalently by a map $\phi = -i\,\mathrm{e}^{-r}\,\tau + \frac{1}{2}\,\mathrm{e}^{-2r}\,\nabla_\xi\tau + o(\mathrm{e}^{-2r})$ *from* $T^{0,1}_{J_0}$ *to* $T^{1,0}_{J_0}$.

2.2. **Theorem** ([4]). *There is on* $]r_0,+\infty[\times X$ *a (formal series) Kähler-Einstein metric* $\overline{g}$. *The Kähler form* ω *of* $\overline{g}$ *is formally determined up to order* 2 *as follows*

$$\begin{aligned}\omega \;=\; & \mathrm{e}^r\,(dr\wedge\eta + d\eta) - \frac{R}{2}\,d\eta \\ & + \frac{4}{3}\left(\frac{i}{8}R_{,\bar 1}\vartheta^0\wedge\theta^{\bar 1} - \frac{i}{8}R_{,1}\vartheta^{\bar 0}\wedge\theta^1 - \frac{1}{2}\tau^1_{\bar 1,1}\vartheta^0\wedge\theta^{\bar 1} - \frac{1}{2}\tau^{\bar 1}_{1,\bar 1}\vartheta^{\bar 0}\wedge\theta^1\right) - \frac{\Delta R}{2}\,\mathrm{e}^{-r}\,d\eta \\ & - \frac{2}{3}\left(\frac{R^2}{8} - |\tau|^2 - \frac{\Delta R}{6} + \frac{2i}{3}(\tau^1_{\bar 1,11} - \tau^{\bar 1}_{1,\bar 1\bar 1})\right)\mathrm{e}^{-r}\,dr\wedge\eta \\ & + \frac{2}{3}\left(\frac{R^2}{8} - |\tau|^2 + \frac{\Delta R}{12} - \frac{i}{3}(\tau^1_{\bar 1,11} - \tau^{\bar 1}_{1,\bar 1\bar 1})\right)\mathrm{e}^{-r}\,d\eta + o(\mathrm{e}^{-2r}).\end{aligned}$$

Moreover, if g *is an* ACHE *metric with the same boundary at infinity, then there exists an anti-*J_0*-invariant symmetric bilinear form* k *on* H *and a unique diffeomorphism* ψ *asymptotic to identity at infinity such that* $\psi^*g = \overline{g} + k\,\mathrm{e}^{-r} + o(\mathrm{e}^{-2r})$.

2.3. **Corollary** ([4]). *The Kähler metric* $\overline{g}$ *is explicitly given by*

$$\begin{aligned}\overline{g} \;=\; & (dr^2 + \mathrm{e}^{2r}\,\eta^2 + \mathrm{e}^r\,\gamma) - \frac{R}{2}\gamma + 2\gamma(J_0\tau\cdot,\cdot) + \frac{1}{6}(R_{,1}\theta^1\circ\vartheta^{\bar 0} + R_{,\bar 1}\theta^{\bar 1}\circ\vartheta^0) \\ & + \frac{2i}{3}(\tau^1_{\bar 1,1}\vartheta^0\circ\theta^{\bar 1} - \tau^{\bar 1}_{1,\bar 1}\vartheta^{\bar 0}\circ\theta^1) - \mathrm{e}^{-r}\,R\gamma(J_0\tau\cdot,\cdot) - \mathrm{e}^{-r}\,\gamma(\nabla_\xi\tau(\cdot),\cdot) \\ & - \frac{2}{3}\left(\frac{R^2}{8} - |\tau|^2 - \frac{\Delta R}{6} + \frac{2i}{3}(\tau^1_{\bar 1,11} - \tau^{\bar 1}_{1,\bar 1\bar 1})\right)\mathrm{e}^{-2r}(dr^2 + \mathrm{e}^{2r}\,\eta^2) \\ & + \frac{2}{3}\left(\frac{R^2}{8} - |\tau|^2 + \frac{\Delta R}{12} - \frac{i}{3}(\tau^1_{\bar 1,11} - \tau^{\bar 1}_{1,\bar 1\bar 1})\right)\mathrm{e}^{-r}\,\gamma + o(\mathrm{e}^{-2r}),\end{aligned}$$

where $\alpha\circ\beta = \alpha\otimes\beta + \beta\otimes\alpha$ *is the symmetrized product of forms.*

The main conclusion of these facts is the following : given any ACHE metric g and any choice of pseudo-hermitian structure at infinity realizing the CR structure, there is a unique diffeomorphism ψ asymptotic to the identity on X such that, up to strictly lower order terms, ψ^*g can be written as the sum of a formally determined Kähler-Einstein metric and a *formally undetermined* term of order 2 (notice that $k\,\mathrm{e}^{-r}$ decays like e^{-2r}). From now on, we will forget the diffeomorphism ψ and, if the metric g is written this way in such coordinates, we will say that it is "in the Kähler gauge associated to the choice of pseudo-hermitian structure at infinity".

Using the results of [4] that we have just recalled, we can now define the renormalized volume:

2.4. **Proposition.** *Let* g *be an ACHE metric on* M, *written in a Kähler gauge associated to some choice of pseudo-hermitian structure at infinity. Then the volume of large coordinate balls* $B(r)$ *of radius* r (*complement of* $]r,+\infty[\times X$ *in* M) *has an asymptotic expansion:* $\mathrm{vol}_g\,B(r) = \pi^2\,\mathrm{e}^{2r} + v_1\,\mathrm{e}^r + V + o(1)$. *the number* V *is the* renormalized volume *of the metric* g *associated to the choice of pseudo-hermitian structure at infinity*

Proof. – To check the proposition, just notice that the volume form of g only differs from that of $\overline{g}$ at order $\frac{5}{2}$ since k is trace-free, and, in the volume form of the Kähler

form ω, order $\frac{3}{2}$ terms do not exist whereas order 2 terms are of zero integral from the CR Stokes' formula [5]: whenever $\alpha = \alpha_1\theta^1$ is a $(1,0)$-form on X (given in a local orthonormal coframe), one has

$$d\alpha = \alpha_{1,\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + \alpha_{1,0}\eta \wedge \theta^1 + \alpha_1\tau^1_{\bar{1}}\eta \wedge \theta^{\bar{1}}$$

(recall $\alpha_{.,.}$ denotes the components of $\nabla\alpha$ in the local coframe), and

$$\int_X \alpha_{1,\bar{1}}\eta \wedge \theta^1 \wedge \theta^{\bar{1}} = \int_X (d\alpha) \wedge \eta = -\int_X \alpha \wedge d\eta = 0,$$

since X is closed and $d\eta = i\theta^1 \wedge \theta^{\bar{1}}$. This achieves the proof of Theorem 1.1. □

2.5. *Remark.* In the asymptotically *real* hyperbolic Einstein case (AHE), the renormalized volume is similarly defined [8], but with the help of a different gauge. It is proved in [8] that it is always possible, given a metric h^∞ in the conformal class of the boundary at infinity, to find coordinates such that $g = dr^2 + h(r)$ on $]r_0, +\infty[\times X$ (this is the *geodesic gauge*, the function r being the *geodesic defining function* associated to a choice of metric h^∞ in the conformal class at infinity). The metric $h(r)$ has an expansion in powers of e^r and V is defined as above as the constant coefficient in the expansion of $\mathrm{vol}(B(r))$. It is is in itself an invariant of g. The reader might hence think that the "misbehaviour" of the renormalized volume in the ACHE case comes from a bad choice of gauge. However, the standard metric of $\mathbb{C}\mathbf{H}^2$ is both in the Kähler and geodesic gauges, and Theorem 1.1 yields that the boundary term in the renormalized Gauss-Bonnet formula (which necessarily is an invariant of g) *is different* than the renormalized volume.

2.6. *Remark.* The most important fact to be noted in the previous Proposition is that *there is no term in* $\mathrm{vol}_g B(r)$ *that is linear in* r. In the AHE case [8], when the boundary at infinity is *odd-dimensional*, an analogous phenomenon occurs: linear terms cancel in the asymptotic expansion of large balls. In the AHE case again but when the boundary at infinity is *even-dimensional* (the case that is considered to be the closest to the ACHE case, although dimensions of the boundaries at infinity differ), the situation is different: some non-zero linear term appears in the expansion of the volume, with a coefficient related to the integral term in the Gauss-Bonnet-Chern formula for the boundary at infinity X itself [8].

If one believes in this analogy (between *even*-dimensional conformal geometry and CR geometry), one might then wonder why there is indeed no linear term in Proposition 2.4. However, reasoning by analogy again, one would assert from [8] that the coefficient of the linear term should be a multiple of the *integral Q-curvature* of X [7], but it has been proved in [7] that this integral always vanishes in 3-dimensional CR geometry, thus the absence of any linear term, a phenomenon that might be purely 3-dimensional.

3. Proof of Theorem 1.2.

We first choose a contact form (or pseudo-hermitian structure) at infinity (*i.e.* on X) realizing the CR structure and we put the ACHE metric in the associated Kähler

gauge around infinity. The basic element of the proof then is the Gauss-Bonnet-Chern formula for the Euler characteristic of the compact domain with boundary $B(r)$ delimited by what we shall call the *coordinate sphere* $S(r) = \{r\} \times X$:

$$\begin{aligned}\chi(B(r)) \;&= \frac{1}{8\pi^2}\int_{B(r)}\left(|W|^2 - \frac{1}{24}\,\mathrm{Scal}^2\right) \;+\; \frac{1}{96\pi^2}\int_{B(r)}\mathrm{Scal}^2 \\ &\quad +\; \frac{1}{12\pi^2}\int_{S(r)} \mathcal{T}(\mathbb{I}\wedge\mathbb{I}\wedge\mathbb{I}) + 3\,\mathcal{T}(\mathbb{I}\wedge\mathcal{R}),\end{aligned} \tag{3.1}$$

where the $\wedge$ operation provides a $p+q$-form with values in $\otimes^{r+s}TM$ from a p-form with values in $\otimes^r TM$ and a q-form with values in $\otimes^s TM$, and we have denoted by $\mathcal{T}$ the contraction between the volume form of $S(r)$ and elements of $\otimes^3 TM$. Moreover, $\mathbb{I}$ is the shape operator of $S(r)$ in (M,g) and $\mathcal{R}$ is the curvature (2-form with values in 2-forms) of (M,g), with W and Scal denoting its Weyl and scalar curvature (trace-free Ricci curvature is zero as g is Einstein). Notice also the difference in notation between the (scalar) curvature R of the 3-dimensional CR manifold X and the curvature tensor $\mathcal{R}$ of the 4-dimensional Einstein manifold M.

It is proven in [4] that the integral involving $|W|^2 - \frac{1}{24}\,\mathrm{Scal}^2$ on $B(r)$ converges for an ACHE metric when r goes to infinity. Moreover, it is clear that both the scalar curvature integral (which is, up to a constant, $\mathrm{vol}\,B(r)$) and the boundary integrals have an asymptotic expansion in powers of $\mathrm{e}^{-\frac{r}{2}}$ (there are no polynomial terms as they cancel in the volume expansion, as noted above). Convergence of all the other terms implies that divergent terms cancel pairwise, whereas the limit as r goes to infinity of

$$\chi(M) - \int_{B(r)}\left(|W|^2 - \frac{1}{24}\,\mathrm{Scal}^2\right) - \frac{3}{8\pi^2}V$$

is given by the constant terms in the asymptotic expansion of the boundary integrals. Our task then reduces to a careful computation of these terms. For this, the following facts will be useful:

Fact 1. It is proven in [4] that replacing g by $\overline{g}$ in the boundary integrals introduces terms that are $o(\mathrm{e}^{-2r})$ only, hence do not contribute in the limit as the volume form of each sphere is $O(\mathrm{e}^{2r})$ at most. Hence all computations can be done using the formal Kähler-Einstein metric $\overline{g}$ rather than the ACHE metric g.

Fact 2. More precisely, the highest-order term in the expansion of the volume form of $S(r)$ is $\mathrm{e}^{2r}\,\eta\wedge d\eta$ (where η is the contact form underlying the chosen pseudo-hermitian structure chosen at infinity). Hence we will only need to track the order e^{-2r} terms in the proof below. Every asymptotic expansion we will meet in the course of the computations is of the following type:

$$A_0\,\mathrm{e}^{2r} \;+\; A_1\,\mathrm{e}^{-r} \;+\; A_{\frac{3}{2}}\,\mathrm{e}^{-\frac{3}{2}r} \;+\; A_2\,\mathrm{e}^{-2r} \;+\; o(\mathrm{e}^{-2r}).$$

As a result, order 2 terms may arise during the computation only when putting together an order 2 term with order 0 terms or two order 1 terms with order 0 terms. Order $\frac{3}{2}$ terms can hence be forgotten during the whole computation, *unless* when some differentiation is involved, as doing so along directions in X raises the order possibly by a factor $\frac{1}{2}$.

Fact 3. Our final computation involves integration along X, hence each exact term can be forgotten. Using the CR Stokes' formula already described in the proof of Proposition 2.4, see also [5], this means that every term in $R_{,1\bar{1}}$, $R_{,\bar{1}1}$, $\Delta R = R_{,1\bar{1}} + R_{,\bar{1}1}$, $\tau^1_{\bar{1},11}$ or $\tau^{\bar{1}}_{1,\bar{1}\bar{1}}$ drops out. In what follows, occurrence of such a term will be denoted by $\mathcal{O}$.

Fact 4. From [4] again, the curvature tensor $\mathcal{R}$ of the formal Kähler-Einstein metric (seen as a 2-form with values in 2-forms) is, up to order 2, given by the sum of the model curvature tensor (*i.e.* that has the same expression w.r.t. $\overline{g}, J$ as the constant holomorphic sectional curvature has w.r.t. $g^{\mathbb{C}\mathbf{H}^2}, J_0$) and of an order 2 term, called W_2^- and controlled by the Cartan tensor of the CR-structure at infinity. Said shortly, one writes: $\mathcal{R} = \mathcal{R}_0 + W_2^- \, \mathrm{e}^{-2r}$.

From now on, the task can be divided into three steps: computation of the outer unit normal and intrinsic volume form of $S(r)$, computation of the shape operator (the only step that involves differentiation) and estimation of the order 2 terms in $\mathcal{T}(\mathbb{I} \wedge \mathbb{I} \wedge \mathbb{I})$ and $\mathcal{T}(\mathbb{I} \wedge \mathcal{R})$. As the computations involved are rather long, we shall give here the main intermediate results only, indicating at each stage which are the key steps and facts that lead to them.

From the explicit expansion of $\overline{g}$, we can get immediately the *outer unit normal* of $S(r)$:

$$\nu(r) \;=\; \left(1 + \frac{1}{3}\,\mathrm{e}^{-2r}\left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O}\right)\right)\partial_r \;+\; \nu^T \;+\; o(\mathrm{e}^{-2r}) \tag{3.2}$$

where ν^T is an order $\frac{3}{2}$ term tangent to X, involving linearly $R_{,1}$, $R_{,\bar{1}}$, $\tau^1_{\bar{1},1}$ and $\tau^{\bar{1}}_{1,\bar{1}}$. It will be proved below that it is not necessary to detail further the expression of this term.

The *volume form* ϖ of $S(r)$ is then (up to forgotten order $\frac{3}{2}$ terms, see Fact 2 above):

$$\begin{aligned}
\varpi \;&=\; \frac{1}{2}\,\omega^2(\nu(r), ., ., .) \\
&=\; \mathrm{e}^{2r}(1 + \mathrm{e}^{-r}\,\varpi_1 + \mathrm{e}^{-2r}\,\varpi_2)\,\eta \wedge d\eta \;+\; o(\mathrm{e}^{-2r}) \\
&=\; \mathrm{e}^{2r}\left(1 - \frac{R}{2}\,\mathrm{e}^{-r} + \frac{1}{3}\,\mathrm{e}^{-2r}\left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O}\right)\right)\eta \wedge d\eta \;+\; o(\mathrm{e}^{-2r}).
\end{aligned} \tag{3.3}$$

The *shape operator* $\mathbb{I}$ is obtained by taking the extrinsic covariant derivative of the unit outer normal $\bar{\nabla}\nu(r)$ (where $\bar{\nabla}$ here denotes the Levi-Civita connection of $\overline{g}$. As ν^T is an order $\frac{3}{2}$ term, only its derivatives in the direction of H might contribute to order 2 terms in $\mathbb{I}$, but it is an easy task to convince oneself that these would add only terms linear in $R_{,1\bar{1}}$, $R_{,\bar{1}1}$, $\tau^1_{\bar{1},11}$ and $\tau^{\bar{1}}_{1,\bar{1}\bar{1}}$, hence of vanishing integral from Fact 3. As a result, they can be forgotten.

It remains to compute the derivative of the radial term in $\nu(r)$, seen first as a bilinear symmetric form. Keeping only symmetric terms in the usual 6-term formula for the covariant derivative, the only significant term is

$$\frac{1}{2}\left(1 + \frac{1}{3}\,\mathrm{e}^{-2r}\left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O}\right)\right)\partial_r \overline{g}.$$

This is easily evaluated from the expansion of $\overline{g}$ recalled above and one gets

$$\begin{aligned}\overline{g}(\bar{\nabla}\nu(r),\cdot) \;&=\; \mathrm{e}^{2r}\,\eta^2 + \frac{1}{2}\,\mathrm{e}^{r}\,\gamma + \frac{R}{2}\,\mathrm{e}^{-r}\,\gamma(J_0\tau\cdot,\cdot) + \frac{1}{2}\,\mathrm{e}^{-r}\,\gamma(\nabla_\xi\tau\cdot,\cdot)\\ &\quad + \frac{1}{3}\left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O}\right)\eta^2 - \frac{1}{6}\left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O}\right)\mathrm{e}^{-r}\,\gamma.\end{aligned} \tag{3.4}$$

One step further, this yields the (endomorphism) shape operator, which we shall develop as

$$\mathbb{I} = \mathbb{I}_0 + \mathrm{e}^{-r}\,\mathbb{I}_1 + \mathrm{e}^{-\frac{3}{2}r}\,\mathbb{I}_{\frac{3}{2}} + \mathrm{e}^{-2r}\,\mathbb{I}_2 + o(\mathrm{e}^{-2r}), \tag{3.5}$$

where $\mathbb{I}_0 = \mathrm{Id}_\xi + \frac{1}{2}\,\mathrm{Id}_H$, $\mathbb{I}_1 \;=\; \frac{R}{4}\,\mathrm{Id}_H \;-\; J_0\tau$, and

$$\mathbb{I}_2 \;=\; \left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O}\right)\mathrm{Id}_\xi \;+\; \left(\frac{R^2}{16} + \frac{5}{2}|\tau|^2 + \mathcal{O}\right)\mathrm{Id}_H \;+\; \nabla_\xi\tau$$

and the precise value of $\mathbb{I}_{\frac{3}{2}}$ is irrelevant as before. As a *last step*, we will obtain below the desired contributions of the integral terms in Formula (3.1) by chasing the order e^{-2r}-terms.

A first easy consequence of the expression (3.5) of $\mathbb{I}$ is that the $\nabla_\xi\tau$-term may be forgotten: it would create a scalar term linear in $\nabla_\xi\tau$, and no such scalar invariant exists.

The contraction $\mathcal{T}$ is now explicitly described as follows: if symmetric endomorphisms A, B, and C are diagonal in a basis $\{e^0, e^1, e^2, e^3\}$ chosen to be $\overline{g}$-orthonormal and J-adapted, with eigenvalues α_r, β_s and γ_t, then:

$$\mathcal{T}(A\wedge B\wedge C) = \mathfrak{S}(\alpha_r\beta_s\gamma_t)\,\varpi, \tag{3.6}$$

where $\mathfrak{S}$ denotes the sum over all permutations of $\{r,s,t\}$; moreover, if ρ is a curvature term (endomorphism on 2-forms) with constant coefficients in the same basis with diagonal entries $K_{rs} =< \rho(e^r\wedge e^s), e^r\wedge e^s >$, and A is as above, then:

$$\mathcal{T}(A\wedge\rho) = \mathfrak{A}(K_{rs}\lambda_t)\,\varpi, \tag{3.7}$$

where $\mathfrak{A}$ denotes the sum over circular (not all) permutations of $\{r,s,t\}$.

These formulae make easy the evaluation of all possible order 2 terms but the one involving W_2^-: the computations are done in a basis $(\partial_r, \mathrm{e}^{-r}\,\xi, \mathrm{e}^{-\frac{r}{2}}\,h, \mathrm{e}^{-\frac{r}{2}}\,J_0h)$, with h a γ-unit element of H chosen to be an eigenvector of $J_0\tau$. This basis is orthogonal for $\overline{g}$ except for order $\frac{3}{2}$-terms, which we can neglect as usual, and for an order 2-term due to the presence of $\nabla_\xi\tau$ in the expression of $\overline{g}$. Taking into account this last term yields a scalar correction linear in $\nabla_\xi\tau$, which must necessarily vanish, hence one may also forget it. We summarize below the results, using the following notations: for any geometric quantity s, s_k denotes the k-th order term in its asymptotic expansion; we also denote the order 2 term in $\mathbb{I}$ as:

$$\mathbb{I}_2 \;=\; A\;\mathrm{Id}_\xi \;+\; B\;\mathrm{Id}_H \;+\; \nabla_\xi\tau.$$

The term $\mathcal{T}(\mathbb{I}\wedge\mathbb{I}\wedge\mathbb{I})$ is the sum of contributions of type $<\varpi_a,\mathbb{I}_b\wedge\mathbb{I}_c\wedge\mathbb{I}_d>$, with $a+b+c+d=2$; the results are:

involved terms	result
$\varpi_2,\mathbb{I}_0\wedge\mathbb{I}_0\wedge\mathbb{I}_0$	$1/16\,R^2-1/2\,\lvert\tau\rvert^2$
$\varpi_1,\mathbb{I}_1\wedge\mathbb{I}_0\wedge\mathbb{I}_0$ (3 terms)	$-3/4\,R^2$
$\varpi_0,\mathbb{I}_1\wedge\mathbb{I}_1\wedge\mathbb{I}_0$ (3 terms)	$3/2\,R^2-6\,\lvert\tau\rvert^2$
$\varpi_0,\mathbb{I}_2\wedge\mathbb{I}_0\wedge\mathbb{I}_0$ (3 terms)	$3/2\,A+6B$

And for the $\mathcal{T}(\mathbb{I}\wedge\mathcal{R})$-term, the results are:

involved terms	result
$\varpi_2,\mathbb{I}_0\wedge\mathcal{R}_0$	$-5/96\,R^2+5/12\,\lvert\tau\rvert^2$
$\varpi_1,\mathbb{I}_1\wedge\mathcal{R}_0$	$1/16\,R^2$
$\varpi_0,\mathbb{I}_2\wedge\mathcal{R}_0$	$-A-1/2\,B$

For the last term, *i.e.* $\mathcal{T}(\mathbb{I}_0\wedge W_2^-)$, we have to rely on the following explicit expression of the second order correction to the curvature, extracted from [4]. If $\omega_-^2=e^0\wedge e^2-e^1\wedge e^3$, and $\omega_-^3=e^0\wedge e^3-e^1\wedge e^2$, then

$$W_2^-=a\,\mathrm{e}^{-2r}((\omega_-^2)^2-(\omega_-^3)^2)+b\,\mathrm{e}^{-2r}(\omega_-^2\omega_-^3+\omega_-^3\omega_-^2),$$

a and b being reals. The definition of $\mathcal{T}$ then yields $\mathcal{T}(\mathbb{I}_0\wedge W_2^-)=0$. Putting together all the results obtained so far yields easily the expected Theorem. □

Appendix: variations of R and τ.

We give here a quick glimpse on the computations leading to the expression of the variation of the curvature quantity $T(\eta)=\frac{R^2}{16}-\frac{5}{2}|\tau|^2$ of the Tanaka-Webster connection under a conformal deformation of the contact form η.

Let η be a compatible contact form on X and f a positive function. We denote by $(\theta^1,\theta^{\bar1})$ a local orthonormal (complex) coframe for η, *i.e.* $d\eta=i\theta^1\wedge\theta^{\bar1}$. The variation of the Tanaka-Webster curvature R is well-known in dimension 3 [11]; if $\widehat{R}$ is the curvature for $f^2\eta$, and $\Delta u=u_{,\bar11}+u_{,1\bar1}$ is the sub-elliptic Laplacian, then

$$\widehat{R}=f^{-3}\,(2\,\Delta f+R\,f)\,. \tag{3.8}$$

For the torsion τ and in lack of a precise reference, we will detail the computation a little bit. Starting from $(\theta^1,\theta^{\bar1})$, a local orthonormal coframe for $f^2\eta$ is then given by $\hat\theta^1=f\left(\theta^1+2i\,f_{,\bar1}\,\eta\right)$ and its complex conjugate. The Tanaka-Webster connection 1-form ω_1^1 and torsion endomorphism $\tau_{\bar1}^1$ for η (resp. $f^2\eta$) are defined by $d\theta^1=-\omega_1^1\wedge\theta^1+\tau_{\bar1}^1\eta\wedge\theta^{\bar1}$ (resp. the same formula in the hatted version). Computing at a point where ω_1^1 is zero, one gets

$$\begin{aligned}d\hat\theta^1&=df\wedge\theta^1+f\,d\theta^1+2if_{,\bar1}\,d\eta+2i\,d\left(f_{,\bar1}\right)\wedge\eta\\&=3f_{,\bar1}\,\theta^{\bar1}\wedge\theta^1+2i\,(f_{,\bar11}+\frac{i}{2}f_{,0})\,\theta^1\wedge\eta+(2if_{,\bar1\bar1}-f\tau_{\bar1}^1)\,\theta^{\bar1}\wedge\eta.\end{aligned} \tag{3.9}$$

Identifying this with $-\hat\omega_1^1\wedge\hat\theta^1+\hat\tau_{\bar1}^1\,\hat\theta^{\bar1}\wedge f^2\eta$, it comes finally:

$$\hat\tau_{\bar1}^1=f^{-2}\,\left(\tau_{\bar1}^1-2i\,f^{-1}f_{,\bar1\bar1}-6if^{-2}f_{,1}\,f_{,\bar1}\right). \tag{3.10}$$

From these computations, the interested reader can easily derive the variation of $T(\eta)$ under conformal changes in η.

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