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# VARIATIONS SUR UN THÈME DE ALDAMA ET SHELAH

CÉDRIC MILLIET

ABSTRACT. We study the definability of certain subgroups of a group  $G$  that does not have the independence property. If a (type) definable subset  $\mathbf{X}$  of an elementary extension  $\mathbf{G}$  of  $G$  has property  $P$ , we call its trace  $\mathbf{X} \cap G$  over  $G$  an *externally (type) definable  $P$  set*. We show the following. Centralisers of subsets of  $G$  are externally definable subgroups. Cores of externally definable subgroups and iterated centres of externally definable subgroups are externally definable subgroups. Normalisers of externally definable subgroups are externally type definable subgroups and externally definable (as sets). A soluble subgroup  $S$  of derived length  $\ell$  is contained in an  $S$ -invariant externally type definable soluble subgroup of  $G$  of derived length  $\ell$ . The subgroup  $S$  is also contained in an externally definable subset  $\mathbf{X} \cap G$  of  $G$  such that  $\mathbf{X}$  generates a soluble subgroup of  $\mathbf{G}$  of derived length  $\ell$ . Analogue results are discussed when  $G$  is merely a type definable group in a structure that does not have the independence property. A soluble subgroup  $S$  of  $G$  of derived length  $\ell$  is contained in an externally type definable soluble subgroup of derived length  $\ell$ .

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Given a group  $G$ , a subset  $X$  of  $G$  is definable *in*  $G$  if there exist a first-order formula  $\varphi(x, \bar{y})$  and parameters  $\bar{a}$  from  $G$  such that  $X$  consists of all  $g$  in  $G$  such that  $\varphi(g, \bar{a})$  holds in  $G$ . A subset  $X$  of  $G$  is *externally definable* if there is an elementary extension  $\mathbf{G}$  of  $G$  and parameters  $\bar{\mathbf{a}}$  in  $\mathbf{G}$  such that  $X$  consists of all  $g$  in  $G$  such that  $\varphi(g, \bar{\mathbf{a}})$  holds in  $\mathbf{G}$ . We write  $\varphi(G, \bar{\mathbf{a}})$  for such a set  $X$  if we want to stress on the defining formula  $\varphi$ , otherwise we write  $\mathbf{X} \cap G$  where  $\mathbf{X}$  stands for  $\varphi(\mathbf{G}, \bar{\mathbf{a}})$ . Definable subsets and externally definable ones coincide for the field  $\mathbf{R}$  of real numbers (L. Van den Dries [vdD86]), for the field  $\mathbf{Q}_p$  of  $p$ -adic numbers (F. Delon [Del89]), for an algebraically closed field and more generally for stable structures (it follows from the definability of types).

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*Key words and phrases.* Model theory; independence property; shattering type; VC-dimension; Abelian, nilpotent, and soluble subgroups; nice group; definable and type definable envelope.

They do not coincide in general: in the ordered Abelian group  $(\mathbf{Q}, +, \leq)$ , the (convex) interval  $]\sqrt{2}, +\infty[$  is not definable in  $\mathbf{Q}$ , but externally definable using the irrational parameter  $\sqrt{2}$ . Externally definable sets play an important role in structures that do not have the independence property, such as  $(\mathbf{Q}, +, \leq)$ . They correspond to finite unions of convex subsets in the particular case of o-minimal and weakly o-minimal structures. Expanding the language of a weakly o-minimal structure by unary predicates interpreting finite unions of convex subsets preserves weak o-minimality (B. Baizhanov [Bai01]). Expanding the language of a structure that does not have the independence property by predicates interpreting externally definable subsets preserves the absence of the independence property (S. Shelah [She09]).

A group  $G$  *does not have the independence property* if for every first order formula  $\varphi(x, \bar{y})$ , the Vapnik-Chervonenkis dimension of the family  $\{\varphi(G, \bar{g}) : \bar{g} \subset G\}$  is finite. We shall also use the short hand NIP group. S. Shelah [She09] and R. de Aldama [dA13] began investigating definable subgroups of  $G$  using *external* parameters lying in a fixed elementary extension  $\mathbf{G}$  of  $G$ . S. Shelah showed that if  $G$  has an infinite Abelian subgroup  $A$ , there exists an externally definable Abelian subgroup of  $G$  that contains infinitely many elements of  $A$ . R. de Aldama went on showing that any nilpotent subgroup  $N$  of  $G$  is contained in a externally definable nilpotent subgroup of  $G$  that has the same nilpotency class as  $N$ , and that any soluble subgroup  $S$  of  $G$  that is normalised by a  $|G|^+$ -saturated elementary extension of  $G$  is contained in an externally definable soluble subgroup of  $G$  that has the same derived length as  $S$ . As we were further investigating the soluble case, we had to cope with subgroups closely related to the infinitesimal numbers, in the following way: in a non-principal ultrapower  $\mathfrak{R} = \mathbf{R}^{\mathbf{N}}/\mathcal{U}$  of the field  $\mathbf{R}$  of real numbers, the subgroup  $\mathfrak{I}$  of infinitesimal numbers is not definable in  $\mathfrak{R}$ . There is an external parameter  $\varepsilon$  in an elementary extension  $\mathfrak{R}^*$  of  $\mathfrak{R}$  such that  $\mathfrak{I} = ]-\varepsilon, \varepsilon[$ , so that  $\mathfrak{I}$  is externally definable as a set.  $\mathfrak{I}$  is not externally definable as a group, however it is the conjunction of the uniform filtering family of definable sets  $]-\frac{1}{n}, \frac{1}{n}[$  that defines a group both in  $\mathfrak{R}$  and  $\mathfrak{R}^*$ .

We call a subgroup  $H$  of  $G$  a *discernible subgroup* if there is a subgroup  $\mathbf{H}$  of  $\mathbf{G}$  that is the intersection of a uniform filtering family of definable subsets of  $\mathbf{G}$  such that  $H = \mathbf{H} \cap G$ . Discernible subgroups are examples of externally definable subsets, and in the particular case when  $G$  is a stable group, they coincide with definable subgroups (see Lemma 3.7). Our main results are the following.

**Theorem 0.1** (finding discernible subgroups). *Let  $G$  be a NIP group,  $\mathbf{G}$  a  $|G|^+$ -saturated elementary extension of  $G$  and  $H = \mathbf{H} \cap G$  an externally definable subgroup of  $G$ .*

- (1) *There is a natural number  $n$  such that for every subset  $A \subset G$ , there are elements  $\mathbf{a}_1, \dots, \mathbf{a}_n$  in  $\mathbf{G}$  such that*

$$C_G(A) = C_G(\mathbf{a}_1, \dots, \mathbf{a}_n).$$

- (2) *For every natural number  $n$ , there is a definable subgroup  $\mathbf{K}$  of  $\mathbf{G}$  such that*

$$H = \mathbf{K} \cap G \quad \text{and} \quad Z_n(H) = Z_n(\mathbf{K}) \cap G.$$

- (3) *There is a natural number  $n$  such that for every subset  $A \subset G$ , there are elements  $\mathbf{a}_1, \dots, \mathbf{a}_n$  in  $\mathbf{G}$  such that*

$$\bigcap_{a \in A} H^a = \mathbf{H}^{\mathbf{a}_1} \cap \dots \cap \mathbf{H}^{\mathbf{a}_n} \cap G.$$

- (4) *For every subset  $A \subset G$  and every discernible subgroup  $H$  of  $G$ , the group  $\bigcap_{a \in A} H^a$  is a discernible subgroup of  $G$ .*
- (5) *The normaliser of a discernible subgroup of  $G$  is a discernible subgroup of  $G$ .*
- (6) *For every natural number  $n$  and subset  $A \subset G$ , the iterated centraliser  $C_G^n(A)$  is a discernible subgroup of  $G$ .*

**Theorem 0.2** (discernible soluble envelope). *Let  $G$  be a NIP group and  $S$  a soluble subgroup of  $G$  of derived length  $\ell$  and  $\mathbf{G}$  a  $|G|^+$ -saturated elementary extension of  $G$ .*

- (1) *There is an  $S$ -invariant discernible soluble subgroup of  $G$  of derived length  $\ell$  that contains  $S$ .*
- (2) *There is a definable subset  $\mathbf{X} \subset \mathbf{G}$  such that  $\mathbf{X} \cap G$  is a subgroup of  $G$  containing  $S$ , and such that  $\mathbf{X}$  generates a soluble subgroup of  $\mathbf{G}$  of derived length  $\ell$ .*
- (3) *If  $S$  is in addition normal in  $G$ , there is a normal soluble subgroup of  $\mathbf{G}$  of derived length  $\ell$  that contains  $S$ .*

NIP groups include finite groups, Abelian groups in the pure language of groups (W. Szmielew [Szm55]), Abelian ordered groups (Y. Gurevich and P. Schmitt [GS84]), groups definable in a stable structure (*e.g.* linear algebraic groups over separably closed fields, C. Wood [Woo79]) and groups definable in an o-minimal structure (*e.g.* linear algebraic groups over the field of real numbers) ; these are trivial ones for most of the considerations of this paper, as both stable and o-minimal groups satisfy strong descending chain conditions, either on uniformly definable subgroups [BS76] or on all definable subgroups [Pil88]. In particular, centralisers of subsets, as well as cores of definable subgroups are definable, and these properties remain true of quotients by normal definable subgroups.

Other examples include linear algebraic groups over a field  $k$  that does not have the independence property, and more generally groups *interpretable* therein, *e.g.* quotients  $H_1/H_2$  where  $H_2 \triangleleft H_1$  are definable subgroups (not necessarily Zariski-closed) of the general linear group  $\mathrm{GL}_n(k)$  in a field structure  $(k, L)$  where  $L$  is an expansion of the field language such that the structure  $(k, L)$  is NIP. This holds in particular with  $k$  equal to (a finite algebraic extension of) the pure field  $\mathbf{Q}_p$  of  $p$ -adic numbers (L. Matthews [Mat93], see also [Bél12]) and more generally to a Henselian valued field of characteristic 0 whose residue field is NIP (F. Delon [Del81]), or with  $k$  equal to the valued field  $\bigcup_{n \geq 1} \mathbf{F}_p^{\mathrm{alg}}((t^{1/n}))$  of Puiseux series over  $\mathbf{F}_p^{\mathrm{alg}}$  and more generally to a valued field of characteristic  $p > 0$  with perfect NIP residue field,  $p$ -divisible value group and no proper algebraic valuated extension having ramification index 1 and residue degree 1 (I. Kaplan, T. Scanlon and F. Wagner [KSW11]). Note that in an algebraic group  $G(k)$  over a field, every descending chain of Zariski-closed subgroups has finite length. In particular, centralisers are Zariski-closed (hence definable), cores of Zariski-closed subgroups are Zariski-closed, but cores of definable subgroups may not be definable.

NIP groups also include general linear groups over an infinite NIP ring  $R$ , which may be a domain (such as the valuation rings of the valued fields cited above) or not (such as any non-principal ultraproduct  $(\prod_{n \in \mathbf{N}} \mathbf{Z}/p^n \mathbf{Z})/\mathcal{U}$ ).

Two more examples of a less algebraic nature. The universal covering group  $\tilde{G}$  of a definably connected group  $G$  that is interpretable in an o-minimal expansion  $M$  of the field  $\mathbf{R}$  of real numbers is a NIP group:  $\tilde{G}$  is interpretable in the two sorted structure  $((\pi_1(G), +), M)$  (E. Hrushovski et al. [HPP11]) hence NIP (A. Conversano and A. Pillay [CP12]). An ultraproduct of groups that are uniformly interpretable in a NIP structure (D. Macpherson and K. Tent [MT12]) is NIP.

## 1. PRELIMINARIES ON THE INDEPENDENCE PROPERTY

Before discussing the particular case of groups, we consider an arbitrary first-order language  $L$ , a complete theory  $T$ , one of its models  $M$  and a subset  $A \subset M$ . Let  $\bar{x}$  and  $\bar{y}$  be disjoint tuples of variables of respective length  $p \geq 1$  and  $q \geq 1$ . Given a formula  $\phi(\bar{x})$  and a partial type  $\pi(\bar{x})$ , we write  $\phi(A)$  for the subset  $\{(x_1, \dots, x_p) \in A^p : \phi(x_1, \dots, x_p) \text{ holds in } M\}$  of  $M^p$ ,  $\pi(A)$  for the family  $\{\phi(A) : \phi \in \pi\}$  and  $\bigcap \pi(A)$  the intersection of its members.

**1.1. Shattering formulas.** Let  $\varphi(\bar{x}, \bar{y})$  be a formula in  $p + q$  variables with possible parameters in  $M$ . Given a natural number  $n \geq 1$ , we say that the formula  $\varphi(\bar{x}, \bar{y})$  *shatters*  $n$  in  $T$  if there are parameters  $\bar{a}_1, \dots, \bar{a}_n$  in  $M^p$  and  $(\bar{b}_J)_{J \subset \{1, \dots, n\}}$  in  $M^q$  such that

$$\varphi(\bar{a}_i, \bar{b}_J) \text{ holds in } M \text{ if and only if } i \in J.$$

In other words,  $\varphi(\bar{x}, \bar{y})$  shatters  $n$  in  $T$  if there is a finite subset  $A$  of  $M^p$  with  $n$  elements whose subsets are all of the form  $A \cap \varphi(M, \bar{b})$  for some  $\bar{b}$  varying in  $M^q$ . As shattering  $n$  is a first order property, it does not depend on the model  $M$  of  $T$  chosen. We call *Vapnik-Chervonenkis dimension of  $\varphi(\bar{x}, \bar{y})$  in  $T$* , sometimes omitting to precise  $T$  when the ambient theory is obvious, the maximal natural number  $n$  that is shattered by  $\varphi(\bar{x}, \bar{y})$  in  $T$  if such a number exists, or  $\infty$  otherwise. We write it  $VC(\varphi)$ . Note that  $VC(\varphi)$  equals  $VC(\neg\varphi)$ . In these definitions, the tuples of variables  $\bar{x}$  and  $\bar{y}$  do not play the same role. We write  $\varphi^*(\bar{x}, \bar{y})$  for the *dual formula of  $\varphi(\bar{x}, \bar{y})$* , obtained by interchanging the role of  $\bar{x}$  and  $\bar{y}$ . We say that  $\varphi(\bar{x}, \bar{y})$  *has the independence property in  $T$* , if it<sup>1</sup> has infinite VC-dimension in  $T$ . The structure  $M$  or its theory  $T$  *do not have the independence property (i.e. are NIP)* if no formula has the independence property in  $T$ , i.e. if every formula has a finite VC-dimension. The relation between Shelah's independence property in [She90] and Vapnik-Chervonenkis' dimension in [VC71] is pointed out in [Las92]. We refer to [She90] and [Adl14] the reader willing to know more about structures that do not have the independence property.

**1.2. Shattering types.** We extend the previous definitions to partial types. Let  $\lambda$  and  $\mu$  be two cardinal numbers, with  $\mu \leq \lambda^+$ . If  $\pi(\bar{x}, \bar{y})$  is a partial type in  $p + q$  variables, we say that  $\pi(\bar{x}, \bar{y})$  *shatters  $\lambda$  up to  $\mu$  in  $T$*  if there is a subset  $A$  of  $M^p$  of cardinal  $\lambda$  for an elementary

<sup>1</sup>According to Shelah's definition in [She90, Definition 4.2],  $\varphi(\bar{x}, \bar{y})$  has the independence property if the dual formula  $\varphi^*(\bar{x}, \bar{y})$  has infinite VC-dimension. The two statements are equivalent as  $VC(\varphi) \leq n$  implies  $VC(\varphi^*) \leq 2^n$  by [Poi85, Lemme 12.16].

extension  $\mathbf{M}$  of  $M$  such that every subset  $B$  of  $A$  with  $|B| < \mu$  equals  $A \cap \bigcap \pi(\mathbf{M}, \bar{b})$  for some  $\bar{b}$  in  $\mathbf{M}^q$ . Equivalently, there is an elementary extension  $\mathbf{M}$  of  $M$ , a family  $\{\bar{a}_i : i < \lambda\}$  of elements of  $\mathbf{M}^p$ , and a family  $\{\bar{b}_J : J \subset \lambda, \text{ and } |J| < \mu\}$  of elements of  $\mathbf{M}^q$  such that

$$\pi(\bar{a}_i, \bar{b}_J) \text{ holds in } \mathbf{M} \text{ if and only if } i \in J$$

We say that  $\pi(\bar{x}, \bar{y})$  *co-shatters*  $\lambda$  in  $T$  up to  $\mu$  if there is an elementary extension  $\mathbf{M}$  of  $M$  and families  $\{\bar{a}_i : i < \lambda\}$  and  $\{\bar{b}_J : J \subset \lambda, \text{ and } |J| < \mu\}$  whose elements lie respectively in  $\mathbf{M}^p$  and  $\mathbf{M}^q$ , such that

$$\pi(\bar{a}_i, \bar{b}_J) \text{ holds in } \mathbf{M} \text{ if and only if } i \notin J,$$

or equivalently if there is a subset  $A$  of  $\mathbf{M}^p$  of cardinal  $\lambda$  whose subsets  $B$  with  $|A \setminus B| < \mu$  are all of the form  $A \cap \bigcap \pi(\mathbf{M}, \bar{b})$  for some  $\bar{b}$  in  $\mathbf{M}^q$ .

If  $\pi(\bar{x}, \bar{y})$  shatters  $\lambda$  up to  $\lambda^+$ , then it shatters and co-shatters  $\lambda$  up to every  $\mu \leq \lambda^+$ . In this case, we simply say that  $\pi(\bar{x}, \bar{y})$  *shatters*  $\lambda$ .

If a formula  $\varphi(\bar{x}, \bar{y})$  shatters every natural number  $n$  in  $T$ , by the Compactness Theorem,  $\varphi(\bar{x}, \bar{y})$  shatters  $\lambda$  for every cardinal number  $\lambda$ . If a partial type  $\pi(\bar{x}, \bar{y})$  shatters every natural number in  $T$ , by the Compactness Theorem, for every natural number  $n$ , there is a finite conjunction of formulas in  $\pi(\bar{x}, \bar{y})$  that has VC-dimension at least  $n$ . However, it is possible that  $T$  be NIP.

*Example 1.1* (The Cantor ternary set in  $\mathbf{R}$ ). The Cantor ternary set  $C$  is the intersection of the closed sets  $I_n$  defined inductively by

$$I_1 = [0, 1] \quad \text{and} \quad I_{n+1} = \frac{I_n}{3} \cup \left( \frac{2}{3} + \frac{I_n}{3} \right).$$

It consists of the elements of  $[0, 1]$  having at least one ternary representation whose digits belong to  $\{0, 2\}$ . The partial type  $\{x + y \in I_n : n \geq 1\}$  shatters  $\aleph_0$ . For every natural number  $i$  and subset  $J$  of  $\aleph_0$ , we define

$$a_i = \left( \frac{1}{3} + \frac{1}{3^2} \right) \times \frac{1}{3^{2i}} \quad \text{and} \quad b_J = \sum_{j \in J} \frac{2}{3^{2j+2}},$$

so that we have

$$a_i + b_J = \frac{1}{3^{2i+1}} + \frac{1}{3^{2i+2}} + \sum_{j \in J} \frac{2}{3^{2j+2}}.$$

On the one hand, if  $i$  does not belong to  $J$ , then  $a_i + b_J$  has occurrences of 1 in every ternary representation. On the other hand, if  $i$  belongs to  $J$ , then

$$a_i + b_J = \frac{2}{3^{2i+1}} + \sum_{j \in J \setminus \{i\}} \frac{2}{3^{2j+2}}.$$

*Example 1.2* (The Cantor ternary set in  $\mathbf{Q}_3$ ). The 3-adic ternary Cantor set  $C_3$  is the intersection of the closed subsets  $K_n \subset \mathbf{Q}_3$  defined inductively by

$$K_1 = \mathbf{Z}_3 \quad \text{and} \quad K_{n+1} = 3K_n \cup (2 + 3K_n).$$

It consists of the 3-adic integers whose canonical expansion has coefficients in  $\{0, 2\}$  (M. Lapidus and H. Lũ [LL08]). The ring of 3-adic integers is defined by the formula  $(\exists y)(y^2 = 1 + 3x^2)$  and  $C_3$  is a type definable subset of  $\mathbf{Q}_3$ . The type  $x - y \in C_3$  shatters  $\aleph_0$  in  $\mathbf{Q}_3$ , taking

$a_i = 3^{2^i} + 3^{2^{i+1}}$  and  $b_J = \sum_{j \in J} 2 \cdot 3^{2^j}$ . Note that the field  $\mathbf{Q}_3$  does not have the independence property by [Mat93] (see also [Bél12, Section 4.2]).

Given a cardinal  $\lambda$ , here is an example of a language  $L_\lambda$ , an  $L_\lambda$ -structure  $M$  and partial type  $\pi_\lambda(x, y)$  that shatters  $\lambda$  in  $M$ . Let  $\Gamma$  be an ordered Abelian divisible group containing a copy of the semi-group  $\lambda$ . Consider the Hahn field  $\mathbf{Q}_3((\Gamma))$  of generalised power series with 3-adic coefficients. Consider the structure  $(\mathbf{Q}_3((\Gamma)), +, 0, (P_n^\mu)_{\mu < \lambda, n \geq 0})$  where  $P_n^\mu$  is a unary predicate interpreting the subgroup of  $\mathbf{Q}_3((\Gamma))$  whose elements are of the form  $\sum_{i \in I} a_i t^i$  with  $a_\mu \in 3^n \mathbf{Z}_3$ . In this language,  $\mathbf{Q}_3((\Gamma))$  is an *Abelian structure* in the sense of E. Fisher (see [Fis77], or [Wag97, Example 0.3.1]). Any definable subset of the Cartesian product  $\mathbf{Q}_3((\Gamma))^n$  is a Boolean combination of cosets of  $acl(\emptyset)$ -definable subgroups of  $\mathbf{Q}_3((\Gamma))^n$  by [Wei93] (see also [Wag97, Theorem 4.2.8]): the structure  $(\mathbf{Q}_3((\Gamma)), +, 0, (P_n^\mu)_{\mu < \lambda, n \geq 0})$  is stable.

Let  $C_3^\lambda$  denote the partial type defined by

$$C_3^\lambda = \bigcap_{\mu < \lambda} \bigcap_{n \geq 1} K_n^\mu, \quad \text{where} \quad K_1^\mu = P_0^\mu \quad \text{and} \quad K_{n+1}^\mu = 3K_n^\mu \cup (2 + 3K_n^\mu).$$

The realisations of  $C_3^\lambda$  in  $\mathbf{Q}_3((\Gamma))$  are the elements of the form  $\sum_{i \in I} a_i t^i$  where  $a_\mu \in C_3$  for each cardinal number  $\mu < \lambda$  belonging to  $I$ . The families  $a_i = 4t^i$  and  $b_J = \sum_{j \in J} 2t^j$  witness that the type  $x + y \in C^\lambda$  shatters  $\lambda$  in  $\mathbf{Q}_3((\Gamma))$ .

*Example 1.3* (A type that shatters every  $n$ , but not  $\aleph_0$  up to 4). In  $(\mathbf{R}, +, \leq)$ , here is a sequence of definable subsets  $A_n \subset [0, 1]$  such that the partial type  $\{x - y \in A_n : n \geq 1\}$  shatters every natural number  $n$  but does not shatter  $\aleph_0$  up to 4. Define for all  $n \geq 1$ , a definable subset  $B_n$  of  $[n, n + 1]$  of the form

$$B_n = ]n, n + 1[ \setminus \{c_{n,1}, \dots, c_{n,2^n}\} \quad \text{with} \quad n < c_{n,1} < \dots < c_{n,2^n} < n + 1$$

such that, there are  $a_{n,i}$  and  $b_{J,n}$  in  $\mathbf{R}$  such that  $a_i - b_J \in ]n, n + 1[$  and  $a_i - b_J \in B_n$  if and only if  $i \in J$ . We put  $C = \bigcup \{c_{i,j} : i, j \geq 1\}$ , and we may build each  $B_n$  so that the map mapping a 2 element subset  $\{x, y\}$  of  $C$  to  $|x - y|$  has finite (unbounded) fibres (using a  $\mathbf{Q}$ -basis of  $\mathbf{R}$  for instance). We put for all  $n \geq 1$ ,

$$A_n = ]-\infty, 1] \cup B_1 \cup B_2 \cup \dots \cup B_n \cup [n + 1, +\infty[$$

It follows that, for every  $n$ ,  $\pi(a_{n,i}, b_{J,n})$  holds if and only if  $i \in J$ , so  $\pi(x, y)$  shatters every natural number  $n$ . If  $A$  is an infinite set shattered by  $\pi(x, y)$ , then there is a real number  $\ell$  and infinitely many 3 element subsets  $B$  of  $A$  with the property that  $|x - y| = \ell$  for some  $x$  and  $y$  in  $B$ . This shows that  $\pi(x, y)$  does not shatter  $\aleph_0$  up to 4.

**1.3. Nice sets.** The previous examples show that shattering types can occur in a theory that does not have the independence property. We go on by giving two elementary conditions for a shattering type in a theory  $T$  to yield a formula having the independence property in  $T$ .

Given a partial type  $\rho(\bar{x})$  (where  $\bar{x}$  is a  $p$ -tuple of variables), we call  $\bigcap \rho(M)$  a  $\rho$ -definable set in  $M$ , and we say that a subset  $X$  of  $M^p$  is *type definable in  $M$*  if there is a partial  $p$ -type  $\rho(\bar{x})$  such that  $X$  is  $\rho$ -definable.

We say that  $\rho$  is *uniform* if there is a  $q$ -tuple of variables  $\bar{y}$ , a formula  $\varphi(\bar{x}, \bar{y})$ , an elementary extension  $\mathbf{M}$  of  $M$ , and a parameter subset  $A$  of  $\mathbf{M}^q$  such that

$$\rho(\bar{x}) = \left\{ \varphi(\bar{x}, \bar{a}) : \bar{a} \in A \right\}.$$

Let  $\mathfrak{F}$  be a family of subsets of  $M^p$ . We say that  $\mathfrak{F}$  is *uniformly definable* if there is a uniform  $p$ -type  $\rho(\bar{x})$ , such that  $\mathfrak{F} = \rho(M)$ .

If  $X$  is any set, we say that a family  $\mathfrak{F}$  of subsets of  $X$  is a *filter* if for every  $F_1$  and  $F_2$  in  $\mathfrak{F}$ , there is an  $F$  in  $\mathfrak{F}$  such that  $F \subset F_1 \cap F_2$ . We say that  $\rho$  is a *filter in  $M$*  if the family  $\rho(M)$  is a filter. Note that if  $\rho$  is a filter in  $M$ , then it is a filter in every structure  $N$  elementary equivalent to  $M$ .

**Definition 1.4** (nice set). Let  $M$  be any structure, and  $\rho(\bar{x})$  a partial  $p$ -type. We say that  $\rho(\bar{x})$  is *nice in  $M$*  if  $\rho(\bar{x})$  is a uniform filter in  $M$ . If  $X$  is a type definable subset of  $M^p$ , we say that  $X$  is *nice in  $M$*  if there is a partial nice type  $\rho(\bar{x})$  in  $M$ , such that  $X$  is  $\rho$ -definable.

**Lemma 1.5.** *If there is a nice partial type that shatters every natural number  $n$  in  $T$ , then  $M$  has the independence property.*

*Proof.* Let  $\pi(\bar{x}, \bar{y})$  be this type. There is an  $r$ -tuple of variables  $z$ , a formula  $\varphi(\bar{x}, \bar{y}, \bar{z})$ , an elementary extension  $\mathbf{M}$  of  $M$  and a set of tuples of parameters  $A \subset \mathbf{M}^r$  such that

$$\pi(\bar{x}, \bar{y}) = \left\{ \varphi(\bar{x}, \bar{y}, \bar{a}) : \bar{a} \in A \right\}.$$

Let  $n \geq 1$  be a natural number. There are parameters  $\bar{a}_1, \dots, \bar{a}_n$  in  $\mathbf{M}^p$  and  $(\bar{b}_J)_{J \subset \{1, \dots, n\}}$  in  $\mathbf{M}^q$  such that for every  $J \subset \{1, \dots, n\}$  and every  $i \in \{1, \dots, n\}$ ,

$$\pi(\bar{a}_i, \bar{b}_J) \text{ holds in } \mathbf{M} \text{ if and only if } i \in J.$$

It follows that the partial type  $\bigvee_{i \notin J} \pi(\bar{a}_i, \bar{b}_J)$  is inconsistent. By compactness, there is a finite conjunction  $\phi_n(\bar{x}, \bar{y})$  of formulas in  $\pi(\bar{x}, \bar{y})$  such that  $\phi_n(\bar{a}_i, \bar{b}_J)$  does not hold whenever  $i \notin J$ . As  $\pi(\bar{x}, \bar{y})$  is a filter, there is an element  $\bar{a}_n$  in  $A$  such that  $\varphi(\bar{x}, \bar{y}, \bar{a}_n)$  implies  $\phi_n(\bar{x}, \bar{y})$ . Because  $\varphi(\bar{x}, \bar{y}, \bar{a}_n)$  belongs to  $\pi(\bar{x}, \bar{y})$ , it follows that for every  $J \subset \{1, \dots, n\}$  and  $i \in \{1, \dots, n\}$

$$\varphi(\bar{a}_i, \bar{b}_J, \bar{a}_n) \text{ holds in } \mathbf{M} \text{ if and only if } i \in J.$$

Calling  $\bar{z}$  any  $q + r$  tuple of variables, the formula  $\varphi(\bar{x}, \bar{z})$  shatters  $n$  in  $M$ . This holds for every natural number  $n \geq 1$ .  $\square$

*Remark 1.6.* In Lemma 1.5, one cannot drop the assumption that the partial type is uniformly definable: Example 1.3 furnishes a type shattering every natural number  $n$  in  $(\mathbf{R}, +, \leq)$ . Nor can one drop the assumption that  $\pi(\bar{x}, \bar{y})$  is a filter: the type constructed in Example 1.3 is of the form  $\{x - y \in A_n : n \geq 1\}$  with every  $A_n$  being the complement of finitely many points, so the same type can be written in the form  $\{x - y \in B_n : n \geq 1\}$  where  $B_n$  is the complement of one point only, so that the formulas  $x - y \in B_n$  are uniformly definable, but do not form a filter. More generally, in an o-minimal structure, as every definable set is the conjunction of uniformly definable sets, every type is equivalent to a uniform type. Every type is also equivalent to a filter, but need not be equivalent to a uniform filter.

**Corollary 1.7.** *Let  $M$  be a NIP structure,  $N$  a substructure whose domain is a nice subset and  $E$  a type definable equivalence relation on  $N$  defined by a nice partial type that preserves the language. Then no existential formula  $(\exists \bar{z})\varphi(\bar{x}, \bar{y}, \bar{z})$  has the independence property in the structure  $N/E$ .*



*Proof.* If  $\varphi(\bar{x}, \bar{y}, \bar{z})$  is quantifier free, putting

$$E(x, y) = \bigwedge_{i \in I} E_i(x, y) \quad \text{and} \quad N = \bigwedge_{i \in I} N_i,$$

then  $\exists \bar{z} \varphi(\bar{a}_E, \bar{b}_E, \bar{z})$  holds in  $N/E$  if and only if the nice partial type

$$\left\{ (\exists \bar{x}_1)(\exists \bar{x}_2)(\exists \bar{z}) \varphi(\bar{x}_1, \bar{x}_2, \bar{z}) \wedge E_i(\bar{a}, \bar{x}_1) \wedge E_i(\bar{b}, \bar{x}_2) \wedge (\bar{a}, \bar{b}, x_1, \bar{x}_2, \bar{z}) \in N_i : i \in I \right\}$$

holds in  $M$ . □

With no assumption on the type  $\pi(\bar{x}, \bar{y})$ , but under large cardinals assumptions, we get the following Lemma. A cardinal  $\lambda$  is  $\omega$ -Erdős if for every partition of the set  $[\lambda]^{< \aleph_0}$  of finite subsets of  $\lambda$  into two equivalence classes, there is a subset of  $\lambda$  of order type  $\omega$  whose finite subsets lie in the same class.

**Lemma 1.8.** *If there is a countable partial type  $\pi(\bar{x}, \bar{y})$  and an  $\omega$ -Erdős cardinal  $\lambda$  such that  $\pi(\bar{x}, \bar{y})$  co-shatters  $\lambda$  up to  $\aleph_0$  in  $M$ , then  $M$  has the independence property.*

*Proof.* Assume that  $\pi(\bar{x}, \bar{y})$  is the conjunction of the formulas  $\{\varphi_n(\bar{x}, \bar{y}) : n \geq 1\}$ . Without loss of generality, we may inductively replace  $\varphi_{n+1}$  by  $\varphi_n \wedge \varphi_{n+1}$  and assume that  $\varphi_{n+1}(\bar{x}, \bar{y})$  implies  $\varphi_n(\bar{x}, \bar{y})$  for every  $n$ . For every finite  $J \subset \lambda$  and every  $i$  in  $J$ , the partial type  $\pi(\bar{a}_i, \bar{b}_J)$  is inconsistent. By compactness, there is a least  $n(J)$  depending on  $J$  such that, for every  $i$  in  $J$ , the formula  $\varphi_n(\bar{a}_i, \bar{b}_J)$  does not hold. We consider the equivalence relation defined by  $n(J) = n(I)$  for finite subsets  $I$  and  $J$  of  $\lambda$ . By [Jec03, Lemma 17.30], for every partition of the set  $[\lambda]^{< \aleph_0}$  in countably many classes  $\{C_i : i \geq 1\}$ , there is a class  $C_i$  and a subset of  $\lambda$  of order type  $\omega$  whose finite subsets lie in the same class  $C_i$ . So, there is a countable subset  $I$  of  $\lambda$  and a natural number  $m$  such that  $n(J) = m$  for every finite subset  $J$  of  $I$ . This shows that for all finite  $J \subset I$ , and all  $i$  in  $J$ , the formula  $\varphi_m(\bar{a}_i, \bar{b}_J)$  does not hold, so  $\varphi_m(\bar{x}, \bar{y})$  has the independence property. □

**Question.** *In particular, assuming the existence of an  $\omega$ -Erdős cardinal, if a countable partial type shatters every cardinal in  $M$ , then  $M$  has the independence property. Does the statement hold in ZFC ?*

## 2. PRELIMINARIES ON NIP GROUPS

Let us consider a first-order structure  $M$  that does not have the independence property and  $G$  a group definable in  $M$ .

**2.1. Descending chain conditions.** We begin by the Baldwin-Saxl descending chain condition for uniformly definable subgroups.

**Baldwin-Saxl chain condition 2.1** (see [BS76] or [Poi87]). *Let  $\{H_i : i \in I\}$  be a family of uniformly definable subgroups of  $G$ . There is a natural number  $n \geq 1$  such that for all finite subsets  $J$  of  $I$ , there exists a finite subset  $J_n$  of  $J$  of size  $n$  such that*

$$\bigcap_{j \in J_n} H_j = \bigcap_{j \in J} H_j.$$

We shall need the following stronger version. For any subset  $X$  of the group  $G$  and natural number  $n$ , we write  $X^n$  for the set of products  $x_1 \dots x_n$  of any  $n$  elements of  $X$ , and  $X^{\times n}$  for the Cartesian product  $X \times \dots \times X$ . For a set  $Y$ , when there is no ambiguity, we go on writing  $Y^n$  for the Cartesian product  $Y \times \dots \times Y$ .

**Lemma 2.2** (a Baldwin Saxl chain condition for subsets). *Let  $\mathfrak{X}$  be a family of subsets of  $G$ . For all  $X$  in  $\mathfrak{X}$ , let  $X^{1/3}$  stand for a definable subset of  $G$  such that  $(X^{1/3})^3 \subset X$ . Assume that the family  $\{X^{1/3} : X \in \mathfrak{X}\}$  is uniformly definable by a formula  $\varphi(x, y)$ , and that the formula  $\varphi^*(x, y)$  has VC-dimension  $n$ . For all  $X$  in  $\mathfrak{X}$ , assume that there is a subset  $X^{1/3n}$  of  $G$  such that  $(X^{1/3n})^n \subset X^{1/3}$ . Assume that  $(X^{1/3n})^{-1}$  equals  $X^{1/3n}$  and contains 1. Then, for every  $X_1, \dots, X_{n+1}$  in  $\mathfrak{X}$ , there exists  $j$  in  $\{1, \dots, n+1\}$  such that*

$$X_1^{1/3n} \cap \dots \cap X_{j-1}^{1/3n} \cap X_{j+1}^{1/3n} \cap \dots \cap X_{n+1}^{1/3n} \subset X_1 \cap \dots \cap X_{n+1}.$$

*Proof.* Otherwise, there are elements  $b_1, \dots, b_{n+1}$  in  $G$  such that for all  $j \in \{1, \dots, n+1\}$ ,

$$(1) \quad b_j \in \left( X_1^{1/3n} \cap \dots \cap X_{n+1}^{1/3n} \right) \setminus X_j.$$

Let  $J$  be any finite subset of  $\{1, \dots, n+1\}$  with elements  $j_1 < \dots < j_k$ . We write  $b_J$  for the ordered product  $b_{j_1} b_{j_2} \dots b_{j_k}$ . If  $j$  belongs to  $\{1, \dots, n+1\} \setminus J$ , then  $J$  has at most  $n$  elements, and  $b_J$  belongs to  $(X_j^{1/3n})^n \subset X_j^{1/3}$ . On the other hand, if  $J$  has elements  $j_1 < \dots < j_{i-1} < j < j_{i+1} < \dots < j_k$ , then  $b_J$  does not belong to  $X_j^{1/3}$ , for otherwise, we would have

$$b_j = (b_{j_{i-1}}^{-1} \dots b_{j_1}^{-1}) b_J (b_{j_{i+1}}^{-1} \dots b_{j_m}^{-1})$$

and thus  $b_j \in X_j$ , a contradiction with (1). This shows that the formula  $\varphi^*(x, y)$  has VC-dimension at least  $n+1$ , contradicting the hypothesis.  $\square$

*Remark 2.3.* In the particular case where the sets  $X_i$  are subgroups of  $G$ , Lemma 2.2 is the usual Baldwin-Saxl Descending chain condition.

**2.2. Nice groups.** Let  $H$  be a subgroup of  $G$  and  $\pi$  a partial 1-type with parameters in  $A \subset G$ .  $H$  is called a  $\pi$ -definable subgroup of  $G$  if  $H$  is a  $\pi$ -definable set in  $G$ , and for any structure  $\mathbf{G}$  elementary equivalent to  $G$  over  $A$ , the set  $\bigcap \pi(\mathbf{G})$  is a subgroup of  $\mathbf{G}$ .  $H$  is a  $\pi$ -definable subgroup of  $G$  if there is a partial type  $\pi$  such that  $H$  is a  $\pi$ -definable subgroup of  $G$ .

**Definition 2.4** (nice subgroup). Let  $H \leq G$ . We say that  $H$  is a *nice subgroup* of  $G$  if there is a nice partial type  $\pi$  such that  $H$  is a  $\pi$ -definable subgroup of  $G$ .

*Example 2.5.* A definable subgroup  $H$  of  $G$  is nice. By the Baldwin Saxl chain condition, any intersection of uniformly definable subgroups of  $G$  is nice. In particular, for any subset  $A$  of  $G$  and subgroup  $\mathfrak{A} \leq \text{Aut}(G)$ , the subgroups  $\bigcap_{a \in A} H^a$ ,  $C_G(A)$  and  $\bigcap_{\sigma \in \mathfrak{A}} \sigma(H)$  are nice.

*Counter example 2.6* (A centraliser that is not nice). In an infinite extraspecial 3-group  $K$ , which is supersimple of rank 1 (see [MS08]) and whose conjugacy classes are all finite, choose  $(a_n)_{n \geq 1}$  such that the chain of centralisers  $C_K(a_1) > C_K(a_1, a_2) > C_K(a_1, a_2, a_3) > \dots$  is strictly decreasing. The partial type  $\bigcap_{n \geq 1} C_K(a_n)$  is not nice as  $[K : C_K(a_n)] \leq 3$  for every  $n$ .

Nor is it equivalent to a nice partial type, for otherwise, by the Compactness theorem, one could find a definable infinite subset  $X \subset K$  with infinitely many pairwise disjoint left translates, contradicting the fact that  $K$  has rank 1.

*Example 2.7.* In an  $\aleph_0$ -saturated elementary extension  $\mathfrak{R}$  of the field  $\mathbf{R}$ , the subgroup of infinitesimal numbers is nicely defined in the language  $(+, \leq)$ . In the language of fields, the intersection of the Euclidian balls  $\{x \in \mathfrak{R}^n : \|x\|_2 < 1/k\}$  is a nice subgroup of  $\mathfrak{R}^n$ . It is also the intersection of the family  $\mathfrak{H}$  of half hyperplanes of equations  $a_1x_1 + \dots + a_nx_n \leq a_{n+1}$  where  $a_1, \dots, a_{n+1}$  range over  $\mathbf{Q}$  with  $a_{n+1} > 0$ . In  $\text{GL}_n(\mathfrak{R})$  (considered as a group interpretable in the ring  $M_n(\mathfrak{R})$ ) the subgroup of elements that are infinitesimally close to 1 is nice, being the intersection of the neighbourhoods  $\{1 + x : \|x\|_2 < 1/k\}$ .

*Example 2.8.* Let  $\mathfrak{Z}_p$  be an  $\aleph_0$ -saturated elementary extension of the ring  $\mathbf{Z}_p$  of  $p$ -adics integers. The infinitesimal numbers form a nice subgroup of  $\mathfrak{Z}_p$ , defined by the intersection of the subgroups  $p^k\mathfrak{Z}_p$ . In  $\text{GL}_n(\mathfrak{Z}_p)$  (viewed as a group interpretable in the ring  $M_n(\mathfrak{Z}_p)$ ), the intersection of the congruence subgroups  $1 + p^kM_n(\mathfrak{Z}_p)$  is a nice subgroup.

*Counter example 2.9.* One (semi)group that is not a filter but uniformly defined, in a theory that does not have the independence property ?

For any two subgroups  $H$  and  $K$  of  $G$ , let us write

$$H^K = \bigcap_{g \in K} H^g$$

for the  $K$ -core of  $H$ . When  $G$  is stable, if  $H$  is definable, then  $H^K$  is definable, and hence  $N_G(H^K)$ . The situation is far less straightforward when  $G$  does not have the independence property;  $H^K$  is merely a type definable subgroup over  $|K|$  parameters, and its normaliser has no obvious reason to be even type definable.

**Theorem 2.10** (normalising a nice envelope). *Let us assume that  $G$  is  $\kappa$ -saturated for some uncountable cardinal  $\kappa$  and let  $H$  be a nice subgroup of  $G$ . Let  $A$  be a subgroup of  $H$  and  $N_A$  a subgroup of  $N_G(A)$  such that  $|A| < \kappa$  and  $|N_A| < \kappa$  hold. Then there is a nice subgroups  $K$  of  $H$  and a nice subgroup  $N_K$  of  $N_G(K)$  such that  $K$  contains  $A$  and  $N_K$  contains  $N_A$  (the types defining  $K$  and  $N$  are countable).*

*Proof.* Assume that  $H$  is the intersection of a family  $\mathfrak{F}$  of uniformly definable sets  $\{\varphi(G, b) : b \in B\}$ , such that  $\{\varphi(G, b) : b \in B\}$  is a filter. We may replace  $\varphi(x, y)$  by the formula  $\varphi(x, y) \wedge \varphi(x^{-1}, y)$  and assume that  $\varphi(G, b)$  is a symmetric subset of  $G$  for every  $b$ . As  $H^3 \subset H$ , by the Compactness theorem, for every element  $X$  of  $\mathfrak{F}$ , there are finitely many  $X_1, \dots, X_m$  in  $\mathfrak{F}$  such that  $(X_1 \cap \dots \cap X_m)^3 \subset X$ . As  $\mathfrak{F}$  is a filter, there is an element of  $\mathfrak{F}$ , which we write  $X^{1/3}$ , such that  $X^{1/3} \subset X_1 \cap \dots \cap X_m$ . Similarly, for every natural number  $n \geq 1$ , there is an element  $X^{1/3^n}$  of  $\mathfrak{F}$  such that  $(X^{1/3^n})^n \subset X^{1/3} \subset X$ . Iterating Lemma 2.2, for every  $b$  in  $B$ , there is a natural number  $n \geq 1$  such that for all finite subset  $J$  of  $G$ , there exists a finite subset  $J_n$  of  $J$  of size  $n$  and a parameter  $c_J$  in  $B$  such that

$$(2) \quad \bigcap_{g \in J_n} \varphi(G, c_J)^g \subset \bigcap_{g \in J} \varphi(G, b)^g \quad \text{and} \quad \varphi(G, c_J) \subset \varphi(G, b)^{1/3}.$$

**Claim 1.** *For any  $b$  in  $B$ , there are elements  $a_1, \dots, a_n$  in  $G$ , a countable subset  $B \subset G$  and  $c$  in  $B$  such that  $\bigcap \{\varphi(G, b) : b \in B\}$  is a nice subgroup of  $G$  containing  $A$ , the set  $\bigcap_{i=1}^n \varphi(G, c)^{a_i}$  contains  $A$  and for all element  $g$  of  $N_A$ , one has*

$$\left( \bigcap_{i=1}^n \varphi(G, c)^{a_i} \right)^g \subset \varphi(G, b) \quad \text{and} \quad \varphi(G, c)^3 \subset \varphi(G, b).$$

*Proof of Claim 1.* Let us consider the following partial type  $\pi(x_1, \dots, x_n, y)$  with parameters in  $A \cup N_A \cup \{b\}$ , defined by

$$\left\{ \bigcap_{i=1}^n \varphi(G, y)^{x_i} \subset \varphi(G, b)^g, \quad a \in \bigcap_{i=1}^n \varphi(G, y)^{x_i}, \quad \varphi(G, y)^3 \subset \varphi(G, b) : g \in N_A, a \in A \right\}.$$

By (2), the type  $\pi(x_1, \dots, x_n, y)$  is finitely satisfiable in  $(N_A)^{\times n} \times B$ . The conclusion follows from the fact that the condition ‘ $\{\varphi(G, b) : b \in B\}$  is a nice subgroup of  $G$ ’ is expressible by a partial type in  $B$ .  $\square$

We fix an element  $b$  in  $B$  and apply the previous Claim. We put  $a_0 = 1$  and

$$X_1 = \varphi(G, b) \quad \text{and} \quad X_2 = \bigcap_{i=0}^n \varphi(G, c)^{a_i}.$$

One has  $X_2^g \subset X_1$  for any  $g$  in  $N_A$ . Note that  $X_2 \subset \varphi(G, c)$  so that  $X_2 X_2 \subset X_1$ . Applying inductively the Claim, one can find an infinite decreasing chain of definable subsets  $X_1 \supset X_2 \supset X_3 \supset \dots$  of  $G$  such that for all natural number  $i \geq 1$ , and all element  $g$  of  $N_A$ , one has

$$X_{i+1}^g \subset X_i, \quad A \subset X_i \quad \text{and} \quad X_{i+1} X_{i+1} \subset X_i.$$

Note that, because  $\mathfrak{F}$  consists of uniformly definable sets, by Lemma 2.2, the number  $n$  does not depend on  $b$ , but only on the formula  $\varphi(x, y)$  defining  $\mathfrak{F}$ . It follows that every  $X_i$  is uniformly defined by the formula

$$\psi(x, y_0, \dots, y_{n+1}) = \bigwedge_{0 \leq i \leq n} \varphi(y_i x y_i^{-1}, y_{n+1}),$$

where  $y_0, \dots, y_{n+1}$  are replaced by parameters. By compactness, there is family  $(Y_i)_{i \in \mathbf{Q}}$  of uniformly definable subsets of  $G$ , defined by the formula  $\psi(x, y_0, \dots, y_{n+1})$ , such that for all rational numbers  $p < q$ , all element  $g$  of  $N_A$ , and all element  $X$  of  $\mathfrak{F}$ , one has

$$Y_p^g \subset Y_q, \quad A \subset Y_p \subset X, \quad \text{and} \quad Y_p Y_p \subset Y_q.$$

We put  $Y_p = \psi(G, b_p)$  for some tuple  $b_p$ . By compactness and Ramsey’s Theorem, we may assume that the sequence  $(b_p)_{p \in \mathbf{Q}}$  is indiscernible over the empty set. We define

$$K = \bigcap_{p \in \mathbf{Q}} Y_p \quad \text{and} \quad N_K = \bigcap_{\substack{(p,q) \in \mathbf{Q}^2 \\ p < q}} \{x \in G : Y_p^x \subset Y_q \text{ and } Y_p^{x^{-1}} \subset Y_q\}.$$

It is straightforward that  $K$  is a nice subgroup contained in  $\varphi(G, b)$ , that the elements of  $N_K$  normalise  $K$ , that  $N_K$  contains  $N_A$ , and that if  $x$  belongs to  $N_K$ , then so does  $x^{-1}$ . If  $p < q$  are two rational numbers, then for any  $r$  such that  $p < r < q$ , one has

$$\left( \{x \in G : Y_p^x \subset Y_r\} \cap \{x \in G : Y_r^x \subset Y_q\} \right)^2 \subset \{x \in G : Y_p^x \subset Y_q\}.$$

It follows that  $N_K$  is a subgroup of  $G$ . To finish the proof of Theorem 2.10, we only need to show that  $N_K$  is a nice subgroup. For any  $p < q$ , we define  $\{p, q\}$  putting

$$\{p, q\} = \{x \in G : Y_p^x \subset Y_q \text{ and } Y_p^{x^{-1}} \subset Y_q\},$$

and for any  $r_0 < r_1 < \dots < r_m$ , we define  $\{r_0, \dots, r_m\}$  by

$$\{r_0, \dots, r_m\} = \{r_0, r_1\} \cap \{r_1, r_2\} \cdots \cap \{r_{m-1}, r_m\}.$$

Note that  $\{r_0, \dots, r_m\}^m \subset \{r_0, r_m\}$ . Let  $\psi(x, y)$  be the formula defining uniformly the sets  $\{p, q\}$ , and let  $m$  be the VC-dimension of  $g^*(x, y)$ . Let  $r_0 < r_1 < \dots < r_{2m+1}$  be an ordered sequence of  $2m + 2$  rational numbers. By Lemma 2.2, there is  $i \leq 2m + 1$  such that

$$\begin{aligned} \{r_0, r_1\}^{1/3m} \cap \dots \cap \{r_{i-2}, r_{i-1}\}^{1/3m} \cap \{r_{i+2}, r_{i+3}\}^{1/3m} \cap \dots \cap \{r_{2m}, r_{2m+1}\}^{1/3m} \\ \subset \{r_0, r_1\} \cap \{r_2, r_3\} \cap \dots \cap \{r_{2m}, r_{2m+1}\}. \end{aligned}$$

To simplify notations, let us assume that  $i = 2$ . The equation bellow yields in particular

$$\{r_0, r_1\}^{1/3m} \cap \{r_4, r_5\}^{1/3m} \cap \dots \cap \{r_{2m}, r_{2m+1}\}^{1/3m} \subset \{r_2, r_3\}$$

By indiscernability of  $(b_p)_{p \in \mathbf{Q}}$ , it follows that for any rational numbers  $p_0 < p_1 < \dots < p_{2m+1}$ , and any  $p_i = p_0^{i, i+1} < p_1^{i, i+1} < \dots < p_{3m-1}^{i, i+1} < p_{3m}^{i, i+1} = p_{i+1}$ , one has

$$\{p_0^{0,1}, \dots, p_{3m}^{0,1}\} \cap \{p_0^{4,5}, \dots, p_{3m}^{4,5}\} \cap \dots \cap \{p_0^{2m, 2m+1}, \dots, p_{3m}^{2m, 2m+1}\} \subset \{p_2, p_3\}$$

In particular, any finite intersection of subsets of the form  $\{p, q\}$  for  $p < q$  contains an intersection of  $3m^2$  sets of the same form, so that if  $\mathfrak{F}$  denotes the family  $\left\{ \{p, q\} : p < q \right\}$ , then the family  $\left\{ \bigcap \mathfrak{F}_m : \mathfrak{F}_m \subset \mathfrak{F} \text{ and } |\mathfrak{F}_m| \leq 3m^2 \right\}$  is a uniform filter defining  $N_K$  as well.  $\square$

*Remark 2.11.* In Theorem 2.10, if  $H$  is definable, or merely the intersection of uniformly definable groups (hence nice), then the above proof provides that  $K$  is the intersection of uniformly definable groups. However, we do not see any obvious reason why  $N_K$  would be the intersection of uniformly definable groups.

With a similar proof, we get the stronger result:

**Theorem 2.12.** *Let us assume that  $G$  is  $\kappa$ -saturated for some uncountable cardinal  $\kappa$  and let  $H$  be a nice subgroup of  $G$ . Let  $A$  be a subgroup of  $H$  and  $N_A$  a subgroup of  $N_G(A)$  such that  $A$  is the reunion of a uniform family of definable subsets  $\{\alpha(G, a_i) : i < \kappa\}$  of  $G$  and  $N_A$  is the reunion of a uniform family of definable subsets  $\{\nu(G, b_i) : i < \kappa\}$  of  $G$ . Then there is a  $\pi_\alpha$ -definable subgroups  $K$  of  $H$  and a  $\pi_\nu$ -definable subgroup  $N_K$  of  $N_G(K)$  for some countable nice types  $\pi_\alpha$  and  $\pi_\nu$  such that for all  $i < \kappa$ ,*

$$\alpha(x, a_i) \vdash \pi_\alpha(x) \quad \text{and} \quad \nu(x, b_i) \vdash \pi_\nu(x).$$

### 3. EXTERNAL SETS, DISCERNIBLE GROUPS

Let  $M$  be any first order structure, and  $G$  a group definable in  $M$ .

**Definition 3.1** (external set [She09]). A subset  $X$  of  $M^n$  is *externally definable* or *external* for short if  $X$  equals  $\mathbf{X} \cap M^n$  for some set  $\mathbf{X}$  definable (with parameters) in an elementary extension  $\mathbf{M}$  of  $M$ . Equivalently, there is a formula  $\varphi(x, y)$  and parameters  $\mathbf{c}$  in  $\mathbf{M}$  such that  $X$  equals  $\{x \in M^n : \varphi(x, \mathbf{c}) \text{ holds in } \mathbf{M}\}$ .

**Definition 3.2** (discernible set). A subset  $X$  of  $M^n$  is *externally nice* or *discernible* for short if  $X$  equals  $\bigcap \pi(\mathbf{M}) \cap M$  for some elementary extension  $\mathbf{M}$  of  $M$  and some nice type  $\pi$  in  $\mathbf{M}$ .

Note from [Pil07] that if  $\varphi(M, \mathbf{c})$  is an external subset of  $M$ , then every  $\mathbf{b}$  sharing the same  $\varphi^*$ -type as  $\mathbf{c}$  over  $M$  satisfies  $\varphi(M, \mathbf{c}) = \varphi(M, \mathbf{b})$ . In particular, any external subset of  $M$  is definable with parameters in a fixed  $|M|^+$ -saturated extension  $\mathbf{M}$  of  $M$ . External subsets of  $M$  are discernible. Conversely,

**Lemma 3.3.** *A discernible subset of  $M$  is external.*

*Proof.* Let  $B = \bigcap \rho(M)$  be a discernible subset of  $M$  with  $\rho(x) = \{\varphi(x, a) : a \in A\}$  a uniform filter. Then the partial type

$$\pi(y) = \left\{ \varphi(b, y), \forall x (\varphi(x, y) \rightarrow \varphi(x, a)) : b \in B, a \in A \right\}$$

is finitely satisfiable in  $A$  hence realised by some  $\mathbf{b}$  belonging to a saturated extension of  $M$ . It follows that  $B = \varphi(M, \mathbf{b})$ .  $\square$

Note that the first order properties of an external set  $X = \mathbf{X} \cap M$  have no reasons to be lifted up to  $\mathbf{X}$ , but may be lifted up to a partial type.

**Definition 3.4** (external subgroup). An *external subgroup* of  $G$  is a subgroup  $H$  of  $G$  of the form  $\mathbf{H} \cap G$  where  $\mathbf{G}$  is an elementary extension of  $\mathbf{G}$  and  $\mathbf{H}$  a definable subgroup of  $\mathbf{G}$ . If both  $H$  and  $\mathbf{H}$  share a set  $P$  of first order properties (with parameters in  $M$ ), we say that  $H$  is *external as a  $P$ -group*.

**Definition 3.5** (discernible subgroup). A *discernible subgroup* of  $G$  is a subgroup  $H$  of  $G$  of the form  $\mathbf{H} \cap G$  where  $\mathbf{G}$  is an elementary extension of  $\mathbf{G}$  and  $\mathbf{H}$  a nice subgroup of  $\mathbf{G}$ . If  $H$  and  $\mathbf{H}$  share a set  $P$  of first order properties (with parameters in  $M$ ), we say that  $H$  is *discernible as a  $P$ -group*.

If  $H$  is a discernible  $P$ -subgroup of  $G$  defined by the type  $\pi(x) = \{\varphi(x, \mathbf{a}_i) : i \in \mathbf{N}\}$  over countably many external parameters  $\mathbf{a}_1, \mathbf{a}_2, \dots$ , then replacing  $\mathbf{a}_1, \mathbf{a}_2, \dots$  by parameters  $\mathbf{b}_1, \mathbf{b}_2, \dots$  sharing the same type over  $M$  changes neither  $H$  nor the first order consequences (with parameters in  $M$ ) of  $\pi(x)$ . In particular, a discernible  $P$ -subgroup of  $G$  is the trace over  $G$  of a nice  $P$ -subgroup of a  $|M|^+$ -saturated extension  $\mathbf{G}$  of  $G$ . An external subgroup of  $G$  is a discernible subgroup of  $G$ . The converse fails ; the subgroup of infinitesimal numbers in an elementary extension  $\mathfrak{R}$  of the field  $\mathbf{R}$  is discernible as a group, external as a set, but not external as a group.

**Lemma 3.6.** *A discernible subgroup of  $G$  is external as a group provided that it be the conjunction of uniformly external subgroups of  $G$ .*

*Proof.* Same proof as Lemma 3.3, adding to the partial type  $\pi(y)$  a formula  $\psi(y)$  saying that  $\varphi(x, y)$  defines a subgroup of  $G$ .  $\square$

**Lemma 3.7.** *Let the structure  $M$  be stable. Any discernible subgroup  $H$  of  $G$  is definable (if  $H$  is the trace over  $G$  of a group  $\mathbf{H}$  defined by the nice type  $\{\varphi(x, \mathbf{a}) : \mathbf{a} \in A\}$ , there are  $a_1, \dots, a_n$  in  $M$  such that  $\varphi(G, a_1) \cap \dots \cap \varphi(G, a_n)$  is a subgroup of  $H$  of finite index).*

*Proof.* As  $\mathbf{G}$  does not have the order property, there is  $\mathbf{a}$  in  $A$  such that  $\mathbf{H} = \varphi(\mathbf{G}, \mathbf{a})$ . Let  $\psi(y)$  be a formula stating that  $\varphi(G, y)$  is a subgroup of  $G$ . The  $\varphi^* \wedge \psi$ -type of  $\mathbf{a}$  over  $G$  is definable by a positive Boolean combination of formulas of the form  $\varphi(x, m) \wedge \psi(m)$  for  $m$  in  $M$  by [HH84, Corollary 2.8], hence covered by a finite union of subgroups of  $G$ . By Neumann's Lemma [Neu54], one of these subgroups must have finite index in  $H$ .  $\square$

**Theorem 3.8** (finding discernible subgroups). *Let  $G$  be a group definable in a structure  $M$  that does not have the independence property, and  $\mathbf{G}$  a  $|G|^+$ -saturated elementary extension of  $G$ .*

- (1) *The centralisers of subsets of  $G$  are (uniformly) external (as groups). There is a natural number  $n \geq 1$  such that for every  $A \subset G$ , there are  $\mathbf{a}_1, \dots, \mathbf{a}_n$  in  $\mathbf{G}$  such that*

$$C_G(A) = C_G(\mathbf{a}_1, \dots, \mathbf{a}_n).$$

- (2) *The centre of an external group is external (as an Abelian group),*  
 (3) *The iterated centers of an external group  $H$  are external (as iterated centres). For every natural number  $n$ , there is a definable subgroup  $\mathbf{H}$  of  $\mathbf{G}$  such that*

$$H = \mathbf{H} \cap G \quad \text{and} \quad Z_n(H) = Z_n(\mathbf{H}) \cap G.$$

- (4) *The cores of an external subgroup  $H = \mathbf{H} \cap G$  are (uniformly) external (as groups). There is a natural number  $n \geq 1$  such that for every  $A \subset G$ , there are  $\mathbf{a}_1, \dots, \mathbf{a}_n$  in  $\mathbf{G}$  such that*

$$H^A = \mathbf{H}^{\mathbf{a}_1} \cap \dots \cap \mathbf{H}^{\mathbf{a}_n} \cap G.$$

- (5) *The core of a discernible subgroup is discernible (as a group).*  
 (6) *The normaliser of a discernible subgroup is discernible (as a group).*  
 (7) *The iterated centralisers of subsets of  $G$  are discernible (as groups).*

*Proof.* (1) By the Baldwin Saxl descending chain condition, the centraliser of a subset of  $G$  is defined by a nice partial type consisting of uniformly definable subgroups. It is thus an external subgroup by Lemma 3.6.

(2) Let  $H = \mathbf{H} \cap G$  be an external subgroup of  $G$  where  $\mathbf{G}$  is  $|G|^+$ -saturated. Its centre  $Z(H)$  equals  $G \cap C_{\mathbf{H}}(H)$ , hence is external as a group by (1). By the Baldwin Saxl condition, there is a natural number  $n$  such that the centraliser of any finite subset of  $\mathbf{G}$  is the centraliser of  $n$  elements. By the Compactness theorem and the saturation assumption, there is a finite  $n$ -tuple  $\mathbf{h}$  in  $\mathbf{G}$  such that  $Z(H) \subset C_{\mathbf{H}}(\mathbf{h}) \subset C_{\mathbf{H}}(H)$ . It follows that  $Z(C_{\mathbf{H}}(\mathbf{h}))$  contains  $Z(H)$ , hence  $Z(H) = G \cap Z(C_{\mathbf{H}}(\mathbf{h}))$  hold, so  $Z(H)$  is external as an Abelian group.

(3) The following Claim is inspired by [dA13, Lemma 2.1]:

**Claim 2.** *Let  $A$  and  $B$  be two subgroups of  $G$  with  $|A| < |G|$  and  $|B| < |G|$ . Let  $\mathbf{D}$  be a definable subgroup of  $\mathbf{G}$  normalised by both  $A$  and  $B$  such that  $[A, B] \subset \mathbf{D}$ . Then there are two definable subgroups  $\mathbf{A}$  and  $\mathbf{B}$  of  $\mathbf{G}$  containing  $A$  and  $B$  respectively and such that  $[\mathbf{A}, \mathbf{B}] \subset \mathbf{D}$ .*

*Proof of Claim 2.* For any parameter subset  $C \subset \mathbf{G}$ , we define the subgroups  $\mathbf{A}(C)$  and  $\mathbf{B}(C)$  of  $\mathbf{G}$  by

$$\mathbf{A}(C) = \bigcap_{\substack{c \in C \\ [A, c] \subset \mathbf{D}}} \{x \in N_{\mathbf{G}}(\mathbf{D}) : [x, c] \subset \mathbf{D}\},$$

$$\mathbf{B}(C) = \bigcap_{\substack{c \in C \\ [c, B] \subset \mathbf{D}}} \{y \in N_{\mathbf{G}}(\mathbf{D}) : [c, y] \subset \mathbf{D}\},$$

and claim that there is a finite set  $C$  such that  $[\mathbf{A}(A \cup B \cup C), \mathbf{B}(A \cup B \cup C)] \subset \mathbf{D}$ . Otherwise, one could inductively build two sequences  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 1}$  such that for every  $n \geq 1$ ,  $a_n \in \mathbf{A}(A \cup B \cup \{a_k, b_k : k < n\})$  and  $b_n \in \mathbf{B}(A \cup B \cup \{a_k, b_k : k < n\})$  but  $[a_n, b_n] \notin \mathbf{D}$ . It would follow that  $[a_i, b_j] \in \mathbf{D}$  if and only if  $i \neq j$ , so that the chain  $(C_{\mathbf{G}}(a_n \mathbf{D}))_{n \geq 1}$  would not the Baldwin Saxl chain condition. By the Compactness theorem, there is a finite tuple  $c$  such that  $[\mathbf{A}(c), \mathbf{B}(c)] \subset \mathbf{D}$ . We consider  $\mathbf{A} = \mathbf{A}(c)$  and  $\mathbf{B} = \mathbf{B}(c)$ .  $\square$

We prove (3) inductively on  $n$ . For  $n = 1$ , by Claim 2, there are definable subgroups  $\mathbf{H}_1$  and  $\mathbf{Z}_1$  of  $\mathbf{G}$  containing  $H$  and  $Z(H)$  respectively such that  $[\mathbf{H}_1, \mathbf{Z}_1] = 1$ . We may assume that  $\mathbf{Z}_1 \leq \mathbf{H}_1$  replacing  $\mathbf{Z}_1$  by  $\mathbf{H}_1 \cap \mathbf{Z}_1$  if need be. As  $Z(H)$  is an external Abelian group by (1), we may also assume that  $\mathbf{Z}_1$  is Abelian. It follows that  $C_{\mathbf{G}}(\mathbf{Z}_1)$  contains  $H$ , and  $Z(C_{\mathbf{G}}(\mathbf{Z}_1))$  contains  $Z(H)$ . In particular, one has

$$H = G \cap C_{\mathbf{H}}(\mathbf{Z}_1) \quad \text{and} \quad Z(H) = G \cap Z(C_{\mathbf{H}}(\mathbf{Z}_1)).$$

We assume now that there is a definable subgroup  $\mathbf{H} \leq \mathbf{G}$  such that  $H = \mathbf{H} \cap G$  and  $Z_n(H) = Z_n(\mathbf{H}) \cap G$ . As  $[Z_{n+1}(H), H] \subset Z_n(\mathbf{H})$ , according to Claim 2, there are two definable subgroups  $\mathbf{Z}_{n+1}$  and  $\mathbf{H}_{n+1}$  of  $\mathbf{G}$  containing  $Z_{n+1}(H)$  and  $H$  respectively such that  $[\mathbf{Z}_{n+1}, \mathbf{H}_{n+1}] \subset Z_n(\mathbf{H})$ . Replacing  $\mathbf{H}_{n+1}$  by  $\mathbf{H}_{n+1} \cap \mathbf{H}$  and  $\mathbf{Z}_{n+1}$  by  $\mathbf{Z}_{n+1} \cap \mathbf{H}_{n+1}$ , we may assume that  $\mathbf{H}_{n+1} \leq \mathbf{H}$  and  $\mathbf{Z}_{n+1} \leq \mathbf{H}_{n+1}$ . It follows that  $[\mathbf{Z}_{n+1}, \mathbf{H}_{n+1}] \subset Z_n(\mathbf{H}_{n+1})$ , so that  $Z_{n+1}(\mathbf{H}_{n+1})$  contains  $\mathbf{Z}_{n+1}$ , hence  $Z_{n+1}(H)$ . One has thus  $H = \mathbf{H}_{n+1} \cap G$  and  $Z_{n+1}(H) = Z_{n+1}(\mathbf{H}_{n+1}) \cap G$ .

(4) Follows from the Baldwin Saxl descending chain condition and Lemma 3.6.

(5) If  $H$  is a discernible subgroup of  $G$  and  $S$  any subgroup of  $G$ , by Theorem 2.10,  $H^S$  is contained in a discernible subgroup  $K$  of  $H$  such that  $K^S = K$ . From the inclusion  $H^S \leq K \leq H$ , it follows that  $K = H^S$ , so that  $H^S$  is discernible.

(6) Let  $H$  be a discernible subgroup of  $G$ . By Theorem 2.10, there is a discernible subgroup  $N$  of  $G$  that contains  $N_G(H)$  and normalises  $H$ , so that  $N = N_G(H)$ .

(7) Recall that the  $n$ th centraliser  $C_G^n(A)$  of  $A$  is defined inductively putting  $C_G^0(A) = \{1\}$  and  $C_G^{n+1}(A) = \bigcap_{k < n+1} N_G(C_G^k(A)) \cap \{g \in G : [g, A] \subset C_G^n(A)\}$ , so (7) follows inductively from (1), Theorem 2.10, and (6).  $\square$

#### 4. ENVELOPES IN A DEFINABLE GROUP

In this section,  $G$  stands for a group that does not have the independence property.

**Theorem 4.1** (Abelian envelope). *Any Abelian subgroup  $A$  of  $G$  is contained in an  $A$ -invariant external Abelian subgroup that is normalised by  $N_G(A)$ .*



*First proof* (Adapted from [dA13, Lemma 2.1]). Assume that  $A$  is infinite and let  $\mathbf{G}$  be an  $|A|^+$ -saturated elementary extension of  $G$ . For any parameter set  $\mathbf{B} \subset \mathbf{G}$ , we put

$$C(\mathbf{B}) = \bigcap_{\substack{b \in \mathbf{B} \\ [b, A] = 1}} C_{\mathbf{G}}(b).$$

We claim that there is a finite subset  $B \subset \mathbf{G}$  such that  $C(A \cup \mathbf{B})$  is Abelian, for otherwise we could construct inductively two sequences  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 1}$  such that for every  $n$ , both  $a_n$  and  $b_n$  belong to  $C(A \cup \{a_k, b_k : k < n\})$  and  $[a_n, b_n] \neq 1$ . It would follow that  $[a_i, b_j] = 1$  if and only if  $i = j$ , so that the family  $(C_{\mathbf{G}}(a_i))_{i \geq 1}$  would not satisfy the Baldwin-Saxl chain condition 2.1. By the Compactness Theorem, there is a finite tuple  $c_1, \dots, c_n$  in  $\mathbf{B}$  such that  $C_{\mathbf{G}}(c_1, \dots, c_n)$  is Abelian, and contains  $A$ . By Lemma 3.6, one can even assume that  $C_G(c_1, \dots, c_n)$  is  $A$ -invariant and normalised by  $N_G(A)$ .  $\square$

*Second proof.*  $Z(C_G(A))$  is external (as an Abelian group) by Theorem 3.8.1 and 3.8.2 that is  $A$ -invariant and normalised by  $N_G(A)$ .  $\square$

*Remark 4.2.* The second proof provides an Abelian envelope that is externally defined by the formula  $Z(C_G(x_1, \dots, x_n))$ . The first proof provides an envelope defined by the simpler formula  $C_G(x_1, \dots, x_n)$ . Note that in a stable group  $G$ , an Abelian subgroup  $A \leq G$  is contained in an Abelian centraliser:  $Z(C_G(A))$  is Abelian and equals  $C_G(C_G(A))$ , hence is the centraliser of a finite tuple by the descending chain condition.

**Theorem 4.3** (nilpotent envelope). *Any nilpotent subgroup  $N$  of class  $n$  of  $G$  is contained in an  $N$ -invariant external nilpotent subgroup of class  $n$ .*

*Proof.* By [dA13, Theorem 2.2], there is a external nilpotent envelope  $H$  of  $N$  of nilpotency class  $n$ . By Lemma 3.6,  $\bigcap_{\sigma \in \text{Aut}(G/\{N\})} \sigma(H)$  is an  $N$ -invariant external one.  $\square$

**Theorem 4.4** (soluble envelope 1). *Let  $S$  be a soluble subgroup of  $G$  of derived length  $\ell$ . There is an  $S$ -invariant discernible subgroup  $N$  of  $G$  that contains  $N_G(S)$ , and an external normal soluble subgroup of  $N$  of derived length  $\ell$  that contains  $S$ , is  $S$ -invariant and normalised by  $N_G(S)$ .*

*Proof.* Let  $\mathbf{G} \succ G$  be a  $|G|^+$ -saturated elementary extension of  $G$ . We show more precisely that there is a nice subgroup  $\mathbf{N}$  of  $\mathbf{G}$  containing  $N_G(S)$  and a relatively definable subgroup  $\mathbf{H}$  of  $\mathbf{N}$  that is soluble of derived length  $\ell$  and contains  $S$ . When  $\ell$  is 1 we apply Theorem 4.1 and Theorem 2.10. If the result holds for every  $\ell$ -soluble subgroups of  $G$  and if  $S$  is soluble of derived length  $\ell + 1$ , there is a nice subgroup  $\mathbf{N}$  of  $\mathbf{G}$  that contains  $N_G(S)$  and a relatively definable  $\ell$ -soluble subgroup  $\mathbf{H}$  of  $\mathbf{N}$  that contains  $S'$ . We thus have

$$S\mathbf{H}/\mathbf{H} \leq \mathbf{N}/\mathbf{H}.$$

As  $(S\mathbf{H})' \leq H$  holds, the group  $S\mathbf{H}/\mathbf{H}$  is Abelian. By Corollary 1.7, the formula  $yx = xy$  does not have the independence property in the group  $\mathbf{N}/\mathbf{H}$ . As  $\mathbf{N}/\mathbf{H}$  is  $|S|^+$ -saturated, according to Theorem 4.1 there are  $\mathbf{a}_1, \dots, \mathbf{a}_n$  in  $\mathbf{N}$  such that  $C_{\mathbf{N}/\mathbf{H}}(\mathbf{a}_1\mathbf{H}, \dots, \mathbf{a}_n\mathbf{H})$  is Abelian and contains  $S\mathbf{H}/\mathbf{H}$ . It follows that the group

$$\mathbf{K} = \bigcap_{i=1}^n \{x \in \mathbf{N} : [x, \mathbf{a}_i] \subset \mathbf{H}\}$$

is relatively definable in  $\mathbf{N}$ , soluble of derived length  $\ell+1$  and contains  $S$ . Putting  $K = \mathbf{K} \cap G$  the subgroup

$$\bigcap_{g \in N_G(S)} \bigcap_{\sigma \in \text{Aut}(G/\{S\})} \sigma(K^g)$$

of  $K$  is in addition  $S$ -invariant and normalised by  $N_G(S)$ , and its normaliser is discernible by Theorem 2.10.  $\square$

**Corollary 4.5** (soluble envelope 2). *Let  $S$  a soluble subgroup of  $G$  of derived length  $\ell$ . There is an external subset  $X = \mathbf{X} \cap G$  that is a subgroup of  $G$  containing  $S$ , and such that  $\mathbf{X}$  generates a soluble subgroup of derived length  $\ell$ .*

*Proof.* By Zorn's Lemma, we may assume that  $S$  is a maximal soluble subgroup of  $G$  of derived length  $\ell$ . By Theorem 4.4, there is a formula  $\varphi(x, y)$  and a parameter set  $\mathbf{A}$  such that  $\bigcap \{\varphi(\mathbf{G}, \mathbf{a}) : \mathbf{a} \in \mathbf{A}\}$  defines a nice soluble subgroup containing  $S$  in an elementary  $|S|^+$ -saturated extension  $\mathbf{G}$  of  $G$ . We write  $\varphi(\mathbf{G}, \mathbf{a})^n$  for the set of products of  $n$  elements of  $\varphi(\mathbf{G}, \mathbf{a})$ . We say that a subset  $\mathbf{X}$  of  $\mathbf{G}$  is *soluble of derived length  $\ell$*  if  $\mathbf{X}$  satisfies all the commutator identities satisfied by a soluble group of derived length  $\ell$ . For a definable set  $\mathbf{X}$ , being soluble of derived length is a first order property. Thus, the following partial type over  $S$

$$\pi_S(x) = \left\{ \varphi(s, x), \varphi(\mathbf{G}, x)^n \text{ is soluble of derived length } \ell : s \in S, n \geq 1 \right\}$$

is finitely satisfiable in  $\mathbf{A}$ , hence consistent. For any of its realisation  $\mathbf{s}$  in  $\mathbf{G}$ , take  $\mathbf{X} = \varphi(\mathbf{G}, \mathbf{s})$ . The set  $\mathbf{X} \cap G$  is a subgroup by maximality of  $S$ .  $\square$

*Remark 4.6.* In an arbitrary group  $G$ , if a subset  $X$  satisfies all the commutator identity satisfied by a nilpotent group of class  $n$ , we call  $X$  a *nilpotent subset of class  $n$* . If a subset  $X$  is nilpotent of class  $n$ , then  $X$  generates a nilpotent subgroup of class  $n$ . If  $X$  is in addition definable, then it is contained in a definable nilpotent subgroup of class  $n$ . This can be shown taking  $Z(C_G(X))$  for  $n = 1$ , and  $Z_n(E_n)$  for arbitrary  $n$ , with  $E_n$  defined inductively by  $E_0 = G$  and  $E_{k+1} = \{x \in E_k : [x, C_{E_k}^{k+1}(X)] \subset C_{E_k}^k(X)\}$  (see [AB14]). However, if  $X$  is merely soluble of derived length 2, then  $X$  may not even generate a soluble subgroup. Consider for instance two generators  $a$  and  $b$  of the alternating group  $A_5$ . The set  $\{a, b\}$  obviously satisfies the equation  $[[x, y], [z, t]] = 1$ , but  $A_5$  is not solvable.

**Theorem 4.7** (normal soluble envelope). *Let  $S$  be a normal soluble subgroup of  $G$  of derived length  $\ell$ . There is an external normal, soluble of derived length  $\ell$  subgroup of  $G$  that contains  $S$  and is  $S$ -invariant.*

*Proof.* The following proof is due to Frank Wagner. For  $\ell = 1$ , the group  $S$  is Abelian and normal. For any elementary extension  $\mathbf{G}$  of  $G$  and  $s_1, \dots, s_n$  in  $S$ , the conjugacy classes  $s_1^{\mathbf{G}}, \dots, s_n^{\mathbf{G}}$  generate an Abelian subgroup of  $\mathbf{G}$ . By the Compactness Theorem, there are  $\mathbf{a}_1, \dots, \mathbf{a}_n$  in a  $|S|^+$ -saturated elementary extension  $\mathbf{G}$  of  $G$  such that  $S \subset C_{\mathbf{G}}(\mathbf{a}_1^{\mathbf{G}}, \dots, \mathbf{a}_n^{\mathbf{G}}) \subset C_{\mathbf{G}}(S)$ . It follows that  $Z(C_{\mathbf{G}}(\mathbf{a}_1^{\mathbf{G}}, \dots, \mathbf{a}_n^{\mathbf{G}}))$  is normal in  $\mathbf{G}$ , Abelian and contains  $S$ . We go on inductively.  $\square$

*Remark 4.8.* Not only is the external envelope normal in  $G$ , it is also the trace over  $G$  of a definable normal subgroup of  $\mathbf{G}$ .

5. FURTHER CHAIN CONDITIONS *à la* BALDWIN SAXL

We now consider a group  $G \subset M^n$  defined by a conjunction of infinitely many formulas, in a structure  $M$  that does not have the independence property. Two new difficulties appear :  $G$  need not be the intersection of definable groups, and the quotient  $G/H$  by an infinitely definable normal subgroup  $H$  might have the independence property unless the formulas defining  $H$  relatively to  $G$  are not controlled.

**5.1. Relatively nice subgroups.** Given a structure  $M$ , a subset  $B$  of some finite Cartesian product  $M^n$  that is definable by a formula  $\varphi$ , and a partial  $n$ -type  $\pi(x_1, \dots, x_n)$ , we say that  $G \subset M^n$  is a  $\pi$ -definable group in  $M$  of base  $B$  if there is a definable function  $\times$  from  $B^2$  to  $M^n$  such that  $G$  equals  $\bigcap \pi(M)$ , and for every structure  $\mathbf{M}$  that is an elementary extension of  $M$ , the set  $\bigcap \pi(\mathbf{M})$  is contained in  $\varphi(\mathbf{M})$  and is a group for the law induced by  $\times$ . We say that  $G$  is a *type definable group of base  $B$*  if there is a partial type  $\pi$  such that  $G$  is  $\pi$ -definable of base  $B$ . Assuming the group law  $\times$  to be definable rather than type definable is no restriction by a compactness argument (see [Poi85, page 170]).

Throughout the section, we consider a fixed structure  $M$  that does not have the independence property,  $G$  a  $\pi$ -definable group of base  $B$  in  $M$  for some partial type  $\pi$ . If  $H$  is a type definable subgroup of  $G$ , we say that  $H$  is *relatively definable in  $G$*  if there is a formula  $\varphi(x)$  such that  $H$  is  $\pi \cup \{\varphi\}$ -definable subgroup of  $G$ . We call  $\varphi$  the *defining formula of  $H$* . More generally:

**Definition 5.1** (relatively nice subgroup). A type definable subgroup  $H$  of  $G$  is *relatively nice in  $G$*  if there is a formula  $\varphi(x, y)$  and tuples  $(a_i)_{i \in I}$  of parameters in  $M$  such that  $H$  is a  $\pi(x) \cup \{\varphi(x, a_i) : i \in I\}$ -definable subgroup of  $G$  and the family  $\{\bigcap \pi(\mathbf{M}) \cap \varphi(\mathbf{M}, a_i) : i \in I\}$  is a filter for every elementary extension  $\mathbf{M}$  of  $M$ . We call  $\varphi(x, y)$  the *defining formula of  $H$* .

**Definition 5.2** (uniform family of rel. nice subgroups). A family  $\mathfrak{H}$  of relatively nice subgroups of  $G$  is *uniform* if its members have the same defining formula.

**Lemma 5.3** (Baldwin Saxl chain condition for relatively nice subgroups). *If  $\mathfrak{H}$  is a uniform family of relatively nice subgroups of  $G$ , then there is a natural number  $n$  such that any finite intersection of members of  $\mathfrak{H}$  is the intersection of at most  $n$  of them.*

*Proof.* Otherwise, by the usual Baldwin-Saxl argument, for every natural number  $n$  one would find  $H_1, \dots, H_n$  in  $\mathfrak{H}$  and tuples  $(b_J)_{J \subset \{1, \dots, n\}}$  of elements in  $G$  such that  $b_J \in H_i$  if and only if  $i \in J$ . Let  $\varphi(x, y)$  be the common defining formula of the members of  $\mathfrak{H}$ . By the Compactness Theorem, there are  $(a_1, \dots, a_n)$  in  $G$  such that  $\varphi(b_J, a_i)$  holds if and only if  $i \in J$ , a contradiction.  $\square$

**Corollary 5.4.** *Let  $\mathfrak{H}$  be a family of relatively nice subgroups of  $G$  and  $\{a_H : H \in \mathfrak{H}\}$  a family of elements from  $G$  such that  $a_H$  belongs to  $K$  if and only if  $K \neq H$ . Then  $\mathfrak{H}$  is not a uniform family.*

*Proof.* For any natural number  $n \geq 1$ , the intersection of  $n$  members of  $\mathfrak{H}$  does not contain any proper subintersection.  $\square$

**Corollary 5.5** (uniform families are closed under intersections). *If  $\mathfrak{H}$  is a uniform family of relatively nice subgroups of  $G$ , then  $\bigcap_{H \in \mathfrak{H}} H$  is relatively nice and the family  $\{\bigcap_{H \in \mathfrak{H}} H : \mathfrak{K} \subset \mathfrak{H}\}$  is uniform.*

*Proof.* Let  $n$  be the natural number provided by Lemma 5.3 and for every  $H$  in  $\mathfrak{H}$ , let  $\{\varphi(x, a_i^H) : i \in I\}$  be a type defining  $H$  relatively to  $G$ . Let  $A$  be the set  $\{a_i^H : H \in \mathfrak{H}, i \in I\}$ . Calling  $\psi(x, y)$  the formula  $\varphi(x, y_1) \wedge \dots \wedge \varphi(x, y_n)$ , the group  $\bigcap_{H \in \mathfrak{H}} H$  is  $\pi \cup \{\psi(x, b) : b \in A^n\}$ -definable and the family  $\{\bigcap \pi(M) \cap \psi(x, b) : b \in A^n\}$  is a filter by Lemma 5.3.  $\square$

**5.2. Uniformity.** A third difficulty arises. It is not obvious to us that being a relatively nice subgroup with parameters  $A$  is expressible by a partial type in  $A$ , which prevents applying Compactness arguments. We introduce therefor a strengthening of the preceding notions. If  $N$  is relatively nice in  $G$ , the Compactness Theorem ensures that for all  $i$  in  $I$ , there is some  $j$  in  $I$  and some definable set  $X_i$  containing  $G$  such that

$$\left(X_i \cap \varphi(M, a_j)\right)^2 \subset \varphi(M, a_i).$$

**Definition 5.6** (uniformly relatively nice subgroup). We say that  $N$  is *uniformly relatively nice in  $G$*  if it is relatively nice in  $G$  and there is a definable set  $X$  containing  $G$  such that for all  $i$  in  $I$ , there is  $j$  in  $I$  such that

$$\left(X \cap \varphi(M, a_j)\right)^2 \subset \varphi(M, a_i).$$

We call this set  $X$  a *second base* for  $N$ .

**Definition 5.7** (uniform family of unif. rel. nice subgroups). A family  $\mathfrak{H}$  of uniformly relatively nice subgroups of  $G$  is *uniform* if its members have the same defining formula and share a common second base.

**Lemma 5.8.** *If  $\mathfrak{H}$  is a uniform family of uniformly relatively nice subgroups of  $G$ , then  $\bigcap_{H \in \mathfrak{H}} H$  is uniformly relatively nice, and the family  $\{\bigcap_{H \in \mathfrak{H}} H : \mathfrak{K} \subset \mathfrak{H}\}$  is uniform.*

*Example 5.9.* A relatively definable subgroup of  $G$  is uniformly relatively nice in  $G$ . For any  $g \in G$ , the centraliser  $C_G(g)$  is relatively definable in  $G$ , and the family  $\{C_G(g) : g \in G\}$  is uniform. In particular,  $C_G(A)$  is uniformly relatively nice in  $G$  for any  $A \subset G$ .

*Example 5.10.* If  $G$  is type definable using parameters in  $A$ , if  $\mathfrak{A} \leq \text{Aut}(G/A)$  and  $H \leq G$  is uniformly relatively nice in  $G$  with a second base  $X$  that is definable with parameters in  $A$ , then the family  $\{\sigma(H) : \sigma \in \mathfrak{A}\}$  shares  $X$  has a second base, and  $\bigcap_{\sigma \in \mathfrak{A}} \sigma(H)$  is uniformly relatively nice in  $G$ .

*Example 5.11.* Let  $G$  be a type definable group with parameters in  $A$ , and  $H$  a relatively nice subgroup of  $G$ , type definable by the partial type  $\{\varphi(x, a_i) : i \in I\}$  relatively to  $G$ , where  $I$  is a linearly ordered set. If  $(a_i)_{i \in I}$  is an indiscernible sequence over  $A$ , then  $H$  is uniformly relatively nice in  $G$ .

**Lemma 5.12.** *If  $H$  is a uniformly relatively nice subgroup of  $G$  and  $g \in G$ , then so is  $H^g$ , and the family  $\{H^g : g \in G\}$  is uniform. In particular  $H^A$  is uniformly relatively nice in  $G$  for any  $A \subset G$ .*

*Proof.* We need only find a common second base for the family  $\{H^g : g \in G\}$ . A second base  $X \subset B$  for  $H$  is such that for all  $i$  in  $I$ , there is  $j$  in  $I$  such that

$$\left(X \cap \varphi(M, a_j)\right)^2 \subset \varphi(M, a_i).$$

By the Compactness theorem, there is a definable set  $X^{1/3}$  such that  $G \subset X^{1/3} \subset X$  and  $(X^{1/3})^3 \subset X$  hold. It follows that for every  $g, g_1, \dots, g_n$  in  $G$ , one has

$$\left(X^{1/3} \cap \varphi(M, a_j)^g\right)^2 \subset \varphi(M, a_i)^g.$$

□

**Lemma 5.13** (Baldwin Saxl chain condition for subsets). *Let  $\mathfrak{X}$  be a family of subsets of  $B$ . Assume that  $\{X^{1/3} : X \in \mathfrak{X}\}$  is a corresponding family of symmetric cube roots, uniformly definable by a formula  $\varphi(x, y)$ , and let  $n$  be the VC-dimension of  $\varphi^*(x, y)$  in  $M$ . For all  $X$  in  $\mathfrak{X}$ , assume that there is a symmetric subset  $X^{1/3n}$  of  $B$  containing 1 with  $(X^{1/3n})^n \subset X^{1/3}$ . Then, for any elements  $X_1, \dots, X_{n+1}$  of  $\mathfrak{X}$ , there is  $j$  in  $\{1, \dots, n+1\}$  with*

$$X_1^{1/3n} \cap \dots \cap X_{j-1}^{1/3n} \cap X_{j+1}^{1/3n} \cap \dots \cap X_{n+1}^{1/3n} \subset X_1 \cap \dots \cap X_{n+1}.$$

**Corollary 5.14.** *If  $N \leq G$  is a  $\pi \cup \{\varphi(x, a_i) : i \in I\}$ -definable subgroup and if there is a definable subset  $Y \subset B$  such that for all  $i$  in  $I$  there is  $j$  in  $I$  with*

$$\left(X \cap \varphi(M, a_j)\right)^2 \subset \varphi(M, a_i),$$

*then  $N$  is uniformly relatively nice in  $G$ .*

*Proof.* There are definable sets  $X_1, X_2, \dots$  such that  $X_1 = B$  and  $X_{n+1}^2 \subset X_n$  for all  $n$ . By Lemma 5.13, there is an  $n$  such that the family  $\{X_n \cap \bigcap_{i \in J} \varphi(M, a_i) : J \subset I, |J| \leq n\}$  is a filter. □

If  $H \leq G$  is a type definable subgroup that normalises  $N$ , by the Compactness Theorem, for all  $i$  in  $I$ , there is some  $j$  in  $I$ , a definable set  $X_i \supset G$  and a definable  $H_i \supset H$  such that for all  $g$  in  $H_i$ ,

$$\left(X_i \cap \varphi(M, a_j)\right)^g \subset \varphi(M, a_i).$$

**Definition 5.15** (uniformly normal subgroup). If  $N$  is relatively nice in  $G$  and normalised by  $H$ , we say that  $N$  is *uniformly normalised by  $H$*  if there are definable sets  $X \supset G$  and  $K \supset H$  such that for all  $i$  in  $I$ , there is some  $j$  in  $I$  such that for all  $g$  in  $K$ ,

$$\left(Y \cap \varphi(M, a_j)\right)^g \subset \varphi(M, a_i).$$

**Theorem 5.16** (normalising a uniformly relatively nice envelope). *Assume that  $M$  is  $\kappa$ -saturated for some uncountable cardinal  $\kappa$ . Let  $A \leq G$  and  $N_A \leq N_G(A)$  be two subgroups with  $|A| < \kappa$  and  $|N_A| < \kappa$ . Assume that  $H$  is a uniformly relatively nice (resp. relatively definable) subgroup of  $G$  enveloping  $A$ . Then there is a subgroup  $E \leq H$  that is uniformly relatively nice (resp. a conjunction of a uniform family of relatively definable subgroups) in  $G$ , and a type definable subgroup  $N \leq N_G(E)$  such that  $E$  contains  $A$ ,  $N$  contains  $N_A$  and normalises  $E$  uniformly.*

*Proof.* Let  $H = G \cap \bigcap \pi(M, (a_i)_{i \in I})$  with  $\pi(x) = \{\varphi(x, a_i) : i \in I\}$  and  $I$  an ordinal. By assumption, there is a definable set  $X$  containing  $G$  such that for all  $i$  in  $I$ , there is a  $j$  in  $I$  with

$$(X \cap \varphi(M, a_j))^2 \subset \varphi(M, a_i).$$

There are also definable sets  $X = X_1 \supset \dots \supset X_n \supset \dots$  such that  $X_{n+1}^2 \subset X_n$  for all  $n$ . We may assume without loss of generality that  $G$  equals  $\bigcap_{n \geq 0} X_n$  and that the formula  $\forall y (\varphi(x, y) \leftrightarrow \varphi(x^{-1}, y))$  holds. By Lemma 5.13, one may also assume that the family  $\{X \cap \varphi(M, a_i) : i \in I\}$  is a filter.

**Claim 3.** *There is a natural number  $n$  such that for all  $a_i$ , there exists a sequence  $(b_i)_{i \in I}$  in  $G$ , a finite subset  $G_n \subset N_G(A)$  of size  $n$  and an element  $j$  in  $I$  such that  $G \cap \bigcap \pi(M, (b_i)_{i \in I})$  is uniformly relatively nice in  $G$ , contains  $A$  and such that*

$$A \subset X_n \cap \bigcap_{g \in G_n} \varphi(M, b_j)^g \subset \bigcap_{g \in N_G(A)} \varphi(M, a_i)^g \quad \text{and} \quad (X \cap \varphi(M, b_j))^2 \subset \varphi(M, a_i).$$

*Proof of Claim 3.* Fix  $i$  in  $I$ . By the Compactness Theorem, there is a natural number  $n$  bounding the VC-dimension in  $M$  of the formulas  $\{\varphi(y^x, a) : a \in G\}$ . By Lemma 5.12, the family  $\{H^g : g \in G\}$  is uniform so there exists  $j$  in  $I$  such that for all  $g$  in  $G$ ,

$$(3) \quad (X_n \cap \varphi(M, a_j)^g)^{3n} \subset \varphi(M, a_i)^g \quad \text{and} \quad (X \cap \varphi(M, a_j))^2 \subset \varphi(M, a_i).$$

By iterating Lemma 5.13 and (3) in turn, for all finite subset  $J$  of  $G$ , there is a subset  $J_n \subset J$  of size  $n$  and some  $k$  in  $I$  such that

$$(4) \quad X_n \cap \bigcap_{g \in J_n} \varphi(M, a_k)^g \subset \bigcap_{g \in J} \varphi(M, a_i)^g \quad \text{and} \quad (X \cap \varphi(M, a_k))^2 \subset \varphi(M, a_i).$$

We consider the following partial types over  $N_G(A) \cup \{a_i : i \in I\} \cup \text{dom}(G)$  defined by

$$\rho(x_1, \dots, x_n, (z_i)_{i \in I}) = \Phi((z_i)_{i \in I}) \cup \left\{ X_n \cap \bigcap_{k=1}^n \varphi(M, z_0)^{x_k} \subset \varphi(M, a_i)^g, \right. \\ \left. (X \cap \varphi(M, z_0))^2 \subset \varphi(M, a_i), x_1 \in G, \dots, x_n \in G : g \in N_G(A) \right\}$$

where  $\Phi$  is a set of formulas stating that  $G \cap \bigcap \pi(M, (z_i)_{i \in I})$  is a subgroup that contains  $A$  and that is uniformly relatively nice in  $G$  (of base  $X$ ). By (4), the type  $\rho$  is finitely satisfiable in  $N_G(A)^{\times n} \times \{a_i : i \in I\}^I$ .  $\square$

We call  $\psi(x, y_1, \dots, y_{n+1})$  the formula  $\varphi(y_1^{-1}xy_1, y_{n+1}) \wedge \dots \wedge \varphi(y_n^{-1}xy_n, y_{n+1})$ .

**Claim 4.** *There exists a sequence  $(\mathbf{b}_k)_{k \in \mathbf{N}}$  of  $n+1$  tuples in  $G$ , such that for all  $g$  in  $N_G(a)$ , all  $k$  in  $\mathbf{N}$  and all  $i$  in  $I$ ,*

$$X \cap \psi(M, \mathbf{b}_k) \subset \varphi(M, a_i), \\ A \subset (X_n \cap \psi(M, \mathbf{b}_{k+1}))^g \subset \psi(M, \mathbf{b}_k) \quad \text{and} \quad (X \cap \psi(M, \mathbf{b}_{k+1}))^2 \subset \psi(M, \mathbf{b}_k).$$

*Proof of Claim 4.* One may first fix some  $i$  in  $I$  and take  $\varphi(M, a_i)$  for  $\psi(M, \mathbf{b}_1)$  and iterate Claim 3 to get a family  $(\mathbf{b}_k)_{k \in \mathbf{N}}$  depending on  $i$ . As the family  $\{X \cap \varphi(M, a_i) : i \in I\}$  is a filter, the conclusion follows from the Compactness Theorem.  $\square$

**Claim 5.** *There exists an indiscernible sequence  $(\mathbf{b}_q)_{q \in \mathbf{Q}}$  of  $n+1$  tuples in  $G$ , such that for all  $g$  in  $N_G(a)$ , all rational numbers  $p < q$ , and all  $i$  in  $I$ ,*

$$X \cap \psi(M, \mathbf{b}_p) \subset \varphi(M, a_i),$$

$$A \subset \left( X_n \cap \psi(M, \mathbf{b}_p) \right)^g \subset \psi(M, \mathbf{b}_q) \quad \text{and} \quad \left( X \cap \psi(M, \mathbf{b}_p) \right)^2 \subset \psi(M, \mathbf{b}_q).$$

*Proof of Claim 5.* From Claim 4 by the Compactness Theorem and Ramsey's.  $\square$

We may now finish the proof of Theorem 5.16. Note that the sequence  $(X_n^{X_m})_{m > n}$  is increasing and that  $X_n^{X_m}$  contains  $G$  for all  $m > n+1$ . One also has  $(X_n^{X_m})^g \subset X_n^{X_{m+1}}$  for all  $g$  in  $X_{m+1}$ .

Let  $E_q$  stand for  $\psi(M, \mathbf{b}_q)$  and let  $N_{p,q}^m$  be defined by

$$N_{p,q}^m = \left\{ g \in X_m : \left( X_n^{X_m} \cap E_p \right)^g \subset E_q \right\},$$

and let  $N$  and  $E$  by

$$N = \bigcap_{m > n} \bigcap_{p < q} N_{p,q}^m \quad \text{and} \quad E = G \cap \bigcap_{q \in \mathbf{Q}} E_q.$$

As one has for every rational numbers  $p < r < q$  and natural number  $m > n$ ,

$$\left( N_{p,r}^{m+1} \cap N_{r,q}^{m+1} \right)^2 \subset N_{p,q}^m,$$

it follows that  $N$  is a group. As  $X_n^{X_m}$  contains  $G$  for all  $m > n+1$ , the group  $N$  normalises  $E$ . Applying Lemma 5.13 and indiscernability of  $(\mathbf{b}_q)_{q \in \mathbf{Q}}$ , for any fixed  $m$ , there is a natural number  $k(m)$  such that every finite intersection of  $N_{p,q}^m$  contains an intersection of at most  $3k^2$  sets of the form  $N_{s,t}^{m+3k}$ . By the Compactness theorem, one can find a countable descending chain  $N_1 \supset N_2 \cdots \supset N_\ell \supset \cdots$  of definable subsets of  $X$  such that for all  $\ell \geq 1$ ,

$$N_G(A) \subset N_\ell \subset \bigcap_{p < q} N_{p,q}^{m+\ell} \quad \text{and} \quad N_{\ell+1}^2 \subset N_\ell$$

It follows that  $\bigcap_{\ell \geq 1} N_\ell$  is a type definable subgroup of  $G$  that uniformly normalises  $E$ . The group  $E$  is uniformly relatively nice by Remark 5.11.  $\square$

**5.3. External and discernible subgroups.**  $G$  still stands for a type definable group of type  $\pi$  and base  $B$  in a structure  $M$  that does not the independence property. We fix  $\mathbf{M}$  a  $|G|^+$ -saturated elementary extension of  $M$  and we write  $\mathbf{G}$  for  $\bigcap \pi(\mathbf{M})$ .

**Definition 5.17** (external subgroup). A subgroup  $H \leq G$  is an *external  $P$ -subgroup* of  $G$  if there is a relatively definable subgroup  $\mathbf{H} \leq \mathbf{G}$  (a *witness*) having property  $P$  such that  $H = \mathbf{H} \cap G$ . A family of external subgroups is *uniform* if the corresponding family of relatively definable witnesses have a common defining formula and share a common second base.

**Lemma 5.18.** *The conjunction of a uniform family  $\mathfrak{H}$  of external subgroups of  $G$  is an external subgroup of  $G$ , and the family  $\{\bigcap_{H \in \mathfrak{K}} H : \mathfrak{K} \subset \mathfrak{H}\}$  is uniform.*

*Proof.* Let  $(\mathbf{a}_i)_{i \in I}$  be a family of parameters of  $\mathbf{M}$  such that the types  $\pi \cup \{\varphi(x, \mathbf{a}_i)\}$  define a uniform family of relatively definable subgroups of  $\mathbf{G}$  with second base  $X = \psi(\mathbf{M})$ . Let  $H$  be the trace over  $G$  of their conjunction. Assume that  $\forall x \forall y (\varphi(x, y) \leftrightarrow \varphi(x^{-1}, y))$  holds. By Lemma 5.13, we may replace the formula  $\varphi(x, y)$  by a finite conjunction of formulas  $\varphi(x, y_1), \dots, \varphi(x, y_n)$  intersected with a symmetric definable  $n$ th root of  $X$  containing  $G$ , and assume that  $\{\varphi(x, \mathbf{a}_i) : i \in I\}$  is filter. It follows that the type

$$\rho(y) = \left\{ \forall x (\varphi(x, y) \rightarrow \varphi(x, a_i)), \varphi(h, y), \forall x \left( (\varphi(x, y) \cap \psi(x))^2 \rightarrow \varphi(x, y) \right) : i \in I, h \in H \right\}$$

is finitely satisfiable in  $\{a_i : i \in I\}$ . For any of its realisations  $\mathbf{a}$  in  $\mathbf{M}$ , the type  $\pi \cup \{\varphi(x, \mathbf{a})\}$  defines a relatively definable subgroup of  $\mathbf{G}$  with  $\varphi(M, \mathbf{a}) \cap G = H$ .  $\square$

**Definition 5.19** (discernible subgroup). A subgroup  $H \leq G$  is a *discernible  $P$ -subgroup* of  $G$  if there is a uniformly relatively nice  $P$ -subgroup  $\mathbf{H} \leq \mathbf{G}$  (a *witness*) such that  $H = \mathbf{H} \cap G$ . A family of discernible subgroups of  $G$  is *uniform* if the corresponding family of uniformly relatively nice witnesses is uniform.

**Lemma 5.20.** *The conjunction of a uniform family  $\mathfrak{H}$  of discernible subgroups of  $G$  is a discernible subgroup of  $G$ , and the family  $\{\bigcap_{H \in \mathfrak{K}} H : \mathfrak{K} \subset \mathfrak{H}\}$  is uniform.*

*Proof.* By Lemma 5.8.  $\square$

**Theorem 5.21.** *Let us assume that  $M$  does not have the independence property and that  $G \subset M^n$  is a type definable group in  $M$ .*

- (1) *The centralisers of subsets of  $G$  are (a uniform family of) external subgroups.*
- (2) *The iterated centers of an external subgroup are external nilpotent subgroups.*
- (3) *The iterated centers of a discernible subgroup are discernible nilpotent subgroups.*
- (4) *The cores of an external subgroup are (a uniform family of) external subgroups.*
- (5) *The cores of a discernible subgroup are (a uniform family of) discernible subgroups.*
- (6) *The normaliser of a discernible subgroup is the trace over  $G$  of a type definable subgroup of  $\bigcap \pi(\mathbf{M})$  for some  $|G|^+$ -saturated elementary extension  $\mathbf{M} \succ M$ .*

*Proof.* (1) By Example 5.9 and Lemma 5.18.

(2) Similar to the proof of Theorem 3.8.3 using the following claim instead of Claim 2.

**Claim 6.** *Let  $A$  and  $B$  be two subgroups of  $G$  with  $|A| < |G|$  and  $|B| < |G|$ . Let  $\mathbf{D}$  be a relatively definable subgroup of  $\mathbf{G}$  normalised by both  $A$  and  $B$  such that  $[A, B] \subset \mathbf{D}$ . Assume that  $A$  and  $B$  are contained in a relatively definable subgroup  $\mathbf{N}_{\mathbf{D}}$  of  $\mathbf{G}$  that normalises  $\mathbf{D}$ . Then there are two relatively definable subgroups  $\mathbf{A}$  and  $\mathbf{B}$  of  $\mathbf{G}$  containing  $A$  and  $B$  respectively and such that  $[\mathbf{A}, \mathbf{B}] \subset \mathbf{D}$ .*

*Proof of Claim 6.* Similar to the proof of Claim 2, defining

$$\mathbf{A}(C) = \bigcap_{\substack{c \in C \\ [A, c] \subset \mathbf{D}}} \{x \in \mathbf{N}_{\mathbf{D}} : [x, c] \subset \mathbf{D}\}, \quad \mathbf{B}(C) = \bigcap_{\substack{c \in C \\ [c, B] \subset \mathbf{D}}} \{y \in \mathbf{N}_{\mathbf{D}} : [c, y] \subset \mathbf{D}\},$$

and using Corollary 5.4.  $\square$

(3) Similar to Theorem 3.8.3, using the following claim instead of Claim 2.



**Claim 7.** *Let  $A$  and  $B$  be two subgroups of  $G$  with  $|A| < |G|$  and  $|B| < |G|$ . Let  $\mathbf{D}$  be a uniformly relatively nice subgroup of  $\mathbf{G}$  such that  $[A, B] \subset \mathbf{D}$ . Assume that  $A$  and  $B$  are contained in a uniformly relatively nice subgroup  $\mathbf{N}_{\mathbf{D}}$  of  $\mathbf{G}$  that uniformly normalises  $\mathbf{D}$ . Then there are two uniformly relatively nice subgroups  $\mathbf{A}$  and  $\mathbf{B}$  of  $\mathbf{G}$  containing  $A$  and  $B$  respectively and such that  $[\mathbf{A}, \mathbf{B}] \subset \mathbf{D}$ .*

*Proof of Claim 7.* As in Claim 2, there exists a finite parameter set  $C \subset \mathbf{N}_{\mathbf{D}}$  such that, defining for any parameter subset  $S \subset \mathbf{G}$

$$\mathbf{A}(S) = \bigcap_{\substack{c \in S \\ [A, c] \subset \mathbf{D}}} \{x \in \mathbf{N}_{\mathbf{D}} : [x, c] \subset \mathbf{D}\} \quad \text{and} \quad \mathbf{B}(S) = \bigcap_{\substack{c \in S \\ [c, B] \subset \mathbf{D}}} \{y \in \mathbf{N}_{\mathbf{D}} : [c, y] \subset \mathbf{D}\},$$

one has  $[\mathbf{A}(A \cup B \cup C), \mathbf{B}(A \cup B \cup C)] \subset \mathbf{D}$ . Let  $\{\varphi(x, a_i) : i \in I\}$  be the type that defines  $\mathbf{D}$  relatively to  $\mathbf{G}$  and let  $\mathbf{X}$  be a base for  $\mathbf{N}_{\mathbf{D}}$ . Let us show that, for any subset  $S \subset \mathbf{G}$ , putting

$$\mathbf{H}_i(s) = \{g \in \mathbf{X} : [g, s] \subset \varphi(\mathbf{M}, a_i)\},$$

the family  $\mathfrak{H} = \{\bigcap_{i \in I} \mathbf{N}_{\mathbf{D}} \cap \mathbf{H}_i(s) : s \in S\}$  is a uniform family of uniformly relatively nice subgroups in  $\mathbf{N}_{\mathbf{D}}$ . By Lemma 5.8, this will show that  $\mathbf{A}(A \cup B \cup C)$  and  $\mathbf{B}(A \cup B \cup C)$  are uniformly relatively nice in  $\mathbf{N}_{\mathbf{D}}$ , hence in  $\mathbf{G}$ . We need only find a common second base for the members of  $\mathfrak{H}$ . As  $\mathbf{D}$  is uniformly relatively nice in  $\mathbf{G}$ , there is a definable set  $\mathbf{Y}$  containing  $\mathbf{G}$  such that for all  $i \in I$  there is  $j \in I$  with

$$(\mathbf{Y} \cap \varphi(\mathbf{M}, a_j))^2 \subset \varphi(\mathbf{M}, a_i).$$

As  $\mathbf{D}$  is uniformly normal in  $\mathbf{N}_{\mathbf{D}}$ , there is a definable subset  $\mathbf{Z} \subset \mathbf{X}$  containing  $\mathbf{N}_{\mathbf{D}}$  such that for all  $j \in I$ , there is  $k \in I$  such that for all  $g \in \mathbf{Z}$ ,

$$(\mathbf{Z} \cap \varphi(\mathbf{M}, a_k))^g \subset \varphi(\mathbf{M}, a_j).$$

There are also definable sets  $\mathbf{X}^{1/2}$ ,  $\mathbf{Y}^{1/6}$  and  $\mathbf{Z}^{1/4}$  containing  $\mathbf{N}_{\mathbf{D}}$  with

$$(\mathbf{X}^{1/2})^2 \subset \mathbf{X}, \quad (\mathbf{Y}^{1/6})^6 \subset \mathbf{Y} \quad \text{and} \quad (\mathbf{Z}^{1/4})^4 \subset \mathbf{Z}.$$

We put  $\mathbf{W} = \mathbf{X}^{1/2} \cap \mathbf{Y}^{1/6} \cap \mathbf{Z}^{1/4}$  and claim that  $(\mathbf{W} \cap \mathbf{H}_k(s))^2 \subset \mathbf{H}_i(s)$  holds.

For any  $g$  and  $h$  in  $\mathbf{W} \cap \mathbf{H}_k(s)$ , we have

$$[g, s]^h \in \left( (\mathbf{Z}^{1/4})^4 \cap \varphi(\mathbf{M}, a_k) \right)^h \quad \text{and} \quad [h, s] \in (\mathbf{Z}^{1/4})^4 \cap \varphi(\mathbf{M}, a_k),$$

so that

$$[g, s]^h \in \varphi(\mathbf{M}, a_j) \quad \text{and} \quad [h, s] \in \varphi(\mathbf{M}, a_j),$$

hence

$$[gh, s] = [g, s]^h [h, s] \in \left( (\mathbf{Y}^{1/6})^6 \cap \varphi(\mathbf{M}, a_j) \right)^2 \subset \varphi(\mathbf{M}, a_i).$$

This shows that  $gh$  belongs to  $\mathbf{H}_i(s)$  and that  $\mathbf{W}$  is the desired second base.  $\square$

(4) By Lemma 5.12 and Lemma 5.18.

(5) By Lemma 5.12 and Lemma 5.8.

(6) By Theorem 5.16  $\square$

## 6. ENVELOPES IN TYPE DEFINABLE GROUPS

We consider a  $\pi$ -definable group  $G$  of base  $D$  in a structure  $M$  that does not have the independence property, and look for type definable envelopes around an Abelian, nilpotent or soluble subgroup of  $G$ . We fix a  $|G|^+$ -saturated extension  $\mathbf{M}$  of  $M$  and write  $\mathbf{G}$  for  $\cap \pi(\mathbf{M})$ .

**Theorem 6.1** (Abelian envelope). *Any Abelian subgroup  $A$  of  $G$  is contained in an  $A$ -invariant external Abelian subgroup of  $G$  that is normalised by  $N_G(A)$*

*Proof.* The  $A$ -invariant subgroup  $Z(C_G(A))$  contains  $A$  and is normalised by  $N_G(A)$ . It is an external Abelian subgroup by Theorem 5.21.1 and 5.21.2.  $\square$

**Theorem 6.2** (nilpotent envelope). *Any nilpotent subgroup of  $G$  of class  $n$  is contained in an  $N$ -invariant external nilpotent of class  $n$  subgroup of  $G$  that is normalised by  $N_G(N)$ .*

*Proof.* We build inductively on  $k \leq n$  a chain  $\mathbf{Z}_0 \triangleleft \cdots \triangleleft \mathbf{Z}_k$  of relatively definable subgroups of  $\mathbf{G}$  such that

$$\mathbf{Z}_0 = \{1\}, \quad Z_k(N) \subset \mathbf{Z}_k \quad \text{and} \quad [\mathbf{Z}_k, N] \subset \mathbf{Z}_{k-1}.$$

If  $\mathbf{Z}_k$  is built, as one has

$$[Z_{k+1}(N), N] \subset Z_k(N) \subset \mathbf{Z}_k,$$

and as  $N$  is contained in the subgroup  $C_{\mathbf{G}}(\mathbf{Z}_k/\mathbf{Z}_{k-1})$  which normalises  $\mathbf{Z}_k$ , by Claim 6, there is a relatively definable subgroup  $\mathbf{Z}_{k+1}$  of  $\mathbf{G}$  containing  $Z_{k+1}(N)$  such that

$$[\mathbf{Z}_{k+1}, N] \subset \mathbf{Z}_k.$$

For every  $k \leq n$ , the group

$$\mathbf{H}_k = \{x \in \mathbf{Z}_n : [\mathbf{Z}_k, x] \subset \mathbf{Z}_{k-1}\}$$

is external by Lemma 5.18, and the group  $\mathbf{H}_1 \cap \cdots \cap \mathbf{H}_n$  is a nilpotent group of class  $n$  that contains  $N$ .  $\square$

**Theorem 6.3** (soluble envelope). *Let  $S$  be a soluble subgroup of  $G$  of derived length  $\ell$ . There is a type definable subgroup  $\mathbf{N}$  of  $\mathbf{G}$  containing  $N_G(S)$  and a uniformly relatively nice and uniformly normal subgroup  $\mathbf{H}$  of  $\mathbf{N}$  that contains  $S$  and is soluble of derived length  $\ell$ . More precisely,  $\mathbf{H}$  is the intersection of a uniform family of relatively definable subgroups of  $\mathbf{N}$ .*

*Proof.* We consider the derived series  $S \triangleright S^{(1)} \triangleright \cdots \triangleright S^{(\ell)}$ , we call  $\mathbf{H}_1$  a relatively definable Abelian subgroup of  $\mathbf{G}$  that contains  $S^{(\ell-1)}$  and that is normalised by  $N_G(S)$ , we write  $\mathbf{N}_1$  for a type definable subgroup of  $\mathbf{G}$  that normalises  $\mathbf{H}_1$  and contains  $N_G(S)$  and we build inductively on  $k \leq \ell$  two chains  $1 \triangleleft \mathbf{H}_1 \triangleleft \cdots \triangleleft \mathbf{H}_k$  and  $\mathbf{N}_1, \dots, \mathbf{N}_k$  of subgroups of  $\mathbf{G}$  such that for every  $k \leq \ell$ , the group  $\mathbf{N}_k$  is a type definable subgroup of  $\mathbf{G}$  that contains  $N_G(S)$  and  $\mathbf{H}_k$  is a uniformly relatively nice and uniformly normal subgroup of  $\mathbf{N}_k$  that satisfies

$$S^{(\ell-k)} \subset \mathbf{H}_k \quad \text{and} \quad [\mathbf{H}_k, \mathbf{H}_k] \subset \mathbf{H}_{k-1}.$$

If  $\mathbf{N}_k$  and  $\mathbf{H}_k$  are built, one has

$$[S^{(\ell-k-1)}, S^{(\ell-k-1)}] \subset S^{(\ell-k)} \subset \mathbf{H}_k.$$

By Claim 7, there is a uniformly relatively nice subgroup  $\mathbf{K}_{k+1}$  of  $\mathbf{N}_k$  such that

$$S^{(\ell-k-1)} \subset \mathbf{K}_{k+1} \quad \text{and} \quad [\mathbf{K}_{k+1}, \mathbf{K}_{k+1}] \subset \mathbf{H}_k.$$

By Theorem 5.16 there is a type definable subgroup  $\mathbf{N}_{k+1}$  of  $\mathbf{G}$  that contains  $N_G(S)$  and a uniformly relatively nice subgroup  $\mathbf{H}_{k+1}$  of  $\mathbf{K}_{k+1}$  that contains  $S^{(\ell-k-1)}$  and is uniformly normalised by  $\mathbf{N}_{k+1}$ . We put

$$\mathbf{N} = \mathbf{N}_1 \cap \cdots \cap \mathbf{N}_\ell \quad \text{and} \quad \mathbf{H} = \mathbf{H}_\ell.$$

□

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