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A REMARK ON RENORMALIZED VOLUME AND EULER CHARACTERISTIC FOR ACHE 4-MANIFOLDS

MARC HERZLICH

ABSTRACT. This note computes the “renormalized volume” and a renormalized Gauss-Bonnet-Chern formula for asymptotically complex hyperbolic Einstein (the so-called ACHE) 4-manifolds.

1. INTRODUCTION

Asymptotically symmetric Einstein metrics exhibit many interesting phenomena [3, 8]. They were especially studied in the asymptotically *real hyperbolic* case, which enjoys fruitful relationships with physics through the ADS-CFT correspondence. In this setting, an intriguing invariant, called *renormalized volume*, has been defined by C. R. Graham [7] (after works by physicists such as Henningson and Skenderis [9]), and its role in the formula for the Euler characteristic of the manifold has been pointed out by M. T. Anderson [1], with applications in the study of the moduli space of asymptotically real hyperbolic Einstein metrics [2]. This formula is called “the renormalized Gauss-Bonnet-Chern formula” because all divergent terms due to non-compactness are shown to cancel, whereas renormalized volume appears as a finite limit contribution.

The goal of this short note is to point out an analogous formula in the case of asymptotically *complex* hyperbolic Einstein (ACHE) 4-manifolds. Unfortunately, the situation is less pleasant than in the real case, as we are unable to define a renormalized volume as an invariant of the the complete Einstein metric. It rather depends on the choice of some contact form (*i.e.* equivalently, a *pseudo-hermitian structure*) realising the CR-structure at infinity. Adding some well-chosen local quantity at infinity yields however an invariant, and some renormalized Gauss-Bonnet characteristic formula as well.

1.1. Theorem. *Let (M, g) be a 4-dimensional Einstein asymptotically complex hyperbolic manifold, with boundary at infinity a compact CR manifold X . If V is the renormalized volume of g associated to a choice of contact form η at infinity, and if R and τ are the Webster-Tanaka curvature and torsion of that structure, then*

$$\mathcal{V} = \frac{3}{2} V - \int_X \left(\frac{R^2}{16} - \frac{5}{2} |\tau|^2 \right)$$

is an invariant of the ACHE metric and

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(|W^g|^2 - \frac{1}{24} (\text{Scal}^g)^2 \right) + \frac{1}{4\pi^2} \mathcal{V}.$$

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As the model case of the complex hyperbolic plane shows, the appearance of the integral factor on the boundary seems unavoidable; see section 2 for further details. This shows than, rather than giving rise to a global invariant, the renormalized volume gives birth to a *conformal anomaly*, i.e. a formula relating the renormalized volume for some choice of pseudo-hermitian structure at infinity to its expression for some other choice at infinity, through a *local* differential expression. Namely, if we let $V(\eta)$ be the renormalized volume for a choice of contact form (pseudo-hermitian structure) at infinity,

1.2. Corollary. *For a contact form η and a function f on X which never vanishes,*

$$V(f\eta) - V(\eta) = \int_X P_\eta(f) \eta \wedge d\eta,$$

where P_η is a differential operator on X .

In the *real hyperbolic* case [7], the conformal anomaly is given by conformally invariant differential operators, such as the Paneitz operator or its generalizations in higher even dimensions. Our result in the *complex hyperbolic case* strongly suggests that it should be interesting to study the operator arising from the variation of the quantity

$$Q(\eta) = \frac{R^2}{16} - \frac{5}{2}|\tau|^2$$

under deformations of the contact form in the same contact structure.

2. DEFINITIONS AND NOTATIONS

Let (X^3, H, J_0) be a strictly pseudo-convex 3-dimensional CR manifold, i.e. a contact manifold with contact distribution H and almost complex structure J_0 on H . If η in any choice of compatible contact form, an associated metric γ may be defined on H by $\gamma = d\eta(\cdot, J_0\cdot)$. This induces a Reeb field ξ and a unique (Tanaka-Webster) connection whose torsion with respect to the Reeb field will be denoted by τ .

Let M be a 4-manifold such that the complement of some compact set is diffeomorphic to $]r_0, +\infty[\times X$. We consider first the metric $g_0 = dr^2 + e^{2r} \eta^2 + e^r \gamma$ on $]r_0, +\infty[\times X$ and let C_δ^∞ be the space of smooth functions on M such that $e^{\delta r} \nabla^k f$ is bounded for any k . Any metric g on M such that $g - (dr^2 + e^{2r} \eta^2 + e^r \gamma)$ belongs to C_δ^∞ for some $\delta > 0$ will be called an *asymptotically complex hyperbolic* metric. Moreover, (M, g) is said to be ACHE if g is an Einstein metric.

A lot of such metrics arise on pseudo-convex domains in \mathbb{C}^2 (and are Kähler-Einstein in this case [6]) whereas another important family was constructed by O. Biquard in [3] in the case the boundary at infinity X is endowed with a non-embeddable CR structure. In [4], the following asymptotic expansions were obtained for an ACHE metric g on a neighbourhood of infinity $]r_0, +\infty[\times X$, if a contact form η is given, $\tau_{b,c}^a$ are the matrix coefficients of its torsion τ and its Tanaka-Webster derivatives in a coframe $(\theta^1, \theta^{\bar{1}})$, and $\vartheta^0 = e^{-r} dr + i\eta$, $\vartheta^1 = \theta^1 - \phi_{\perp}\theta^{\bar{1}}$.

2.1. Proposition. *(i) there exists an integrable complex structure J given by a (not necessarily convergent) formal series, determined solely from data at infinity, whose expansion is given by $J = J_0 - 2e^{-r} \tau + e^{-2r}(2|\tau|^2 - J_0 \nabla_\xi \tau) + o(e^{-2r})$, or equivalently by a map $\phi = -ie^{-r} \tau + \frac{1}{2}e^{-2r} \nabla_\xi \tau + o(e^{-2r})$ from $T_{J_0}^{0,1}$ to $T_{J_0}^{1,0}$.*

(ii) there is a (formal series) Kähler-Einstein metric \bar{g} , whose Kähler form ω of \bar{g} is formally determined up to order 2 as follows

$$\begin{aligned} \omega &= e^r (dr \wedge \eta + d\eta) - \frac{R}{2} d\eta \\ &+ \frac{4}{3} \left(\frac{i}{8} R_{,\bar{1}} \vartheta^0 \wedge \theta^{\bar{1}} - \frac{i}{8} R_{,1} \vartheta^{\bar{0}} \wedge \theta^1 - \frac{1}{2} \tau_{\bar{1},1}^1 \vartheta^0 \wedge \theta^{\bar{1}} - \frac{1}{2} \tau_{1,\bar{1}}^{\bar{1}} \vartheta^{\bar{0}} \wedge \theta^1 \right) - \frac{\Delta R}{2} e^{-r} d\eta \\ &- \frac{2}{3} \left(\frac{R^2}{8} - |\tau|^2 - \frac{\Delta R}{6} + \frac{2i}{3} (\tau_{\bar{1},11}^1 - \tau_{1,\bar{1}\bar{1}}^{\bar{1}}) \right) e^{-r} dr \wedge \eta \\ &+ \frac{2}{3} \left(\frac{R^2}{8} - |\tau|^2 + \frac{\Delta R}{12} - \frac{i}{3} (\tau_{\bar{1},11}^1 - \tau_{1,\bar{1}\bar{1}}^{\bar{1}}) \right) e^{-r} d\eta + o(e^{-2r}). \end{aligned}$$

(iii) there exists an anti- J_0 -invariant symmetric bilinear form k on H and a unique diffeomorphism ψ asymptotic to identity at infinity such that $\psi^*g = \bar{g} + k e^{-r} + o(e^{-2r})$.

2.2. Corollary. *The Kähler metric \bar{g} is explicitly given by*

$$\begin{aligned} \bar{g} &= dr^2 + e^{2r} \eta^2 + e^r \gamma - \frac{R}{2} \gamma + 2\gamma(J_0 \tau \cdot, \cdot) + \frac{1}{6} (R_{,1} \theta^1 \circ \vartheta^{\bar{0}} + R_{,\bar{1}} \theta^{\bar{1}} \circ \vartheta^0) \\ &+ \frac{2i}{3} (\tau_{\bar{1},1}^1 \vartheta^0 \circ \theta^{\bar{1}} - \tau_{1,\bar{1}}^{\bar{1}} \vartheta^{\bar{0}} \circ \theta^1) - e^{-r} R \gamma(J_0 \tau \cdot, \cdot) - e^{-r} \gamma(\nabla_{\xi} \tau(\cdot), \cdot) \\ &- \frac{2}{3} \left(\frac{R^2}{8} - |\tau|^2 - \frac{\Delta R}{6} + \frac{2i}{3} (\tau_{\bar{1},11}^1 - \tau_{1,\bar{1}\bar{1}}^{\bar{1}}) \right) e^{-2r} (dr^2 + e^{2r} \eta^2) \\ &+ \frac{2}{3} \left(\frac{R^2}{8} - |\tau|^2 + \frac{\Delta R}{12} - \frac{i}{3} (\tau_{\bar{1},11}^1 - \tau_{1,\bar{1}\bar{1}}^{\bar{1}}) \right) e^{-r} \gamma + o(e^{-2r}), \end{aligned}$$

where $\alpha \circ \beta = \alpha \otimes \beta + \beta \otimes \alpha$ is the symmetrized product of forms.

The main drawback of these facts is the following : given any ACHE metric g and any choice of pseudo-hermitian structure at infinity realizing the CR structure induced by g , there is a unique diffeomorphism ψ asymptotic to the identity on X such that ψ^*g can be written as the sum of a formal Kähler-Einstein metric and a (not formally determined) term of order 2, plus lower order terms. From now on, we will forget the diffeomorphism ψ and, if the metric g is written this way in such coordinates, we will say that it is “in the Kähler gauge associated to the choice of pseudo-hermitian structure at infinity”.

2.3. Proposition. *Let g be an ACHE metric on M , written in a Kähler gauge associated to some choice of pseudo-hermitian structure at infinity. Then the volume of large coordinate balls $B(r)$ of radius r (complement of $]r, +\infty[\times X$ in M) has an asymptotic expansion: $\text{vol}_g B(r) = \pi^2 e^{2r} + v_1 e^r + V + o(1)$.*

The term V is called the *renormalized volume* of the metric g associated to the choice of pseudo-hermitian structure at infinity (it depends only on the choice of a structure at infinity). To check the proposition, just notice that the volume form of g only differs from that of \bar{g} at order $\frac{5}{2}$ since k is trace-free, and, in the volume form of the Kähler form ω , order $\frac{3}{2}$ terms do not exist whereas order 2 terms are of zero integral from Stokes’ formula.

2.4. *Remark.* In the asymptotically *real* hyperbolic Einstein case, the renormalized volume is similarly defined [7], but with the help of a different gauge. It is proved in [7] that it is always possible, for any such metric, to find coordinates so that $g = dr^2 + h(r)$ on $]r_0, +\infty[\times X$ (this is the “geodesic gauge”). The metric h on X has an expansion in powers of e^r and V is defined as above as the constant coefficient in the expansion of $\text{vol}(B(r))$ and is in itself an invariant of g . The reader might hence think that the “misbehaviour” of the renormalized volume in the *complex* case comes from a bad choice of gauge. However, the standard metric of \mathbf{CH}^2 is both in the Kähler and geodesic gauges, and Theorem 1.1 yields that the boundary term in the renormalized Gauss-Bonnet formula (which necessarily is an invariant of g) is *different* of the renormalized volume.

3. THE PROOF

We first choose a contact form (or pseudo-hermitian structure) at infinity realizing the CR structure and we put the ACHE metric in the associated Kähler gauge around infinity. The basic element of the proof then is the Gauss-Bonnet-Chern formula for a coordinate ball $B(r)$, whose boundary is the coordinate sphere $S(r) = \{r\} \times X$:

$$(3.1) \quad \begin{aligned} \chi(B(r)) &= \frac{1}{8\pi^2} \int_{B(r)} \left(|W|^2 - \frac{1}{24} \text{Scal}^2 \right) + \frac{1}{96\pi^2} \int_{B(r)} \text{Scal}^2 \\ &+ \frac{1}{12\pi^2} \int_{S(r)} \mathcal{T}(\mathbb{I} \wedge \mathbb{I} \wedge \mathbb{I}) + 3 \mathcal{T}(\mathbb{I} \wedge R), \end{aligned}$$

where \mathbb{I} is the shape operator of $S(r)$ and R is the curvature (2-form with values in 2-forms) of (M, g) , \wedge provides a $p + q$ -form with values in $\otimes^{r+s} TM$ from a p -form with values in $\otimes^r TM$ and a q -form with values in $\otimes^s TM$, and we have denoted by \mathcal{T} the contraction between the volume form of $S(r)$ and elements of $\otimes^3 TM$.

It is proven in [4] that the integral involving $|W|^2 - \frac{1}{24} \text{Scal}^2$ on $B(r)$ converges for an ACHE metric when r goes to infinity. Moreover, it is clear that both the scalar curvature integral (which is, up to a constant, $\text{vol} B(r)$) and the boundary integrals have an asymptotic expansion in powers of $e^{-\frac{r}{2}}$ (there are no polynomial terms as they cancel in the volume expansion, as noted above). Convergence of all the other terms implies that divergent terms cancel pairwise, whereas the limit as r goes to infinity of

$$\chi(M) - \int_{B(r)} \left(|W|^2 - \frac{1}{24} \text{Scal}^2 \right) - \frac{3}{8\pi^2} V$$

is given by the constant terms in the asymptotic expansion of the boundary integrals. Our task then reduces to a careful computation of these terms. For this, the following facts will be useful:

(i) it is proven in [4] that replacing g by \bar{g} in the boundary integrals the boundary integrals introduces terms that are $o(e^{-2r})$ only, hence do not contribute in the limit as the volume form of each sphere is $O(e^{2r})$ at most. Hence all computations can be done using the formal Kähler-Einstein metric \bar{g} rather than the ACHE metric g .

(ii) as the highest-order term in the volume form of $S(r)$ is $e^{2r} \eta \wedge d\eta$ (where η is the contact form underlying the chosen pseudo-hermitian structure chosen at infinity),

we will only need to track the order 2 terms in the computations below. Every asymptotic expansion we will use is of the following type:

$$A = A_0 e^{2r} + A_1 e^{-r} + A_{\frac{3}{2}} e^{-\frac{3}{2}r} + A_2 e^{-2r} + o(e^{-2r}).$$

As a result, order 2 terms may arise only when putting together an order 2 term with order 0 terms or two order 1 terms with order 0 terms. Order $\frac{3}{2}$ terms can hence be forgotten during the whole computation, *unless* when some differentiation is involved, as doing so along directions in X raises the order possibly by a factor $\frac{1}{2}$.

(iii) our final computation involves integration along X , hence each exact term can be forgotten. Using the CR Stokes' formula [5], this will be the case of every term involving $R_{,1\bar{1}}$, $R_{,\bar{1}1}$, $\Delta R = R_{,1\bar{1}} + R_{,\bar{1}1}$, $\tau_{1,1}^1$ or $\tau_{1,\bar{1}\bar{1}}^1$. In what follows, occurrence of such a term will be denoted by \mathcal{O} .

(iv) from [4] again, the curvature tensor R of any ACHE metric (seen as a 2-form with values in 2-forms) is, up to order 2, given by the sum of the model curvature tensor (*i.e.* that has the same values as the one of the model space in any basis $\{\partial_r, e^{-r}\xi, e^{-\frac{r}{2}}e, e^{-\frac{r}{2}}J_0e\}$ of the tangent space of $]r_0, +\infty[\times X$) and an order 2 term, called W_2^- and controlled by the Cartan tensor of the CR-structure at infinity.

From now on, the task can be divided into three steps: computation of the outer unit normal to $S(r)$ and its intrinsic volume form, computation of the shape operators (the only step that involves differentiation) and estimation of the order 2 terms in $\mathcal{T}(\mathbb{I} \wedge \mathbb{I} \wedge \mathbb{I})$ and $\mathcal{T}(\mathbb{I} \wedge R)$.

From the explicit expansion of \bar{g} , we get the *outer unit normal* of $S(r)$:

$$(3.2) \quad \nu(r) = \left(1 + \frac{1}{3} e^{-2r} \left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O} \right) \right) \partial_r + \nu^T + o(e^{-2r})$$

where ν^T is an order $\frac{3}{2}$ term tangent to X , involving $R_{,1}$, $R_{,\bar{1}}$, $\tau_{1,1}^1$ and $\tau_{1,\bar{1}}^1$. The *volume form* ϖ of $S(r)$ is then (up to forgotten order $\frac{3}{2}$ terms):

$$(3.3) \quad \begin{aligned} \varpi &= \frac{1}{2} \omega^2(\nu(r), \dots) = e^{2r} (1 + e^{-r} \varpi_1 + e^{-2r} \varpi_2) \eta \wedge d\eta + o(e^{-2r}) \\ &= e^{2r} \left(1 - \frac{R}{2} e^{-r} + \frac{1}{3} e^{-2r} \left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O} \right) \right) \eta \wedge d\eta + o(e^{-2r}). \end{aligned}$$

The *shape operator* is obtained by taking the extrinsic covariant derivative of the unit outer normal $\nabla \nu(r)$. As ν^T is an order $\frac{3}{2}$ term, only its derivatives in the direction of H might contribute to order 2 terms in \mathbb{I} , but it is an easy task to convince oneself that these would add only terms of vanishing integral, hence can be forgotten. It remains to compute the derivative of the first term in $\nu(r)$, seen first as a bilinear symmetric form. Keeping only symmetric terms in the usual 6-term formula for the covariant derivative, the only significant term is

$$\frac{1}{2} \left(1 + \frac{1}{3} e^{-2r} \left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O} \right) \right) \partial_r \bar{g}.$$

This is easily evaluated from the expansion of g recalled above and one gets

$$(3.4) \quad \begin{aligned} \bar{g}(\nabla\nu(r), \cdot) &= e^{2r} \eta^2 + \frac{1}{2} e^r \gamma + \frac{R}{2} e^{-r} \gamma(J_0\tau, \cdot) + \frac{1}{2} e^{-r} \gamma(\nabla_\xi\tau, \cdot) \\ &+ \frac{1}{3} \left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O} \right) \eta^2 - \frac{1}{6} \left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O} \right) e^{-r} \gamma. \end{aligned}$$

One step further, this yields the (endomorphism) shape operator, which we shall denote by $\mathbb{I} = \mathbb{I}_0 + e^{-r} \mathbb{I}_1 + e^{-\frac{3}{2}r} \mathbb{I}_\frac{3}{2} + e^{-2r} \mathbb{I}_2 + o(e^{-2r})$, where $\mathbb{I}_0 = \text{Id}_\xi + \frac{1}{2} \text{Id}_H$, $\mathbb{I}_1 = \frac{R}{4} \text{Id}_H - J_0\tau$, and

$$\mathbb{I}_2 = \left(\frac{R^2}{8} - |\tau|^2 + \mathcal{O} \right) \text{Id}_\xi + \left(\frac{R^2}{16} + \frac{5}{2} |\tau|^2 + \mathcal{O} \right) \text{Id}_H + \nabla_\xi\tau$$

and the precise value of $\mathbb{I}_\frac{3}{2}$ is irrelevant as before.

The *last step* is then easily done by using the following explicit description of the contraction \mathcal{T} : if a curvature term ρ (endomorphism on 2-forms) and symmetric endomorphisms A , B , and C are diagonal in a basis $\{e^0, e^1, e^2, e^3\}$ chosen to be $\{\partial_r, e^{-r} \xi, e^{-\frac{r}{2}} e, e^{-\frac{r}{2}} J_0 e\}$, with eigenvalues K_{rs} , α_r , β_s and γ_t , then:

$$(3.5) \quad \mathcal{T}(A \wedge \rho) = \mathfrak{A}(K_{rs} \lambda_t) \varpi, \quad \mathcal{T}(A \wedge B \wedge C) = \mathfrak{S}(\alpha_r \beta_s \gamma_t) \varpi,$$

where \mathfrak{A} (resp. \mathfrak{S}) denotes the sum over circular (resp. all) permutations of $\{r, s, t\}$.

This formula makes easy the evaluation of all possible order 2 term, but $\mathcal{T}(\mathbb{I}_0 \wedge W_2^-)$. We then rely on the following explicit expression of the second order correction to the curvature [4]. If $\omega_-^2 = e^0 \wedge e^2 - e^1 \wedge e^3$, and $\omega_-^3 = e^0 \wedge e^3 - e^1 \wedge e^2$, then

$$W_2^- = a e^{-2r} ((\omega_-^2)^2 - (\omega_-^3)^2) + b e^{-2r} (\omega_-^2 \omega_-^3 + \omega_-^3 \omega_-^2),$$

a and b being reals. The definition of \mathcal{T} then yields $\mathcal{T}(\mathbb{I}_0 \wedge W_2^-) = 0$. \square

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