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**MULTI-VALUED BOUNDARY VALUE PROBLEMS  
INVOLVING LERAY-LIONS OPERATORS  
AND DISCONTINUOUS NONLINEARITIES**

SIMONA DĂBULEANU – VICENȚIU RĂDULESCU

We prove an existence result for a class of Dirichlet boundary value problems with discontinuous nonlinearity and involving a Leray-Lions operator. The proof combines monotonicity methods for elliptic problems, variational inequality techniques and basic tools related to monotone operators. Our work generalizes a result obtained in Carl [4].

*Key words:* sub- and super-solution, Leray-Lions operator, maximal monotone graph, pseudo-monotone operator, variational inequality.

*2000 Mathematics Subject Classification:* 35J65, 47H10, 47J25, 58J32.

### 1. Introduction and the main result.

Let  $\Omega \subset \mathbf{R}^N$  be a bounded domain with smooth boundary. Consider the boundary value problem

$$(P) \begin{cases} -\operatorname{div}(a(x, \nabla u(x))) = f(u(x)), & \text{if } x \in \Omega \\ u = 0, & \text{on } \partial \Omega, \end{cases}$$

where  $a : \Omega \times \mathbf{R}^N \rightarrow \mathbf{R}^N$  is a Carathéodory function having the properties

- (a<sub>1</sub>) there exist  $p > 1$  and  $\lambda > 0$  such that  $a(x, \xi) \cdot \xi \geq \lambda \cdot \|\xi\|^p$ , for a.e.  $x \in \Omega$  and for any  $\xi \in \mathbf{R}^N$ ;
- (a<sub>2</sub>)  $(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0$ , for any  $\xi, \eta \in \mathbf{R}^N$ ,  $\xi \neq \eta$ ;
- (a<sub>3</sub>) there exist  $\alpha \in \mathbf{R}^+$  and  $k \in L^{p'}(\Omega)$  such that  $|a(x, \xi)| \leq \alpha(k(x) + |\xi|^{p-1})$ , for a.e.  $x \in \Omega$  and for any  $\xi \in \mathbf{R}^N$ .

Assume that the nonlinearity  $f : \mathbf{R} \rightarrow \mathbf{R}$  satisfies the hypothesis

- (H<sub>1</sub>) there exist nondecreasing functions  $f, g : \mathbf{R} \rightarrow \mathbf{R}$  such that  $f = g - h$ .

Let  $\beta : \mathbf{R} \rightarrow 2^{\mathbf{R}}$  be the maximal monotone graph associated with the nondecreasing function  $h$  (see Brezis [3]). More exactly,

$$\beta(s) := [h^-(s), h^+(s)], \quad \text{for all } s \in \mathbf{R},$$

where

$$h^-(s) = \lim_{\varepsilon \rightarrow 0^+} h(s - \varepsilon), \quad h^+(s) = \lim_{\varepsilon \rightarrow 0^+} h(s + \varepsilon).$$

Under this assumption we reformulate the problem  $(P)$  as follows

$$(P') \begin{cases} -\operatorname{div}(a(x, \nabla(x))) + \beta(u(x)) \ni g(u(x)), & \text{if } x \in \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Denote by  $G$  the Nemitskii operator associated with  $g$ , that is,  $G(u)(x) = g(u(x))$ .

DEFINITION 1. A function  $u \in W_0^{1,p}(\Omega)$  is called a solution of the problem  $(P')$  if there exists  $v \in L^{p'}(\Omega)$  such that

- i)  $v(x) \in \beta(u(x))$  a.e. in  $\Omega$ ,
- ii)  $\int_{\Omega} a(x, \nabla u) \cdot \nabla w \, dx + \int_{\Omega} v \cdot w \, dx = \int_{\Omega} G(u) \cdot w \, dx$ , for any  $w \in W_0^{1,p}(\Omega)$ .

Let  $L_+^p$  be the set of nonnegative elements of  $L^p(\Omega)$ . For any  $v, w \in \Omega$  such that  $v \leq w$ , we set

$$[v, w] = \{u \in L^p(\Omega) / v \leq u \leq w\}.$$

DEFINITION 2. A function  $\bar{u} \in W^{1,p}(\Omega)$  is called an upper solution of the problem  $(P')$  if there exists a function  $\bar{v} \in L^{p'}(\Omega)$  such that

- i)  $\bar{v}(x) \in \beta(\bar{u}(x))$  a.e. in  $\Omega$ ,
- ii)  $\bar{u} \geq 0$  on  $\partial\Omega$ ,
- iii)  $\int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla w \, dx + \int_{\Omega} \bar{v} \cdot w \, dx \geq \int_{\Omega} G(\bar{u}) \cdot w \, dx$  for all  $w \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$ .

DEFINITION 3. A function  $\underline{u} \in W^{1,p}(\Omega)$  is called a lower solution of the problem  $(P')$  if there exists a function  $\underline{v} \in L^{p'}(\Omega)$  such that

- i)  $\underline{v}(x) \in \beta(\underline{u}(x))$  a.e. in  $\Omega$ ,

- ii)  $\underline{u} \leq 0$  on  $\partial\Omega$ ,
- iii)  $\int_{\Omega} a(x, \nabla \underline{u}) \cdot \nabla w \, dx + \int_{\Omega} \underline{v} \cdot w \, dx \leq \int_{\Omega} G(\underline{u}) \cdot w \, dx$  for any  $w \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$ .

In the sequel the following hypothesis will be needed:

- (H<sub>2</sub>) There exist an upper solution  $\bar{u}$  and a lower solution  $\underline{u}$  of the problem (P') such that  $\underline{u} \leq \bar{u}$ , and  $G(\underline{u}), G(\bar{u}), H^+(\bar{u}), H^-(\underline{u}) \in L^{p'}(\Omega)$ .

The following is a generalization of the main result in Carl [4].

**THEOREM 1.** *Assume hypothesis (H<sub>1</sub>) and (H<sub>2</sub>) hold and that  $g$  is right (resp. left) continuous. Then there exists a maximal (resp. minimal) solution  $u \in [\underline{u}, \bar{u}]$  of the problem (P').*

## 2. Proof of Theorem 1.

We first reformulate the problem (P') in terms of variational inequalities using the subdifferential theory in the sense of convex analysis.

Let  $j : \mathbf{R} \rightarrow (-\infty, \infty]$  be a convex, proper and lower semicontinuous function. Let  $\partial j$  be the subdifferential of  $j$ , that is

$$(1) \quad \partial j(r) = \{\hat{r} \in \mathbf{R} : j(s) \geq j(r) + \hat{r}(s - r) \quad \text{for all } s \in \mathbf{R}\}.$$

We recall the following result concerning maximal monotone graphs in  $\mathbf{R}^2$  (see Brezis [3] [Corollary 2.10], p. 43)

**LEMMA 1.** *Let  $\beta : \mathbf{R} \rightarrow 2^{\mathbf{R}}$  be a maximal monotone graph in  $\mathbf{R}^2$ . Then there exists a convex, proper and lower semicontinuous function  $j : \mathbf{R} \rightarrow (-\infty, +\infty]$  such that  $\beta = \partial j$ . Moreover, the function  $j$  is uniquely determined up to an additive constant.*

We observe that the function  $h$  appearing in (H<sub>1</sub>) can always be chosen so that  $h(0) = 0$ . Then the maximal monotone graph  $\beta$  has the properties

$$(2) \quad D(\beta) = \mathbf{R} \quad \text{and} \quad 0 \in \beta(0).$$

Since the function  $j$  related to  $\beta$  according to Lemma 1 is uniquely determined up to an additive constant we can assume that

$$(3) \quad j(0) = 0.$$

So, by (1), (2) and (3) it follows that

$$(4) \quad j(s) \geq 0 \quad \text{for all } s \in \mathbf{R}.$$

Define  $J : L^p(\Omega) \rightarrow (-\infty, +\infty]$  by

$$J(v) = \begin{cases} \int_{\Omega} j(v(x)) dx, & \text{for } j(v(\cdot)) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then  $J$  is convex, proper and lower semicontinuous (see Barbu [1]).

Under the above assertions we can reformulate the problem ( $P'$ ) in terms of variational inequalities as follows: find  $u \in W_0^{1,p}(\Omega)$  such that

$$(5) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) dx + J(w) - J(u) \geq \int_{\Omega} G(u)(w - u) dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

LEMMA 2. *Let hypotheses ( $H_1$ ) and ( $H_2$ ) be fulfilled. Then  $u \in [\underline{u}, \bar{u}]$  is a solution of (5) if and only if  $u$  is a solution of the problem ( $P'$ ).*

*Proof.* Let  $u \in [\underline{u}, \bar{u}]$  satisfy the variational inequality (5). Then

$$J(w) \geq J(u) + \int_{\Omega} G(u) \cdot (w - u) dx - \int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) dx.$$

It follows that

$$(6) \quad \operatorname{div}(a(x, \nabla u)) + G(u) \in \partial J(u) \quad \text{in } W^{-1,p'}(\Omega).$$

It follows by Brezis [2] [Corollaire 1] that any subgradient  $v \in \partial J(u)$  of the functional  $J : W_0^{1,p}(\Omega) \rightarrow (-\infty, +\infty]$  at  $u \in W_0^{1,p}(\Omega)$  belongs to  $L^1(\Omega)$  and satisfies

$$(7) \quad v(x) \in \partial j(u(x)) = \beta(u(x)) \quad \text{a.e. in } \Omega.$$

Furthermore

$$h^-(\underline{u}(x)) \leq h^-(u(x)) \leq \beta(u(x)) \leq h^+(\bar{u}(x)) \leq h^+(\bar{u}(x)) \quad \text{a.e. in } \Omega.$$

Thus

$$(8) \quad |v| \leq |H^+(\bar{u})| + |H^-(\underline{u})|.$$

By ( $H_2$ ), the right-hand side of (8) belongs to  $L^{p'}(\Omega)$ . It follows that  $v \in L^{p'}(\Omega)$ . Thus there exists  $v \in L^{p'}(\Omega)$  such that

$$\operatorname{div}(a(x, \nabla u)) + G(u) = v \quad \text{in } W^{-1,p'}(\Omega)$$

or, equivalently,

$$(9) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla w \, dx + \int_{\Omega} v \cdot w \, dx = \int_{\Omega} G(u)w \, dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

Relations (7) and (9) imply that  $u \in W_0^{1,p}(\Omega)$  is a solution of the problem  $(P')$ .

Conversely, let  $u \in [\underline{u}, \bar{u}]$  be a solution of the problem  $(P')$ . Then there exists  $v \in L^p(\Omega)$  such that  $v \in \beta(u) = \partial j(u(x))$  and the relation (9) is fulfilled. Since  $v(x) \in \partial j(u(x))$  we have

$$(10) \quad j(s) \geq j(u(x)) + v(x)(s - u(x)).$$

Taking  $s = 0$  in (10) we obtain, by means of (3) and (4) that  $0 \leq j(u(x)) \leq v(x)u(x)$ . Thus

$$(11) \quad j(u(\cdot)) \in L^1(\Omega) \quad \text{and} \quad J(u) = \int_{\Omega} j(u(x)) \, dx.$$

Let  $w \in W_0^{1,p}(\Omega)$ . Taking  $s = w(x)$  in (10) we obtain

$$(12) \quad \int_{\Omega} j(w(x)) \, dx - \int_{\Omega} j(u(x)) \, dx \geq \int_{\Omega} v(x)(w(x) - u(x)) \, dx.$$

From (9), substituting  $w$  by  $w - u \in W_0^{1,p}(\Omega)$  we get, by means of (12)

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) \, dx + J(w) - J(u) \geq \int_{\Omega} G(u) \cdot (w - u) \, dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

This means that  $u$  is a solution of the variational inequality (5).  $\square$

*Remark 1.* If  $u$  is a solution of  $(P')$  then, by (11),  $J(u) < +\infty$ . The result also holds also if we replace  $u$  by a super-solution  $\bar{u}$  or by a sub-solution  $\underline{u}$ .

Set  $v^+ = \max\{v, 0\}$ .

LEMMA 3. Let  $u, v \in L^p(\Omega)$  such that  $J(u)$  and  $J(v)$  are finite. Then

$$(13) \quad J(u - (u - v)^+) - J(u) + J(v + (u - v)^+) - J(v) = 0.$$

*Proof.* Let  $\Omega_+ := \{x \in \Omega \mid u > v\}$  and  $\Omega_- := \{x \in \Omega \mid u \leq v\}$ . Since  $(u - v)^+ = 0$  in  $\Omega_-$  and  $(u - v)^+ = u - v$  in  $\Omega_+$  we obtain

$$(14) \quad J(u - (u - v)^+) = \int_{\Omega_+} j(v) dx + \int_{\Omega_-} j(u) dx \leq \infty$$

$$(15) \quad J(v + (u - v)^+) = \int_{\Omega_+} j(u) dx + \int_{\Omega_-} j(v) dx \leq \infty$$

By (14) and (15) we obtain (13).

Consider now the following variational inequality: given  $z \in L^p(\Omega)$ , find  $u \in W_0^{1,p}(\Omega)$  such that

$$(16) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) + J(w) - J(u) \geq \int_{\Omega} G(z)(w - u) dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

The variational inequality (16) defines a mapping  $T : z \rightarrow u$  and each fixed point of  $T$  yields a solution of (5) and conversely.

**LEMMA 4.** *Let hypotheses  $(H_1)$  and  $(H_2)$  be satisfied. Then for each  $z \in [\underline{u}, \bar{u}]$  the variational inequality (16) has a unique solution  $u = Tz \in [\underline{u}, \bar{u}]$ . Moreover, there is a constant  $C > 0$  such that  $\|Tz\|_{W_0^{1,p}(\Omega)} \leq C$ , for any  $z \in [\underline{u}, \bar{u}]$ .*

*Proof. Existence.* Let  $z \in [\underline{u}, \bar{u}]$  be arbitrarily given. Then  $G(z)$  is measurable and  $G(z) \in L^{p'}(\Omega)$ , due to the estimate

$$|G(z)| \leq |G(\bar{u})| + |G(\underline{u})|$$

and after observing that the right-hand side of the above inequality is in  $L^{p'}(\Omega)$ , by  $(H_2)$ .

We now apply Theorem II.8.5 in Lions [5]. We first observe that the above assertions show that the mapping  $W_0^{1,p}(\Omega) \ni u \rightarrow \int_{\Omega} G(z)u$  is in  $W^{-1,p'}(\Omega)$ .

Consider the Leray-Lions operator  $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$  defined by

$$\langle Au, w \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla w dx .$$

We show that  $A$  is a pseudo-monotone operator. For this aim it is enough to prove that  $A$  is bounded, monotone and hemi-continuous (see Lions [5] [Prop. II.2.5]).

Condition  $(a_3)$  yields the boundedness of  $A$ . Indeed

$$\|Au\|_{W^{-1,p'}(\Omega)} \leq C(\|k\|_{L^{p'}(\Omega)} + \|\nabla u\|_{L^p(\Omega)}^{p-1}).$$

We also observe that  $(a_2)$  implies that  $A$  is a monotone operator.

In order to justify the hemi-continuity of  $A$ , let us consider a sequence  $(\lambda_n)_{n \geq 1}$  converging to  $\lambda$ . Then, for given  $u, v, w \in W_0^{1,p}(\Omega)$ , we have

$$a(x, \nabla(u + \lambda_n v)) \cdot \nabla w \rightarrow a(x, \nabla(u + \lambda v)) \cdot \nabla w \quad \text{a.e. in } \Omega.$$

From the boundedness of  $\{\lambda_n\}$  and condition  $(a_3)$  we obtain that the sequence  $\{|a(x, \nabla(u + \lambda_n v)) \nabla w|\}$  is bounded by a function which belongs to  $L^1(\Omega)$ . Using the Lebesgue dominated convergence theorem it follows that

$$\langle A(u + \lambda_n v), w \rangle \rightarrow \langle A(u + \lambda v), w \rangle \quad \text{as } n \rightarrow \infty.$$

Hence the application  $\lambda \rightarrow \langle A(u + \lambda v), w \rangle$  is continuous.

It follows that all assumptions of Theorem II.8.5 in [5] are fulfilled, so the problem (16) has at least a solution.

*Uniqueness.* Let  $u_1$  and  $u_2$  be two solutions of (16). Then taking  $w = u_2$  as a test function for the solution  $u_1$ , we obtain

$$\int_{\Omega} a(x, \nabla u_1) \cdot \nabla(u_2 - u_1) dx + J(u_2) - J(u_1) \geq \int_{\Omega} G(z)(u_2 - u_1) dx.$$

Similarly we find

$$\int_{\Omega} a(x, \nabla u_2) \cdot \nabla(u_1 - u_2) dx + J(u_1) - J(u_2) \geq \int_{\Omega} G(z)(u_1 - u_2) dx.$$

Therefore

$$\int_{\Omega} (a(x, \nabla u_1) - a(x, \nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) dx \leq 0.$$

So, by  $(a_2)$ , it follows that  $\nabla u_1 = \nabla u_2$ , so  $u_1 = u_2 + C$  in  $\Omega$ . Since  $u_1 = u_2 = 0$  on  $\partial\Omega$ , it follows that  $u_1 = u_2$  in  $\Omega$ .

From (3) and (4) we deduce that  $J(0) = 0$  and  $J(u) \geq 0$ . Moreover, the variational inequality (16) implies

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla(-u) dx + J(0) - J(u) \geq - \int_{\Omega} G(z)u dx.$$



Thus

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla u \, dx \leq \int_{\Omega} G(z)u \, dx .$$

This last inequality, assumption  $(a_1)$  and Hölder's inequality yield

$$\begin{aligned} \lambda \cdot \|u\|_{W_0^{1,p}(\Omega)}^p &\leq \int_{\Omega} G(z)u \, dx \leq \|G(z)\|_{L^{p'}(\Omega)} \cdot \|u\|_{L^p} \\ &\leq C_1 (\|G(\bar{u})\|_{L^{p'}(\Omega)} + \|G(\underline{u})\|_{L^{p'}(\Omega)}) \|u\|_{W_0^{1,p}(\Omega)} . \end{aligned}$$

Thus  $u = Tz$  verifies

$$\|u\|_{W_0^{1,p}(\Omega)}^{p-1} \leq C_1 (\|G(\bar{u})\|_{L^{p'}(\Omega)} + \|G(\underline{u})\|_{L^{p'}(\Omega)}) = C_2 .$$

This implies that there exists a universal constant  $C > 0$  such that

$$\|u\|_{W_0^{1,p}(\Omega)} \leq C .$$

So, in order to conclude our proof, it is enough to show that  $u \in [\underline{u}, \bar{u}]$ . But, by the definition of an upper solution, there exists  $\bar{v} \in L^{p'}(\Omega)$  such that  $\bar{v} \in \beta(\bar{u}(x))$  and

$$(17) \quad \int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla w \, dx + \int_{\Omega} \bar{v} \cdot w \, dx \geq \int_{\Omega} G(\bar{u})w \, dx ,$$

for all  $w \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$ .

The solution  $u = Tz$  of the variational inequality (16) satisfies

$$(18) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla (w - u) \, dx + J(w) - J(u) \geq \int_{\Omega} G(z)(w - u) \, dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

Setting  $\bar{v} \in \beta(\bar{u}) = \partial j(\bar{u})$ , we have

$$(19) \quad j(s) \geq j(\bar{u}(x)) + \bar{v}(x)(s - \bar{u}(x)) \quad \text{for all } s \in \mathbf{R} .$$

Taking  $s := \bar{u}(x) + (u(x) - \bar{u}(x))^+$  in (19) we find by integration

$$(20) \quad J(\bar{u} + (u - \bar{u})^+) \geq J(\bar{u}) + \int_{\Omega} \bar{v}(u - \bar{u})^+ \, dx .$$

Choosing now  $w = (u - \bar{u})^+$  in (17) we obtain

$$(21) \quad \int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+ dx + \int_{\Omega} \bar{v} \cdot (u - \bar{u})^+ dx \geq \int_{\Omega} G(\bar{u}) \cdot (u - \bar{u})^+ dx.$$

Relations (20) and (21) yield

$$(22) \quad \int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+ dx + J(\bar{u} + (u - \bar{u})^+) - J(\bar{u}) \geq \int_{\Omega} G(\bar{u}) \cdot (u - \bar{u})^+ dx.$$

Taking  $w = u - (u - \bar{u})^+$  in (18), we obtain

$$\int_{\Omega} a(x, \nabla u) \cdot (-\nabla (u - \bar{u})^+) dx + J(u - (u - \bar{u})^+) - J(u) \geq - \int_{\Omega} G(z)(u - \bar{u})^+ dx.$$

Since  $z \in [\underline{u}, \bar{u}]$  and  $G : L^p(\Omega) \rightarrow L^p(\Omega)$  is nondecreasing, it follows that

$$(23) \quad \begin{aligned} & \int_{\Omega} a(x, \nabla u) \cdot \nabla (u - \bar{u})^+ dx + J(u - (u - \bar{u})^+) - J(u) \\ & \geq - \int_{\Omega} G(\bar{u})(u - \bar{u})^+ dx. \end{aligned}$$

From (22), (23) and Lemma 3 we have

$$(24) \quad \int_{\Omega} (a(x, \nabla u) - a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+) dx \leq 0.$$

Let  $\Omega_+ = \{x \in \Omega \mid u \leq \bar{u}\}$  and  $\Omega_- = \{x \in \Omega \mid u > \bar{u}\}$ . Since  $(u - \bar{u})^+ = 0$  in  $\Omega_+$  and  $(u - \bar{u})^+ = u - \bar{u}$  in  $\Omega_-$ , it follows by (24) that

$$\int_{\Omega_-} (a(x, \nabla u) - a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+) dx \leq 0.$$

So, by  $(a_2)$  and the definition of  $\Omega_-$ , we obtain  $\text{meas}(\Omega_-) = 0$ , hence  $u \leq \bar{u}$  a.e. in  $\Omega$ . Proceeding in the same way we prove that  $\underline{u} \leq u$ .  $\square$

**LEMMA 5.** *The operator  $T$  defines a monotone nondecreasing mapping from  $[\underline{u}, \bar{u}]$  to  $[\underline{u}, \bar{u}]$ .*

*Proof.* Let  $z_1, z_2 \in [\underline{u}, \bar{u}]$  be such that  $z_1 \leq z_2$ . By Lemma 4, we obtain that  $Tz_1, Tz_2 \in [\underline{u}, \bar{u}]$  and

$$(25) \quad \int_{\Omega} a(x, \nabla Tz_1) \cdot \nabla(w - Tz_1) dx + J(w) - J(Tz_1) \geq \int_{\Omega} G(z_1)(w - Tz_1) dx$$

$$(26) \quad \int_{\Omega} a(x, \nabla Tz_2) \cdot \nabla(w - Tz_2) dx + J(w) - J(Tz_2) \geq \int_{\Omega} G(z_2)(w - Tz_2) dx.$$

Taking  $w = Tz_1 - (Tz_1 - Tz_2)^+$  in (25) and  $w = Tz_2 + (Tz_1 - Tz_2)^+$  in (26), we get

$$- \int_{\Omega} a(x, \nabla Tz_1) \cdot \nabla(Tz_1 - Tz_2)^+ dx + J(Tz_1 - (Tz_1 - Tz_2)^+) - J(Tz_1) \geq \int_{\Omega} G(z_1)(-(Tz_1 - Tz_2)^+) dx$$

$$\int_{\Omega} a(x, \nabla Tz_2) \cdot \nabla(Tz_1 - Tz_2)^+ dx + J(Tz_2 + (Tz_1 - Tz_2)^+) - J(Tz_2) \geq \int_{\Omega} G(z_2)(Tz_1 - Tz_2)^+ dx.$$

Summing up these inequalities we get, by means of (13),

$$\int_{\Omega} (a(x, \nabla Tz_1) - a(x, \nabla Tz_2)) \cdot \nabla(Tz_1 - Tz_2)^+ dx \leq \int_{\Omega} (G(z_1) - G(z_2))(Tz_1 - Tz_2)^+ dx.$$

But  $G(z_1) \leq G(z_2)$ , since  $G$  is a nondecreasing operator. Therefore, by the

above inequality we obtain

$$\int_{\Omega} a(x, (\nabla T z_1) - a(x, \nabla T z_2)) \cdot \nabla (T z_1 - T z_2)^+ dx \leq 0.$$

With the same argument as for proving (24) we obtain  $T z_1 \leq T z_2$ .  $\square$

*Proof of Theorem 1 completed.* Assume that  $g$  is right continuous. Define

$$(27) \quad u^{n+1} = T u^n,$$

where  $u^0 = \bar{u}$ . Then, by Lemma 4,  $\{u^n\}$  is nondecreasing,  $u^n \in [\underline{u}, \bar{u}]$ , and there is a constant  $C$  such that

$$(28) \quad \|u^n\|_{W_0^{1,p}(\Omega)} \leq C.$$

The compact embedding  $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$  and (28) ensure that there exists  $u \in W_0^{1,p}(\Omega)$  such that, up to a subsequence,

$$\begin{aligned} u^n &\rightarrow u \quad \text{strongly in } L^p(\Omega) \\ u^n &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega) \\ u_n &\rightarrow u \quad \text{a.e. in } \Omega. \end{aligned}$$

By Lemma 4, there exists  $u' \in W_0^{1,p}(\Omega)$ ,  $u' \in [\underline{u}, \bar{u}]$  such that  $u' = T u$ . We prove in what follows that  $u$  is a fixed point of  $T$  i.e.  $u' = u$ .

From (27) and by the definition of  $T$  we obtain

$$(29) \quad \int_{\Omega} a(x, \nabla u^{n+1}) \nabla (w - u^{n+1}) dx + J(w) - J(u^{n+1}) \geq \int_{\Omega} G(u^n) (w - u^{n+1}) dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

Also, from  $T u = u'$ , we have

$$(30) \quad \int_{\Omega} a(x, \nabla u') \nabla (w - u') dx + J(w) - J(u') \geq \int_{\Omega} G(u) \cdot (w - u') dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

Taking  $w = u'$  in (29) and  $w = u^{n+1}$  in (30), we get

$$\int_{\Omega} a(x, \nabla u^{n+1}) \nabla (u' - u^{n+1}) dx + J(u') - J(u^{n+1}) \geq \int_{\Omega} G(u^n) \cdot (u' - u^{n+1}) dx$$

$$\int_{\Omega} a(x, \nabla u') \nabla(u^{n+1} - u') dx + J(u^{n+1}) - J(u') \geq \int_{\Omega} G(u) \cdot (u^{n+1} - u') dx.$$

So, by (29) and (30),  $J(u') < \infty$  and  $J(u^{n+1}) < \infty$ . Summing up the last two inequalities we obtain

$$(31) \quad \begin{aligned} & \int_{\Omega} (a(x, \nabla u') - a(x, \nabla u^{n+1})) \cdot \nabla(u' - u^{n+1}) dx \\ & \leq \int_{\Omega} (G(u) - G(u^{n+1})) (u' - u^{n+1}) dx. \end{aligned}$$

Since  $G$  is right continuous we have  $G(u^n) \rightarrow G(u)$  in  $\Omega$ . We also have

$$|G(u) - G(u^n)| (u - u^{n+1}) \leq 2 (|G(\underline{u})| + |G(\bar{u})|) (|\underline{u}| + |\bar{u}|) \in L^1(\Omega).$$

By  $(a_2)$  and the Lebesgue dominated convergence theorem, we deduce from (31) that

$$(32) \quad \int_{\Omega} (a(x, \nabla u') - a(x, \nabla u^n)) \cdot \nabla(u' - u^n) dx \rightarrow 0.$$

This implies that  $\nabla u^n \rightarrow \nabla u'$  a.e. in  $\Omega$ .

Relation (32) implies that (up to a subsequence)

$$(33) \quad (a(x, \nabla u') - a(x, \nabla u^n)) \cdot \nabla(u' - u^n) \rightarrow 0 \quad \text{a.e. } x \in \Omega.$$

This leads to  $\nabla u^n \rightarrow \nabla u'$  a.e. in  $\Omega$ . Indeed, if not, there exists  $x \in \Omega$  such that (up to a subsequence),  $\nabla u^n(x) \rightarrow \xi \in \overline{\mathbf{R}^N}$  for  $\xi \neq \nabla u'$ . Passing to the limit in (33) we obtain

$$(a(x, \nabla u') - a(x, \xi)) \cdot (\nabla u' - \xi) = 0,$$

which contradicts  $(a_2)$ . So, we have proved that  $\nabla u^n \rightarrow \nabla u$ . Using the fact that  $u^n \rightharpoonup u$  weakly in  $W_0^{1,p}(\Omega)$ , we conclude that  $\nabla u' = \nabla u$ , thus  $u' = u$ . Replacing  $u'$  by  $u$  in (30) we get

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) dx + J(w) - J(u) \geq \int_{\Omega} G(u)(w - u) dx$$

for all  $w \in W_0^{1,p}(\Omega)$ .

Hence  $u$  is a fixed point of  $T$  and a solution for the problem  $(P')$ .

In order to prove that  $u$  is a maximal solution of (3) with respect to the order interval  $[\underline{u}, \bar{u}]$ , take any other solution  $\hat{u} \in [\underline{u}, \bar{u}]$  of the problem  $(P')$ .

Then  $\hat{u}$  is in particular a sub-solution satisfying  $\hat{u} \leq \bar{u}$ . Starting again the iteration (27) with  $u^0 = \bar{u}$  we obtain

$$\hat{u} \leq \dots \leq u^{n+1} \leq u^n \leq \dots \leq u^0 = \bar{u}.$$

It follows that  $\hat{u} \leq u$ , which concludes our proof.  $\square$

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Simona Dăbuleanu – Vicențiu Rădulescu  
*Department of Mathematics*  
*University of Craiova*  
*1100 Craiova, Romania*  
*E-mail: vicrad@yahoo.com*