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# Fractional coloring of triangle-free planar graphs\*

Zdeněk Dvořák<sup>†</sup>    Jean-Sébastien Sereni<sup>‡</sup>    Jan Volec<sup>§</sup>

## Abstract

We prove that every planar triangle-free graph on  $n$  vertices has fractional chromatic number at most  $3 - \frac{1}{n+1/3}$ .

## 1 Introduction

Coloring of triangle-free planar graphs is an attractive topic. It started with Grötzsch's theorem [7], stating that such graphs are 3-colorable. Since then, several simpler proofs have been given, e.g., by Thomassen [13, 14]. Algorithmic questions have also been addressed: while most proofs readily yield quadratic algorithms to 3-color such graphs, it takes considerably more effort to obtain asymptotically faster algorithms. Kowalik [11] proposed an algorithm running in time  $O(n \log n)$ , which relies on the design of an advanced data structure. More recently, Dvořák *et al.* [2] managed to obtain a linear-time algorithm, yielding at the same time a yet simpler proof of Grötzsch's theorem.

The fact that all triangle-free planar graphs admit a 3-coloring implies that all such graphs have an independent set containing at least one third of the vertices. Albertson *et al.* [1] had conjectured that there is always a larger independent set, which was confirmed by Steinberg and Tovey [12] even in a stronger sense: all triangle-free planar  $n$ -vertex graphs admit a 3-coloring

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where not all color classes have the same size, and thus at least one of them forms an independent set of size at least  $\frac{n+1}{3}$ . This bound turns out to be tight for infinitely many triangle-free graphs, as Jones [9] showed. As an aside, let us mention that the graphs built by Jones have maximum degree 4: this is no coincidence as Heckman and Thomas later established that all triangle-free planar  $n$ -vertex graphs with maximum degree at most 3 have an independent set of order at least  $\frac{3n}{8}$ , which again is a tight bound—actually attained by planar graphs of girth 5.

All these considerations naturally lead to investigate the fractional chromatic number of triangle-free planar graphs. Indeed, as we shall later see, this invariant actually corresponds to a weighted version of the independence ratio. In addition, since  $\chi_f(G) \leq \chi(G)$  for every graph  $G$ , Grötzsch's theorem implies that  $\chi_f(G) \leq 3$  whenever  $G$  is triangle-free and planar. On the other hand, Jones's construction shows the existence of triangle-free planar graphs with fractional chromatic number arbitrarily close to 3. Thus one wonders whether there exists a triangle-free planar graph with fractional chromatic number exactly 3. Let us note that this happens for the circular chromatic number  $\chi_c$ , which is a different relaxation of ordinary chromatic number such that  $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$ .

The purpose of this work is to answer this question. We do so by establishing the following upper bound on the fractional chromatic number of triangle-free planar  $n$ -vertex graphs, which depends on  $n$ .

**Theorem 1.** *Every planar triangle-free graph on  $n$  vertices has fractional chromatic number at most  $3 - \frac{1}{n+1/3}$ .*

Consequently, no (finite) triangle-free planar graph has fractional chromatic number equal to 3. We also note that the bound provided by Theorem 1 is tight up to the multiplicative factor. Indeed, the aforementioned construction of Jones [9] yields, for each  $n \geq 2$  such that  $n \equiv 2 \pmod{3}$ , a triangle-free planar graph  $G_n$  with  $\alpha(G_n) = \frac{n+1}{3}$ . Consequently,  $\chi_f(G_n) \geq \frac{3n}{n+1} = 3 - \frac{3}{n+1}$ .

Our result can be improved for triangle-free planar graphs with maximum degree at most four, giving an exact bound for such graphs.

**Theorem 2.** *Every planar triangle-free  $n$ -vertex graph of maximum degree at most four has fractional chromatic number at most  $\frac{3n}{n+1}$ .*

Furthermore, the graphs of Jones's construction contain a large number of separating 4-cycles (actually, all their faces have length five). We show that planar triangle-free graphs of *maximum degree 4 and without separating 4-cycles* cannot have fractional number arbitrarily close to 3.

**Theorem 3.** *There exists  $\delta > 0$  such that every planar triangle-free graph of maximum degree at most four and without separating 4-cycles has fractional chromatic number at most  $3 - \delta$ .*

Dvořák and Mnich [5] proved that there exists  $\beta > 0$  such that all planar triangle-free  $n$ -vertex graphs without separating 4-cycles contain an independent set of size at least  $n/(3 - \beta)$ . This gives an evidence that the restriction on the maximum degree in Theorem 3 might not be necessary.

**Conjecture 1.** *There exists  $\delta > 0$  such that every planar triangle-free graph without separating 4-cycles has fractional chromatic number at most  $3 - \delta$ .*

Faces of length four are usually easy to deal with in the proofs by collapsing; thus the following seemingly simpler variant of Conjecture 1 is likely to be equivalent to it.

**Conjecture 2** (Dvořák and Mnich [5]). *There exists  $\delta > 0$  such that every planar graph of girth at least five has fractional chromatic number at most  $3 - \delta$ .*

## 2 Notation and auxiliary results

Let  $\mu$  be the Lebesgue measure on real numbers. Let  $G$  be a graph. If a function  $\varphi$  assigns to each vertex of  $G$  a measurable subset of  $[0, 1]$  and  $\varphi(u) \cap \varphi(v) = \emptyset$  for all edges  $uv$  of  $G$ , we say that  $\varphi$  is a *fractional coloring* of  $G$ . Let  $f: V(G) \rightarrow \mathbb{Q} \cap [0, 1]$  be a function with rational values. If the fractional coloring  $\varphi$  satisfies  $\mu(\varphi(v)) \geq f(v)$  for every  $v \in V(G)$ , then we say that  $\varphi$  is an  *$f$ -coloring* of  $G$ . If  $\mu(\varphi(v)) = f(v)$  for every  $v \in V(G)$ , then we say that  $\varphi$  is a *tight  $f$ -coloring*. Note that if  $G$  has an  $f$ -coloring, then it also has a tight one. For  $x \in \mathbb{Q} \cap [0, 1]$ , let  $c_x$  denote the constant function assigning the value  $x$  to each vertex of  $G$ . The *fractional chromatic number* of  $G$  is defined as

$$\chi_f(G) = \frac{1}{\sup \{x \in \mathbb{Q} \cap [0, 1] : G \text{ has a } c_x\text{-coloring}\}}.$$

Let  $w: V(G) \rightarrow \mathbf{R}^+$  be an arbitrary function. For a set  $X \subseteq V(G)$ , by  $w(X)$  we mean  $\sum_{v \in X} w(v)$ . Let  $w(f) = \sum_{v \in V(G)} f(v)w(v)$ . An integer  $N \geq 1$  is a *common denominator* of  $f$  if  $Nf(v)$  is an integer for every  $v \in V(G)$ . Setting  $[N] = \{1, \dots, N\}$ , a function  $\psi: V(G) \rightarrow \mathcal{P}([N])$  is an  *$(f, N)$ -coloring* of  $G$  if  $\psi(u) \cap \psi(v) = \emptyset$  for every  $uv \in E(G)$  and  $|\psi(v)| \geq Nf(v)$  for every  $v \in V(G)$ . The  $(f, N)$ -coloring is *tight* if  $|\psi(v)| = Nf(v)$  for every  $v \in V(G)$ .

The fractional chromatic number of a graph can be expressed in various equivalent ways, based on its well known linear programming formulation and duality. The proof of the following lemma can be found e.g. in Dvořák *et al.* [6, Theorem 2.1].

**Lemma 4.** *Let  $G$  be a graph and  $f: V(G) \rightarrow Q \cap [0, 1]$  a function. The following statements are equivalent.*

- *The graph  $G$  has an  $f$ -coloring.*
- *There exists a common denominator  $N$  of  $f$  such that  $G$  has an  $(f, N)$ -coloring.*
- *For every  $w: V(G) \rightarrow \mathbf{R}^+$ , there exists an independent set  $X \subseteq V(G)$  with  $w(X) \geq w(f)$ .*

We need several results related to Grötzsch's theorem. The following lemma was proved for vertices of degree at most three by Steinberg and Tovey [12]. The proof for vertices of degree four follows from the results of Dvořák and Lidický [4], as observed by Dvořák *et al.* [3].

**Lemma 5.** *If  $G$  is a triangle-free planar graph and  $v$  is a vertex of  $G$  of degree at most four, then there exists a 3-coloring of  $G$  such that all neighbors of  $v$  have the same color.*

In fact, Dvořák *et al.* [3] proved the following stronger statement.

**Lemma 6.** *There exists an integer  $D \geq 4$  with the following property. Let  $G$  be a triangle-free planar graph without separating 4-cycles and let  $X$  be a set of vertices of  $G$  of degree at most four. If the distance between every two vertices in  $X$  is at least  $D$ , then there exists a 3-coloring of  $G$  such that all neighbors of vertices of  $X$  have the same color.*

Let  $G$  be a triangle-free plane graph. A 5-face  $f = v_1v_2v_3v_4v_5$  of  $G$  is *safe* if  $v_1, v_2, v_3$  and  $v_4$  have degree exactly three, their neighbors  $x_1, \dots, x_4$  (respectively) not incident with  $f$  are pairwise distinct and non-adjacent, and

- the distance between  $x_2$  and  $v_5$  in  $G - \{v_1, v_2, v_3, v_4\}$  is at least four, and
- $G - \{v_1, v_2, v_3, v_4\}$  contains no path of length exactly three between  $x_3$  and  $x_4$ .

**Lemma 7** (Dvořák *et al.* [2, Lemma 2.2]). *If  $G$  is a plane triangle-free graph of minimum degree at least three and all faces of  $G$  have length five, then  $G$  has a safe face.*

Finally, let us recall the folding lemma, which is frequently used in the coloring theory of planar graphs.

**Lemma 8** (Klostermeyer and Zhang [10]). *Let  $G$  be a planar graph with odd-girth  $g > 3$ . If  $C = v_0v_1 \dots v_{r-1}$  is a facial circuit of  $G$  with  $r \neq g$ , then there is an integer  $i \in \{0, \dots, r-1\}$  such that the graph  $G'$  obtained from  $G$  by identifying  $v_{i-1}$  and  $v_{i+1}$  (where indices are taken modulo  $r$ ) is also of odd-girth  $g$ .*

### 3 Proofs

First, let us show a lemma based on the idea of Hilton *et al.* [8].

**Lemma 9.** *Let  $G$  be a planar triangle-free graph and let  $w : V(G) \rightarrow \mathbf{R}^+$  be an arbitrary function. If  $v \in V(G)$  has degree at most 4, then  $G$  contains an independent set  $X$  such that  $w(X) \geq \frac{w(V(G))+w(v)}{3}$ .*

*Proof.* Lemma 5 implies that there exists a 3-coloring of  $G$  such that all neighbors of  $v$  have the same color. Consequently,  $G$  has an  $f_v$ -coloring for the function  $f_v$  such that  $f_v(z) = 1/3$  for  $z \in V(G) \setminus \{v\}$  and  $f_v(v) = 2/3$ . By Lemma 4, there exists an independent set  $X \subseteq V(G)$  such that  $w(X) \geq w(f_v) = \frac{w(V(G))+w(v)}{3}$ .  $\square$

Theorem 2 now readily follows.

*Proof of Theorem 2.* Let  $G$  be a planar triangle-free  $n$ -vertex graph of maximum degree at most four. Consider any function  $w : V(G) \rightarrow \mathbf{R}^+$ , and let  $v$  be the vertex to which  $w$  assigns the maximum value. We have  $w(v) \geq w(V(G))/n$ . By Lemma 9, there exists an independent set  $X$  such that  $w(X) \geq \frac{w(V(G))+w(v)}{3} \geq \frac{n+1}{3n}w(V(G))$ . Therefore, for every  $w : V(G) \rightarrow \mathbf{R}^+$ , there exists an independent set  $X$  with  $w(X) \geq w(c_{(n+1)/(3n)})$ . By Lemma 4, it follows that the fractional chromatic number of  $G$  is at most  $\frac{3n}{n+1}$ .  $\square$

Similarly, Lemma 6 implies Theorem 3.

*Proof of Theorem 3.* Let  $D$  be the constant of Lemma 6, let  $\delta_0 = \frac{1}{3 \cdot 4^D}$  and  $\delta = \frac{9\delta_0}{3\delta_0+1} = \frac{3}{4^D+1}$ . Let  $G$  be a planar triangle-free graph of maximum degree

at most four and without separating 4-cycles. Clearly, it suffices to prove that  $G$  has a  $c_{1/3+\delta_0}$ -coloring.

Let  $G'$  be the graph obtained from  $G$  by adding edges between all pairs of vertices at distance at most  $D - 1$ . The maximum degree of  $G'$  is less than  $4^D$ , and thus  $G'$  has a coloring by at most  $4^D$  colors. Let  $C_1, \dots, C_{4^D}$  be the color classes of this coloring. For  $i \in \{1, \dots, 4^D\}$ , let  $f_i$  be the function defined by  $f_i(v) = 2/3$  for  $v \in C_i$  and  $f_i(v) = 1/3$  for  $v \in V(G) \setminus C_i$ . Lemma 6 ensures that  $G$  has an  $f_i$ -coloring.

Consider any function  $w : V(G) \rightarrow R^+$ . There exists  $i \in \{1, \dots, 4^D\}$  such that  $w(C_i) \geq w(V(G))/4^D$ . By Lemma 4 applied for  $f_i$ , we conclude that  $G$  contains an independent set  $X$  such that  $w(X) \geq w(f_i) = \frac{w(V(G))+w(C_i)}{3} \geq (1/3 + \delta_0)w(V(G)) = w(c_{1/3+\delta_0})$ . Since the choice of  $w$  was arbitrary, Lemma 4 implies that  $G$  has a  $c_{1/3+\delta_0}$ -coloring.  $\square$

The proof of Theorem 1 is somewhat more involved. Let  $\varepsilon = 1/9$  and for  $n \geq 1$ , let  $b(n) = 1/3 + \varepsilon/n$ . Let  $G$  be a plane triangle-free graph. We say that  $G$  is a *counterexample* if  $G$  does not have a  $c_{b(|V(G)|)}$ -coloring. We say that  $G$  is a *minimal counterexample* if  $G$  is a counterexample and no plane triangle-free graph with fewer than  $|V(G)|$  vertices is a counterexample. Since  $b$  is a decreasing function, every minimal counterexample is connected.

**Lemma 10.** *If  $G$  is a minimal counterexample, then  $G$  is 2-connected. Consequently, the minimum degree of  $G$  is at least two.*

*Proof.* Since  $b(n) \leq 1/2$ , every counterexample has at least three vertices; hence, it suffices to prove that  $G$  is 2-connected, and the bound on the minimum degree will follow. Let  $n$  be the number of vertices of  $G$ .

Suppose that  $G$  is not 2-connected, and let  $G_1$  and  $G_2$  be subgraphs of  $G$  such that  $G = G_1 \cup G_2$ , the graph  $G_1$  intersects  $G_2$  in exactly one vertex  $v$ , and both  $n_1 = |V(G_1)|$  and  $n_2 = |V(G_2)|$  are greater than 1. Since  $n = n_1 + n_2 - 1$ , we have  $n_1, n_2 < n$ , and thus neither  $G_1$  nor  $G_2$  is a counterexample. Consequently,  $G_i$  has a  $c_{b(n_i)}$ -coloring for  $i \in \{1, 2\}$ . Since  $b$  is a decreasing function, we deduce that  $G_i$  has a  $c_{b(n)}$ -coloring and hence, by Lemma 4, there exists  $N \geq 1$  such that  $G_i$  has a  $(c_{b(n)}, N)$ -coloring  $\varphi_i$ . By permuting the colors if necessary, we can assume that  $\varphi_1(v) = \varphi_2(v)$ , and thus  $\varphi_1 \cup \varphi_2$  is a  $(c_{b(n)}, N)$ -coloring of  $G$ . This contradicts the assumption that  $G$  is a counterexample.  $\square$

**Lemma 11.** *If  $G$  is a minimal counterexample, then every face of  $G$  has length exactly 5.*

*Proof.* Let  $n$  be the number of vertices of  $G$ . Suppose that  $G$  has a face  $f$  of length other than 5. Since  $G$  is triangle-free, it has odd girth at least five, and by Lemma 8, there exists a path  $v_1v_2v_3$  in the boundary of  $f$  such that the graph  $G'$  obtained by identifying  $v_1$  with  $v_3$  to a single vertex  $z$  has odd girth at least five as well. It follows that  $G'$  is triangle-free. Since  $G$  is a minimal counterexample,  $G'$  has a  $c_{b(n-1)}$ -coloring, and by giving both  $v_1$  and  $v_3$  the color of  $z$ , we obtain a  $c_{b(n-1)}$ -coloring of  $G$ . Since  $b(n) < b(n-1)$ , this contradicts the assumption that  $G$  is a counterexample.  $\square$

Given a counterexample  $G$  on  $n$  vertices, a function  $w: V(G) \rightarrow \mathbf{R}^+$  is a *witness* if  $G$  has no independent set  $X$  satisfying  $w(X) \geq w(c_{b(n)})$ . By Lemma 4, every counterexample has a witness. Let us now state a useful special case of Lemma 9.

**Lemma 12.** *If  $G$  is a counterexample on  $n$  vertices,  $w$  is a witness and  $v \in V(G)$  has degree at most three, then  $w(v) < 3\epsilon w(V(G))/n$ .*

*Proof.* Let  $n$  be the number of vertices of  $G$ . By Lemma 9, there exists an independent set  $X \subseteq V(G)$  with  $w(X) \geq \frac{w(V(G)) + w(v)}{3}$ . On the other hand, since  $w$  is a witness, we have  $w(X) < w(c_{b(n)}) = \frac{w(V(G))}{3} + \frac{\epsilon}{n}w(V(G))$ . The claim of this lemma follows.  $\square$

**Lemma 13.** *If  $G$  is a minimal counterexample, then  $G$  has minimum degree at least three.*

*Proof.* Let  $n$  be the number of vertices of  $G$  and let  $w: V(G) \rightarrow \mathbf{R}^+$  be a witness for  $G$ . By Lemma 10, the graph  $G$  has minimum degree at least two. Suppose that  $v \in V(G)$  has degree two. By Lemma 12, we have  $w(v) < 3\epsilon w(V(G))/n$ .

Since  $G$  is a minimal counterexample, there exists  $N \geq 1$  and a tight  $(c_{b(n-1)}, N)$ -coloring  $\psi$  of  $G - v$ . Let  $f(x) = b(n-1)$  for  $x \in V(G - v)$  and  $f(v) = 1 - 2b(n-1)$ . Clearly,  $\psi$  extends to an  $(f, N)$ -coloring of  $G$ . By



Lemma 4, there exists an independent set  $X \subseteq V(G)$  such that

$$\begin{aligned}
w(X) &\geq w(f) \\
&= b(n-1)w(V(G)) - (3b(n-1) - 1)w(v) \\
&> b(n-1)w(V(G)) - \frac{3(3b(n-1) - 1)\varepsilon}{n}w(V(G)) \\
&= \left[ b(n-1) - \frac{9\varepsilon^2}{n(n-1)} \right] w(V(G)) \\
&= \left[ b(n) + \frac{\varepsilon}{n(n-1)} - \frac{9\varepsilon^2}{n(n-1)} \right] w(V(G)) \\
&= b(n)w(V(G)) = w(c_{b(n)}).
\end{aligned}$$

This contradicts that  $w$  is a witness for  $G$ .  $\square$

**Lemma 14.** *No minimal counterexample contains a safe 5-face.*

*Proof.* Let  $G$  be a minimal counterexample containing a safe 5-face  $f = v_1v_2v_3v_4v_5$ , and let  $x_1, \dots, x_4$  be the neighbors of  $v_1, \dots, v_4$ , respectively, that are not incident with  $f$ . Let  $n$  be the number of vertices of  $G$  and let  $w: V(G) \rightarrow \mathbf{R}^+$  be a witness for  $G$ . By Lemma 12, we have  $w(v_i) < 3\varepsilon w(V(G))/n$  for  $1 \leq i \leq 4$ .

Let  $G'$  be the graph obtained from  $G - \{v_1, v_2, v_3, v_4\}$  by identifying  $x_2$  with  $v_5$  into a new vertex  $u_1$ , and  $x_3$  with  $x_4$  into a new vertex  $u_2$ . Since  $f$  is safe,  $G'$  is triangle-free. Since  $G$  is a minimal counterexample, there exists  $N \geq 1$  and a tight  $(c_{b(n-6)}, N)$ -coloring  $\psi$  of  $G'$ . Let  $f(x) = b(n-6)$  for  $x \in V(G - \{v_1, v_2, v_3, v_4\})$  and  $f(v_i) = 1 - 2b(n-6)$  for  $1 \leq i \leq 4$ . We use  $\psi$  to design an  $(f, N)$ -coloring of  $G$ .

Let  $\psi(x_2) = \psi(v_5) = \psi(u_1)$  and  $\psi(x_3) = \psi(x_4) = \psi(u_2)$ . Let  $\psi(v_1)$  be a subset of  $[N] \setminus (\psi(x_1) \cup \psi(v_5))$  of size  $f(v_1)N$ , and let  $\psi(v_2)$  be a subset of  $[N] \setminus (\psi(x_2) \cup \psi(v_1))$  of size  $f(v_2)N$ . Let  $M_3 = [N] \setminus (\psi(v_2) \cup \psi(x_3))$  and  $M_4 = [N] \setminus (\psi(v_5) \cup \psi(x_4))$ . Note that  $|M_3| \geq f(v_3)N$  and  $|M_4| \geq f(v_4)N$ . Furthermore, since  $\psi(x_3) = \psi(x_4)$  and  $\psi(v_2) \cap \psi(v_5) = \emptyset$ , we have  $|M_3 \cup M_4| = 1 - |\psi(x_3)| = 1 - b(n-6) \geq f(v_3) + f(v_4)$ . Therefore, we can choose disjoint sets  $\psi(v_3) \subseteq M_3$  and  $\psi(v_4) \subseteq M_4$  of size  $f(v_3)N = f(v_4)N$ . This gives an  $(f, N)$ -coloring of  $G$ .

By Lemma 4, there exists an independent set  $X \subseteq V(G)$  such that

$$\begin{aligned}
w(X) &\geq w(f) \\
&= b(n-6)w(V(G)) - (3b(n-6) - 1) \sum_{i=1}^4 w(v_i) \\
&> b(n-6)w(V(G)) - \frac{12(3b(n-6) - 1)\varepsilon}{n} w(V(G)) \\
&= \left[ b(n-6) - \frac{36\varepsilon^2}{n(n-6)} \right] w(V(G)) \\
&= \left[ b(n) + \frac{6\varepsilon}{n(n-6)} - \frac{36\varepsilon^2}{n(n-6)} \right] w(V(G)) \\
&\geq b(n)w(V(G)) = w(c_{b(n)}).
\end{aligned}$$

This contradicts that  $w$  is a witness for  $G$ . □

We can now establish Theorem 1.

*Proof of Theorem 1.* Note that  $\frac{1}{3 - \frac{1}{n+1/3}} = b(n)$ . Suppose that there exists a planar triangle-free graph  $G$  on  $n$  vertices with fractional chromatic number greater than  $3 - \frac{1}{n+1/3}$ . Then  $G$  has no  $c_{b(n)}$ -coloring, and thus  $G$  is a counterexample. Therefore, there exists a minimal counterexample  $G_0$ . Lemmas 13, 11 and 7 imply that  $G_0$  has a safe 5-face. However, that contradicts Lemma 14. □

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