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LIFTS OF  $C_\infty$  AND  $L_\infty$ -MORPHISMS TO  $G_\infty$ -MORPHISMSGRÉGORIE GINOT<sup>(a)</sup> , GILLES HALBOUT<sup>(b)</sup><sup>(a)</sup> Laboratoire Analyse Géométrie et Applications, Université Paris 13 et Ecole Normale Supérieure de Cachan, France<sup>(b)</sup> Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, Strasbourg, France

ABSTRACT. Let  $\mathfrak{g}_2$  be the Hochschild complex of cochains on  $C^\infty(\mathbb{R}^n)$  and  $\mathfrak{g}_1$  be the space of multivector fields on  $\mathbb{R}^n$ . In this paper we prove that given any  $G_\infty$ -structure (i.e. Gerstenhaber algebra up to homotopy structure) on  $\mathfrak{g}_2$ , and any  $C_\infty$ -morphism  $\varphi$  (i.e. morphism of commutative, associative algebra up to homotopy) between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$ , there exists a  $G_\infty$ -morphism  $\Phi$  between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  that restricts to  $\varphi$ . We also show that any  $L_\infty$ -morphism (i.e. morphism of Lie algebra up to homotopy), in particular the one constructed by Kontsevich, can be deformed into a  $G_\infty$ -morphism, using Tamarkin's method for any  $G_\infty$ -structure on  $\mathfrak{g}_2$ . We also show that any two of such  $G_\infty$ -morphisms are homotopic.

## 0-Introduction

Let  $M$  be a differential manifold and  $\mathfrak{g}_2 = (C^\bullet(A, A), b)$  be the Hochschild cochain complex on  $A = C^\infty(M)$ . The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of  $\mathfrak{g}_2$  is the graded Lie algebra  $\mathfrak{g}_1 = \Gamma(M, \wedge^\bullet TM)$  of multivector fields on  $M$ . There is also a graded Lie algebra structure on  $\mathfrak{g}_2$  given by the Gerstenhaber bracket. In particular  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are also Lie algebras up to homotopy ( $L_\infty$ -algebra for short). In the case  $M = \mathbb{R}^n$ , using different methods, Kontsevich ([Ko1] and [Ko2]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms “up to homotopy” ( $L_\infty$ -morphisms) from  $\mathfrak{g}_1$  to  $\mathfrak{g}_2$ . Kontsevich's proof uses graph complex and is related to multizeta functions whereas Tamarkin's construction uses the existence of Drinfeld's associators. In fact Tamarkin's  $L_\infty$ -morphism comes from the restriction of a Gerstenhaber algebra up to homotopy homomorphism ( $G_\infty$ -morphism) from  $\mathfrak{g}_1$  to  $\mathfrak{g}_2$ . The  $G_\infty$ -algebra structure on  $\mathfrak{g}_1$  is induced by its classical Gerstenhaber algebra structure and a far less trivial  $G_\infty$ -structure on  $\mathfrak{g}_2$  was proved to exist by Tamarkin [Ta] and relies on a Drinfeld's associator. Tamarkin's  $G_\infty$ -morphism also restricts into a commutative, associative up to homotopy morphism ( $C_\infty$ -morphism for short). The  $C_\infty$ -structure on  $\mathfrak{g}_2$  (given by

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restriction of the  $G_\infty$ -one) highly depends on Drinfeld's associator, and any two choices of a Drinfeld associator yields *a priori* different  $C_\infty$ -structures. When  $M$  is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). A connection between the two approaches has been given in [KS] but the morphisms given by Kontsevich and Tamarkin are not the same. The aim of this paper is to show that, given any  $G_\infty$ -structure on  $\mathfrak{g}_2$  and any  $C_\infty$ -morphism  $\varphi$  between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$ , there exists a  $G_\infty$ -morphism  $\Phi$  between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  that restricts to  $\varphi$ . We also show that any  $L_\infty$ -morphism can be deformed into a  $G_\infty$ -one.

In the first section, we fix notation and recall the definitions of  $L_\infty$  and  $G_\infty$ -structures. In the second section we state and prove the main theorem. In the last section we show that any two  $G_\infty$ -morphisms given by Tamarkin's method are homotopic.

**Remark :** In the sequel, unless otherwise is stated, the manifold  $M$  is supposed to be  $\mathbb{R}^n$  for some  $n \geq 1$ . Most results could be generalized to other manifolds using techniques of Kontsevich [Ko1] (also see [TS], [CFT]).

### 1- $C_\infty$ , $L_\infty$ and $G_\infty$ -structures

For any graded vector space  $\mathfrak{g}$ , we choose the following degree on  $\wedge^\bullet \mathfrak{g}$  : if  $X_1, \dots, X_k$  are homogeneous elements of respective degree  $|X_1|, \dots, |X_k|$ , then

$$|X_1 \wedge \dots \wedge X_k| = |X_1| + \dots + |X_k| - k.$$

In particular the component  $\mathfrak{g} = \wedge^1 \mathfrak{g} \subset \wedge^\bullet \mathfrak{g}$  is the same as the space  $\mathfrak{g}$  with degree shifted by one. The space  $\wedge^\bullet \mathfrak{g}$  with the deconcatenation cobracket is the cofree cocommutative coalgebra on  $\mathfrak{g}$  with degree shifted by one (see [LS], Section 2). Any degree one map  $d^k : \wedge^k \mathfrak{g} \rightarrow \mathfrak{g}$  ( $k \geq 1$ ) extends into a derivation  $d^k : \wedge^\bullet \mathfrak{g} \rightarrow \wedge^\bullet \mathfrak{g}$  of the coalgebra  $\wedge^\bullet \mathfrak{g}$  by cofreeness property.

**Definition 1.1.** A vector space  $\mathfrak{g}$  is endowed with a  $L_\infty$ -algebra (Lie algebras “up to homotopy”) structure if there are degree one linear maps  $m^{1, \dots, 1}$ , with  $k$  ones :  $\wedge^k \mathfrak{g} \rightarrow \mathfrak{g}$  such that if we extend them to maps  $\wedge^\bullet \mathfrak{g} \rightarrow \wedge^\bullet \mathfrak{g}$ , then  $d \circ d = 0$  where  $d$  is the derivation

$$d = m^1 + m^{1,1} + \dots + m^{1, \dots, 1} + \dots .$$

For more details on  $L_\infty$ -structures, see [LS]. It follows from the definition that a  $L_\infty$ -algebra structure induces a differential coalgebra structure on  $\wedge^\bullet \mathfrak{g}$  and that the map  $m^1 : \mathfrak{g} \rightarrow \mathfrak{g}$  is a differential. If  $m^{1, \dots, 1} : \wedge^k \mathfrak{g} \rightarrow \mathfrak{g}$  are 0 for  $k \geq 3$ , we get the usual definition of (differential if  $m^1 \neq 0$ ) graded Lie algebras.

For any graded vector space  $\mathfrak{g}$ , we denote  $\underline{\mathfrak{g}^{\otimes n}}$  the quotient of  $\mathfrak{g}^{\otimes n}$  by the image of all shuffles of length  $n$  (see [GK] or [GH] for details). The graded vector space  $\bigoplus_{n \geq 0} \underline{\mathfrak{g}^{\otimes n}}$  is a quotient coalgebra of the tensor coalgebra  $\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ . It is well known that this coalgebra  $\bigoplus_{n \geq 0} \underline{\mathfrak{g}^{\otimes n}}$  is the cofree Lie coalgebra on the vector space  $\mathfrak{g}$  (with degree shifted by minus one).

**Definition 1.2.** A  $C_\infty$ -algebra (commutative and associative “up to homotopy” algebra) structure on a vector space  $\mathfrak{g}$  is given by a collection of degree one linear maps  $m^k : \underline{\mathfrak{g}^{\otimes k}} \rightarrow \mathfrak{g}$  such that if we extend them to maps  $\bigoplus \underline{\mathfrak{g}^{\otimes \bullet}} \rightarrow \bigoplus \underline{\mathfrak{g}^{\otimes \bullet}}$ , then  $d \circ d = 0$  where  $d$  is the derivation

$$d = m^1 + m^2 + m^3 + \dots .$$

In particular a  $C_\infty$ -algebra is an  $A_\infty$ -algebra.

For any space  $\mathfrak{g}$ , we denote  $\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$  the graded space

$$\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}} = \bigoplus_{m \geq 1, p_1 + \dots + p_n = m} \underline{\mathfrak{g}^{\otimes p_1}} \wedge \dots \wedge \underline{\mathfrak{g}^{\otimes p_n}}.$$

We use the following grading on  $\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$ : for  $x_1^1, \dots, x_n^{p_n} \in \mathfrak{g}$ , we define

$$|\underline{x_1^1 \otimes \dots \otimes x_1^{p_1}} \wedge \dots \wedge \underline{x_n^1 \otimes \dots \otimes x_n^{p_n}}| = \sum_{i_1}^{p_1} |x_1^{i_1}| + \dots + \sum_{i_n}^{p_n} |x_n^{i_n}| - n.$$

Notice that the induced grading on  $\underline{\wedge^\bullet \mathfrak{g}} \subset \underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$  is the same than the one introduced above. The cobracket on  $\underline{\oplus \mathfrak{g}^{\otimes \bullet}}$  and the coproduct on  $\underline{\wedge^\bullet \mathfrak{g}}$  extend to a cobracket and a coproduct on  $\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$  which yield a Gerstenhaber coalgebra structure on  $\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$ . It is well known that this coalgebra structure is cofree (see [Gi], Section 3 for example).

**Definition 1.3.** A  $G_\infty$ -algebra (Gerstenhaber algebra “up to homotopy”) structure on a graded vector space  $\mathfrak{g}$  is given by a collection of degree one maps

$$m^{p_1, \dots, p_n} : \underline{\mathfrak{g}^{\otimes p_1}} \wedge \dots \wedge \underline{\mathfrak{g}^{\otimes p_n}} \rightarrow \mathfrak{g}$$

indexed by  $p_1, \dots, p_n \geq 1$  such that their canonical extension:  $\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}} \rightarrow \underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$  satisfies  $d \circ d = 0$  where

$$d = \sum_{m \geq 1, p_1 + \dots + p_n = m} m^{p_1, \dots, p_n}.$$

Again, as the coalgebra structure of  $\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$  is cofree, the map  $d$  makes  $\underline{\wedge^\bullet \mathfrak{g}^{\otimes \bullet}}$  into a differential coalgebra. If the maps  $m^{p_1, \dots, p_n}$  are 0 for  $(p_1, p_2, \dots) \neq (1, 0, \dots), (1, 1, 0, \dots)$  or  $(2, 0, \dots)$ , we get the usual definition of (differential if  $m^1 \neq 0$ ) Gerstenhaber algebra.

The space of multivector fields  $\mathfrak{g}_1$  is endowed with a graded Lie bracket  $[-, -]_S$  called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product:  $(\alpha, \beta) \mapsto \alpha \wedge \beta$

Setting  $d_1 = m_1^{1,1} + m_1^2$ , where  $m_1^{1,1} : \underline{\wedge^2 \mathfrak{g}_1} \rightarrow \mathfrak{g}_1$ , and  $m_1^2 : \underline{\mathfrak{g}_1^{\otimes 2}} \rightarrow \mathfrak{g}_1$  are the extension of the Schouten bracket and the exterior product, we find that  $(\underline{\mathfrak{g}_1}, d_1)$  is a  $G_\infty$ -algebra.

In the same way, one can define a differential Lie algebra structure on the vector space  $\mathfrak{g}_2 = C(A, A) = \bigoplus_{k \geq 0} C^k(A, A)$ , the space of Hochschild cochains (generated by differential  $k$ -linear maps from  $A^k$  to  $A$ ), where  $A = C^\infty(M)$  is the algebra of smooth differential functions over  $M$ . Its bracket  $[-, -]_G$ , called the Gerstenhaber bracket, is defined, for  $D, E \in \mathfrak{g}_2$ , by

$$[D, E]_G = \{D|E\} - (-1)^{|E||D|} \{E|D\},$$

where

$$\{D|E\}(x_1, \dots, x_{d+e-1}) = \sum_{i \geq 0} (-1)^{|E| \cdot i} D(x_1, \dots, x_i, E(x_{i+1}, \dots, x_{i+e}), \dots).$$

The space  $\mathfrak{g}_2$  has a grading defined by  $|D| = k \Leftrightarrow D \in C^{k+1}(A, A)$  and its differential is  $b = [m, -]_G$ , where  $m \in C^2(A, A)$  is the commutative multiplication on  $A$ .

Tamarkin (see [Ta] or also [GH]) stated the existence of a  $G_\infty$ -structure on  $\mathfrak{g}_2$  (depending on a choice of a Drinfeld associator) given by a differential  $d_2 = m_2^1 + m_2^{1,1} + m_2^2 + \dots + m_2^{p_1, \dots, p_n} + \dots$ , on  $\wedge^\bullet \underline{\mathfrak{g}}_2^{\otimes \bullet}$  satisfying  $d_2 \circ d_2 = 0$ . Although this structure is non-explicit, it satisfies the following three properties :

- (a)  $m_2^1$  is the extension of the differential  $b$
- (b)  $m_2^{1,1}$  is the extension of the Gerstenhaber bracket  $[-, -]_G$  and  $m_2^{1,1, \dots, 1} = 0$
- (c)  $m_2^2$  induces the exterior product in cohomology and the collection of the  $(m^k)_{k \geq 1}$  defines a  $C_\infty$ -structure on  $\mathfrak{g}_2$ .(1.1)

**Definition 1.4.** *A  $L_\infty$ -morphism between two  $L_\infty$ -algebras  $(\mathfrak{g}_1, d_1 = m_1^1 + \dots)$  and  $(\mathfrak{g}_2, d_2 = m_2^1 + \dots)$  is a morphism of differential coalgebras*

$$\varphi : (\wedge^\bullet \mathfrak{g}_1, d_1) \rightarrow (\wedge^\bullet \mathfrak{g}_2, d_2). \quad (1.2)$$

Such a map  $\varphi$  is uniquely determined by a collection of maps  $\varphi^n : \wedge^n \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  (again by cofreeness properties). In the case  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are respectively the graded Lie algebra  $(\Gamma(M, \wedge TM), [-, -]_S)$  and the differential graded Lie algebra  $(C(A, A), [-, -]_G)$ , the formality theorems of Kontsevich and Tamarkin state the existence of a  $L_\infty$ -morphism between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $\varphi^1$  is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

**Definition 1.5.** *A morphism of  $C_\infty$ -algebras between two  $C_\infty$ -algebras  $(\mathfrak{g}_1, d_1)$  and  $(\mathfrak{g}_2, d_2)$  is a map  $\phi : (\oplus \underline{\mathfrak{g}}_1^{\otimes \bullet}, d_1) \rightarrow (\oplus \underline{\mathfrak{g}}_2^{\otimes \bullet}, d_2)$  of codifferential coalgebras.*

A  $C_\infty$ -morphism is in particular a morphism of  $A_\infty$ -algebras and is uniquely determined by maps  $\partial^k : \underline{\mathfrak{g}}^{\otimes k} \rightarrow \mathfrak{g}$ .

**Definition 1.6.** *A morphism of  $G_\infty$ -algebras between two  $G_\infty$ -algebras  $(\mathfrak{g}_1, d_1)$  and  $(\mathfrak{g}_2, d_2)$  is a map  $\phi : (\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}, d_1) \rightarrow (\wedge^\bullet \underline{\mathfrak{g}}_2^{\otimes \bullet}, d_2)$  of codifferential coalgebras.*

There are coalgebras inclusions  $\wedge^\bullet \mathfrak{g} \rightarrow \wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$ ,  $\oplus \underline{\mathfrak{g}}^{\otimes \bullet} \rightarrow \wedge^\bullet \underline{\mathfrak{g}}^{\otimes \bullet}$  and it is easy to check that any  $G_\infty$ -morphism between two  $G_\infty$ -algebras  $(\mathfrak{g}, \sum m^{p_1, \dots, p_n})$ ,  $(\mathfrak{g}', \sum m'^{p_1, \dots, p_n})$  restricts to a  $L_\infty$ -morphism  $(\wedge^\bullet \mathfrak{g}, \sum m^{1, \dots, 1}) \rightarrow (\wedge^\bullet \mathfrak{g}', \sum m'^{1, \dots, 1})$  and a  $C_\infty$ -morphism  $(\oplus \underline{\mathfrak{g}}^{\otimes \bullet}, \sum m^k) \rightarrow (\oplus \underline{\mathfrak{g}}'^{\otimes \bullet}, \sum m'^k)$ . In the case  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are as above, Tamarkin's theorem states that there exists a  $G_\infty$ -morphism between the two  $G_\infty$  algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  (with the  $G_\infty$  structure he built) that restricts to a  $C_\infty$  and a  $L_\infty$ -morphism.

## 2-Main theorem

We keep the notations of the previous section, in particular  $\mathfrak{g}_2$  is the Hochschild complex of cochains on  $C^\infty(M)$  and  $\mathfrak{g}_1$  its cohomology. Here is our main theorem.

**Theorem 2.1.** *Given any  $G_\infty$ -structure  $d_2$  on  $\mathfrak{g}_2$  satisfying the three properties of (1.1), and any  $C_\infty$ -morphism  $\varphi$  between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $\varphi^1$  is the Hochschild-Kostant-Rosenberg map, there exists a  $G_\infty$ -morphism  $\Phi : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$  that restricts to  $\varphi$ .*

*Also, given any  $L_\infty$ -morphism  $\gamma$  between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $\gamma^1$  is the Hochschild-Kostant-Rosenberg map, there exists a  $G_\infty$ -structure  $(\mathfrak{g}_1, d'_1)$  on  $\mathfrak{g}_1$  and  $G_\infty$ -morphism  $\Gamma : (\mathfrak{g}_1, d'_1) \rightarrow (\mathfrak{g}_2, d_2)$  that restricts to  $\gamma$ . Moreover there exists a  $G_\infty$ -morphism  $\Gamma' : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_1, d'_1)$ .*

In particular, Theorem 2.1 applies to the formality map of Kontsevich and also to any  $C_\infty$ -map derived (see [Ta], [GH]) from any  $B_\infty$ -structure on  $\mathfrak{g}_2$  lifting the Gerstenhaber structure of  $\mathfrak{g}_1$ .

Let us first recall the proof of Tamarkin's formality theorem (see [GH] for more details):

1. First one proves there exists a  $G_\infty$ -structure on  $\mathfrak{g}_2$ , with differential  $d_2$ , as in (1.1).
2. Then, one constructs a  $G_\infty$ -structure on  $\mathfrak{g}_1$  given by a differential  $d'_1$  together with a  $G_\infty$ -morphism  $\Phi$  between  $(\mathfrak{g}_1, d'_1)$  and  $(\mathfrak{g}_2, d_2)$ .
3. Finally, one constructs a  $G_\infty$ -morphism  $\Phi'$  between  $(\mathfrak{g}_1, d_1)$  and  $(\mathfrak{g}_1, d'_1)$ .

The composition  $\Phi \circ \Phi'$  is then a  $G_\infty$ -morphism between  $(\mathfrak{g}_1, d_1)$  and  $(\mathfrak{g}_2, d_2)$ , thus restricts to a  $L_\infty$ -morphism between the differential graded Lie algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$ .

We suppose now that, in the first step, we take any  $G_\infty$ -structure on  $\mathfrak{g}_2$  given by a differential  $d_2$  and we suppose we are given a  $C_\infty$ -morphism  $\varphi$  and a  $L_\infty$ -morphism  $\gamma$  between  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  satisfying  $\gamma^1 = \varphi^1 = \varphi_{\text{HKR}}$  the Hochschild-Kostant-Rosenberg quasi-isomorphism.

*Proof of Theorem 2.1:*

The Theorem will follow if we prove that steps 2 and 3 of Tamarkin's construction are still true with the extra conditions that the restriction of the  $G_\infty$ -morphism  $\Phi$  (resp.  $\Phi'$ ) on the  $C_\infty$ -structures is the  $C_\infty$ -morphism  $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  (resp. id).

Let us recall (see [GH]) that the constructions of  $\Phi$  and  $d'_1$  can be made by induction. For  $i = 1, 2$  and  $n \geq 0$ , let us set

$$V_i^{[n]} = \bigoplus_{p_1 + \dots + p_k = n} \underline{\mathfrak{g}_i^{\otimes p_1}} \wedge \dots \wedge \underline{\mathfrak{g}_i^{\otimes p_k}}$$

and  $V_i^{[\leq n]} = \sum_{k \leq n} V_i^{[k]}$ . Let  $d_2^{[n]}$  and  $d_2^{[\leq n]}$  be the sums

$$d_2^{[n]} = \sum_{p_1 + \dots + p_k = n} d_2^{p_1, \dots, p_k} \quad \text{and} \quad d_2^{[\leq n]} = \sum_{p \leq n} d_2^{[p]}.$$

Clearly,  $d_2 = \sum_{n \geq 1} d_2^{[n]}$ . In the same way, we denote  $d'_1 = \sum_{n \geq 1} d'_1^{[n]}$  with

$$d'_1^{[n]} = \sum_{p_1 + \dots + p_k = n} d'_1{}^{p_1, \dots, p_k} \quad \text{and} \quad d'_1^{[\leq n]} = \sum_{1 \leq k \leq n} d'_1{}^{[k]}.$$

We know from Section 1 that a morphism  $\Phi : (\wedge^\bullet \underline{\mathfrak{g}_1^{\otimes \bullet}}, d'_1) \rightarrow (\wedge^\bullet \underline{\mathfrak{g}_2^{\otimes \bullet}}, d_2)$  is uniquely determined by its components  $\Phi^{p_1, \dots, p_k} : \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \dots \wedge \underline{\mathfrak{g}_1^{\otimes p_k}} \rightarrow \mathfrak{g}_2$ . Again, we have  $\Phi = \sum_{n \geq 1} \Phi^{[n]}$  with

$$\Phi^{[n]} = \sum_{p_1 + \dots + p_k = n} \Phi^{p_1, \dots, p_k} \quad \text{and} \quad \Phi^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi^{[k]}.$$

We want to construct the maps  $d'_1{}^{[n]}$  and  $\Phi^{[n]}$  by induction with the initial condition

$$d'_1{}^{[1]} = 0 \quad \text{and} \quad \Phi^{[1]} = \varphi_{\text{HKR}},$$

where  $\varphi_{\text{HKR}} : (\mathfrak{g}_1, 0) \rightarrow (\mathfrak{g}_2, b)$  is the Hochschild-Kostant-Rosenberg quasi-isomorphism (see [HKR]) defined, for  $\alpha \in \mathfrak{g}_1$ ,  $f_1, \dots, f_n \in A$ , by

$$\varphi_{\text{HKR}} : \alpha \mapsto ((f_1, \dots, f_n) \mapsto \langle \alpha, df_1 \wedge \dots \wedge df_n \rangle).$$

Moreover, we want the following extra conditions to be true:

$$\Phi^{k \geq 2} = \varphi^k, \quad d'_1{}^2 = d_1^2, \quad d'^{k \geq 3} = 0. \quad (2.3)$$

Now suppose the construction is done for  $n-1$  ( $n \geq 2$ ), i.e., we have built maps  $(d'_1{}^{[i]})_{i \leq n-1}$  and  $(\Phi^{[i]})_{i \leq n-1}$  satisfying conditions (2.3) and

$$\Phi^{[\leq n-1]} \circ d'_1{}^{[\leq n-1]} = d_2^{[\leq n-1]} \circ \Phi^{[\leq n-1]} \text{ on } V_1^{[\leq n-1]} \text{ and } d'_1{}^{[\leq n-1]} \circ d'_1{}^{[\leq n-1]} = 0 \text{ on } V_1^{[\leq n]}. \quad (2.4)$$

In [GH], we prove that for any such  $(d'_1{}^{[i]})_{i \leq n-1}$  and  $(\Phi^{[i]})_{i \leq n-1}$ , one can construct  $d'_1{}^{[n]}$  and  $\Phi^{[n]}$  such that condition (2.4) is true for  $n$  instead of  $n-1$ . To complete the proof of Theorem 2.1 (step 2), we have to show that  $d'_1{}^{[n]}$  and  $\Phi^{[n]}$  can be chosen to satisfy conditions (2.3). In the equation 2.4, the terms  $d'^k$  and  $\Phi^k$  only act on  $V_1^k$ . So one can replace  $\Phi^n$  with  $\varphi^n$ ,  $d'_1{}^2$  with  $d_1^2$  (or  $d_1^i$ ,  $i \geq 3$  with 0) provided conditions (2.4) are still satisfied on  $V_1^n$ . The other terms acting on  $V_1^n$  in the equation (2.4) only involve terms  $\Phi^m = \varphi^m$  and  $d'^m$ . Then conditions (2.4) on  $V_1^{1, \dots, 1}$  are the equations that should be satisfied by a  $C_\infty$ -morphism between the  $C_\infty$ -algebras  $(\mathfrak{g}_1, d_1^{1,1} = d_1^{1,1})$  and  $(\mathfrak{g}_2, \sum_{k \geq 1} d_2^k)$  restricted to  $V_1^n$ . Hence by hypothesis on  $\varphi$  the conditions hold.

Similarly the construction of  $\Phi'$  can be made by induction. Let us recall the proof given in [GH]. Again a morphism  $\Phi' : (\wedge^\bullet \mathfrak{g}_1^{\otimes \bullet}, d_1) \rightarrow (\wedge^\bullet \mathfrak{g}_2^{\otimes \bullet}, d'_1)$  is uniquely determined by its components  $\Phi'^{p_1, \dots, p_k} : \mathfrak{g}_1^{\otimes p_1} \wedge \dots \wedge \mathfrak{g}_1^{\otimes p_k} \rightarrow \mathfrak{g}_1$ . We write  $\Phi' = \sum_{n \geq 1} \Phi'^{[n]}$  with

$$\Phi'^{[n]} = \sum_{p_1 + \dots + p_k = n} \Phi'^{p_1, \dots, p_k} \quad \text{and} \quad \Phi'^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi'^{[k]}.$$

We construct the maps  $\Phi'^{[n]}$  by induction with the initial condition  $\Phi'^{[1]} = \text{id}$ . Moreover, we want the following extra conditions to be true:

$$\Phi'^n = 0 \text{ for } n \geq 2. \quad (2.5)$$

Now suppose the construction is done for  $n-1$  ( $n \geq 2$ ), i.e., we have built maps  $(\Phi'^{[i]})_{i \leq n-1}$  satisfying conditions (2.5) and

$$\Phi'^{[\leq n-1]} d_1^{[\leq n]} = d_1'^{[\leq n]} \Phi'^{[\leq n-1]} \text{ on } V_1^{[\leq n]}. \quad (2.6)$$

In [GH], we prove that for any such  $(\Phi'^{[i]})_{i \leq n-1}$ , one can construct  $\Phi'^{[n]}$  such that condition (2.6) is true for  $n$  instead of  $n-1$  in the following way : making the equation  $\Phi' d_1 = d'_1 \Phi'$  on  $V_1^{[n+1]}$  explicit, we get

$$\Phi'^{[\leq n]} d_1^{[\leq n+1]} = d'_1^{[\leq n+1]} \Phi'^{[\leq n]}. \quad (2.7)$$

If we now take into account that  $d_1^{[i]} = 0$  for  $i \neq 2$ ,  $d'_1^{[1]} = 0$  and that on  $V_1^{[n+1]}$  we have  $\Phi'^{[k]} d_1^{[l]} = d'_1^{[\leq k]} \Phi'^{[l]} = 0$  for  $k+l > n+2$ , the identity (2.7) becomes

$$\Phi'^{[\leq n]} d_1^{[2]} = \sum_{k=2}^{n+1} d'_1^{[k]} \Phi'^{[\leq n-k+2]}.$$

As  $d'_1^{[2]} = d_1^{[2]}$ , (2.7) is equivalent to

$$d_1^{[2]} \Psi'^{[\leq n]} - \Phi'^{[\leq n]} d_1^{[2]} = \left[ d_1^{[2]}, \Phi'^{[\leq n]} \right] = - \sum_{k=3}^{n+1} d'_1^{[k]} \Phi'^{[\leq n-k+2]}.$$

Notice that  $d_1^{[2]} = m_1^{1,1} + m_1^2$ . Then the construction will be possible when the term  $\sum_{k=3}^{n+1} d'_1^{[k]} \psi'^{[\leq n-k+2]}$  is a coboundary in the subcomplex of  $(\text{End}(\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1^{[2]}, -])$  consisting of maps which restrict to zero on  $\oplus_{n \geq 2} \underline{\mathfrak{g}}_1^{\otimes n}$ . It is always a cocycle by straightforward computation (see [GH]) and the subcomplex is acyclic because both  $(\text{End}(\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1^{[2]}, -])$  and the Harrison cohomology of  $\mathfrak{g}_1$  are trivial according to Tamarkin [Ta] (see also [GH] Proposition 5.1 and [Hi] 5.4).

In the case of the  $L_\infty$ -morphism  $\gamma$ , the first step is similar: the fact that  $\gamma$  is a  $L_\infty$ -map enables us to build a  $G_\infty$ -structure  $(\mathfrak{g}_1, d'_1)$  on  $\mathfrak{g}_1$  and a  $G_\infty$ -morphism  $\Gamma : (\mathfrak{g}_1, d'_1) \rightarrow (\mathfrak{g}_2, d_2)$  such that:

$$\Gamma^{1, \dots, 1} = \gamma^{1, \dots, 1}, \quad d'^{1,1} = d_1^{1,1}, \quad d'^{1,1, \dots, 1} = 0. \quad (2.8)$$

For the second step, we have to build a map  $\Gamma'$  satisfying the equation

$$d_1^{[2]} \Gamma'^{[\leq n]} - \Gamma'^{[\leq n]} d_1^{[2]} = \left[ d_1^{[2]}, \Gamma'^{[\leq n]} \right] = - \sum_{k=3}^{n+1} d'_1^{[k]} \Gamma'^{[\leq n-k+2]}$$

on  $V_1^{[n+1]}$  for any  $n \geq 1$ . Again, because Tamarkin has proved that the complex  $(\text{End}(\wedge^\bullet \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1^{[2]}, -])$  is acyclic (we are in the case  $M = \mathbb{R}^n$ ), the result follows from the fact that  $\sum_{k=3}^{n+1} d'_1^{[k]} \Gamma'^{[\leq n-k+2]}$  is a cocycle. The difference with the  $C_\infty$ -case is that the  $\Gamma'^{1, \dots, 1}$  could be non zero.  $\blacksquare$

### 3-The difference between two $G_\infty$ -maps

In this section we investigate the difference between two different  $G_\infty$ -formality maps. We fix once for all a  $G_\infty$ -structure on  $\mathfrak{g}_2$  (given by a differential  $d_2$ ) satisfying the conditions (1.1) and a morphism of  $G_\infty$ -algebras  $T : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$  such that  $T^1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is  $\varphi_{\text{HKR}}$ . Let  $K : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$  be any other  $G_\infty$ -morphism with  $K^1 = \varphi_{\text{HKR}}$  (for example any lift of a Kontsevich formality map or any  $G_\infty$ -maps lifting another  $C_\infty$ -morphism).



**Theorem 3.1.** *There exists a map  $h : \wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet} \rightarrow \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}$  such that*

$$T - K = h \circ d_1 + d_2 \circ h.$$

In other words the formality  $G_\infty$ -morphisms  $K$  and  $T$  are homotopic.

The maps  $T$  and  $K$  are elements of the cochain complex  $(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta)$  with differential given, for all  $f \in \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$ ,  $|f| = k$ , by

$$\delta(f) = d_2 \circ f - (-1)^k f \circ d_1.$$

We first compare this cochain complex with the complexes  $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1; -])$  and  $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), [d_2; -])$  (where  $[-; -]$  is the graded commutator of morphisms). There are morphisms

$$T_* : \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}) \rightarrow \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \quad T^* : \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}) \rightarrow \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$$

defined, for  $f \in \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$  and  $g \in \text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$ , by

$$T_*(f) = T \circ f, \quad T^*(g) = g \circ T.$$

**Lemma 3.2.** *The morphisms*

$$T_* : (\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1; -]) \rightarrow (\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta) \leftarrow (\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), [d_2; -]) : T^*$$

*of cochain complexes are quasi-isomorphisms.*

**Remark:** This lemma holds for every manifold  $M$  and any  $G_\infty$ -morphism  $T : (\mathfrak{g}_1, d_1) \rightarrow (\mathfrak{g}_2, d_2)$ .

*Proof*  $\therefore$  First we show that  $T_*$  is a morphism of complexes. Let  $f \in \text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$  with  $|f| = k$ , then

$$\begin{aligned} T_*([d_1; f]) &= T \circ d_1 \circ f - (-1)^k T \circ f \circ d_1 \\ &= d_2 \circ (T \circ f) - (-1)^k (T \circ f) \circ d_1 \\ &= \delta(T_*(f)). \end{aligned}$$

Let us prove now that  $T_*$  is a quasi-isomorphism. For any graded vector space  $\mathfrak{g}$ , the space  $\wedge^{\bullet} \underline{\mathfrak{g}}^{\otimes \bullet}$  has the structure of a filtered space where the  $m$ -level of the filtration is  $F^m(\wedge^{\bullet} \underline{\mathfrak{g}}^{\otimes \bullet}) = \bigoplus_{p_1 + \dots + p_n - 1 \leq m} \underline{\mathfrak{g}}^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}^{\otimes p_n}$ . Clearly the differential  $d_1$  and  $d_2$  are compatible with the filtrations on  $\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}$  and  $\wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}$ , hence  $\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet})$  and  $\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet})$  are filtered cochain complex. This yields two spectral sequences (lying in the first quadrant)  $E^{\bullet, \bullet}$  and  $\tilde{E}^{\bullet, \bullet}$  which converge respectively toward the cohomology  $H^\bullet(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}))$  and  $H^\bullet(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}))$ . By standard spectral sequence techniques it is enough to

prove that the map  $T_*^0 : E_0^{\bullet,\bullet} \rightarrow \tilde{E}_0^{\bullet,\bullet}$  induced by  $T_*$  on the associated graded is a quasi-isomorphism.

The induced differentials on  $E_0^{\bullet,\bullet}$  and  $\tilde{E}_0^{\bullet,\bullet}$  are respectively  $[d_1^1, -] = 0$  and  $d_2^1 \circ (-) - (-) \circ d_1^1 = b \circ (-)$  where  $b$  is the Hochschild coboundary. By cofreeness property we have the following two isomorphisms

$$E_0^{\bullet,\bullet} \cong \text{End}(\mathfrak{g}_1), \quad \tilde{E}_0^{\bullet,\bullet} \cong \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2).$$

The map  $T_*^0 : E_0^{\bullet,\bullet} \rightarrow \tilde{E}_0^{\bullet,\bullet}$  induced by  $T_*$  is  $\varphi_{\text{HKR}} \circ (-)$ . Let  $p : \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$  be the projection onto the cohomology, *i.e.*  $p \circ \varphi_{\text{HKR}} = \text{id}$ . Let  $u : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  be any map satisfying  $b(u) = 0$  and set  $v = p \circ u \in \text{End}(\mathfrak{g}_1)$ . One can choose a map  $w : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  which satisfies for any  $x \in \mathfrak{g}_1$  the following identity

$$\varphi_{\text{HKR}} \circ p \circ u(x) - u(x) = b \circ w(x).$$

It follows that  $\varphi_{\text{HKR}}(v)$  has the same class of homology as  $u$  which proves the surjectivity of  $T_*^0$  in cohomology. The identity  $p \circ \varphi_{\text{HKR}} = \text{id}$  implies easily that  $T_*^0$  is also injective in cohomology which finish the proof of the lemma for  $T_*$ .

The proof that  $T^*$  is also a quasi-isomorphism is analogous.

*Proof of Theorem 3.1:*

It is easy to check that  $T - K$  is a cocycle in  $(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta)$ . The complex of cochain  $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1, -]) \cong (\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \mathfrak{g}_1), [d_1, -])$  is trigraded with  $||_1$  being the degree coming from the graduation of  $\mathfrak{g}_1$  and any element  $x$  lying in  $\underline{\mathfrak{g}}_1^{\otimes p_1} \wedge \dots \wedge \underline{\mathfrak{g}}_1^{\otimes p_q}$  satisfies  $|x|_2 = q - 1$ ,  $|x|_3 = p_1 + \dots + p_q - q$ . In the case  $M = \mathbb{R}^n$ , the cohomology  $H^\bullet(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1, -])$  is concentrated in bidegree  $(||_2, ||_3) = (0, 0)$  (see [Ta], [Hi]). By Lemma 3.2, this is also the case for the cochain complex  $(\text{Hom}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}, \wedge^{\bullet} \underline{\mathfrak{g}}_2^{\otimes \bullet}), \delta)$ . Thus, its cohomology classes are determined by complex morphisms  $(\mathfrak{g}_1, 0) \rightarrow (\mathfrak{g}_2, d_2^1)$  and it is enough to prove that  $T$  and  $K$  determine the same complex morphism  $(\mathfrak{g}_1, 0) \rightarrow (\mathfrak{g}_2, d_2^1 = b)$  which is clear because  $T^1$  and  $K^1$  are both equal to the Hochschild-Kostant-Rosenberg map.  $\blacksquare$

**Remark.** *It is possible to have an explicit formula for the map  $h$  in Theorem 3.1. In fact the quasi-isomorphism coming from Lemma 3.2 can be made explicit using explicit homotopy formulae for the Hochschild-Kostant-Rosenberg map (see [Ha] for example) and deformation retract techniques (instead of spectral sequences) as in [Ka]. The same techniques also apply to give explicit formulae for the quasi-isomorphism giving the acyclicity of  $(\text{End}(\wedge^{\bullet} \underline{\mathfrak{g}}_1^{\otimes \bullet}), [d_1, -])$  in the proof of theorem 3.1 (see [GH] for example)*

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