Lift of C_∞ and L_∞ morphisms to G_∞ morphisms

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Lifts of L_∞-morphisms to G_∞-morphisms

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Abstract

Let $\mathfrak{g}_2$ be the Hochschild complex of cochains on $C^\infty(\mathbb{R}^n)$ and $\mathfrak{g}_1$ be the space of multivector fields on $\mathbb{R}^n$. In this paper we prove that given any G_∞-structure (i.e. Gerstenhaber algebra up to homotopy structure) on $\mathfrak{g}_2$, and any morphism φ of Lie algebra up to homotopy between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there exists a G_∞-morphism Φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ that restricts to φ. In particular, the morphism constructed by Kontsevich can be obtained using Tamarkin's method for any G_∞-structure on $\mathfrak{g}_2$. We also show that any two of such G_∞-morphisms are homotopic in a certain sense.

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Let M be a differential manifold and $\mathfrak{g}_2 = (C^\cdot(A,A), b)$ be the Hochschild cochain complex on $A = C^\infty(M)$. The classical Hochschild-Kostant-Rosenberg theorem states that the cohomology of $\mathfrak{g}_2$ is the graded Lie algebra $\mathfrak{g}_1 = \Gamma^\cdot(M, \wedge^\cdot TM)$ of multivector fields on M. There is also a graded Lie algebra structure on $\mathfrak{g}_2$ given by the Gerstenhaber bracket. In particular $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are also Lie algebras up to homotopy (L_∞-algebra for short). In the case $M = \mathbb{R}^n$, using different methods, Kontsevich ([Ko1] and [Ko2]) and Tamarkin ([Ta]) have proved the existence of Lie homomorphisms "up to homotopy" (L_∞-morphisms) form $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Kontsevich's proof uses graph complex and is related to multizeta functions whereas Tamarkin's construction uses the existence of Drinfeld's associators. In fact Tamarkin's L_∞-morphism comes from the restriction of a Gerstenhaber algebra up to homotopy homomorphism (G_∞-morphism) from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. The G_∞-algebra structure on $\mathfrak{g}_1$ is induced by its classical Gerstenhaber algebra

structure and a far less trivial G_∞-structure on $\mathfrak{g}_2$ was proved to exist by Tamarkin [Ta] and relies on a Drinfeld's associator. When M is a Poisson manifold, Kontsevich and Tamarkin homomorphisms imply the existence of a star-product (see [BFFLS1] and [BFFLS2] for a definition). A connection between the two approaches has been given in [KS] but the morphisms given by Kontsevich and Tamarkin are not the same. The aim of this paper is to show explicitelly that Kontsevich's homomorphism can be obtained using Tamarkin's method, hence can be extended into a G_∞-morphism. More precisely, we show that given any G_∞-structure on $\mathfrak{g}_2$, and any L_∞-morphism φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there exists a G_∞-morphism Φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ that restricts to φ.

In the first section, we fix notations and recall the definitions of L_∞ and G_∞-structures. In the second section we state and prove the main Theorem. In the last section we show that any two G_∞-morphisms given by Tamarkin's method are homotopic in a certain sense.

Remark : in the sequel, unless otherwise is stated, the manifold M is supposed to be $\mathbb{R}^n$ for some $n \geq 1$. Most results could be generalized to other manifolds using techniques of Kontsevich [Ko1] (also see [TS], [CFT]).

1 L_∞ and G_∞-structures

For any graded vector space $\mathfrak{g}$, we choose the following degree on $\wedge^\cdot\mathfrak{g}$: if $X_1,\dots,X_k$ are homogeneous elements of respective degree $|X_1|,\dots|X_k|$, then

$$|X_1 \wedge \cdots \wedge X_k| = |X_1| + \cdots + |X_k| - k.$$

In particular the component $\mathfrak{g} = \wedge^1\mathfrak{g} \subset \wedge^\cdot\mathfrak{g}$ is the same as the space $\mathfrak{g}$ with degree shifted by one. The space $\wedge^\cdot\mathfrak{g}$ with the deconcatenation cobracket is the cofree cocommutative coalgebra on $\mathfrak{g}$ with degree shifted by one. Any degree one map $d^k : \wedge^k\mathfrak{g} \to \mathfrak{g}$ ($k \geq 1$) extends into a derivation $d^k : \wedge^\cdot\mathfrak{g} \to \wedge^\cdot\mathfrak{g}$ of the coalgebra $\wedge^\cdot\mathfrak{g}$ (by cofreeness property).

Definition 1.1. *A vector space $\mathfrak{g}$ is endowed with a L_∞-algebra (Lie algebras "up to homotopy") structure if there are degree one linear maps $m^{1,\dots,1} : \wedge^k\mathfrak{g} \to \mathfrak{g}$ such that if we extend them to maps $\wedge^\cdot\mathfrak{g} \to \wedge^\cdot\mathfrak{g}$, then $d \circ d = 0$ where d is the derivation*

$$d = m^1 + m^{1,1} + \cdots + m^{1,\dots,1} + \cdots .$$

For more details on L_∞-structures, see [LS]. It follows from the definition that a L_∞-algebra structure induces a differential coalgebra structure on $\wedge^\cdot\mathfrak{g}$ and that the map $m^1 : \mathfrak{g} \to \mathfrak{g}$ is a differential. If $m^{1,\dots,1} : \wedge^k\mathfrak{g} \to \mathfrak{g}$ are 0 for $k \geq 3$, we get the usual definition of (differential if $m^1 \neq 0$) graded Lie algebras.

For any graded vector space $\mathfrak{g}$, we denote $\underline{\mathfrak{g}^{\otimes n}}$ the quotient of $\mathfrak{g}^{\otimes n}$ by the image of all shuffles of length n (see [GK] or [GH] for details). The graded vector space $\oplus_{n\geq 0}\underline{\mathfrak{g}^{\otimes n}}$ is a quotient coalgebra of the tensor coalgebra $\oplus_{n\geq 0}\mathfrak{g}^{\otimes n}$. It is well known that this coalgebra $\oplus_{n\geq 0}\underline{\mathfrak{g}^{\otimes n}}$ is the cofree Lie coalgebra on the vector space $\mathfrak{g}$ (with degree shifted by

minus one). For any space $\mathfrak{g}$, we denote $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$ the graded space $\bigoplus_{m\geq 1,\ p_1+\cdots+p_n=m}\underline{\mathfrak{g}^{\otimes p_1}}\wedge\cdots\wedge\underline{\mathfrak{g}^{\otimes p_n}}$. We use the following grading on $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$: for $x_1^1,\cdots,x_n^{p_n}\in\mathfrak{g}$, we define

$$|\underline{x_1^1\otimes\cdots\otimes x_1^{p_1}}\wedge\cdots\wedge\underline{x_n^1\otimes\cdots\otimes x_n^{p_n}}|=\sum_{i_1}^{p_1}|x_1^{i_1}|+\cdots+\sum_{i_n}^{p_1}|x_n^{i_n}|-n.$$

Notice that the induced grading on $\wedge^{\cdot}\mathfrak{g}\subset\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$ is the same than the one introduced above. The cobracket on $\oplus\underline{\mathfrak{g}^{\otimes\cdot}}$ and the coproduct on $\wedge^{\cdot}\mathfrak{g}$ extend to a cobracket and a coproduct on $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$ which yield a Gerstenhaber coalgebra structure on $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$. It is well known that this coalgebra structure is cofree (see [Gi], Section 3 for example).

Definition 1.2. *A G_∞-algebra (Gerstenhaber algebra "up to homotopy") structure on a graded vector space $\mathfrak{g}$ is given by a collection of degree one maps*

$$m^{p_1,\cdots,p_n}\ :\ \underline{\mathfrak{g}^{\otimes p_1}}\wedge\cdots\wedge\underline{\mathfrak{g}^{\otimes p_n}}\to\mathfrak{g}$$

indexed by $p_1,\dots p_n\geq 1$ such that their canonical extension: $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}\to\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$ satisfies $d\circ d=0$ where

$$d=\sum_{m\geq 1,\ p_1+\cdots p_n=m}m^{p_1,\cdots,p_n}.$$

Again, as the coalgebra structure of $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$ is cofree, the map d makes $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$ into a differential coalgebra. If the maps $m^{p_1,\cdots,p_n}$ are 0 for $(p_1,p_2,\dots)\neq(1,0,\dots),(1,1,0,\dots)$ or $(2,0,\dots)$, we get the usual definition of (differential if $m^1\neq 0$) Gerstenhaber algebra.

The space of multivector fields $\mathfrak{g}_1$ is endowed with a graded Lie bracket $[-,-]_S$ called the Schouten bracket (see [Kos]). This Lie algebra can be extended into a Gerstenhaber algebra, with commutative structure given by the exterior product: $(\alpha,\beta)\mapsto\alpha\wedge\beta$

Setting $d_1=m_1^{1,1}+m_1^2$, where $m_1^{1,1}:\wedge^2\mathfrak{g}_1\to\mathfrak{g}_1$, and $m_1^2:\underline{\mathfrak{g}_1^{\otimes 2}}\to\mathfrak{g}_1$ are the extension of the Schouten bracket and the exterior product, we find that $(\mathfrak{g}_1,d_1)$ is a G_∞-algebra.

In the same way, one can define a differential Lie algebra structure on the vector space $\mathfrak{g}_2=C(A,A)=\bigoplus_{k\geq 0}C^k(A,A)$, the space of Hochschild cochains (generated by differential k-linear maps from A^k to A), where $A=C^\infty(M)$ is the algebra of smooth differential functions over M. Its bracket $[-,-]_G$, called the Gerstenhaber bracket, is defined, for $D,E\in\mathfrak{g}_2$, by

$$[D,E]_G=\{D|E\}-(-1)^{|E||D|}\{E|D\},$$

where

$$\{D|E\}(x_1,\dots,x_{d+e-1})=\sum_{i\geq 0}(-1)^{|E|\cdot i}D(x_1,\dots,x_i,E(x_{i+1},\dots,x_{i+e}),\dots).$$

The space $\mathfrak{g}_2$ has a grading defined by $|D|=k\Leftrightarrow D\in C^{k+1}(A,A)$ and its differential is $b=[m,-]_G$, where $m\in C^2(A,A)$ is the commutative multiplication on A.

Tamarkin (see [Ta] or also [GH]) stated the existence of a G_∞-structure on $\mathfrak{g}_2$ given by a differential $d_2 = m_2^1 + m_2^{1,1} + m_2^2 + \cdots + m_2^{p_1,\ldots,p_n} + \cdots$, on $\wedge^\cdot \underline{\mathfrak{g}_2^{\otimes^\cdot}}$ satisfying $d_2 \circ d_2 = 0$. Although this structure is non-explicit, it satisfies the following three properties :

$$\begin{aligned} &(a) \quad m_2^1 \text{ is the extension of the differential } b \\ &(b) \quad m_2^{1,1} \text{ is the extension of the Gerstenhaber bracket } [-,-]_G \\ &\qquad \text{and } m_2^{1,1,\ldots,1} = 0 \\ &(c) \quad m_2^2 \text{ induces the exterior product in cohomology.} \end{aligned} \tag{1.1}$$

Definition 1.3. *A L_∞-morphism between two L_∞-algebras $(\mathfrak{g}_1, d_1 = m_1^1 + \ldots)$ and $(\mathfrak{g}_2, d_2 = m_2^1 + \ldots)$ is a morphism of differential coalgebras*

$$\varphi \,:\, (\wedge^\cdot \mathfrak{g}_1, d_1) \to (\wedge^\cdot \mathfrak{g}_2, d_2). \tag{1.2}$$

Such a map φ is uniquely determined by a collection of maps $\varphi^n : \wedge^n \mathfrak{g}_1 \to \mathfrak{g}_2$ (again by cofreeness properties). In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are respectively the graded Lie algebra $(\Gamma(M, \wedge TM),\ [-,-]_S)$ and the differential graded Lie algebra $(C(A,A),\ [-,-]_G)$, the formality theorems of Kontsevich and Tamarkin state the existence of a L_∞-morphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that φ^1 is the Hochschild-Kostant-Rosenberg quasi-isomor phism.

Definition 1.4. *A morphism of G_∞-algebras between two G_∞-algebras $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$ is a map $\phi : (\wedge^\cdot \underline{\mathfrak{g}_1^{\otimes^\cdot}}, d_1) \to (\wedge^\cdot \underline{\mathfrak{g}_2^{\otimes^\cdot}}, d_2)$ of codifferential coalgebras.*

There is a coalgebra inclusion $\wedge^\cdot \mathfrak{g} \to \wedge^\cdot \underline{\mathfrak{g}^{\otimes^\cdot}}$, and it is easy to check that any G_∞-morphism between two G_∞-algebras $(\mathfrak{g}, \sum \underline{m^{p_1,\ldots,p_n}})$, $(\mathfrak{g}', \sum {m'}^{p_1,\ldots,p_n})$ restricts to a L_∞-morphism $\left(\wedge^\cdot \mathfrak{g}, \sum m^{1,\ldots,1}\right) \to \left(\wedge^\cdot \mathfrak{g}', \sum {m'}^{1,\ldots,1}\right)$. In the case $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are as above, Tamarkin's theorem states that there exists a G_∞-morphism between the two G_∞ algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (with the G_∞ structure he built) that restricts to a L_∞-morphism.

2 Main theorem

We keep the notations of the previous section, in particular $\mathfrak{g}_2$ is the Hochschild complex of cochains on $C^\infty(M)$ and $\mathfrak{g}_1$ its cohomology. Here is our main theorem.

Theorem 2.1. *Given any G_∞-structure d_2 on $\mathfrak{g}_2$ satisfying the three properties of (1.1), and any L_∞-morphism φ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that φ^1 is the Hochschild-Kostant-Rosenberg map, there exists a G_∞-morphism $\Phi : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ that restricts to φ.*

In particular, Theorem 2.1 implies that the formality map of Kontsevich lifts into a G_∞-morphism from $\mathfrak{g}_1$ (with its classical G_∞-structure) to $\mathfrak{g}_2$ (endowed with Tamarkin's G_∞-structure).

Let us first recall the proof of Tamarkin's formality theorem (see [GH] for more details):

1. First one proves there exists a G_∞-structure on $\mathfrak{g}_2$, given by a differential d_2, as in (1.1).

2. Then, one constructs a G_∞-structure on $\mathfrak{g}_1$ given by a differential d_1' together with a G_∞-morphism Φ between $(\mathfrak{g}_1, d_1')$ and $(\mathfrak{g}_2, d_2)$.

3. Finaly, one constructs a G_∞-morphism Φ' between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_1, d_1')$.

The composition $\Phi \circ \Phi'$ is then a G_∞-morphism between $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$,thus restricts to a L_∞-morphism between the differential graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

We suppose now that, in the first step, we take any G_∞-structure on $\mathfrak{g}_2$ given by a differential d_2 and we suppose we are given a L_∞-morphism φ between the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ satisfying $\varphi^1 = \varphi_{\mathrm{HKR}}$ the Hochschild-Kostant-Rosenberg quasi-isomorphism.

Proof of Theorem 2.1:

The Theorem will follow if we prove that Items 2 and 3 of Tamarkin's construction are still true with the extra conditions that the restriction of the G_∞-morphism Φ (resp. Φ') on the Lie structures is the L_∞-morphism φ (resp. id) between the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. As the arguments for Φ and Φ' are very similar, we will only prove the result for Φ.

Let us recall (see [GH]) that the constructions of Φ and d_1' can be made by induction. For $i = 1, 2$ and $n \geq 0$, let us set

$$V_i^{[n]} = \bigoplus_{p_1+\cdots+p_k=n} \underline{\mathfrak{g}_i^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_i^{\otimes p_k}}$$

and $V_i^{[\leq n]} = \sum_{k \leq n} V_i^{[k]}$. Let $d_2^{[n]}$ and $d_2^{[\leq n]}$ be the sums

$$d_2^{[n]} = \sum_{p_1+\cdots+p_k=n} d_2^{p_1,\ldots,p_k} \qquad \text{and} \qquad d_2^{[\leq n]} = \sum_{p \leq n} d_2^{[p]}.$$

Clearly, $d_2 = \sum_{n \geq 1} d_2^{[n]}$. In the same way, we denote $d_1' = \sum_{n \geq 1} d'^{[n]}_1$ with

$$d'^{[n]}_1 = \sum_{p_1+\cdots+p_k=n} d_1'^{p_1,\ldots,p_k} \qquad \text{and} \qquad d'^{[\leq n]}_1 = \sum_{1 \leq k \leq n} d'^{[k]}_1.$$

We know from Section 1 that a morphism $\Phi : (\wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, d_1') \to (\wedge^{\cdot} \underline{\mathfrak{g}_2^{\otimes \cdot}}, d_2)$ is uniquely determined by its components $\Phi^{p_1,\ldots,p_k} : \underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_1^{\otimes p_k}} \to \mathfrak{g}_2$. Again, we have $\Phi = \sum_{n \geq 1} \Phi^{[n]}$ with

$$\Phi^{[n]} = \sum_{p_1+\cdots+p_k=n} \Phi^{p_1,\ldots,p_k} \qquad \text{and} \qquad \Phi^{[\leq n]} = \sum_{1 \leq k \leq n} \Phi^{[k]}.$$

We want to construct the maps $d'^{[n]}_1$ and $\Phi^{[n]}$ by induction with the initial condition

$$d'^{[1]}_1 = 0 \qquad \text{and} \qquad \Phi^{[1]} = \varphi_{\mathrm{HKR}},$$

where $\varphi_{\mathrm{HKR}} : (\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, b)$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism (see [HKR]) defined, for $\alpha \in \mathfrak{g}_1$, $f_1, \cdots, f_n \in A$, by

$$\varphi_{\mathrm{HKR}} \ : \ \alpha \mapsto \big((f_1, \ldots, f_n) \mapsto \ \langle \alpha, d f_1 \wedge \cdots \wedge d f_n \rangle\big).$$

Moreover, we want the following extra conditions to be true:

$$\Phi^{1,\ldots,1} = \varphi^{1,\ldots,1}, \qquad {d'}_1^{1,1} = d_1^{1,1}, \qquad {d'}_1^{1,1,\ldots,1} = 0. \tag{2.3}$$

Now suppose the construction is done for $n-1$ ($n \geq 2$), i. e., we have built maps $({d'}_1^{[i]})_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$ satisfying conditions (2.3) and

$$\Phi^{[\leq n-1]} \circ {d'}_1^{[\leq n-1]} = d_2^{[\leq n-1]} \circ \ \Phi^{[\leq n-1]} \text{ on } V_1^{[\leq n-1]} \text{ and } {d'}_1^{[\leq n-1]} \circ {d'}_1^{[\leq n-1]} = 0 \text{ on } V_1^{[\leq n]}. \tag{2.4}$$

In [GH], we prove that for any such $({d'}_1^{[i]})_{i \leq n-1}$ and $(\Phi^{[i]})_{i \leq n-1}$, one can construct ${d'}_1^{[n]}$ and $\Phi^{[n]}$ such that condition (2.4) is true for n instead of $n-1$. To complete the proof we have to show that ${d'}_1^{[n]}$ and $\Phi^{[n]}$ also satisfy conditions (2.3). In the equation 2.4, the terms ${d'}_1^{\overbrace{1,\ldots,1}^{n}}$ and $\Phi^{\overbrace{1,\ldots,1}^{n}}$ only act on $V_1^{\overbrace{1,\ldots,1}^{n}}$. So one can replace $\Phi^{1,\ldots,1}$ with $\varphi^{1,\ldots,1}$, ${d'}_1^{1,1}$ with $d_1^{1,1}$ (or ${d'}_1^{\overbrace{1,\ldots,1}^{i}}, i \geq 3$ with 0) provided conditions (2.4) are still satisfied on $V_1^{1,\ldots,1}$. The other terms acting on $V_1^{1,\ldots,1}$ in the equation 2.4 only involves terms $\Phi^{1,\ldots,1} = \varphi^{1,\ldots,1}$ and ${d'}_1^{1,\ldots,1}$. Then conditions (2.4) on $V_1^{1,\ldots,1}$ is the equation that should be satisfied by a L_∞-morphism between the L_∞-algebras $(\mathfrak{g}_1, {d'}_1^{1,1} = d_1^{1,1})$ and $(\mathfrak{g}_2, d_2^{1,1})$ restricted to $V_1^{1,\ldots,1}$. Hence by hypothesis on φ the conditions hold. This proves the theorem. ■

3 The difference between two G_∞-maps

In this section we investigate the difference between Tamarkin's formality maps and the lift of Kontsevich map, and more generally between two differents G_∞-formality maps.

We fix once for all a G_∞-structure on $\mathfrak{g}_2$ (given by a differential d_2) satisfying the conditions (1.1) and a morphism of G_∞-algebras $T : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ such that $T^1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is φ_{HKR}. Let $K : (\mathfrak{g}_1, d_1) \to (\mathfrak{g}_2, d_2)$ be any other G_∞-morphism with $K^1 = \varphi_{\mathrm{HKR}}$ (for example any lift of Kontsevich formality map).

Theorem 3.1. *There exists a map $h : \wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}} \to \wedge^{\cdot} \underline{\mathfrak{g}_2^{\otimes \cdot}}$ such that*

$$T - K = h \circ d_1 + d_2 \circ h.$$

In other words the formality morphisms K and T are homotopic.

The maps T and K are elements of the cochain complex $\Big(\mathrm{Hom}(\wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, \wedge^{\cdot} \underline{\mathfrak{g}_2^{\otimes \cdot}}), \delta\Big)$ with differential given, for all $f \in \mathrm{Hom}(\wedge^{\cdot} \underline{\mathfrak{g}_1^{\otimes \cdot}}, \wedge^{\cdot} \underline{\mathfrak{g}_2^{\otimes \cdot}}), |f| = k$, by

$$\delta(f) = d_2 \circ f - (-1)^k f \circ d_1.$$

We first compare this cochain complex with the complexes $\Big(\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}),[d_1;-]\Big)$ and $\Big(\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}),[d_2;-]\Big)$ (where $[-;-]$ is the graded commutator of morphisms). There are morphisms

$$T_*:\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}})\to\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}),\ T^*:\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}})\to\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}})$$

defined, for $f\in\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}})$ and $g\in\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}})$, by

$$T_*(f)=T\circ f,\ T^*(g)=g\circ T.$$

Lemma 3.2. *the morphisms*

$$T_*:\Big(\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}),[d_1;-]\Big)\to\Big(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}),\delta\Big)\leftarrow\Big(\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}),[d_2;-]\Big):T^*$$

of cochain complexes are quasi-isomorphisms.

Remark: this lemma holds for every manifold M and any G_∞-morphism $T:(\mathfrak{g}_1,d_1)\to(g_2,d_2)$.

Proof: First we show that T_* is a morphism of complexes. Let $f\in\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}})$ with $|f|=k$, then

$$\begin{aligned}T_*([d_1;f]) &= T\circ d_1\circ f-(-1)^kT\circ f\circ d_1\\ &= d_2\circ(T\circ f)-(-1)^k(T\circ f)\circ d_1\\ &= \delta(T_*(f)).\end{aligned}$$

Let us prove now that T_* is a quasi-isomorphism. For any graded vector space $\mathfrak{g}$, the space $\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}}$ has the structure of a filtered space where the m-level of the filtration is $F^m(\wedge^{\cdot}\underline{\mathfrak{g}^{\otimes\cdot}})=\oplus_{p_1+\cdots+p_n-1\leq m}\underline{\mathfrak{g}^{\otimes p_1}\wedge\ldots\mathfrak{g}^{\otimes p_n}}$. Clearly the differential d_1 and d_2 are compatible with the filtrations on $\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}$ and $\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}$, hence $\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}})$ and $\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}})$ are filtered cochain complex. This yields two spectral sequences (lying in the first quadrant) $E^{\cdot,\cdot}_{\cdot}$ and $\widetilde{E}^{\cdot,\cdot}_{\cdot}$ which converge respectively toward the cohomology $H^{\cdot}(\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}}))$ and $H^{\cdot}(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes\cdot}},\wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes\cdot}}))$. By standard spectral sequence techniques it is enough to prove that the map $T^0_*:E^{\cdot,\cdot}_0\to\widetilde{E}^{\cdot,\cdot}_0$ induced by T_* on the associated grading is a quasi-isomorphism.

The induced differentials on $E^{\cdot,\cdot}_0$ and $\widetilde{E}^{\cdot,\cdot}_0$ are respectively $[d_1^1,-]=0$ and $d_2^1\circ(-)-(-)\circ d_1^1=b\circ(-)$ where b is the Hochschild coboundary. By cofreeness property we have the following two isomorphims

$$E^{\cdot,\cdot}_0\cong\mathrm{End}(\mathfrak{g}_1),\qquad\widetilde{E}^{\cdot,\cdot}_0\cong\mathrm{Hom}(\mathfrak{g}_1,\mathfrak{g}_2).$$

The map $T^0_*:E^{\cdot,\cdot}_0\to\widetilde{E}^0_{\cdot\cdot}$ induced by T_* is $\varphi_{\mathrm{HKR}}\circ(-)$. Let $p:\mathfrak{g}_2\to\mathfrak{g}_1$ be the projection onto the cohomology, *i.e.* $p\circ\varphi_{\mathrm{HKR}}=\mathrm{id}$. Let $u:\mathfrak{g}_1\to\mathfrak{g}_2$ be any map satisfying $b(u)=0$ and set $v=p\circ u\in\mathrm{End}(\mathfrak{g}_1)$. One can choose a map $w:\mathfrak{g}_1\to\mathfrak{g}_2$ which satisfies for any $x\in\mathfrak{g}_1$ the following identity

$$\varphi_{\mathrm{HKR}}\circ p\circ u(x)-u(x)=b\circ w(x).$$

It follows that $\varphi_{\mathrm{HKR}}(v)$ has the same class of homology as u which proves the surjectivity of T_*^0 in cohomology. The identity $p \circ \varphi_{\mathrm{HKR}} = \mathrm{id}$ implies easily that T_*^0 is also injective in cohomology which finish the proof of the lemma for T_*.

The proof that T^* is also a quasi-isomorphim is analogous. ■

Proof of Theorem 3.1:

It is easy to check that $T - K$ is a cocycle in $\Big(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}, \wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}), \delta\Big)$. The complex of cochain $\Big(\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}), [d_1, -]\Big) \cong \Big(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}, \mathfrak{g}_1), [d_1, -]\Big)$ is trigraded with $|\,|_1$ being the degree coming from the graduation of $\mathfrak{g}_1$ and any element x lying in $\underline{\mathfrak{g}_1^{\otimes p_1}} \wedge \cdots \wedge \underline{\mathfrak{g}_1^{\otimes p_q}}$ satisfies $|x|_2 = q - 1$, $|x|_3 = p_1 + \ldots p_q - q$. In the case $M = \mathbb{R}^n$, the cohomology $H^{\cdot}\Big(\mathrm{End}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}), [d_1, -]\Big)$ is concentrated in bidegree $(|\,|_2, |\,|_3) = (0,0)$ (see [Ta], [Hi]). By Lemma 3.2, this is also the case for the cochain complex $\Big(\mathrm{Hom}(\wedge^{\cdot}\underline{\mathfrak{g}_1^{\otimes \cdot}}, \wedge^{\cdot}\underline{\mathfrak{g}_2^{\otimes \cdot}}), \delta\Big)$. Thus, its cohomology classes are determined by complex morphisms $(\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, d_2^1)$ and it is enough to prove that T and K determine the same complex morphism $(\mathfrak{g}_1, 0) \to (\mathfrak{g}_2, d_2^1 = b)$ which is clear because T^1 and K^1 are both equal to the Hochschild-Kostant-Rosenberg map. ■

Remark 3.3. It is possible to have an explicit formula for the map h in Theorem 3.1. In fact the quasi-isomorphism coming from Lemma 3.2 can be made explicit using explicit homotopy formulae for the Hochschild-Kostant-Rosenberg map (see [Ha] for example) and deformation retract techniques (instead of spectral sequences) as in [Ka]. It is also the case for the isomorphism in cohomology in the proof of Theorem 3.1 (see [GH] for example).

References

[BFFLS1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Quantum mechanics as a deformation of classical mechanics*, Lett. Math. Phys. 1 (1975/77), no. 6, 521–530

[BFFLS2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization, I and II*, Ann. Phys. 111 (1977), 61-151

[CFT] A. S. Cattaneo, G. Felder, L. Tomassini, *Fedosov connections on jet bundles and deformation quantization*, math.QA/0111290 (to appear in IRMA Lecture Notes in Mathematics and Theoretical Physics : Deformation Quantization - Proceedings of the Meeting between Mathematicians and Theoretical Physicists, Institut de Recherche Mathématique Avancée de Strasbourg (May 31 - June 2, 2001)

[Gi] G. Ginot, *Homologie et modèle minimal des algèbres de Gerstenhaber*, Preprint Institut de Recherche Mathématique Avancée, Strasbourg (2001), no. 42

[GH] G. Ginot, G. Halbout *A formality theorem for Poisson manifold*, Preprint Institut de Recherche Mathématique Avancée, Strasbourg (2002), no. 42, to appear in Letters in Mathematical Physics

[GK] V. Ginzburg, M. Kapranov, *Koszul duality for operads*, Duke Math. J. 76 (1994), No 1, 203–272

[Ha] G. Halbout, *Formule d'homotopie entre les complexes de Hochschild et de de Rham*, Compositio Math. 126 (2001), No 2, 123–145

[Hi] V. Hinich, *Tamarkin's proof of Kontsevich's formality theorem*, math.QA/9803025

[HKR] G. Hochschild, B. Kostant and A. Rosenberg, *Differential forms on regular affine algebras*, Transactions AMS 102 (1962), 383-408

[Ka] C. Kassel, *Homologie cyclique, caractère de Chern et lemme de perturbation*, J. Reine Angew. Math. 408 (1990), 159–180

[Ko1] M. Kontsevich, *Formality conjecture. Deformation theory and symplectic geometry*, Math. Phys. Stud. 20, Ascona (1996), 139–156

[Ko2] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, q-alg/9709040

[KS] M. Kontsevich, Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Conférence Moshé Flato 1999, Vol. I (Dijon) Math. Phys. Stud. 21 (2000), 255–307

[Kos] J. L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie in "Elie Cartan et les mathémati-ques d'aujourd'hui"*, Astérisque (1985), 257-271

[LS] T. Lada, J. D. Stasheff, *Introduction to SH Lie algebras for physicists*, Internat. J. Theoret. Phys. 32 (1993), No 7, 1087-1103

[Ta] D. Tamarkin, *Another proof of M. Kontsevich's formality theorem*, math.QA/9803025

[TS] D. Tamarkin, B Tsygan, *Noncommutative differential calculus, homotopy BV algebras and formality conjectures*, Methods Funct. Anal. Topology, 6 (2000), no. 2, 85–97