Asymptotic behaviour of multiple scattering on infinite number of parallel demi-planes

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Abstract

The exact solution for the scattering of electromagnetic waves on an infinite number of parallel demi-planes has been obtained by J.F. Carlson and A.E. Heins in 1947 using the Wiener-Hopf method [7]. We analyze their solution in the semiclassical limit of small wavelength and find the asymptotic behaviour of the reflection and transmission coefficients. The results are compared with the ones obtained within the Kirchhoff approximation.

1 Introduction

For most systems classical mechanics corresponds to the $\hbar \to 0$ limit of quantum mechanics and the nature of classical motion to a large extent determines spectral statistics of quantum problems [1], [2]. Nevertheless, for certain models this correspondence breaks down because limiting classical Hamiltonian has singularities and there is no unique way to continue classical trajectories which hit them. Though classical mechanics itself is not complete, quantum mechanics in many cases smoothes singularities and associates with each of them a diffraction coefficient which gives a probability amplitude for the scattering to a given channel. The simple example of such models is given by diffractive models with small-size impurities where the total wave function at large distances from an impurity is a sum of the free wave function plus a reflected field. In two dimension in a convenient normalization

$$\Psi(\vec{r}\,) = e^{i\vec{k}\vec{r}} + \frac{D(\theta_f, \theta_i)}{\sqrt{8\pi kr}} e^{ikr - 3i\pi/4}. \tag{1}$$

$D(\theta_f, \theta_i)$ is the diffraction coefficient for the scattering of the incident plane wave of the direction θ_i to a reflected plane wave of the direction θ_f.

Though in classical mechanics such scatters are negligible, their presence perturbs greatly (and in a calculable way) quantum mechanical problems (see [3] and references therein).

An important class of diffractive systems consists of plane polygonal billirds (see e.g. [4]) with billiard corners playing the role of diffraction centers. The main difficulty in such type of diffractive models is the impossibility to represent everywhere the scattering field as a sum of a free motion plus small corrections as in Eq. (1) which forms the basis of the usual scattering theory.

As an example we present the exact results for the diffraction on a demi-plan derived by A. Sommerfeld in 1896 [5]. The total wave function for this problem obeying the Helmholtz equation

$$(\Delta + k^2)\Psi(z, x) = 0 \tag{2}$$

in the plan (z, x) with the Dirichlet boundary conditions along the semi-infinite screen $x \geq 0$ has the following form (in notations of Fig. 1)

$$\begin{aligned} \Psi(\vec{r}\,) &= e^{-ikr\cos(\theta_f - \theta_i)} F(-\sqrt{2kr}\cos\frac{\theta_f - \theta_i}{2}) \\ &- e^{-ikr\cos(\theta_f + \theta_i)} F(-\sqrt{2kr}\cos\frac{\theta_f + \theta_i}{2}), \end{aligned} \tag{3}$$

where $F(u)$ is the Fresnel integral

$$F(u) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_u^\infty e^{it^2} dt. \tag{4}$$

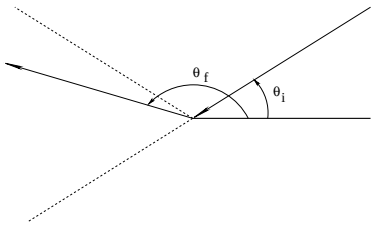

Figure 1: Scattering on the demi-plan (thick line). The initial and final rays are indicated by thin lines. The dotted lines represent optical boundaries.

From the expansion of $\Psi(\vec{r}\,)$ at large distance it follows (see [5]) that the diffraction coefficient for this problem is

$$D(\theta_f, \theta_i) = \frac{1}{\cos\frac{\theta_f - \theta_i}{2}} - \frac{1}{\cos\frac{\theta_f + \theta_i}{2}}. \tag{5}$$

The important specificity of this diffraction coefficient is that it formally blows up in two directions

$$\theta_f = \pi \pm \theta_i \tag{6}$$

called optical boundaries (see Fig. 1) which separate regions with different number of geometrical optic rays which play the role of classical trajectories. As wave fields are continuous, the separation of the exact field (3) in the sum of free motion plus small reflected fields is not possible in a vicinity of optical boundaries which manifests itself as the divergence of the diffraction coefficient (5).

In practice one can use diffractive coefficient description for all scattering angles except intermediate parabolic regions near optical boundaries where the dimensionless arguments of F-functions in Eq. (3) are of the order of 1

$$u = \sqrt{kr}\sin\frac{\delta\varphi}{2} \sim 1, \tag{7}$$

and $\delta\varphi$ is the deviation angle from the optical boundaries (6).

Difficult problems appear when inside these intermediate regions there are new points of singular diffractions which is inevitable e.g. for plane polygonal billiards. In the semiclassical limit $k \to \infty$ certain cases of multiple diffraction on singular corners have been computed within the Kirchhoff approximation in [6]. The long-range nature of the diffraction on sharp corners leads to a strong interaction between different singular points and no general formulas for exact diffraction coefficients in these cases are known.

In Refs. [7] and [8] the exact solution for the reflection of electromagnetic wave on an infinite number of parallel semi-infinite metallic sheets (equivalent to the scalar Helmholtz equation with the Dirichlet [7] and Neumann [8] boundary conditions) has been obtained by the Wiener-Hopf method (see also [9]). Exact expressions for reflection and transmission coefficients (Eqs. (30), (31), and (32) below) are quite cumbersome. They are given by infinite products of different terms each depending non-trivially on the initial momentum k. The purpose of this paper is the investigation of these results in the semiclassical limit $k \to \infty$.

The plan of the paper is the following. In Section 2 general properties of the reflection from periodic chain of demi-plans are discussed. In Section 3 the exact expressions for diffraction and transmission coefficients obtained in [7] are given and more tractable expressions for the modulus of these coefficients are presented. The most interesting case corresponds to the scattering with small incident angle when in the intermediate region (7) there are many singular points. In Section 4 two first terms of the expansion of the elastic reflection coefficient are derived. In Section 5 the first terms of expansion of the reflection coefficients for the forward scattering in the power of the incident angle are calculated and in Section 6 the same is done for large-angle scattering. In Section 7 these calculations are generalized for all powers of the incident angle provided the condition (7) is fulfilled. The exceptional case when demi-plans are perpendicular to the scattering plan is treated in Section 8. In Section 9 the exact asymptotics are compared with the ones calculated within the Kirchhoff approximation and in Section 10 the summary of obtained results is given.

2 Generalities

The configuration of half-planes considered in Refs. [7], [8] is represented schematically at Fig. 2. The equally spaced half-planes are parallel to the z-axis and their left corners form a scattering plane inclined by an angle α. The initial incident plane wave $\Psi_0(z,x)$ comes from the left with angle of incidence θ with respect to z-axis.

$$\Psi_0(z,x) = e^{ik(z\cos\theta + x\sin\theta)}. \tag{8}$$

In the far field due to the periodicity of the problem only finite number of waves can propagate. The reflected plane waves have the following form

$$\Psi_n^{(ref)}(z,x) = e^{ik(z\cos\theta_n' + x\sin\theta_n')}, \tag{9}$$

where the allowed values of reflected angle θ_n' are determined by the usual grating equation (see e.g. [5])

$$kd(\cos\varphi - \cos\varphi_n') = 2\pi n. \tag{10}$$

Here φ and φ_n' are the incident and reflected angles defined as the angles between the scattering plane and the incident (respectively, reflected) plane wave direction

$$\varphi = \alpha - \theta, \tag{11}$$

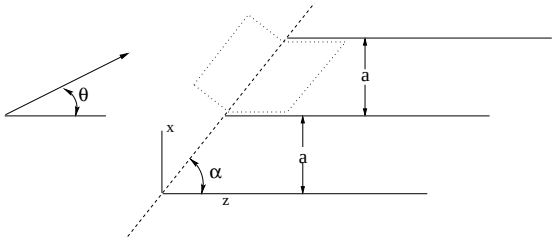

Figure 2: Scattering on half-planes staggered by an angle α. θ is the angle of incidence of the initial plane wave. Dashed line indicates the scattering plane. Dotted line is the contour used in the computation of the current conservation.

and

$$\varphi'_n = \theta'_n - \alpha. \tag{12}$$

$d = a/\sin\alpha$ is the distance along the scattering plane between corners of the half-planes, n is an integer.

In terms of the dimensionless momentum

$$Q = \frac{kd}{\pi} \tag{13}$$

Eq. (10) determines a finite number of possible reflected directions when φ is fixed

$$\cos\varphi'_n = \cos\varphi - \frac{2n}{Q}. \tag{14}$$

The allowed values of n are restricted by the inequalities followed from this expression

$$-Q\sin^2\frac{\varphi}{2} \le n \le Q\cos^2\frac{\varphi}{2}. \tag{15}$$

For later use it is convenient to introduce instead of the initial angle φ a new variable

$$u = \sqrt{Q}\sin\frac{\varphi}{2} \tag{16}$$

which is suitable for the description of intermediate regions (7) where the usual diffraction coefficient description can not be applied. To investigate the multiple

scattering in such regions, we consider below the limiting case $Q \to \infty$ with u kept fixed which corresponds to small initial angles of the order of $1/\sqrt{Q}$

$$\varphi \approx \frac{2u}{\sqrt{Q}}. \tag{17}$$

For reflected waves we shall distinguish between small angle reflection where the variable

$$u_n = \sqrt{Q} \sin \frac{\varphi'_n}{2} \tag{18}$$

remains finite when $Q \to \infty$ and large angle reflection otherwise.

In terms of variables (16) and (18) the grating equation (10) reads

$$u_n = \sqrt{n + u^2} \tag{19}$$

Physical values of reflection angles should obey the inequality $0 \le \varphi'_n \le \pi$ or $0 \le \sin \varphi'_n \le 1$. From (10) it follows that the physical branch is defined by the condition

$$\sin \varphi'_n = 2\sqrt{\frac{n+u^2}{Q}\left(1 - \frac{n+u^2}{Q}\right)} \ge 0. \tag{20}$$

In the notations of Fig. 2 the grating equation (10) takes the form used in [7]

$$k\rho - \omega_n b - a\sqrt{k^2 - \omega_n^2} = 2\pi n, \tag{21}$$

where

$$\rho = a \sin\theta + b \cos\theta = d \cos\varphi, \tag{22}$$

$b = a/\tan\alpha$ is the shift of the half-plane corners along the z-axis, and

$$\omega_n = k \cos\theta'_n = k \cos(\alpha + \varphi'_n) \tag{23}$$

is the projection of the final momentum on the z-axis. The positive (resp. negative) sign of $\sqrt{k^2 - \omega_n^2}$ correspond to forward (resp. backward) scattering.

In particular, the specular reflection corresponds to $n = 0$ in the above equation and

$$\omega_0 = k \cos(2\alpha - \theta) = k \cos(\alpha + \varphi). \tag{24}$$

The possible transmission modes correspond to motion in a straight tube

$$\Psi_m^{(trans)}(z, x) = e^{i\widetilde{\omega}_m z} \sin \frac{\pi}{a} m x, \tag{25}$$

where the transmitted frequencies are

$$\widetilde{\omega}_m = \sqrt{k^2 - \left(\frac{\pi m}{a}\right)^2} = k\sqrt{1 - \left(\frac{m}{Q'}\right)^2}. \tag{26}$$

Here

$$Q' = \frac{ka}{\pi} = Q \sin\alpha \tag{27}$$

and $1 \le m \le Q'$ is an integer.

3 Exact results

At large distances the total reflected field is the sum over the individual allowed propagating modes (9)

$$\Psi^{(ref)}(z,x) = \sum_{-Q\sin^2(\varphi/2)\le n\le Q\cos^2(\varphi/2)} R_n e^{ik[z\cos(\alpha+\varphi'_n)+x\sin(\alpha+\varphi'_n)]}, \qquad (28)$$

with φ'_n determined by Eq. (10) and the coefficients R_n in the sum are reflection coefficients which define the probability amplitudes of reflection to the given channel.

Similarly, the total transmitted field is the sum over individual transmitted modes (25)

$$\Psi^{(trans)}(z,x) = \sum_{1\le m\le Q\sin\alpha} T_m e^{i\widetilde{\omega}_m z}\sin\frac{\pi}{a}mx \qquad (29)$$

with $\widetilde{\omega}_m$ from Eq. (26) and the coefficients T_m are called the transmission coefficients.

We restrict ourselves to the case of the Dirichlet boundary conditions on the demi-plans treated in [7]. From the results of this work one can write down explicit expressions for reflection and transmission coefficients in the form of infinite products.

The reflection coefficients in Eq. (28) with $n = 0$ and $n \neq 0$ are slightly different. For $n = 0$ the reflection coefficient R_0 corresponds to the specular reflection $\varphi_0 = \varphi$ and we shall call it the elastic reflection coefficient. From [7] one gets

$$R_0 = -\frac{K(\omega')}{K(\omega_0)}. \qquad (30)$$

For non-zero n [7]

$$R_n = \frac{(\omega'-\omega_0)}{(\omega_n-\omega')(\omega_n-\omega_0)}\frac{K(\omega')}{K'(\omega_n)}. \qquad (31)$$

The transmission coefficients in the expansion (29) are given by the following expressions [7]

$$T_m = -\frac{\pi m(\omega'-\omega_0)}{a^2\widetilde{\omega}_m(\widetilde{\omega}_m-\omega')(\widetilde{\omega}_m-\omega_0)}\left[1-(-1)^m e^{i(k\rho-\widetilde{\omega}_m b)}\right]\frac{K(\omega')}{K(\widetilde{\omega}_m)}. \qquad (32)$$

In these formulas ω_0 is given by (24), ω_n are defined by (21) and

$$\omega' = k\cos\theta = k\cos(\alpha-\varphi). \qquad (33)$$

The function $K(\omega)$ is the ratio of two functions

$$K(\omega) = \frac{g_+(\omega)}{f_+(\omega)}e^{\chi\omega}, \qquad (34)$$

with

$$\chi = -i\frac{a}{\pi}[(\alpha - \frac{\pi}{2})\frac{1}{\tan\alpha} - \ln(2\sin\alpha)], \tag{35}$$

The denominator $f_+(\omega)$ in (34) is given by the convergent infinite product

$$f_+(\omega) = \prod_{n=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi n}\right)^2} - i\frac{\omega a}{\pi n}\right]e^{i\omega a/\pi n}. \tag{36}$$

The numerator $g_+(\omega)$ in (34) is the product of two functions

$$g_+(\omega) = G_1(\omega)G_2(\omega), \tag{37}$$

where $G_{1,2}(\omega)$ are represented as the following convergent infinite products

$$G_1(\omega) = \prod_{n=1}^{\infty}(\Delta_n - i\Psi_n)e^{(k\rho-\omega b+i\omega a)/(2\pi n)+i(\pi/2-\alpha)}, \tag{38}$$

$$G_2(\omega) = \prod_{n=-\infty}^{-1}(\Delta_n + i\Psi_n)e^{(k\rho-\omega b-i\omega a)/(2\pi n)-i(\pi/2-\alpha)}, \tag{39}$$

and

$$\Delta_n = \sqrt{\sin^2\alpha\left(1-\frac{k\rho}{2\pi n}\right)^2 - \left(\frac{ak}{2\pi n}\right)^2}, \tag{40}$$

$$\Psi_n = \frac{\omega a}{2\pi n\sin\alpha} + \cos\alpha\left(1-\frac{k\rho}{2\pi n}\right). \tag{41}$$

The functions $f_+(\omega)$ and $g_+(\omega)$ have no singularities or zeros in the upper half plane

$$\text{Im }\omega > \text{Im }\omega_0 \tag{42}$$

where one assumes that the momentum k has a small positive imaginary part.

These functions appear in the following Wiener-Hopf type factorization problem

$$f_+(\omega)f_-(\omega) = \frac{\sin[a\sqrt{k^2-\omega^2}]}{a\sqrt{k^2-\omega^2}}, \tag{43}$$

and

$$\frac{a^2+b^2}{2}(\omega-\omega_0)(\omega-\omega')g_+(\omega)g_-(\omega) = \cos[a\sqrt{k^2-\omega^2}] - \cos[k\rho-\omega b]. \tag{44}$$

Functions $f_-(\omega)$ and $g_-(\omega)$ have no singularities in the lower half plane

$$\text{Im }\omega < \text{Im }\omega'. \tag{45}$$

The explicit form of these functions is

$$f_-(\omega) = \prod_{n=1}^{\infty}\left[\sqrt{1-\left(\frac{ka}{\pi n}\right)^2} + i\frac{\omega a}{\pi n}\right]e^{-ia\omega/\pi n}, \tag{46}$$

and

$$g_-(\omega) = \widetilde{G}_1(\omega)\widetilde{G}_2(\omega), \tag{47}$$

where

$$\widetilde{G}_1(\omega) = \prod_{n=1}^{\infty}(\Delta_n + i\Psi_n)e^{(k\rho-\omega b-i\omega a)/(2\pi n)-i(\pi/2-\alpha)}, \tag{48}$$

$$\widetilde{G}_2(\omega) = \prod_{n=-\infty}^{-1}(\Delta_n - i\Psi_n)e^{(k\rho-\omega b+i\omega a)/(2\pi n)+i(\pi/2-\alpha)}. \tag{49}$$

The real zeros of the function $g_+(\omega)$

$$g_+(\omega_n) = 0 \tag{50}$$

correspond to pure imaginary values of Δ_n and coincide with the reflected frequencies (23). Their explicit form is

$$\omega_n = k\cos\alpha\left(\cos\varphi - \frac{2\pi n}{kd}\right) - k\sin\alpha\sqrt{\sin^2\varphi + \frac{4\pi n}{kd}\left(\cos\varphi - \frac{\pi n}{kd}\right)}. \tag{51}$$

In accordance with Eq. (20) the positive branch of the square root has to be chosen. ω_n with positive (resp. negative) n are real zeros of $G_1(\omega)$ (resp. of $G_2(\omega)$).

For the later use we need also the zeros ω_n^* of the function $g_-(\omega)$. They are given by Eq. (51) with the different sign of the square root

$$\omega_n^* = k\cos\alpha\left(\cos\varphi - \frac{2\pi n}{kd}\right) + k\sin\alpha\sqrt{\sin^2\varphi + \frac{4\pi n}{kd}\left(\cos\varphi - \frac{\pi n}{kd}\right)}. \tag{52}$$

The transmitted frequencies $\widetilde{\omega}_m$ are zeros of the function $f_-(\omega)$. They are given by (26). The zeros of $f_+(\omega)$ are just $-\widetilde{\omega}_m$.

The expressions (30), (31), and (32) for the reflection and transmission coefficients include infinite products and are quite cumbersome. Simpler formulas can be obtained for the modulus of these coefficients which is sufficient for many purposes. Using repeatedly the defining relations (43) and (44) one can demonstrate that the values of $|R_n|^2$ and $|T_n|^2$ are expressed by the following finite products

$$|R_0|^2 = \left|\prod_{n'\neq 0}\frac{\omega_0-\omega_{n'}^*}{\omega_0-\omega_{n'}}\prod_{m'\neq 0}\frac{\omega_0+\widetilde{\omega}_{m'}}{\omega_0-\widetilde{\omega}_{m'}}\right||K(\omega')|^2, \tag{53}$$

$$|R_n|^2 = \left|\frac{4\sin^2\alpha\sin^2\varphi}{(\omega_n-\omega')(\omega_n-\omega_0)}\right. \times \left.\prod_{\substack{n'\neq 0\\ n'\neq n}}\frac{\omega_n-\omega_{n'}^*}{\omega_n-\omega_{n'}}\prod_{m'\neq 0}\frac{\omega_n+\widetilde{\omega}_{m'}}{\omega_n-\widetilde{\omega}_{m'}}\right||K(\omega')|^2, \tag{54}$$

$$|T_m|^2 = \left| \frac{4\sin^2\varphi}{(\widetilde{\omega}_m - \omega')(\widetilde{\omega}_m - \omega_0)} \right.$$
$$\left. \times \prod_{n'\neq 0} \frac{\widetilde{\omega}_m - \omega^*_{n'}}{\widetilde{\omega}_m - \omega_{n'}} \prod_{\substack{m'\neq 0 \\ m'\neq m}} \frac{\widetilde{\omega}_m + \widetilde{\omega}_{m'}}{\widetilde{\omega}_m - \widetilde{\omega}_{m'}} \right| |K(\omega')|^2, \tag{55}$$

and

$$|K(\omega')|^2 = \left| \prod_{n'\neq 0} \frac{\omega' - \omega_{n'}}{\omega' - \omega^*_{n'}} \prod_{m'\neq 0} \frac{\omega' - \widetilde{\omega}_{m'}}{\omega' + \widetilde{\omega}_{m'}} \right|. \tag{56}$$

In Eds. (53)-(56) the products are taken over finite number of real reflected (51) and transmitted (26) frequencies.

As $\omega' - \omega_0 = 2\sin\alpha\sin\varphi$ and $\omega^*_0 = \omega'$, Eqs. (53) and (54) can be rewritten in a form which is valid for both $n = 0$ and $n \neq 0$

$$|R_n|^2 = \left| \left(\frac{\omega_0 - \omega'}{\omega_n - \omega'} \right)^2 \prod_{n'\neq n} \frac{\omega_n - \omega^*_{n'}}{\omega_n - \omega_{n'}} \prod_{m'\neq 0} \frac{\omega_n + \widetilde{\omega}_{m'}}{\omega_n - \widetilde{\omega}_{m'}} \right| |K(\omega')|^2. \tag{57}$$

If $n \neq 0$ the first product includes the term with $n' = 0$.

The scattering amplitudes fulfill the current conservation. In the configuration of Fig. 2 it is convenient to consider the current through the surface of a rectangle enclosing the scattering plane whose two sides are parallel to it. Due to the periodicity of the demi-planes this surface is reduced to the one indicated by dotted line in Fig. 2 and the current through boundaries perpendicular to the scattering plane can be ignored. In this case the total current through the parallel parts of this surface is zero and a simple calculation gives the following relation between the reflected and transmission coefficients

$$k\sin\varphi = k\sum_n \sin\varphi'_n \ |R_n|^2 + \sum_m \sin\alpha \ |T_m|^2 \sqrt{k^2 - \left(\frac{\pi m}{a}\right)^2}. \tag{58}$$

All expressions in this Section are exact and suitable for numerical calculations. But for theoretical purposes they are practically intractable, especially in the semiclassical limit $Q \to \infty$, because the number of factors in Eqs. (53)-(55) increases with Q and each factor depends non-trivially on Q.

In the next Sections we investigate these expressions when $Q \to \infty$. In this limit non-trivial results appear when the incident wave forms a small angle φ with the scattering plane (as in (17)) and we focus our attention on this region.

4 Elastic reflection coefficient

Let us consider first the behavior of the elastic reflection coefficient R_0 in Eq. (30) in the limit $\varphi \to 0$. Taking into account only the linear in φ terms direct calculations give

$$R_0 = -(1 + \beta\varphi) \tag{59}$$

and

$$\beta = i\sum_{n=1}^{\infty}\xi_n\left[\frac{2\sin\alpha}{\sqrt{(1-\xi_n^2)}-i\xi_n\cos\alpha}-\frac{1}{\sqrt{(\sin^2\alpha+\xi_n\sin\alpha)}+i\cos\alpha}\right.$$
$$\left.-\frac{1}{\sqrt{(\sin^2\alpha-\xi_n\sin\alpha)}-i\cos\alpha}\right]$$
$$-2iQ'\sin\alpha[(\alpha-\frac{\pi}{2})\frac{1}{\tan\alpha}-\ln(2\sin\alpha)], \qquad (60)$$

where $\xi_n = Q'/n$ and $Q' = ka/\pi = Q\sin\alpha$.

Separating the real and imaginary parts of Eq. (60) one obtains

$$\beta = Q'(C_1 + iC_2), \qquad (61)$$

where

$$C_1 = -2\sin\alpha\sum_{n=1}^{[Q']}\frac{1}{\sqrt{Q'^2-n^2}+Q'\cos\alpha}+\sum_{n=1}^{[Q]}\frac{1}{\sin\alpha\sqrt{(Q-n)n}+n\cos\alpha}$$
$$-\cos\alpha\sum_{n=1}^{[Q']}\frac{1}{n+Q'\sin\alpha}-\cos\alpha\sum_{n=[Q']+1}^{[Q]}\frac{1}{n-Q'\sin\alpha}, \qquad (62)$$

and

$$C_2 = \sin\alpha\left[\sum_{n=[Q']+1}^{\infty}\frac{\sqrt{n^2-Q'^2}}{n^2-Q'^2\sin^2\alpha}-\sum_{n=1}^{\infty}\frac{\sqrt{n+Q}}{\sqrt{n}(n+Q'\sin\alpha)}\right.$$
$$\left.-\sum_{n=[Q]+1}^{\infty}\frac{\sqrt{n-Q}}{\sqrt{n}(n-Q'\sin\alpha)}-2[(\alpha-\frac{\pi}{2})\frac{1}{\tan\alpha}-\ln(2\sin\alpha)]\right]. \qquad (63)$$

Here and below $[x]$ denotes the integer part of x such that $x = [x] + \{x\}$ and the fractional part $\{x\}$ obeys the inequality $0 \le \{x\} < 1$.

In the computations below we assume that $\alpha \neq 0$, $\pi/2$, and π. In the semiclassical limit $k \to \infty$ all sums in the above expressions include many terms which can be estimated by usual methods. The dominant contribution corresponds to the change of the summation to the integration. One gets

$$\widetilde{C}_1 = -2\sin\alpha\int_0^{Q'}\frac{dn}{\sqrt{Q'^2-n^2}+Q'\cos\alpha}+\int_0^{Q}\frac{dn}{\sin\alpha\sqrt{(Q-n)n}+n\cos\alpha}$$
$$-\cos\alpha\int_0^{Q'}\frac{dn}{n+Q'\sin\alpha}-\cos\alpha\int_{Q'}^{Q}\frac{dn}{n-Q'\sin\alpha} \qquad (64)$$

and

$$\widetilde{C}_2 = \sin\alpha\left[\int_{Q'}^{\infty}\frac{\sqrt{n^2-Q'^2}}{n^2-Q'^2\sin^2\alpha}dn-\int_0^{\infty}\frac{\sqrt{n+Q}}{\sqrt{n}(n+Q'\sin\alpha)}dn\right.$$
$$\left.-\int_Q^{\infty}\frac{\sqrt{n-Q}}{\sqrt{n}(n-Q'\sin\alpha)}dn-2[(\alpha-\frac{\pi}{2})\frac{1}{\tan\alpha}-\ln(2\sin\alpha)]\right]. \quad (65)$$

The integrals are elementary and cancel each other, i.e. $\widetilde{C}_1=\widetilde{C}_2=0$. One can prove that the first correction term (of the order of $1/\sqrt{Q}$) to C_1 appears from small n summation in the second term in Eq. (62)

$$C_1 \to \sum_{n=1}^{[Q]}\frac{1}{\sin\alpha\sqrt{(Q-n)n}+n\cos\alpha}+\text{smooth terms}$$
$$\to \frac{1}{\sin\alpha\sqrt{Q}}\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{\sqrt{n}}-\int_0^N\frac{dn}{\sqrt{n}}\right) \quad (66)$$

The integral is subtracted because we know that all integrals over n are canceled by other terms.

The last limit can be computed from the relation

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{\sqrt{n}}-\frac{\sqrt{N}}{2}\right)=\zeta(\frac{1}{2}) \quad (67)$$

where $\zeta(s)$ is the Riemann zeta function ($\zeta(\frac{1}{2})\approx -1.460354$).

Finally when $Q\to\infty$

$$C_1\to\frac{1}{\sin\alpha\sqrt{Q}}\zeta(\frac{1}{2}). \quad (68)$$

Similarly the dominant contribution to C_2 comes from small-n summation in the second term of Eq. (63)

$$C_2\to-\sin\alpha\sum_{n=1}^{\infty}\frac{\sqrt{n+Q}}{\sqrt{n}(n+Q'\sin\alpha)}+\text{smooth terms}\to-\frac{1}{\sin\alpha\sqrt{Q}}\zeta(\frac{1}{2}). \quad (69)$$

From (61) one concludes that for large k and $\alpha\neq 0,\pi/2,\pi$

$$\beta=\sqrt{Q}(1-i)\zeta(\frac{1}{2}). \quad (70)$$

In terms of variable (16) this result states that the two first terms of the expansion of the elastic reflection coefficient into the power of u are the following

$$R_0=-1-2\sqrt{2}e^{-i\pi/4}\zeta(\frac{1}{2})u. \quad (71)$$

In Section 9 we demonstrate that these two terms can be obtained from the Kirchhoff approximation developed in [6].

5 Small angle reflection in the limit $\varphi \to 0$

When the incident angle $\varphi \to 0$ it follows from Eqs. (31) and (32) that the reflection coefficients with $n \neq 0$ and all transmission coefficients are proportional to φ.

In the computation below it is convenient to rescale all frequencies by k i.e. to change

$$\omega \to \frac{\omega}{k}. \tag{72}$$

To simplify the notation we shall from now use the same symbols for the rescaled quantities.

When $\varphi \to 0$ rescaled Eqs. (51), (52), and (26) give

$$\omega_n = \cos\alpha \left(1 - \frac{2n}{Q}\right) - 2\sin\alpha \sqrt{\frac{n}{Q}\left(1 - \frac{n}{Q}\right)}, \tag{73}$$

$$\omega_n^* = \cos\alpha \left(1 - \frac{2n}{Q}\right) + 2\sin\alpha \sqrt{\frac{n}{Q}\left(1 - \frac{n}{Q}\right)}, \tag{74}$$

and

$$\widetilde{\omega}_m = \sqrt{1 - \frac{m^2}{Q^2 \sin^2\alpha}} \tag{75}$$

with all square roots chosen positive.

Using Eqs. (54) and (56) one can compute the reflection coefficient modulus in limit $k \to \infty$ by writing the logarithm of each product as a sum over the corresponding frequencies. E.g.

$$\log \prod_{n' \neq n} |\omega_n - \omega_{n'}| = \sum_{n' \neq n} \log |\omega_n - \omega_{n'}|.$$

When $k \to \infty$ the sum can be substituted by the integral. Exactly as it was done in the precedent Section one can check that all integrals in the full product (54) cancel and the dominant contributions come from regions with small factors.

We are interested first in the behaviour of the reflection coefficients R_n at fixed $n \neq 0$. In this case there are 3 regions with small factors. The first corresponds to $\omega_n - \widetilde{\omega}_{m'} \to 0$, the second appears when $\omega_n - \omega_{n'}^* \to 0$, and the third includes cases when n' is close to n (see Fig. (3)). In the first region one should expand the factors near $Q\sin^2\alpha$, i.e.

$$n' = Q\sin^2\alpha + \delta n, \tag{76}$$

where

$$\delta n = -\{Q\sin^2\alpha\} + q \tag{77}$$

with integer q and the similar expansion for m'. (As above $\{x\}$ denotes the fractional part of x.)

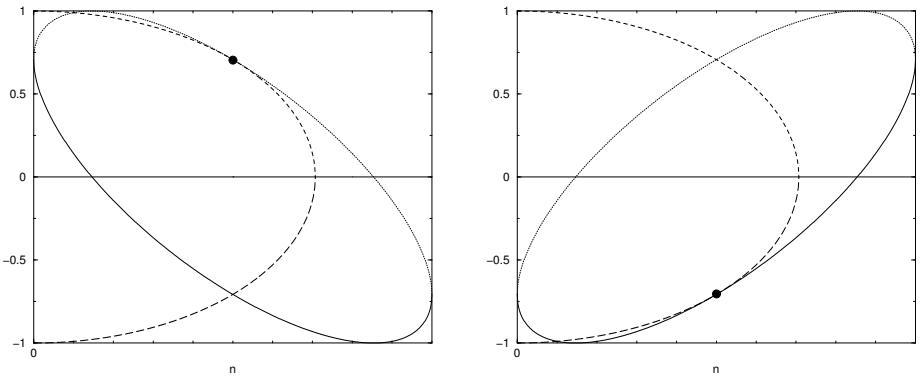

Figure 3: Dependence of different frequencies on n ($0 \le n \le Q$). Left: $0 < \alpha < \pi/2$, right: $\pi/2 < \alpha < \pi$. ω_n is represented by solid line, ω_n^* by dotted line, $\widetilde{\omega}_n$ by dashed line, and $-\widetilde{\omega}_n$ by long-dashed line. The full small circles indicate the point with coordinates $(Q\sin^2\alpha, \cos\alpha)$ where two different frequencies (with $\varphi = 0$) coincide.

The required series when $0 < \alpha < \pi/2$ are easily obtained from Eqs. (73)-(75)

$$\omega_n^*|_{n=n^*+\delta n} = \cos\alpha - \frac{\delta n}{Q\cos\alpha} - \frac{(\delta n)^2}{4Q^2\sin^2\alpha\cos^3\alpha} + \mathcal{O}\left((\frac{\delta n}{Q})^3\right), \qquad (78)$$

and

$$\widetilde{\omega}_m|_{m=n^*+\delta m} = |\cos\alpha| - \frac{\delta m}{Q|\cos\alpha|} - \frac{(\delta m)^2}{2Q^2\sin^2\alpha|\cos^3\alpha|} + \mathcal{O}\left((\frac{\delta m}{Q})^3\right). \quad (79)$$

For completeness we add

$$\omega_n|_{n=n^*+\delta n} = \cos\alpha(1 - 4\sin^2\alpha) + \mathcal{O}(\frac{\delta n}{Q}). \qquad (80)$$

At small $n \ll Q$

$$\omega_n = \cos\alpha - 2\sin\alpha\sqrt{\frac{n}{Q}} + \mathcal{O}(\frac{n}{Q}), \;\; \omega_n^* = \cos\alpha + 2\sin\alpha\sqrt{\frac{n}{Q}} + \mathcal{O}(\frac{n}{Q}) \qquad (81)$$

In the first region when n is small and $0 < \alpha < \pi/2$ one formally has

$$\prod_{m'}(\omega_n - \widetilde{\omega}_{m'}) \approx \prod_q \left(-2\sin\alpha\sqrt{\frac{n}{Q}} + \frac{-\{Q\sin^2\alpha\} + q}{Q\cos\alpha}\right). \qquad (82)$$

Similarly in the second region the expansion of $\omega_n - \omega_{n'}^*$ gives

$$\prod_{n'}(\omega_n - \omega_{n'}^*) \approx \prod_q \left(-2\sin\alpha\sqrt{\frac{n}{Q}} + \frac{-\{Q\sin^2\alpha\} + q}{Q\cos\alpha}\right). \qquad (83)$$

The two products (82) and (83) are identical and cancel each other in the expression for the reflection coefficient. The same remains true when $\pi/2 < \alpha < \pi$. Therefore a dominant contribution comes only from the region of small n' where

$$\prod_{n' \neq n} \frac{\omega_n - \omega^*_{n'}}{\omega_n - \omega_{n'}} \approx \prod_{n' \neq n}^{N} \frac{1 + \sqrt{\frac{n}{n'}}}{1 - \sqrt{\frac{n}{n'}}} \times \exp(-\int_0^N [\log(1 + \sqrt{\frac{n}{n'}}) - \log(1 - \sqrt{\frac{n}{n'}})] dn'). \quad (84)$$

The last term is added because the total integral is canceled by other terms. When $N \to \infty$ the integral equals $2\sqrt{n}\sqrt{N} + \mathcal{O}(1/N)$.

Introducing the convergence factors one gets that this product tends to the following function

$$f_n = \exp\left(2\sqrt{n}(\sum_{n' \neq n}^{N} \frac{1}{\sqrt{n'}} - \sqrt{N}/2)\right) \prod_{\substack{n' \neq n \\ n' \neq 0}}^{N} \frac{1 + \sqrt{\frac{n}{n'}}}{1 - \sqrt{\frac{n}{n'}}} e^{-2\sqrt{\frac{n}{n'}}}. \quad (85)$$

When $N \to \infty$ using Eq. (67) we obtain

$$f_n = e^{2\sqrt{n}\zeta(\frac{1}{2})-2} \prod_{\substack{n' \neq n \\ n' \neq 0}}^{\infty} \frac{1 + \sqrt{\frac{n}{n'}}}{1 - \sqrt{\frac{n}{n'}}} e^{-2\sqrt{\frac{n}{n'}}}. \quad (86)$$

Combining all factors together one concludes that in the limit $Q \to \infty$ the first term of the expansion of the modulus of the reflection coefficient at small $n \neq 0$ has the following asymptotics

$$|R_n|^2 \overset{\varphi \to 0}{\to} Q\varphi^2 |r_n|^2 = 4u^2 |r_n|^2, \quad (87)$$

where u is defined in (16) and

$$|r_n|^2 = \frac{|f_n|}{n}. \quad (88)$$

In Fig. 4 we present the results of numerical calculation of the first term of expansion of reflection coefficient from Eq. (54) together with asymptotic formula (88) for this quantity. The agreement becomes better at larger Q as it should be.

Notice that (when α is not too close to 0, $\pi/2$, and π) the reflection coefficient at small n do not depend on α. One can check that when $Q \to \infty$ the transmission coefficients with small m are always negligible.

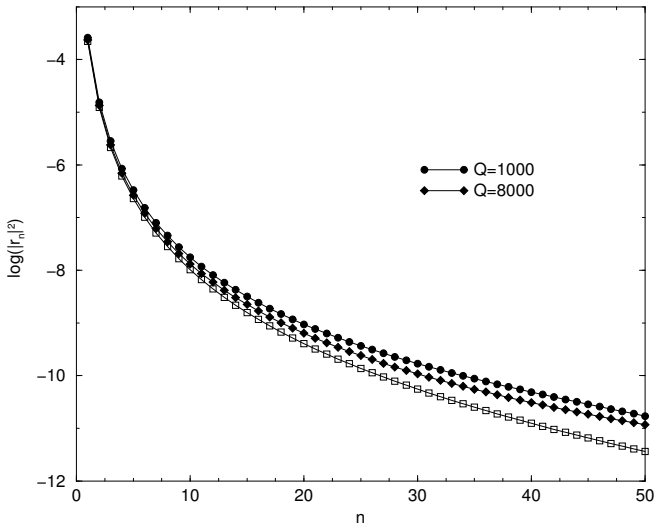

Figure 4: The first term of the expansion of the reflection coefficient at small $n \geq 1$ into a series of u. Two upper curves are $|r_n|^2$ computed at $Q = 1000$ and $Q = 8000$ with $\alpha = \pi/3$. The lower curve is the asymptotic expression (88) for this quantity.

6 Reflection and transmission coefficients at large n and $\varphi \to 0$

One can prove that for large n the only important contribution comes from n close to $n^* = Q\sin^2\alpha$ where two different frequencies coincide (see Fig. 3). When $0 < \alpha < \pi/2$ in this region only the transmission is noticeable and when $\pi/2 < \alpha < \pi$ only the reflection is important.

Let us consider first the case $0 < \alpha < \pi/2$. The expression (55) for the transmission coefficient modulus $|T_m|^2$ can conveniently be rewritten in the following form explicitly separating small factors

$$|T_m|^2 = \sin^2\varphi \left| \frac{2(\widetilde{\omega}_m - \omega_m^*)}{(\widetilde{\omega}_m - \omega_0)(\omega' - \omega_m^*)\widetilde{\omega}_m} \prod_{n' \neq m} \frac{\widetilde{\omega}_m - \omega_{n'}^*}{\omega' - \omega_{n'}^*} \prod_{m' \neq m} \frac{\omega' - \widetilde{\omega}_{m'}}{\widetilde{\omega}_m - \widetilde{\omega}_{m'}} \right.$$
$$\left. \times \prod_{n'} \frac{\omega' - \omega_{n'}}{\widetilde{\omega}_m - \omega_{n'}} \prod_{m'} \frac{\widetilde{\omega}_m + \widetilde{\omega}_{m'}}{\omega' + \widetilde{\omega}_{m'}} \right|. \tag{89}$$

In these formulas n' and m' are non zero and we assume that $m = Q\sin^2\alpha + \delta m$ where δm is of the order of $\sqrt{Q}$.

From Eqs. (78), (79), and (81) formal expansions for the above products can be obtained. The product over n' contains terms with small $n' = n$ and large

$n' = Q\sin^2\alpha + \delta n'$. Putting $\delta n' = \delta m + q$ and taking into account dominant terms one gets

$$\prod_{n'\neq m}(\widetilde{\omega}_m - \omega^*_{n'}) \approx \prod_{n\geq 1}\left(-\frac{\delta m}{Q\cos\alpha} - 2\sin\alpha\sqrt{\frac{n}{Q}}\right) \times \prod_{q\neq 0}\left(\frac{q}{Q\cos\alpha} - \frac{(\delta m)^2}{4Q^2\sin^2\alpha\cos^3\alpha}\right). \quad (90)$$

The first product comes from small n and the second one is due to the contributions with large $n' = Q\sin^2\alpha + \delta m + q$.

Computing other products in the similar way one obtains that the first term of the expansion of the transmission coefficient into a series of u has the form

$$|T_m|^2 \overset{u\to 0}{\to} \frac{8u^2}{Q\sin^2 2\alpha} g(u_f), \quad (91)$$

where the scaled variable u_f is related to m as follows

$$u_f = \frac{m - Q\sin^2\alpha}{\sqrt{Q}\sin 2\alpha}, \quad (92)$$

and the function $g(x)$ is

$$g(x) = e^{2\zeta(\frac{1}{2})x}\prod_{n=1}^{\infty}\left(1 + \frac{x}{\sqrt{n}}\right)^2\left(1 + \frac{x^2}{n}\right)e^{-\frac{2x}{\sqrt{n}}}. \quad (93)$$

The renormalization factor $e^{2\zeta(\frac{1}{2})x}$ has been computed exactly as it was done above for the reflection coefficient at small n.

When $\pi/2 < \alpha < \pi$ one can similarly show that transmission coefficients at small φ are negligible and the reflection coefficients with n close to $n = Q\sin^2\alpha$ differ only by the factor $1/2$ from the transmission ones with the same value of n and $\alpha = \pi - \alpha$

$$|R_n(\alpha)|^2 \overset{u\to 0}{\to} \frac{1}{2}|T_n(\pi - \alpha)|^2. \quad (94)$$

For clarity we explicitly show the dependence of α. The factor $1/2$ with respect to Eq. (91) is related with the different normalization of real transmitted modes (25) and complex reflected ones (9).

The formulas above are valid when u_f is fixed and $Q \to \infty$. The meaning of the variable u_f (92) is most easily seen when $\pi/2 < \alpha < \pi$. The scattering demiplans can be considered as a system of mirrors and the initial ray with $\varphi = 0$ after the specular reflection in these (demi) mirrors will form the angle $2\pi - 2\alpha$ with the scattering plane. For the reflection close to this optical boundary (cf. (6)) it is naturally to introduce a new variable ψ_n which measures the deviation of the reflected angle from the angle $2\pi - 2\alpha$

$$\varphi'_n = 2\pi - 2\alpha - \psi_n \quad (95)$$

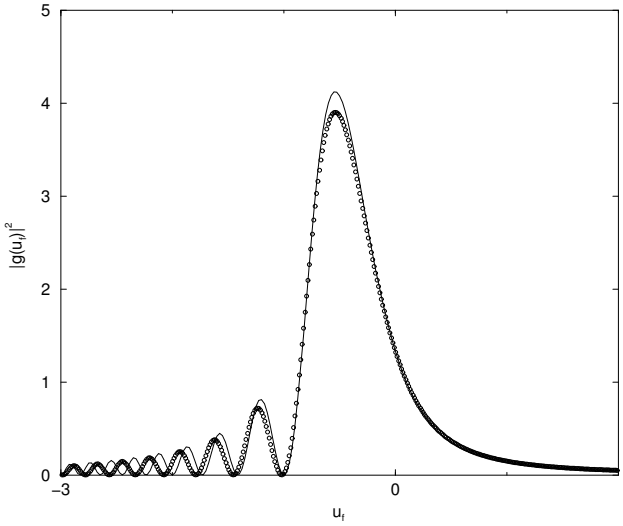

Figure 5: The first term of the expansion of the transmission coefficient for $Q = 1000$ and $\alpha = \pi/3$ into a series of u (dots). The solid line is asymptotic formula (93)

At large Q the variable u_f in (92) (with $m = n$) is related with ψ_n exactly in the same way as the variable u is related to the incident angle φ (see (16))

$$u_f = \sqrt{Q} \sin \frac{\psi_n}{2}. \tag{96}$$

We write u_f without the index n in order to stress that for large angle scattering the discreetness of reflected and transmitted waves plays no role and the asymptotic formulas (91), (94) can be considered as functions of a continuous variable u_f.

In Fig. (5) we present the results of numerical calculations of the function $g(u_f)$ calculated from Eq. (55) together with the asymptotic formula (91). The agreement is quite good. The reflection coefficient for $\pi/2 < \alpha < \pi$ is also well described by the asymptotic relation (94).

7 Reflection and transmission coefficients at finite u

Eqs. (87), (91) and (94) represent the first terms of the expansion of the transmission and reflection coefficients in the power of $u = \sqrt{Q} \sin(\varphi/2)$.

The purpose of this Section is to calculate these coefficients at finite values of u. The calculations are performed exactly as in the precedent Sections. First we note that the frequencies ω_n and ω_n^* depend on u only in the combination

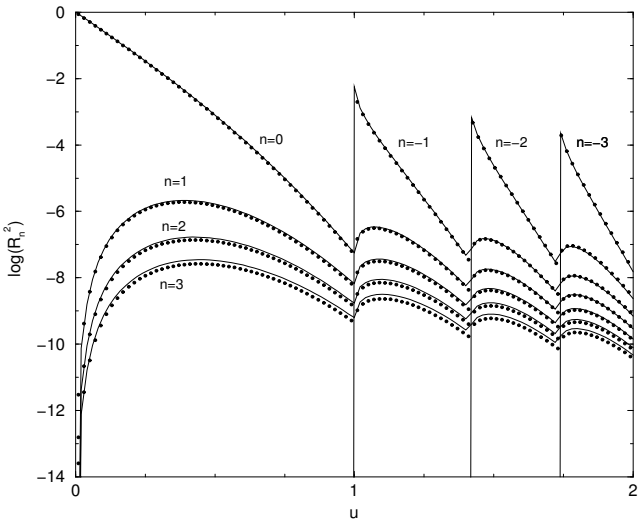

Figure 6: The reflection coefficients with $n = -3, \ldots, 3$ with $Q = 8000$ and $\alpha = \pi/3$ (solid lines) together with the asymptotic formula (97) (dots) for the same values of n.

$n(u) = n + u^2$. Therefore precedent formulas remain valid when n is substituted by $n(u) = n + u^2$ and the dominant contributions come from the same terms as above. We omit the details and present only the final answers.

The reflection coefficients at all small n (for negative $n \geq -[u^2]$, $n = 0$, and positive n) are given by

$$|R_n(u)|^2 = \left(\frac{2u}{u + \sqrt{n + u^2}} \right)^2 G(\sqrt{n + u^2}) G(u) \tag{97}$$

and the function $G(x)$ has the following form

$$G(x) = e^{2x\zeta(\frac{1}{2})} \prod_{n' \geq 1} e^{-2\frac{x}{\sqrt{n'}}} \prod_{\substack{n' \geq 0 \\ n' \neq [x^2]}} \left| \frac{1 + \frac{x}{\sqrt{n' + \{u^2\}}}}{1 - \frac{x}{\sqrt{n' + \{u^2\}}}} \right| \tag{98}$$

The terms with the same value of n' should be grouped together and then the total product converges.

In Fig. 6 the results of numerical calculations for $|R_n(u)|^2$ with $n = -3, \ldots, 3$, $Q = 8000$ and $\alpha = \pi/3$ are compared with the asymptotic formula (97) and the excellent agreement is found.

In the discussion of the scattering at large angles corresponding to n close to $Q \sin^2 \alpha$ it is convenient to introduce reduced transmission $t(u_f, u)$ and reflection

$r(u_f, u)$ coefficients in the following way

$$T_m = \frac{2}{\sqrt{Q}\sin 2\alpha} t(u_f, u), \tag{99}$$

and

$$R_n = \frac{2}{\sqrt{Q}\sin 2\alpha} r(u_f, u), \tag{100}$$

where the variable u defined by Eq. (16) fixes the initial angle and the variable u_f is related with the scattering angle by Eq. (92).

As above one can demonstrate that when $0 < \alpha < \pi/2$ only the transmission coefficients are important and when $\pi/2 < \alpha < \pi$ the reflection coefficients dominate and asymptotically in corresponding regions

$$|r(u_f, u)|^2 = \frac{1}{2}|t(u_f, u,)|^2 = g(u_f, u) \tag{101}$$

where

$$\begin{aligned} g(u_f, u) &= u^2 e^{2(u+u_f)\zeta(\frac{1}{2})} \prod_{n'\geq 1} e^{-2\frac{u+u_f}{\sqrt{n'}}} \prod_{\substack{n'\geq 0 \\ n'\neq [u^2]}} \left| \frac{1 + \frac{u_f}{\sqrt{n'+\{u^2\}}}}{1 - \frac{u}{\sqrt{n'+\{u^2\}}}} \right|^2 \\ &\times \prod_{n'\geq [u^2]+1} \left(1 + \frac{u_f^2 - u^2}{n'}\right). \end{aligned} \tag{102}$$

As above, the terms with the same n' should be combined together for convergence.

The reduced coefficients $t(u_f, u)$ and $r(u_f, u)$ introduced in Eqs. (99), (100) can be considered as the reflection and transmission coefficients for the scattering in the (continuous) interval of 'angles' u_f as it follows from the current conservation (58) rewritten in terms of new variables

$$u = \sum_{n\geq -[u^2]} \sqrt{n+u^2}|R_n(u)|^2 + 2\int_{-\infty}^{\infty} du_f g(u_f, u). \tag{103}$$

Here $R_n(u)$ mean the reflection coefficients at small n.

When $u \to \infty$ the precedent equations can considerably be simplified by noting that in this limit the products in these equations are large when u_f is close to $-u$. Putting $u_f = -u + \delta$ and taking into account the dominant terms one gets

$$g(u_f, u) \approx \left(\frac{\sin(2\pi(u_f + u)u)}{2\pi(u + u_f)}\right)^2. \tag{104}$$

Practically this approximation works well even with $u \geq 1$. The semiclassical derivation of this expression is performed in Section 9.

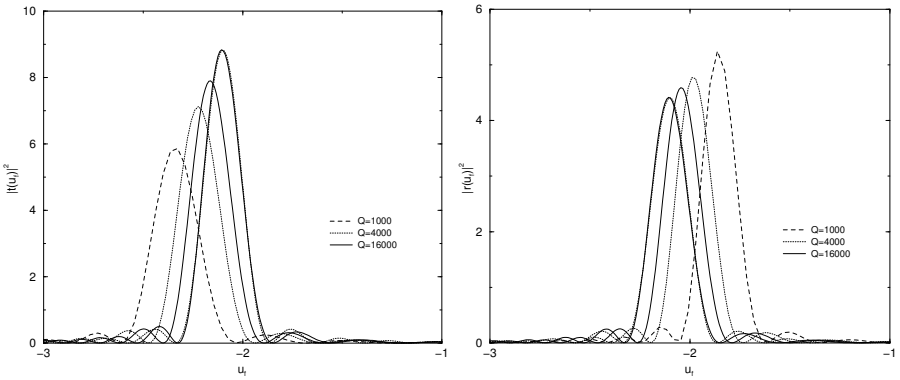

Figure 7: Reduced transmission coefficients with $\alpha = \pi/3$ (left) and the reduced reflection coefficients with $\alpha = 2\pi/3$ (right). In both figures $u = 2.1$. Dashed line: $Q = 1000$, dotted line: $Q = 4000$, solid line: $Q = 16000$. The thick solid line are the asymptotic formulas (102) and (101). The thick dotted line is the approximation (104).

In Fig. 7 we present numerically computed transmission coefficients with $u = 2.1$ and $\alpha = \pi/3$ and reflection coefficients with $\alpha = 2\pi/3$ for $Q = 1000$, $Q = 4000$, and $Q = 16000$ together with the asymptotic formulas (102) and (101). With increasing Q the agreement becomes better and better. The approximation (104) is hardly distinguished from the exact asymptotic formula (102).

8 Scattering when $\alpha = \pi/2$

The formulas of the precedent Sections are not valid when $\alpha = \pi/2$, i.e. when the demi-plans are perpendicular to the scattering plane. Nevertheless, asymptotic expressions for large Q can also be obtained in this case but the results depend on the fractional part of Q and, strictly speaking, the semiclassical limit $Q \to \infty$ does not exist. The main reason for such behaviour is related with the existence of a trivial solution for some particular values of Q. It is easily seen that when $\alpha = \pi/2$ the following function (in the notation of Fig. 2)

$$\Psi^{(0)}(x, y) = \sin(kx \cos\varphi) e^{ikz \sin\varphi} \tag{105}$$

is an exact solution of our problem (i.e. it vanishes at the demi-plans) provided

$$kd \cos\varphi = \pi l, \tag{106}$$

with integer l. In the notations (13) and (16) this relation takes the form

$$\{Q - 2u^2\} = 0, \tag{107}$$

where $0 \le \{x\} < 1$ is the fractional part of x.

Irrespectively how large is the dimensionless momentum Q, close to points when Eq. (107) is fulfilled, the appearance and disappearance of the exact solution (105) strongly perturb the reflection and transmission coefficients.

Though the limit $Q \to \infty$ does not exist, a special limiting case when $\{Q\}$ fixed and $[Q] \to \infty$ can be computed from the approach developed in the preceding Sections. We omit the details and present only the final formulas.

Let us define a function $W(t)$ as follows.

$$W(t) = e^{2(2-\sqrt{2})t\zeta(\frac{1}{2})} \prod_{n'\ge 1} e^{-2(2-\sqrt{2})\frac{t}{\sqrt{n'}}} \times {\prod_{n'\ge 0}}' \left(\frac{1+\frac{t}{\sqrt{n'+\{u^2\}}}}{1-\frac{t}{\sqrt{n'+\{u^2\}}}} \right) \left(\frac{1+\frac{t}{\sqrt{n'+\{Q-u^2\}}}}{1-\frac{t}{\sqrt{n'+\{Q-u^2\}}}} \right) \times \left(\frac{1-\frac{\sqrt{2}t}{\sqrt{n'+\{Q\}}}}{1+\frac{\sqrt{2}t}{\sqrt{n'+\{Q\}}}} \right). \tag{108}$$

We shall need this function at special values of the arguments $t = \sqrt{n+\{u^2\}}$, $t = \sqrt{n+\{Q-u^2\}}$ and $t = -\frac{1}{\sqrt{2}}\sqrt{n+\{Q\}}$ with integer n. The prime in the second product in Eq. (109) means that when $t = \sqrt{n+\{u^2\}}$ the term with $n' = n$ is omitted in the first factor, when $t = \sqrt{n+\{Q-u^2\}}$ the term with $n' = n$ is omitted in the second factor and when $t = -\frac{1}{\sqrt{2}}\sqrt{n+\{Q\}}$ the term with $n' = n$ is absent in the third factor.

The reflection coefficients at small n

$$|R_n(u)|^2 = \left(\frac{2u}{u+u_n} \right)^2 W(u_n)W(u), \tag{109}$$

where $u_n = \sqrt{n+u^2}$ determines the allowed value of small reflection angle (see (18) and (19)).

The reflection coefficients at large $n = [Q]-q$ is given by the same expression (109) but u_n is substituted by u_q

$$|R_q(u)|^2 = \left(\frac{2u}{u+u_q} \right)^2 W(u_q)W(u), \tag{110}$$

where

$$u_q = \sqrt{q+\{Q\}-u^2} \tag{111}$$

has the meaning of the allowed value of small deviation of reflected angle from its maximum possible value. It means that if one writes $\varphi'_n = \pi - \delta\varphi'_n$ then for $n = [Q]-q$ with large $[Q]$ and fixed q

$$u_q \approx \sqrt{Q}\sin\frac{\varphi'_{[Q]-q}}{2}. \tag{112}$$

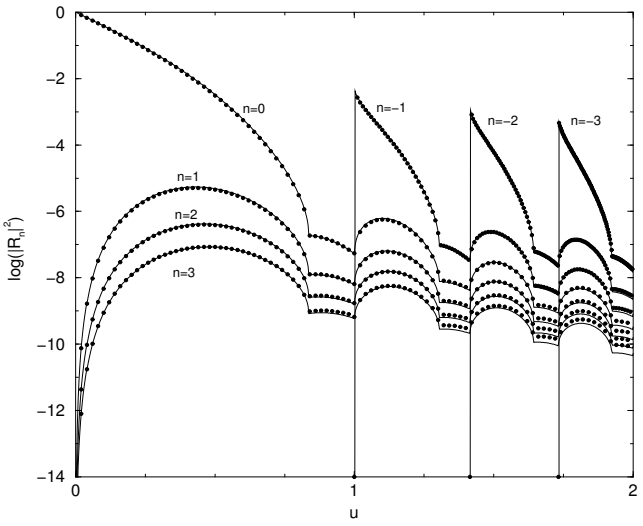

Figure 8: Reflection coefficients with $n = -3, \ldots, 3$ with $\alpha = \pi/2$ and $Q = 8000.7$ (solid lines) together with the asymptotic formula (109) (dots) for the same values of n and $\{Q\}$.

The reflection coefficient R_q with integer $q = 0, 1, \ldots,$ exists for a finite interval of u such that $0 \le u \le \sqrt{q + \{Q\}}$.

The transmission coefficients for $m = [Q] - p$ with $p = 0, 1, \ldots,$ are given by the same expression (109) but with the substitution $u_n \to u_p$

$$|T_p(u)|^2 = \left(\frac{2u}{u + u_p}\right)^2 W(u_p)W(u), \tag{113}$$

where

$$u_p = -\frac{1}{\sqrt{2}}\sqrt{p + \{Q\}}. \tag{114}$$

If for $\alpha = \pi/2$ one determines the angle of transmission ϕ_m from the natural relation $\cos\phi_m = m/Q$ then for $m = [Q] - p$ u_p is related with $\delta\phi_p = \pi - \phi_{m=[Q]-p}$ in the same way as above

$$u_p \approx \sqrt{Q}\sin\frac{\delta\phi_{[Q]-p}}{2}. \tag{115}$$

Ay Figs. 8-10 we present the the results of numerical calculations of the reflection and transmission coefficients together with the above asymptotic formulas. The agreement is very good.

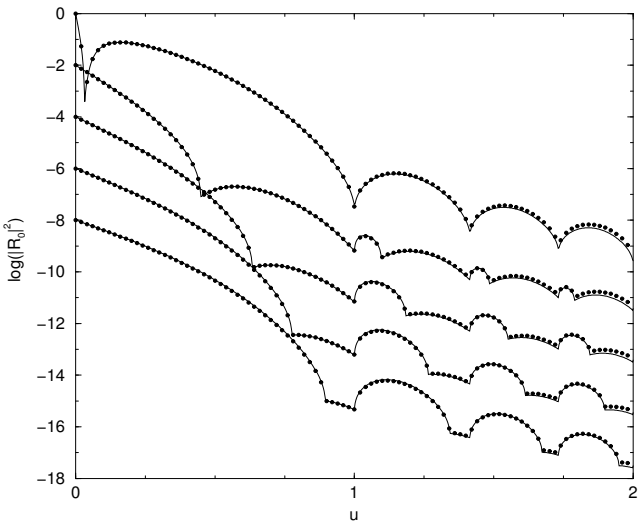

Figure 9: Elastic reflection coefficient with $\alpha = \pi/2$ and $[Q] = 8000$ for different values of fractional part of Q (solid lines). From top to bottom $\{Q\} = .001, 0.201, 0, 401, 0.601, 0.801$. For clarity each curve is lowered with respect to the precedent by 2 units. Dots represent the asymptotic formula (109).

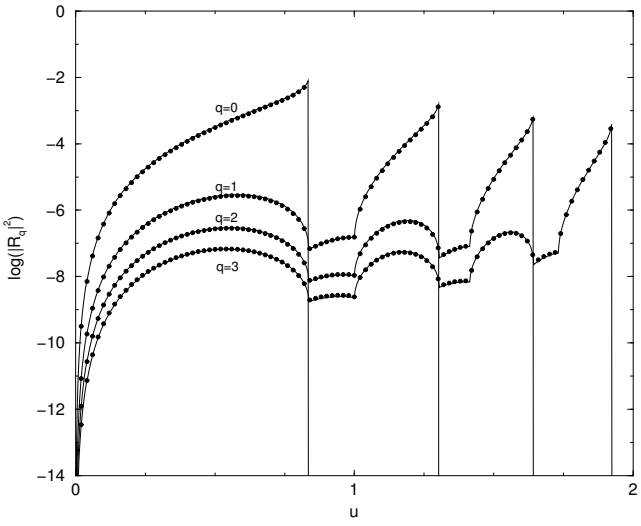

Figure 10: Reflection coefficients with $\alpha = \pi/2$ and $Q = 8000.7$ for the largest values of $n \equiv [Q] - q$ with $q = 0, \ldots, 3$ (solid lines). Dots represent the asymptotic formula (110).

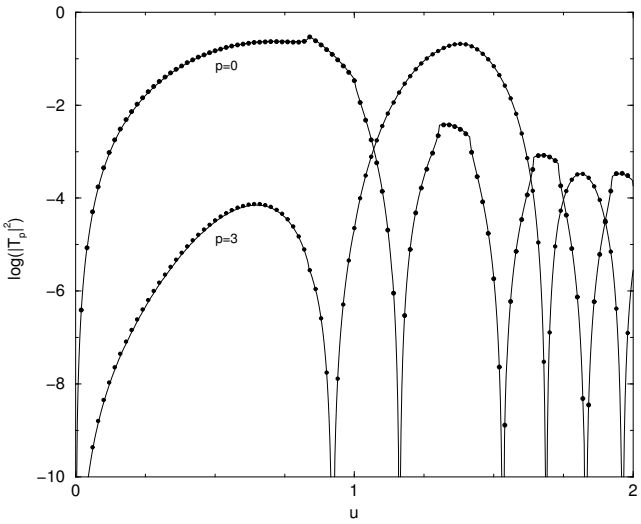

Figure 11: Transmission coefficients with $\alpha = \pi/2$ and $Q = 8000.7$ for the largest values of $m \equiv \{Q\} - p$ with $p = 0$ and $p = 3$ (solid lines). Dots represent the asymptotic formula (113).

9 Kirchhoff approximation

Semiclassical limit of different cases of multiple diffraction near the optical boundaries have been considered in Ref. [6]. In particular in this paper the contribution to the trace formula from diffractive orbits close to n-fold repetition of a primitive diffractive orbit has been calculated.

In the Kirchhoff approximation this contribution is given by the diagram of Fig. 12. Each line in this figure corresponds to the free Green function

$$G_0(\vec{x}, \vec{x}') = \frac{e^{ikl - 3\pi i/4}}{\sqrt{8\pi kl}}, \tag{116}$$

where $l = |\vec{x} - \vec{x}'|$ is the distance between two points $\vec{x}$, $\vec{x}'$ and $k = \sqrt{E}$ is the momentum. Each circle at Fig. 12 describes the convolution of two Green functions and in the Kirchhoff approximation it gives the factor $-2ik$. The role of the obstacles (corners) consists in the restriction of the integration over y_i to the half line.

In Ref. [6] it was demonstrated that the contribution to the trace formula from such trajectories has the form

$$\rho^{(diff)}(E) = -\frac{l}{16\pi k} A_n e^{ikln} + c.c., \tag{117}$$

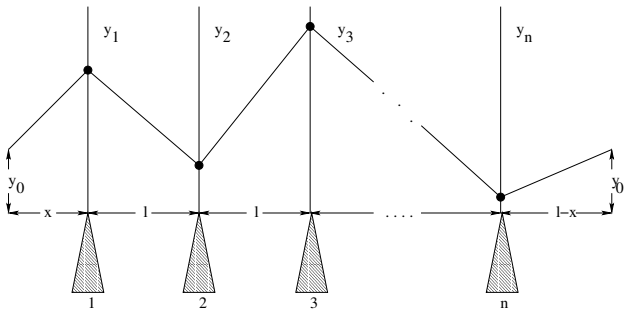

Figure 12: Schematic representation of multiple diffraction near optical boundary of n equally spaced obstacles.

where A_n is given by the following n-fold integral

$$A_n = \frac{4e^{-i\pi n/4}}{\pi^{n/2}} \int_0^\infty ds_1 \left[\int_{-\infty}^\infty ds_2 \ldots \int_{-\infty}^\infty ds_n - \int_0^\infty ds_2 \ldots \int_0^\infty ds_n \right] e^{i\Phi(\vec{s}\,)} \tag{118}$$

with

$$\Phi(\vec{s}\,) = (s_1 - s_2)^2 + (s_2 - s_3)^2 + \ldots + (s_{n-1} - s_n)^2 + (s_n - s_1)^2. \tag{119}$$

The presence a discrete symmetry in this quadratic form permits the analytical computation of this integral and [6]

$$A_n = \frac{1}{\pi} \sum_{q=1}^{n-1} \frac{1}{\sqrt{q(n-q)}}. \tag{120}$$

When $n \to \infty$ the sum over q can be substituted by the integral and

$$\lim_{n\to\infty} A_n = \frac{1}{\pi} \int_0^n \frac{dq}{\sqrt{q(n-q)}} = 1 \tag{121}$$

and as it was noted in Ref. [6] the contribution (117) in this limit coincides with the contribution from the boundary trajectory reflected from a straight mirror with the Dirichlet boundary condition.

Let us consider this point in details. Assume that after each reflection with a mirror a trajectory gets a reflection coefficient R. Then its contribution to

the trace formula computed in the usual manner is

$$\rho^{(boundary)}(E) = -2R\frac{e^{ikL-3\pi i/4}}{16\pi\sqrt{k}}\int_0^L \frac{dl}{\sqrt{l(L-l)}}\int_0^\infty dy e^{i2ky^2(1/l+1/(L-l))} + c.c. = R\frac{Le^{ikL+i\Phi}}{16\pi^2 k} + c.c. \quad (122)$$

Here L is the length of the trajectory and the factor 2 takes into account that each trajectory can be passed in two directions. For simplicity we do not consider the symmetry factor and consider the trajectory as being primitive.

Comparing this result with Eqs. (117) and (121) one concludes that these equations can be interpreted as the reflection from a mirror with the effective reflection coefficient $R = -1$ as from a boundary with the Dirichlet boundary conditions.

To compute the behaviour of the reflection coefficient for large but finite n it is convenient to use Eq. (67) from which it follows that when $n \to \infty$

$$A_n \to 1 + \frac{2\zeta(\frac{1}{2})}{\pi\sqrt{n}}. \quad (123)$$

One can incorporate this result into the above picture of the reflection from a straight mirror by assuming that the reflection coefficient R depends on the reflection angle φ (or transverse momenta $p_y \approx k\varphi$)

$$R(\theta) = -(1 + \beta\varphi). \quad (124)$$

As in the saddle point approximation $\varphi \approx 2y/L$, the only modification of the above calculation is the following integral

$$\int_0^\infty dy(1 + \beta\frac{2y}{L})e^{2iy^2/L} = \frac{\sqrt{\pi L}e^{i\pi/4}}{\sqrt{8k}}(1 + \beta\frac{\sqrt{2}e^{i\pi/4}}{\sqrt{k\pi L}}). \quad (125)$$

Because $L = ln$ one concludes from (123) that

$$\beta = \sqrt{2\frac{kl}{\pi}}e^{-i\pi/4}\zeta(\frac{1}{2}), \quad (126)$$

which up to notations agrees with Eq. (71) obtained by the direct expansion of the exact solution.

The above considerations demonstrate that small-angle singular reflection from a periodic set of demi-plans can be interpreted as much simpler process of the specular reflection from a straight mirror whose reflection coefficient at small angles has behavior as in Eqs. (124) and (126). Of course, at small distances reflection fields for two processes are very different but at large distances they are equivalent.

Similar (but simpler) considerations permit also to understand the approximation (104) for the reflection (and transmission) at large angles.

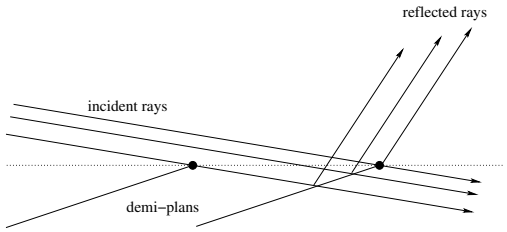

Figure 13: Schematic representation of the scattering at large angles. Black circles are the corners of the scattering demi-plans. The dotted line represents the scattering plane.

At Fig. 13 we represent schematically the configuration important for the large-angle scattering. The rays reflect from small parts of the scattering demi-plans restricted by the indicated corners. The width of each effective mirror is

$$\Delta = d \sin \varphi, \tag{127}$$

where, as above, d is the distance between singular corners along the scattering plane and φ is the scattering angle. After unfolding the amplitude of such reflection can be calculated in the Kirchhoff approximation as the transmission through a slit of the same width by the usual formula (an additional minus sign is due to the reflection from a mirror)

$$D(\epsilon) = 2ik \int_{-\Delta/2}^{\Delta/2} e^{-iky \sin \epsilon} dy = \frac{4i}{\sin \epsilon} \sin \left(\frac{1}{2} k \Delta \sin \epsilon \right), \tag{128}$$

where ϵ is the deviation of the scattering angle from the direction of the specular reflection.

The total reflected field is the sum over all diffracted fields

$$\Psi^{(ref)}(x, y) = \sum_{m=-\infty}^{\infty} e^{ikdm \cos \varphi} D(\delta\varphi') \frac{e^{ikR_m - 3\pi i/4}}{\sqrt{8\pi k R_m}}, \tag{129}$$

where $R_m = \sqrt{(x - md)^2 + y^2}$ is the distance between the m^{th} diffractive center and the point of observation with coordinates (x, y). Using the Poisson

summation formula one gets

$$\Psi^{(ref)}(x,y) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dm e^{iS(m,n)} D(\epsilon) \frac{e^{-3\pi i/4}}{\sqrt{8\pi k R_m}}, \tag{130}$$

where

$$S(m,n) = -2\pi mn + k(dm\cos\varphi + R_m). \tag{131}$$

When $k \to \infty$ one can use the saddle point method. The saddle point is obtained from the condition $dS(m,n)/dm = 0$ which gives the grating equation (10). The computation of the second derivative and the resulting integral leads to

$$\Psi^{(ref)}(x,y) = \sum_n \frac{D(\epsilon_n)}{2ikd\sin\varphi'_n} e^{ik(x\cos\varphi'_n + y\sin\varphi'_n)}, \tag{132}$$

where the reflected angle φ'_n is defined by Eq. (10).

Using Eq. (128) for the diffraction coefficient of the reflection on a small mirror and taking into account that in the important region $\varphi'_n \approx 2\pi - 2\alpha$ one obtains the following expression for reflection coefficient R_n

$$R_n = -\frac{2}{kd\sin 2\alpha \sin\epsilon_n} \sin(\frac{1}{2} kd \sin\varphi \sin\epsilon_n). \tag{133}$$

As $\varphi \approx 2u/\sqrt{Q}$ and $\epsilon_n = -2(u_f + u)/\sqrt{Q}$ with $Q = kd/\pi$, the last expression equals

$$R_n = -\frac{2}{\sqrt{Q}\sin 2\alpha} \left[\frac{\sin 2\pi u(u_f + u)}{2\pi(u_f + u)} \right], \tag{134}$$

whose modulus coincides with Eq. (104).

10 Summary

The exact transmission and reflection coefficients for the scattering on infinite number of parallel demi-plans obtained in Ref. [7] are analyzed in semiclassical limit of large momentum.

The most interesting (and difficult) case of small incident angle is considered. More precisely, the incident angle φ is chosen in such a manner that approximately $\varphi = 2u/\sqrt{Q}$ where Q is the dimensionless momentum. The limit considered corresponds to $Q \to \infty$ with fixed u.

It is demonstrated that at small final angles (of the order of $u_n/\sqrt{Q}$) the transmission is always negligible. The modulus of reflection coefficients in this case are independent of Q and the angle α of inclination of the demi-plans and are given by Eq. (97). The reflection coefficients decay quickly with n (i.e. with increasing of reflection angle) and $\sum_n \sqrt{n}|R_n|^2$ converges. The largest reflection coefficient corresponds to the smallest possible angle of reflection i.e. $n = -[u^2]$. For very small incident angle the elastic scattering corresponded to the specular reflection dominates.

The large angle scattering is noticeable only close to the specular reflection from the demi-plans. When $0 < \alpha < \pi/2$ only the transmission is large and the large-angle reflection can be neglected. The transmission coefficients have the form $t(u_f, u)/(\sqrt{Q}\sin 2\alpha)$ where $|f(u, u_f)|^2 = 2g(u_f, u)$ and $g(u_f, u)$ is given by Eq. (102). Here u_f is the deviation from the angle of mirror reflection magnified by the factor $\sqrt{Q}/2$ exactly as u is related with the incident angle. For $\pi/2 < \alpha < \pi$ the transmission is small and the reflection coefficients have similar asymptotics.

The exceptional case of demi-plans perpendicular to the scattering plane is characterized by the dependence of the fractional part of the momentum. In the limit $\{Q\}$ fixed and $[Q] \to \infty$ the reflection and transmission coefficients are independent on $[Q]$ and are given by Eqs. (109) – (113).

The first two terms of expansion of the exact elastic reflection coefficient into powers of small incident angle can also be obtained from the results of Ref. [6] where the Kirchhoff approximation for multiple scattering was developed. The large angle scattering is well described by the usual Kirchhoff approximation when the initial parameter $u \geq 1$.

References

[1] M.V. Berry and M. Tabor, Level clustering in the regular spectrum, *Proc. R. Soc. Lond.* A **356** (1977) 375-394.

[2] O. Bohigas, M.-J. Giannoni, and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation laws, *Phys. Rev. Lett.* **52** (1984) 1-4.

[3] E. Bogomolny and O. Giraud, Semiclassical calculations of the two-point correlation form factor for diffractive systems, *Nonlinearity* **15** (2002) 993-1018.

[4] P.J. Richens and M.V. Berry, Pseudointegrable systems in classical and quantum mechanics, *Physica* D **2** (1981) 495-512.

[5] A. Sommerfeld, *Optics* (Academic Press, New York, 1954).

[6] E. Bogomolny, N. Pavloff, and C. Schmit, Diffractive corrections in the trace formula for polygonal billiards, *Phys. Rev.* E **61** (2000) 3689-3711.

[7] J. F. Carlson and A. E. Heins, The reflection of an electromagnetic plane wave by an infinite set of plates I, *Quart. Appl. Math.* **4** (1947) 313-379.

[8] A. E. Heins and J. F. Carlson, The reflection of an electromagnetic plane wave by an infinite set of plates II, *Quart. Appl. Math.* **5** (1947) 82-88.

[9] B. Noble, *Methods based on the Wiener-Hopf technique* (Chelsea, New York, 1988).