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► **To cite this version:**

Roland Bacher. The Orchard relation of a generic symmetric or antisymmetric function. 2003. hal-00000317

HAL Id: hal-00000317

<https://hal.science/hal-00000317>

Preprint submitted on 18 Apr 2003

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The Orchard relation of a generic symmetric or antisymmetric function

Roland Bacher

April 18, 2003

Abstract: We associate to certain symmetric or antisymmetric functions on the set $\binom{E}{d+1}$ of $(d+1)$ -subsets in a finite set E an equivalence relation on E and study some of its properties.

1 Definitions and main results

We consider a finite set E and denote by $\binom{E}{d}$ the set of subsets containing exactly d elements of E . In the sequel we move often freely from sets to sequences: we identify a subset $\{x_1, \dots, x_d\} \in \binom{E}{d}$ with the finite sequence (x_1, \dots, x_d) where the order of the elements is for instance always increasing with respect to a fixed total order on E .

A function $\varphi : \binom{E}{d} \rightarrow \mathbf{R}$ is *symmetric* if

$$\varphi(x_1, \dots, x_i, x_{i+1}, \dots, x_d) = \varphi(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+1}, \dots, x_d)$$

for $1 \leq i < d$ and all $\{x_1, \dots, x_d\} \in \binom{E}{d}$.

Similarly, such a function $\varphi : \binom{E}{d} \rightarrow \mathbf{R}$ is *antisymmetric* if

$$\varphi(x_1, \dots, x_i, x_{i+1}, \dots, x_d) = -\varphi(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_d)$$

for $1 \leq i < d$ and all $x_1, \dots, x_d \in E$.

φ is *generic* if $\varphi(x_1, \dots, x_d) \neq 0$ for all subsets $\{x_1, \dots, x_d\} \in \binom{E}{d}$ of d distinct elements in E .

In the sequel of this paper all functions will be generic. We will mainly be concerned with sign properties of generic symmetric or antisymmetric functions: Given any symmetric generic function $\sigma : \binom{E}{d} \rightarrow \mathbf{R}_{>0}$ and a symmetric or antisymmetric generic function $\varphi : \binom{E}{d} \rightarrow \mathbf{R}^d$, the two functions

$$(x_1, \dots, x_d) \mapsto \varphi(x_1, \dots, x_d)$$

and

$$(x_1, \dots, x_d) \mapsto \sigma(x_1, \dots, x_d)\varphi(x_1, \dots, x_d)$$

behave similarly with respect to all properties addressed in this paper.

We have also an obvious sign rule: symmetric or antisymmetric functions on $\binom{E}{d}$ behave with respect to multiplication like the elements of the multiplicative group $\{\pm 1\}$ with symmetric functions corresponding to 1 and antisymmetric functions corresponding to -1 .

We fix now a generic symmetric or antisymmetric function $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$. Consider two elements $a, b \in E$. A subset $\{x_1, \dots, x_d\} \in \binom{E \setminus \{a, b\}}{d}$ not containing a and b separates a from b with respect to φ if

$$\varphi(x_1, \dots, x_d, a) \varphi(x_1, \dots, x_d, b) < 0$$

(this definition is of course independent of the particular linear order (x_1, \dots, x_d) on the set $\{x_1, \dots, x_d\}$).

We denote by $n(a, b) = n_\varphi(a, b)$ the number of subsets in $\binom{E \setminus \{a, b\}}{d}$ separating a from b (with respect to the function φ).

Proposition 1.1 (i) *If $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$ is symmetric and generic then*

$$n(a, b) + n(b, c) + n(a, c) \equiv 0 \pmod{2}$$

for any subset $\{a, b, c\}$ of 3 distinct elements in E .

(ii) *If $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$ is antisymmetric and generic then*

$$n(a, b) + n(b, c) + n(a, c) \equiv \binom{\#(E) - 3}{d - 1} \pmod{2}$$

for any subset $\{a, b, c\}$ of 3 distinct elements in E .

Proof. Consider first a subset $\{x_1, \dots, x_d\}$ not intersecting $\{a, b, c\}$. Such a subset separates no pair of elements in $\{a, b, c\}$ if

$$\varphi(x_1, \dots, x_d, a), \varphi(x_1, \dots, x_d, b) \text{ and } \varphi(x_1, \dots, x_d, c)$$

all have the same sign. Otherwise, consider a reordering $\{a', b', c'\} = \{a, b, c\}$ such that $\varphi(x_1, \dots, x_d, a') \varphi(x_1, \dots, x_d, b') < 0$ and $\varphi(x_1, \dots, x_d, a') \varphi(x_1, \dots, x_d, c') < 0$. The subset $\{x_1, \dots, x_d\}$ contributes in this case 1 to $n(a', b')$, $n(a', c')$ and 0 to $n(b', c')$. Such a subset $\{x_1, \dots, x_d\} \in \binom{E \setminus \{a, b, c\}}{d}$ yields hence always an even contribution (0 or 2) to the sum $n(a, b) + n(a, c) + n(b, c)$.

Consider now a subset $\{x_1, \dots, x_{d-1}\} \in \binom{E \setminus \{a, b, c\}}{d-1}$. We have to understand the contributions of the sets

$$\begin{array}{lll} \{x_1, \dots, x_{d-1}, c\} & \text{to} & n(a, b), \\ \{x_1, \dots, x_{d-1}, b\} & \text{to} & n(a, c), \\ \{x_1, \dots, x_{d-1}, a\} & \text{to} & n(b, c). \end{array}$$

Since the product of the six factors

$$\begin{aligned} & \varphi(x_1, \dots, x_{d-1}, c, a) \varphi(x_1, \dots, x_{d-1}, c, b) \\ & \varphi(x_1, \dots, x_{d-1}, b, a) \varphi(x_1, \dots, x_{d-1}, b, c) \\ & \varphi(x_1, \dots, x_{d-1}, a, b) \varphi(x_1, \dots, x_{d-1}, a, c) \end{aligned}$$

is always positive (respectively negative) for a generic symmetric (respectively antisymmetric) function, such a subset $\{x_1, \dots, x_{d-1}\}$ yields an even contribution to $n(a, b) + n(a, c) + n(b, c)$ in the symmetric case and an odd contribution in the antisymmetric case.

Proposition 1.1 follows now from the fact that $\binom{E \setminus \{a, b, c\}}{d-1}$ has $\binom{\#(E)-3}{d-1}$ elements. \square

Given a generic symmetric or antisymmetric function $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$ on some finite set E we set $x \sim y$ if either $x = y \in E$ or if

$$n(x, y) \equiv 0 \pmod{2} \quad \text{for symmetric } \varphi$$

respectively

$$n(x, y) \equiv \binom{\#(E) - 3}{d - 1} \pmod{2} \quad \text{for antisymmetric } \varphi .$$

We call the relation \sim defined in this way on the set E the *Orchard relation*.

Theorem 1.2 *The Orchard relation is an equivalence relation having at most two classes.*

Proof. Reflexivity and symmetry are obvious. Transitivity follows easily from Proposition 1.1.

If $a \not\sim b$ and $b \not\sim c$ then $n(a, b) + n(b, c)$ is even. It follows then from Proposition 1.1 that $a \sim c$. \square

Example. A *tournament* is a generic antisymmetric function $\binom{E}{1+1} \rightarrow \{\pm 1\}$. It encodes for instance orientations of all edges in the complete graph with vertices E and can be summarized by an antisymmetric matrix A with coefficients in $\{\pm 1\}$.

Given such a matrix A with coefficients $a_{i,j}$, $1 \leq i, j \leq n$, we have

$$n_A(i, j) = \frac{n - 2 - \sum_k a_{ik} a_{jk}}{2} .$$

This implies $i \sim_A j$ if and only if

$$\sum_k a_{ik} a_{jk} \equiv n \pmod{4}$$

for $i \neq j$. In the language of tournaments (cf. for instance [4]), this result can be restated in terms of score vectors: Two elements i and j are Orchard

equivalent if and only if the corresponding coefficients of the score vector (counting the number of 1's in line i respectively j) have the same parities.

Main Example. A finite set $\mathcal{P} = \{P_1, \dots, P_n\} \subset \mathbf{R}^d$ of $n > d$ points in real affine space \mathbf{R}^d is *generic* if the affine span of any subset containing $(d + 1)$ points in \mathcal{P} is all of \mathbf{R}^d . Such a generic set \mathcal{P} is endowed with a generic antisymmetric function by restricting

$$\varphi(x_0, \dots, x_d) = \det(x_1 - x_0, x_2 - x_0, \dots, x_d - x_0)$$

to $\binom{\mathcal{P}}{d+1}$. The Orchard relation partitions hence a generic subset $\mathcal{P} \subset \mathbf{R}^d$ into two (generally non-empty) subsets. Its name originates from the fact that the planar case ($d = 2$) yields a natural rule to plant trees of two different species at specified generic locations in an orchard, see [1] and [2].

Proposition 1.3 *Given a finite set E let φ and ψ be two generic symmetric or antisymmetric functions on $\binom{E}{d+1}$.*

(i) *If the numbers*

$$\varphi(x_0, \dots, x_d) \psi(x_0, \dots, x_d)$$

have the same sign for all $\{x_0, \dots, x_d\} \in \binom{E}{d+1}$ then the two Orchard relations \sim_φ and \sim_ψ induced by φ and ψ coincide.

(ii) *If there exists exactly one subset $\mathcal{F} = \{x_0, \dots, x_d\} \in \binom{E}{d+1}$ such that*

$$\varphi(x_0, \dots, x_d) \psi(x_0, \dots, x_d) < 0$$

then the restrictions of \sim_φ and \sim_ψ to the two subsets \mathcal{F} and $E \setminus \mathcal{F}$ coincide but $a \sim_\varphi b \iff a \not\sim_\psi b$ for $a \in \mathcal{F}$ and $b \in E \setminus \mathcal{F}$.

We call two symmetric or antisymmetric functions φ and ψ satisfying the condition of assertion (ii) above *flip-related*. Colouring the equivalence classes of an Orchard relation with two distinct colours, one can express assertion (ii) by the statement that changing a generic (symmetric or antisymmetric) function by a flip switches the colours in the flip-set $\mathcal{F} = \{x_0, \dots, x_d\}$ and leaves the colours of the remaining elements unchanged.

Assertion (i) shows that we can restrict our attention to symmetric or antisymmetric functions from $\binom{E}{d+1}$ into $\{\pm 1\}$ when studying properties of the Orchard relation.

Proof of Proposition 1.3. Assertion (i) is obvious.

For proving assertion (ii) it is enough to remark that the numbers $n_\varphi(a, b)$ and $n_\psi(a, b)$ of separating sets (with respect to φ and ψ) are identical if either $\{a, b\} \subset \mathcal{F}$ or $\{a, b\} \subset E \setminus \mathcal{F}$ and they differ by exactly one in the remaining cases. \square

2 An easy characterisation in the symmetric case

In this section we give a different and rather trivial description of the Orchard relation in the symmetric case.

Given a generic symmetric function $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$ on some finite set E we consider the function

$$\mu(x) = \#\left(\left\{\{x_1, \dots, x_d\} \in \binom{E \setminus \{x\}}{d} \mid \varphi(x, x_1, \dots, x_d) > 0\right\}\right)$$

from E to \mathbf{N} .

Theorem 2.1 *Two elements $x, y \in E$ are Orchard equivalent with respect to φ if and only if $\mu(x) \equiv \mu(y) \pmod{2}$.*

Proof. The result holds if φ is the constant function

$$\varphi(x_0, \dots, x_d) = 1$$

for all $\{x_0, \dots, x_d\} \in \binom{E}{d+1}$.

Given two generic symmetric functions φ, ψ related by a flip with respect to the set $\mathcal{F} = \{x_0, \dots, x_d\} \in \binom{E}{d+1}$ we have

$$\mu_\varphi(x) = \mu_\psi(x)$$

if $x \notin \mathcal{F}$ and

$$\mu_\varphi(x) = \mu_\psi(x) \pm 1$$

otherwise. Proposition 1.3 implies hence the result since any generic symmetric function can be related by a finite number of flips to the constant function. \square

3 Reducing d

Let $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$ be a generic symmetric or antisymmetric function. Consider the function

$$R\varphi : \binom{E}{d} \rightarrow \mathbf{R}$$

defined by

$$R\varphi(x_1, \dots, x_d) = \prod_{x \in E \setminus \{x_1, \dots, x_d\}} \varphi(x, x_1, \dots, x_d) .$$

$R\varphi$ is generic symmetric if φ is generic symmetric.

For φ generic antisymmetric, the function $R\varphi$ is generic symmetric if $\#(E) \equiv d \pmod{2}$ and $R\varphi$ is generic antisymmetric otherwise.

Dependencies of the Orchard relations associated to φ and $R\varphi$ are described by the following result.

Proposition 3.1 Let $\varphi : \binom{E}{d+1} \longrightarrow \mathbf{R}$ be a generic symmetric or antisymmetric function.

(i) If $d \equiv 0 \pmod{2}$ then the Orchard relation of $R\varphi$ is trivial (i.e. $x \sim_{R\varphi} y$ for all $x, y \in E$).

(ii) If $d \equiv 1 \pmod{2}$ then the Orchard relations \sim_φ and $\sim_{R\varphi}$ coincide on E .

The main ingredient of the proof is the following lemma.

Lemma 3.2 Let $\varphi, \psi : \binom{E}{d+1} \longrightarrow \mathbf{R}$ be two generic symmetric or antisymmetric functions which are flip-related with respect to the set $\mathcal{F} = \{x_0, \dots, x_d\} \in \binom{E}{d+1}$. Then

$$R\varphi(y_1, \dots, y_d) R\psi(y_1, \dots, y_d) < 0$$

if $\{y_1, \dots, y_d\} \subset \mathcal{F}$ and

$$R\varphi(y_1, \dots, y_d) R\psi(y_1, \dots, y_d) > 0$$

otherwise.

Proof of Lemma 3.2. If $\{y_1, \dots, y_d\} \not\subset \mathcal{F}$ then $\varphi(x, y_1, \dots, y_d) = \psi(x, y_1, \dots, y_d)$ for all $x \in E \setminus \{y_1, \dots, y_d\}$ and hence $R\varphi(y_1, \dots, y_d) = R\psi(y_1, \dots, y_d)$. Otherwise, exactly one factor of the product yielding $R\psi(y_1, \dots, y_d)$ changes sign with respect to the factors yielding $R\varphi(y_1, \dots, y_d)$. \square

Proof of Proposition 3.1. We consider first the case where $\varphi : \binom{E}{d+1} \longrightarrow \mathbf{R}$ is generic and symmetric.

Proposition 3.1 holds then for the constant symmetric application $\varphi : \binom{E}{d+1} \longrightarrow \{1\}$.

Two generic symmetric functions φ, ψ on $\binom{E}{d+1}$ which are flip-related with respect to $\mathcal{F} = \{x_0, \dots, x_d\}$ give rise to $R\varphi$ and $R\psi$ which are related through $d + 1$ flips with respect to all $d + 1$ elements in $\binom{\mathcal{F}}{d}$ by Lemma 3.2. Proposition 1.3 implies hence the result since an element of $E \setminus \mathcal{F}$ is contained in no element of $\binom{\mathcal{F}}{d}$ and since all elements of \mathcal{F} are contained in exactly d such sets.

Second case: $\varphi : \binom{E}{d+1} \longrightarrow \mathbf{R}$ generic and antisymmetric. This case is slightly more involved. As in the symmetric case, we prove the result for a particular function φ and use the fact that flips of φ affect the Orchard relation $\sim_{R\varphi}$ only for odd d . This shows that it is enough to prove that $\sim_{R\varphi}$ is trivial for a particular function φ in the case of even d and that $\sim_{R\varphi}$ and \sim_φ coincide (for a particular generic antisymmetric function φ) in the case of odd d .

We consider now the set $E = \{1, \dots, n\}$ endowed with the generic antisymmetric function $\varphi : \binom{E}{d+1} \longrightarrow \{\pm 1\}$ defined by

$$\varphi(i_0, \dots, i_d) = 1$$

for all $1 \leq i_0 < i_1 < \dots < i_d \leq n$.

Each element of $\binom{E \setminus \{i, i+1\}}{d-1}$ separates then i from $i+1$ with respect to the generic function $R\varphi$. We have indeed

$$\begin{aligned} R\varphi(j_1, \dots, j_{d-1}, i) &= \varphi(i+1, j_1, \dots, j_{d-1}, i) \prod_{j \in E \setminus \{j_1, \dots, j_{d-1}, i, i+1\}} \varphi(j, j_1, \dots, j_{d-1}, i) \\ &= -\varphi(i, j_1, \dots, j_{d-1}, i+1) \prod_{j \in E \setminus \{j_1, \dots, j_{d-1}, i, i+1\}} \varphi(j, j_1, \dots, j_{d-1}, i+1) \\ &= -R\varphi(j_1, \dots, j_{d-1}, i+1) \end{aligned}$$

showing that the number $n_{R\varphi}(i, i+1)$ of sets separating i from $i+1$ equals $\binom{n-2}{d-1}$.

The proof splits now into four cases according to the parities of n and d .

If $n \equiv d \equiv 0 \pmod{2}$, then $R\varphi$ is symmetric and $\binom{n-2}{d-1}$ is even (recall that

$$\binom{\sum_{i=0} \nu_i 2^i}{\sum_{i=0} \kappa_i 2^i} \equiv \prod_i \binom{\nu_i}{\kappa_i} \pmod{2}$$

for $\nu_i, \kappa_i \in \{0, 1\}$, cf. for instance Exercice 5.36 in Chapter 5 of [3]). Since $n_{R\varphi}(i, i+1) = \binom{n-2}{d-1}$ is even for all $i < n$, the Orchard relation $\sim_{R\varphi}$ associated to the symmetric function $R\varphi$ is trivial.

If $n \equiv 1 \pmod{2}$, $d \equiv 0 \pmod{2}$, then $R\varphi$ is antisymmetric. We have then $\binom{n-3}{d-1} \equiv 0 \pmod{2}$ and thus $\binom{n-3}{d-2} \equiv \binom{n-3}{d-2} + \binom{n-3}{d-1} = \binom{n-2}{d-1} \pmod{2}$ which implies again the triviality of the Orchard relation $\sim_{R\varphi}$ since we have $n_{R\varphi}(i, i+1) = \binom{n-2}{d-1} \equiv \binom{n-3}{d-2} \pmod{2}$ which shows $i \sim_{R\varphi} (i+1)$ for all i .

If $n \equiv d \equiv 1 \pmod{2}$ then $R\varphi$ is symmetric. Since $\binom{n-3}{d-2} \equiv 0 \pmod{2}$ we have $\binom{n-2}{d-1} = \binom{n-3}{d-2} + \binom{n-3}{d-1} \equiv \binom{n-3}{d-1} \pmod{2}$ proving that the Orchard relations \sim_φ and $\sim_{R\varphi}$ coincide.

If $n \equiv 0 \pmod{2}$, $d \equiv 1 \pmod{2}$, then $R\varphi$ is antisymmetric. The equality $\binom{n-2}{d-1} = \binom{n-3}{d-2} + \binom{n-3}{d-1}$ implies $\binom{n-3}{d-1} \equiv \binom{n-3}{d-2} + \binom{n-2}{d-1} \pmod{2}$. This shows that the Orchard relations \sim_φ and $\sim_{R\varphi}$ coincide. \square

4 Homology

We recall that $R\varphi : \binom{E}{d} \longrightarrow \mathbf{R}$ is defined by

$$R\varphi(x_1, \dots, x_d) = \prod_{x \in E \setminus \{x_1, \dots, x_d\}} \varphi(x, x_1, \dots, x_d)$$

for a given generic symmetric or antisymmetric function $\varphi : \binom{E}{d+1} \longrightarrow \mathbf{R}$.

Lemma 4.1 *We have*

$$R(R\varphi)(x_1, \dots, x_{d-1}) \in \epsilon^{\binom{\#(E)-d+1}{2}} \mathbf{R}_{>0}$$

where $\epsilon = 1$ if φ is generic and symmetric and $\epsilon = -1$ if φ is generic and antisymmetric.

Proof. Setting $\mathcal{S} = \{x_1, \dots, x_{d-1}\}$ we have

$$\begin{aligned} R(R\varphi)(x_1, \dots, x_{d-1}) &= \prod_{y \in E \setminus \mathcal{S}} R\varphi(y, x_1, \dots, x_{d-1}) = \prod_{x \neq y \in E \setminus \mathcal{S}} \varphi(x, y, x_1, \dots, x_{d-1}) \\ &= \prod_{\{x, y\} \in \binom{E \setminus \mathcal{S}}{2}} \varphi(x, y, x_1, \dots, x_{d-1}) \varphi(y, x, x_1, \dots, x_{d-1}) \end{aligned}$$

which is positive if φ is symmetric or if $\binom{\sharp(E)-d+1}{2}$ is even and negative otherwise. \square

Writing as in the beginning $[n] = \{1, \dots, n\}$, the set $\{\pm 1\}^{\binom{[n]}{d+1}}$ (endowed with the the usual product of functions) of all symmetric generic functions $\binom{[n]}{d+1} \rightarrow \{\pm 1\}$ is a vector space of dimension $\binom{n}{d+1}$ over the field \mathbf{F}_2 of 2 elements. The map R considered above defines group homomorphisms between these vector spaces and the above Lemma allows to define homology groups. These groups are however all trivial except for $d = 0$ since one obtains the ordinary (simplicial) homology with coefficients in \mathbf{F}_2 of an $(n - 1)$ dimensional simplex.

5 Increasing d

This section is a close analogue of section 3.

Given a generic symmetric or antisymmetric function $\binom{E}{d+1} \rightarrow \mathbf{R}$ we define a function $A\varphi : \binom{E}{d+2} \rightarrow \mathbf{R}$ by setting

$$A\varphi(x_0, \dots, x_{d+1}) = \prod_{i=0}^{d+1} \varphi(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1}).$$

The function $A\varphi$ is generic symmetric if φ is symmetric. For φ antisymmetric it is generic symmetric if $d \equiv 0 \pmod{2}$ and generic antisymmetric otherwise.

The dependency between the Orchard relations \sim_φ and $\sim_{A\varphi}$ for a generic symmetric or antisymmetric function $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$ is described by the following result.

Proposition 5.1 *Let $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$ be a generic symmetric or antisymmetric function.*

The Orchard relation $\sim_{A\varphi}$ of $A\varphi$ is trivial if $\sharp(E) \equiv d \pmod{2}$. Otherwise, the Orchard relations \sim_φ and $\sim_{A\varphi}$ of φ and $A\varphi$ coincide.

The main ingredient of the proof is the following lemma whose easy proof is left to the reader.

Lemma 5.2 Let $\varphi, \psi : \binom{E}{d+1} \longrightarrow \mathbf{R}$ be two generic symmetric or antisymmetric functions which are flip-related with respect to the set $\mathcal{F} = \{x_0, \dots, x_d\}$. Then

$$A\varphi(y_0, \dots, y_{d+1}) A\psi(y_0, \dots, y_{d+1}) > 0$$

if $\mathcal{F} \not\subset \{y_0, \dots, y_{d+1}\}$ and

$$A\varphi(y_0, \dots, y_{d+1}) A\psi(y_0, \dots, y_{d+1}) < 0$$

otherwise.

Proof of Proposition 5.1. Lemma 5.2 shows that $\sim_{A\varphi}$ is independent of φ if $\sharp(E) \equiv d \pmod{2}$. Otherwise, the Orchard relations of φ and $A\varphi$ behave in a similar way under flips. Indeed, given ψ which is flip-related with flipset $\mathcal{F} = \{x_0, \dots, x_d\}$ to φ the functions $A\psi$ and $A\varphi$ are related through $(\sharp(E) - (d+1))$ flips with flipsets $\mathcal{F} \cup \{x\}$, $x \in E \setminus \mathcal{F}$. Each element of $E \setminus \mathcal{F}$ is hence flipped once and each element of \mathcal{F} is flipped $\sharp(E) - (d+1)$ times.

Proposition 1.3 implies hence that $\sim_{A\varphi}$ is independent of φ if $1 \equiv \sharp(E) - (d+1)$ and that \sim_φ and $\sim_{A\varphi}$ behave similarly under flips otherwise. It is hence enough to proof Proposition 5.1 in a particular case.

If φ is symmetric, then Proposition 5.1 clearly holds for the constant application $\varphi : \binom{E}{d+1} \longrightarrow \{1\}$.

In the antisymmetric case we set $E = \{1, \dots, n\}$ and we consider the generic antisymmetric function $\varphi : \binom{E}{d+1} \longrightarrow \{\pm 1\}$ defined by

$$\varphi(i_0, \dots, i_d) = 1$$

for all $1 \leq i_0 < i_1 < \dots < i_d \leq n$. The function $A\varphi : \binom{E}{d+2} \longrightarrow \{\pm 1\}$ is now given by

$$A\varphi(i_0, \dots, i_d, i_{d+1}) = 1$$

for all $1 \leq i_0 < i_1 < \dots < i_d < i_{d+1} \leq n$. The numbers $n_{A\varphi}(i, i+1)$ of subsets in $\binom{E \setminus \{i, i+1\}}{d+1}$ separating i from $i+1$ are hence all 0 and we split the discussion into several cases according to the parities of $n = \sharp(E)$ and d .

$n \equiv d \equiv 0 \pmod{2}$ implies $A\varphi$ symmetric and hence $\sim_{A\varphi}$ trivial.

$n \equiv 1 \pmod{2}$, $d \equiv 0 \pmod{2}$ implies $A\varphi$ symmetric and hence $\sim_{A\varphi}$ trivial. Since then $\binom{n-3}{d-1} \equiv 0 \pmod{2}$ we have also \sim_φ trivial.

$n \equiv 0 \pmod{2}$, $d \equiv 1 \pmod{2}$ implies $A\varphi$ antisymmetric. We have then $\binom{n-3}{d-1} + \binom{n-3}{d} = \binom{n-2}{d} \equiv 0 \pmod{2}$ proving equality of the two Orchard relations \sim_φ and $\sim_{A\varphi}$.

$n \equiv d \equiv 1 \pmod{2}$ implies $A\varphi$ antisymmetric and $\binom{n-3}{d} \equiv 0 \pmod{2}$ thus proving triviality of the Orchard relation $\sim_{A\varphi}$. \square

Remark 5.3 *One sees easily that the function $A(A\varphi)$ is strictly positive for a generic symmetric or antisymmetric function $\varphi : \binom{E}{d+1} \rightarrow \mathbf{R}$.*

This allows the definition of cohomology groups on the set of generic symmetric functions $\binom{E}{d+1} \rightarrow \{\pm 1\}$. The resulting groups are of course not interesting since this boils down once more to the cohomology groups of the $(\#(E) - 1)$ -dimensional simplex with coefficients in the field \mathbf{F}_2 of 2 elements.

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