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GRAPH THEORY AND PFAFFIAN REPRESENTATIONS OF ISING PARTITION FUNCTION.

THIERRY GOBRON

ABSTRACT. A well known theorem due to Kasteleyn states that the partition function of an Ising model on an arbitrary planar graph can be represented as the Pfaffian of a skew-symmetric matrix associated to the graph. This result both embodies the free fermionic nature of any planar Ising model and eventually gives an effective way of computing its partition functions in closed form. An extension of this result to non planar models expresses the partition function as a sum of Pfaffians whose number is related to the genus of the oriented surface on which the graph can be embedded.

In graph theory, McLane's theorem (1937) gives a characterization of planarity as a property of the cycle space of a graph, and recently, Diestel et al. (2009) extended this approach to embeddings in arbitrary surfaces.

Here we show that McLane's approach naturally leads to Kasteleyn's results: McLane's characterization of planar graphs is just what is needed to turn an Ising partition function into a Pfaffian.

Using this approach, we prove that the Ising partition function on an arbitrary non planar graph can be written as *the real part* of the Pfaffian of a single matrix with coefficients taken in a multicomplex algebra $\mathbb{C}_{\tilde{g}}$, where \tilde{g} is the non-orientable genus, or crosscap number, of the embedding surface.

Known representations as sums of Pfaffians follow from this result. In particular, Kasteleyn's result which involves 4^g matrices with real coefficients, g orientable genus, is also recovered through some algebraic reduction.

1. INTRODUCTION

A few years after Onsager's solution of the Ising model on a square lattice [1], an alternative, combinatorial method has been elaborated through the pioneering works of Kac and Ward [2], Potts and Ward [3] and Hurst and Green [4]. In these works, the evaluation of Ising partition function on a rectangular array was reduced to a combinatorial enumeration of perfect matchings (dimer coverings in the physics literature), leading to an expression in terms of the Pfaffian of a related skew-symmetric matrix. This method acquired a deeper signification about fifty years ago, when Fisher [5] and Kasteleyn [6, 7, 8] showed that it applies equally to an arbitrary planar graph.

Numerous attempts have been made to generalize these results to non-planar graphs. Kasteleyn's landmark result [8] states that the partition function can be written as a sum of Pfaffians, whose number grows exponentially with the orientable genus of the graph. This statement has been given a better mathematical basis only rather recently [9, 10] and also extended to non orientable surfaces [11]. Finding a sensible lower bound on this number is still an open problem.

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Enumeration of dimer coverings and evaluation of Ising partition function are clearly two deeply connected problems, but it should be stressed that they differ somewhat on this point:

In the dimer problem, one counts the number of dimer coverings on a given graph, or, in other words, the number of subgraphs for which the incidence number is exactly one on every vertex. For bipartite graphs, it reduces to the old permanent-determinant problem [12] and there are well known complexity issues [13, 14]. However the graphs for which the dimer covering enumeration can be turned into a Pfaffian are not necessarily planar [15, 16]. Accordingly, some exact results on nonplanar graphs have been obtained in the context of statistical physics [17, 18].

In the Ising case, one computes a weighted sum over all closed curves of a given graph, that is over all subgraphs with an even incidence number on each vertex. In contradistinction with the dimer problem, the enumeration of these subgraphs on an arbitrary connected graph $G = (V, E)$ is trivial and equal to 2^{β_1} where β_1 is the first Betti number, $\beta_1(G) = |E| - |V| + 1$. The complexity in evaluating the partition function comes from the introduction of a weight function on the edge set and there seems to be no exception to Kasteleyn's planarity rule.

In the present paper, we consider again Kasteleyn's combinatorial approach, giving another glimpse to its graph theoretical foundations.

Our starting point consists in considering the Ising model on an arbitrary graph and derive a set of nonlinear algebraic equations in the entries of an associated skew-symmetric matrix, so that any consistent solution to these equations would lead to a representation of the Ising partition function as a Pfaffian. We thus let aside Kasteleyn's edge orientation method, valid when matrix entries are chosen in $\{-1, 0, 1\}$ and look for a solution with coefficients in \mathbb{R} or \mathbb{C} . Rather surprisingly, the existence (or not) of such a solution derives straightforwardly from a classical planarity criterion [19].

Theorem 1.1. *(MacLane 1937): A Graph is planar if and only if its cycle space admits a basis in which every edge appears at most twice.*

This criterion allows us to prove in our setting that planarity is a necessary and sufficient condition for representing the Ising partition function as a single Pfaffian.

Recently, a generalization of MacLane's criterion to non planar graphs has been considered [20], from which we draw some new results on non planar graphs. Our main result is the following: the Ising partition function on a non planar graph can be written as the real part of the Pfaffian of a single matrix, which coefficients are chosen in a multicomplex algebra $\mathbb{C}_{\tilde{g}}$, where \tilde{g} is the non-orientable genus, or crosscap number, of the embedding surface.

Old and new representations as sums of Pfaffians with real or complex coefficients follow from this result as corollaries. Basically, such sums involve $2^{\tilde{g}}$ terms rather than 4^g (g orientable genus). For orientably simple graphs, $\tilde{g} = 2g + 1$, we recover Kasteleyn's results through some further algebraic reduction. For non orientably simple graphs, $\tilde{g} < 2g + 1$ and our result does improve Kasteleyn's one. However the number of Pfaffians remains exponential in \tilde{g} .

In Section 2, we define our setting and present some preliminary results that we use later in this work; when using terminology and concepts from graph theory, we try to follow standard textbooks such as Harary's [21] or Diestel's [22], to which we refer for a thorough exposition. The Pfaffian reduction formula is known since a long time but we found rather few quotations [23]. It is related, if not equivalent, to the more well-known Plücker-Grassmann relations

between determinants. The connection between minor graphs and Pfaffian reduction formula is essential in the present approach as it allows to a substantial reduction of the algebraic problem at hand. In Section 3, we state our main results. In Section 4, we give the proofs. In Section 5, we introduce a Grassman representation [24, 25] which allows us to state the Ising representation problem in terms of a system of algebraic equations. For completeness, we also provide a proof of the Pfaffian reduction formula.

2. PRELIMINARIES

The problem we want to address is the representation of the Ising partition function on an arbitrary finite graph as the Pfaffian of some related matrix. Hereafter, we recall the first steps in this approach, fix some notations and give precise definitions that will be used to state the results of next section. We conclude this section with some preliminary results.

- Ising Partition Function.

Let $G = (V, E)$ be a finite simple graph. To each vertex $v \in V$, we attach a variable $\sigma_v \in \{-1, 1\}$. Given a collection of real valued interaction terms, $\underline{J} = \{J_e \geq 0, e \in E\}$, we define the inhomogeneous Ising Hamiltonian on the space of configurations $\Omega = \{-1, 1\}^{|V|}$, as

$$(2.1) \quad \mathcal{H}(\underline{\sigma}) = - \sum_{\{x,y\} \in E} J_{x,y} \sigma_x \sigma_y$$

and the associated Ising partition function $Z_{\text{Ising}}^\beta(G, \underline{J})$ as

$$(2.2) \quad Z_{\text{Ising}}^\beta(G, \underline{J}) = \sum_{\underline{\sigma} \in \Omega} \exp\{-\beta \mathcal{H}(\underline{\sigma})\}$$

Note that for clarity of the exposition we consider here only simple graphs so that (un-oriented) edges can be identified with (un-ordered) pairs of vertices. Without loss of generality, the graph we will consider in the sequels are 2-connected simple graphs without loops. In more general cases, the partition function either factorizes or can be trivially rewritten so that one falls back into the previous class of graphs. In the same line of thought, we do not introduce explicitly boundary terms or external fields, which can be considered through a modification of the underlying graph and/or a particular choice of interaction strength on some edges. Finally, we also reduce to ferromagnetic interactions ($J_e \geq 0$), since we want to keep with the most standard concept of weight functions, but dealing with “signed weight functions” would work as well.

The well known high temperature expansion of the Ising partition function leads to an expression of $Z_{\text{Ising}}^\beta(G, \underline{J})$ as a sum over a class of subgraphs of G .

$$(2.3) \quad Z_{\text{Ising}}^\beta(G, \underline{J}) = 2^{|V|} \left(\prod_{e \in E} \cosh(\beta J_e) \right) \sum_{C \in \mathcal{C}(G)} \prod_{e \in C} \tanh(\beta J_e)$$

where $\mathcal{C}(G)$ is the set of closed curves on G , that is the set of all edge subsets $C \subset E$ such that each vertex $v \in V$ is incident with an even number of edges in C .

Note that if one introduce the operation of symmetric difference Δ :

$$(2.4) \quad A \Delta B = (A \setminus B) \cup (B \setminus A) \quad \text{for all sets } A, B$$

then $(\mathcal{C}(G), \Delta)$ is the cycle space of G and has the structure of a vector space over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Under this structure, the set of all cycles on G (more precisely their edge sets) is a generating family for $\mathcal{C}(G)$ [22].

Here we leave aside the physical meaning of (2.3) and restrict our interest to the representation of the sum in the right hand side. In the sequels, we consider the following closely related quantity, which we still call a partition function for obvious reasons.

Definition 2.5. *Let $G = (V, E)$ be a finite, non-oriented graph and $w : E \rightarrow \mathbb{R}^+$ a weight function defined on the edge set.*

We define the partition function of G with weight function w , as

$$(2.6) \quad Z_G(w) = \sum_{C \in \mathcal{C}(G)} w(C)$$

where the weight of any closed curve $C \subset E$ is defined as the product of its edge weights:

$$(2.7) \quad w(C) = \begin{cases} 1 & \text{if } C = \emptyset \\ \prod_{e \in C} w(e) & \text{otherwise} \end{cases}$$

The partition function $Z_G(w)$ is clearly related to the Ising partition function on the same graph. Choosing the weight function as,

$$(2.8) \quad w(e) = \tanh\left(\frac{\beta J_e}{2}\right) \quad \forall e \in E$$

we have the correspondence

$$(2.9) \quad Z_{\text{Ising}}^\beta(G, \underline{J}) = 2^{|V|} \left(\prod_{e \in E} \cosh(\beta J_e) \right) Z_G(w)$$

• Dart graphs and perfect matchings.

The next step consists in mapping the expression of the (Ising) partition function, onto a weighted dimer problem on an auxiliary graph. Such graphs have been originally called “terminal graphs” [8], but are named hereafter *dart graphs* in order to emphasize their relation to other classes of derived graphs, such as the “line graphs” [28]. The choice of this new name stems from the fact that definitions of both classes are identical, just changing edges (or lines) by “half-edges” or darts.

Definition 2.10. *Let $G = (V, E)$ a graph. Its dart graph $\mathcal{D}(G) = (V_{\mathcal{D}}, E_{\mathcal{D}})$ is the simple unoriented graph which vertex set $V_{\mathcal{D}}$ identifies with the set of darts on G :*

$$(2.11) \quad V_{\mathcal{D}} = \left\{ (v, e) \in V \times E \text{ such that } v \text{ is incident with } e \right\}$$

and such that there is an edge between two vertices if and only if they have exactly one common element,

$$(2.12) \quad E_{\mathcal{D}} = \left\{ \{(v, e), (v', e')\} \in V_{\mathcal{D}} \times V_{\mathcal{D}} \text{ such that either } v = v' \text{ or } e = e'\right\}$$

A dart graph has necessarily an even number of vertices and not every graph is the dart graph of some other.

The edge set of a dart graph $\mathcal{D}(G)$ splits in two disjoint subsets,

$$(2.13) \quad E_{\mathcal{D}} = E_{\mathcal{D}}^V \cup E_{\mathcal{D}}^E,$$

$$(2.14) \quad E_{\mathcal{D}}^V = \left\{ \{(v, e), (v', e')\} \in V_{\mathcal{D}} \times V_{\mathcal{D}} \text{ such that } v = v' \text{ and } e \neq e'\right\}$$

$$(2.15) \quad E_{\mathcal{D}}^E = \left\{ \{(v, e), (v', e')\} \in V_{\mathcal{D}} \times V_{\mathcal{D}} \text{ such that } v \neq v' \text{ and } e = e'\right\}$$

If G has no isolated point, the graph $(V_{\mathcal{D}}, E_{\mathcal{D}}^V)$ has $|V|$ connected components, each one being a complete graph. Moreover, there is a one to one correspondence which associates every edge $e \in E$ with $((v, e), (v', e)) \in E_{\mathcal{D}}^E$ so that both sets can be identified

$$(2.16) \quad E_{\mathcal{D}}^E \equiv E$$

As noted by Kasteleyn [8], interest in dart graphs lies in the close connection between the set of closed curves on a graph G , and the perfect matchings on $\mathcal{D}(G)$.

Definition 2.17. *Let $G = (V, E)$ be a graph. A perfect matching (or equivalently a dimer configuration) on G is a subset $E' \subset E$, such that each vertex $v \in V$ is incident with exactly one edge in E' .*

For an arbitrary graph, the existence (or not) of a perfect matching can be proven using Tutte's characterization theorem[15]. On a dart graph, it is a straightforward property:

Proposition 2.18. *Let $G = (V, E)$ be a graph. If it is a dart graph, then it admits at least one perfect matching.*

Under the identification (2.16), the connection between closed curves on a graph G and perfect matchings on its dart graph $\mathcal{D}(G)$ can be expressed as follows:

Proposition 2.19. *Let M_1, M_2 be any pair of perfect matchings on $\mathcal{D}(G)$, the set $C = (M_1 \Delta M_2) \cap E_{\mathcal{D}}^E$ is a closed curve on G .*

Denote by $\mathcal{M}(G)$ the set of perfect matchings on $\mathcal{D}(G)$. Then, given a fixed perfect matching $M_0 \in \mathcal{M}(G)$, one can construct a mapping $\Phi_{M_0} : \mathcal{M}(G) \rightarrow \mathcal{C}(G)$, which associates any perfect matching on $\mathcal{D}(G)$ to a closed curve on G , as

$$(2.20) \quad \Phi_{M_0}(M) = (M_0 \Delta M) \cap E_{\mathcal{D}}^E$$

Φ_{M_0} is clearly surjective, but generally not one-to-one. This induces an equivalence relation on $\mathcal{M}(G)$ as

$$(2.21) \quad M_1 \sim M_2 \iff \Phi_{M_0}(M_1) = \Phi_{M_0}(M_2) \quad \forall M_1, M_2 \in \mathcal{M}$$

On an arbitrary connected graph $G = (V, E)$, the number of closed curves is 2^{β_1} where

$$\beta_1 = |E| - |V| + 1$$

is the first Betti number of G . This is obviously the same as counting the number of equivalence classes in $\mathcal{M}(G)$. These equivalence classes have not necessarily the same number of elements and enumeration of perfect matchings on a dart graph remains non trivial in the general case. A particular exception are the 3-regular graphs, on which the mapping Φ_{M_0} is one-to-one [26].

Using this correspondence, the notion of Pfaffian representation can be given a precise meaning.

• **incidence matrices, weights and Pfaffian representations.**

We introduce a generalized notion of incidence matrix with complex valued coefficients.

Definition 2.22. *Let $G = (V, E)$ a connected graph and $\mathcal{D}(G)$ its dart graph. $A^G \in M_{|V_{\mathcal{D}}(G)|}(\mathbb{C})$ is an incidence matrix on $\mathcal{D}(G)$ if A^G is an antisymmetric matrix which entries are in one-to-one correspondence with the elements of $V_{\mathcal{D}}(G)$, so that*

$$(2.23) \quad \{d_1, d_2\} \notin E_{\mathcal{D}}(G) \implies A_{d_1, d_2}^G = 0$$

Note that the coefficients of the matrix need not be taken in $\{-1, 0, 1\}$, but are arbitrary complex numbers. We reserve the name of weighted incidence matrix to the following.

Definition 2.24. *Let $G = (V, E)$ a connected graph, M_0 a fixed perfect matching on $\mathcal{D}(G)$ and A^G an incidence matrix defined as above. Given a weight function w on $E \equiv E_{\mathcal{D}}^E(G)$ with value in $\mathbb{R}^{+,*}$, the matrix $A^{G, M_0}(w) \in M_{|V_{\mathcal{D}}(G)|}(\mathbb{C})$ with coefficients*

$$(2.25) \quad A_{d_1, d_2}^{G, M_0}(w) = \begin{cases} w^{-1}(\{d_1, d_2\})A_{d_1, d_2}^G & \text{if } \{d_1, d_2\} \in E_{\mathcal{D}}^E(G) \cap M_0 \\ w(\{d_1, d_2\})A_{d_1, d_2}^G & \text{if } \{d_1, d_2\} \in E_{\mathcal{D}}^E(G) \setminus M_0 \\ A_{d_1, d_2}^G & \text{otherwise.} \end{cases}$$

is the weighted incidence matrix associated to (G, M_0, A^G, w) .

We now can state a precise definition of a Pfaffian representation:

Definition 2.26. *Let $G = (V, E)$ a connected simple graph. The partition function on G admits a Pfaffian representation if there exists a perfect matching M_0 on $\mathcal{D}(G)$, an incidence matrix A^G and a constant $\Lambda \neq 0$ such that for all weight functions $w : E \rightarrow \mathbb{R}^+$, one has*

$$(2.27) \quad Z_G(w) = \frac{1}{\Lambda} w(M_0 \cap E_{\mathcal{D}}^E(G)) \text{ Pf}(A^{G, M_0}(w))$$

The main motivation for such a definition is the fact that the Pfaffian of a weighted incidence matrix $A^{G, M_0}(w)$, can be always written as a sum over all closed curves on G , so that the identity (2.27) derives from identification of the coefficients of two similar expansions. Starting from the definition of a Pfaffian and setting $|V_{\mathcal{D}}(G)| = 2n$, $n \in \mathbb{N}$, we have

$$(2.28) \quad \text{Pf}(A^{G, M_0}(w)) = \frac{1}{2^n n!} \sum_{\sigma} (-1)^{|\sigma|} A_{\sigma(d_1), \sigma(d_2)}^{G, M_0}(w) \cdots A_{\sigma(d_{2n-1}), \sigma(d_{2n})}^{G, M_0}(w)$$

where the summation runs over the set $S_{V_{\mathcal{D}}(G)}$ of all permutations on $V_{\mathcal{D}}(G)$. Taking into account the above definition of an incidence matrix, each non zero term can be associated to a perfect matching on $\mathcal{D}(G)$ and, collecting all edge weights, the expansion can be written as

$$(2.29) \quad \text{Pf}(A^{G, M_0}(w)) = \sum_{M \in \mathcal{M}(G)} \frac{1}{2^n n!} \sum_{\sigma \in \Pi(M)} (-1)^{|\sigma|} A_{\sigma(d_1), \sigma(d_2)}^G \cdots A_{\sigma(d_{2n-1}), \sigma(d_{2n})}^G \\ \times w(M \cap (E_{\mathcal{D}}^E(G) \setminus M_0)) w^{-1}(M \cap E_{\mathcal{D}}^E(G) \cap M_0))$$

where $\Pi(M)$ is the set of $2^n n!$ permutations of $S_{V_{\mathcal{D}}(G)}$ contributing to the same perfect matching M ,

$$(2.30) \quad \Pi(M) = \{\sigma \in S_{V_{\mathcal{D}}(G)}, \{\{\sigma(d_1), \sigma(d_2)\}, \dots, \{\sigma(d_{2n-1}), \sigma(d_{2n})\}\} = M\}$$

As a consequence of the antisymmetry of A^G , the $2^n n!$ terms of the summation on $\Pi(M)$ are equal for every $M \in \mathcal{M}(G)$. Now the weights appearing in the right hand side of (2.29) can be rewritten as

$$(2.31) \quad \begin{aligned} w(M \cap (E_{\mathcal{D}}^E(G) \setminus M_0)) w^{-1}(M \cap E_{\mathcal{D}}^E(G) \cap M_0) \\ = w^{-1}(M_0 \cap E_{\mathcal{D}}^E(G)) w(E_{\mathcal{D}}^E(G) \cap (M \Delta M_0)) \end{aligned}$$

and the expression (2.29) can be turned into a weighted sum over the closed curves in $\mathcal{C}(G)$,

$$(2.32) \quad Pf(A^{G, M_0}(w)) = w^{-1}(M_0 \cap E_{\mathcal{D}}^E(G)) \sum_{C \in \mathcal{C}(G)} w(C) F_{A^G}(M_0, C)$$

where for all $C \in \mathcal{C}(G)$ and any reference perfect matching M_0 , $F_{A^G}(M_0, C)$ is the sum over all perfect matchings mapped to C by Φ_{M_0} ,

$$(2.33) \quad F_{A^G}(M_0, C) = \sum_{M \in \Phi_{M_0}^{-1}(C)} \frac{1}{2^n n!} \sum_{\sigma \in \Pi(M)} (-1)^{|\sigma|} A_{\sigma(d_1), \sigma(d_2)}^G \cdots A_{\sigma(d_{2n-1}), \sigma(d_{2n})}^G$$

Using this expression and the vector space structure of $\mathcal{C}(G)$, we have the following characterization,

Lemma 2.34. *The partition function of G admits a Pfaffian representation if and only if there exists an incidence matrix A^G , a reference dimer configuration M_0 and a cycle basis \mathcal{B}_G on $(\mathcal{C}(G), \Delta)$ such that $\forall C \in \mathcal{C}(G)$, $F_{A^G}(M_0, C) \neq 0$ and*

$$(2.35) \quad F_{A^G}(M_0, C) = F_{A^G}(M_0, C \Delta \gamma) \quad \forall C \in \mathcal{C}(G), \forall \gamma \in \mathcal{B}_G$$

Remark 2.36. *A change of reference perfect matching just induces a shift in the mapping for any two reference mappings $M_0, M_1 \in \mathcal{M}(G)$, one has*

$$(2.37) \quad \Phi_{M_1}(M) = \Phi_{M_0}(M) \Delta \Phi_{M_1}(M_0) \quad \forall M \in \mathcal{M}(G)$$

The set of equations (2.35) remains globally invariant under such a shift and the existence (or not) of a Pfaffian representation (2.27) is independent on the choice of the reference perfect matching.

Remark 2.38. *When all vertices of G have an even degree, the reference perfect matching M_0 can be chosen so that $E_{\mathcal{D}}^E(G) \cap M_0 = \emptyset$, the dependence on the weight function in definitions (2.25) and (2.27) simplifies and more classical expressions are recovered. In the general case, such a reference perfect matching does not exist and Proposition 2.18 suggests to take instead $E_{\mathcal{D}}^E(G)$ as a reference perfect matching. See also reference [8] for an alternative approach in the later case.*

We end this section with a relation between the notion of minor graph and the Pfaffian reduction formula.

- **Minor graphs and Pfaffian reduction formula.**

Informally a graph G_1 is a minor of another graph G_2 (noted $G_1 \preceq G_2$) if it can be obtained from it by repeating in some arbitrary order the two following elementary transformations: (a) deleting one edge; (b) contracting one edge, i.e. identifying its two end-vertices into a single vertex and deleting any loop or parallel edges that may arise. We refer to [22] for a more formal definition.

In the next two figures, two simple examples of these transformations are shown, which have some relevance in the present context: starting from a planar square lattice, one may form an hexagonal lattice by deleting one edge per site (Figure 1), while the contraction of the same set of edges leads to the triangular lattice (Figure 2).

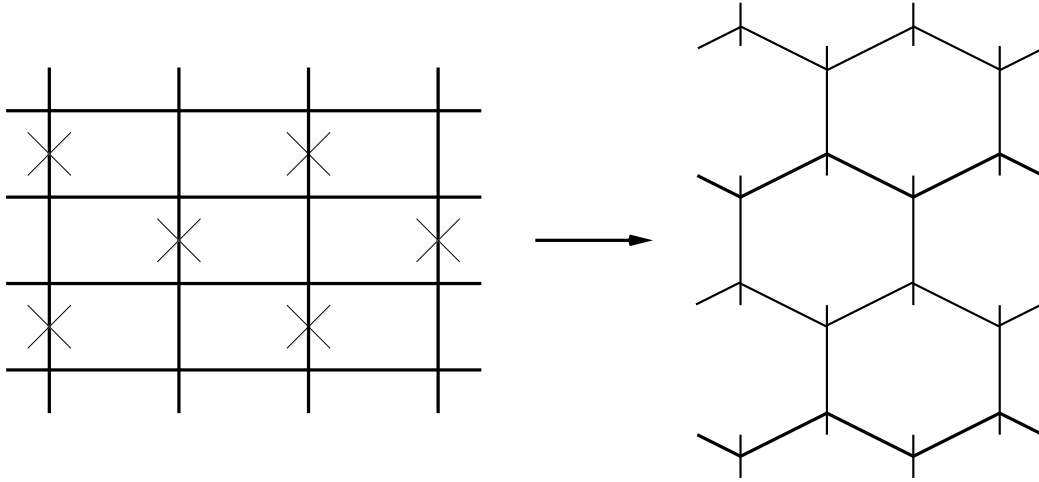


FIGURE 1. Example of minor graph obtained by deleting a subset of edges: an hexagonal lattice is formed out of a square lattice.

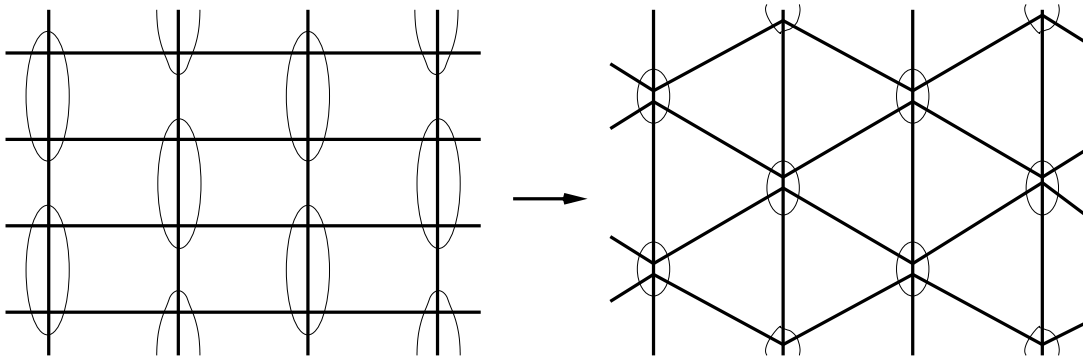


FIGURE 2. Example of minor graph obtained by contracting a subset of edges: a triangular lattice is formed out of a square lattice.

On the other side, the Pfaffian reduction formula [23] is a well known property of Pfaffians and can be stated as follows.

Let $A = (A_{ij})_{i,j \in \{1, \dots, 2n\}}$ a matrix of order $2n$ and K a subset of the set of indices $\{1, \dots, 2n\}$. We denote by A_K the sub-matrix of A obtained by deleting all rows and columns not indexed by an element of K .

$$(2.39) \quad A_K = (A_{ij})_{i,j \in K}$$

Similarly, we denote by $A^{\bar{K}}$ the antisymmetric matrix indexed by the complementary set of indices $\bar{K} = \{1, \dots, 2n\} \setminus K$, which elements are:

$$(2.40) \quad (A^{\bar{K}})_{ij} = -(A^{\bar{K}})_{ji} = \text{Pf}(A_{(K \cup \{i,j\})}) \quad \text{for all } i < j \in \bar{K}$$

We stress that in definitions (2.39)–(2.40), the order on any subset of indices is the one induced by the order on $\{1, \dots, 2n\}$. Pfaffian reduction formula reads:

Proposition 2.41. *Let A an antisymmetric matrix of order $2n$ and K a subset of indices of order $2p$, $0 < p < n$ such that $\text{Pf}(A_K) \neq 0$. We have:*

$$(2.42) \quad \text{Pf}(A) = [\text{Pf}(A_K)]^{-(n-p-1)} \text{Pf}(A^{\bar{K}})$$

The connection between minor graphs and Pfaffian reduction formula is as follows:

Proposition 2.43. *Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two simple graphs such that $G_1 \preceq G_2$. Consider a transformation which reduces G_2 to G_1 and denote by E_2^d (respectively E_2^c) the set of deleted (respectively contracted) edges in this transformation. Let A^{G_2} be an incidence matrix (in the sense of 2.22) on $\mathcal{D}(G_2)$ such that*

$$(2.44) \quad A_{d_1, d_2}^{G_2} = 0 \text{ if either } d_1 \text{ or } d_2 \text{ is incident with an edge in } E_2^d$$

Let K be the set of vertices in $\mathcal{D}(G_2)$ incident with an edge in $E_2^d \cup E_2^c$. Then the set $\bar{K} = V_{\mathcal{D}}(G_2) \setminus K$ can be identified with $V_{\mathcal{D}}(G_1)$, and the matrix $[A^{G_2}]^{\bar{K}}$ defined as in (2.40) is an incidence matrix on $\mathcal{D}(G_1)$.

In the above proposition, edge contraction (respectively edge deletion) is given a counterpart as a transformation of the incidence matrix associated to the dart graph. Contracting one edge means simply integrating out (using the Pfaffian reduction formula) the corresponding matrix entries. Deleting one edge may have various representations depending on the choice of the reference dimer configuration. Here we take $M_0 = E_{\mathcal{D}}^E(G)$, which is consistent with the general case (proposition 2.18), and edge deletion corresponds to factorizing out the associated edge weight. Note in particular that by Proposition 2.41, both Pfaffians are proportional in some functional sense.

Remark 2.45. *The minor relation is a partial order on the set of graphs, and in particular the genus of a graph is always larger or equal than the genus of any of its minors. This property does not transfer to the associated dart graphs and $\mathcal{D}(G_1)$ may have a larger genus than $\mathcal{D}(G_2)$ even if $G_1 \preceq G_2$.*

3. MAIN RESULTS.

In this section, we state our main results on the Pfaffian representations of partition functions on a graph G , starting from definitions 2.5 and 2.26.

Our first result is the well known theorem due to Kasteleyn [8].

Theorem 3.1. *The partition function on a graph G admits a Pfaffian representation if and only if G is planar.*

The proof presented here is independent on the orientability criterion used by Kasteleyn in his original proof. Instead, we emphasize a connection between this representation problem and McLane’s planarity criterion (Theorem 1.1, which is interesting in its own right. Incidentally, we prove also that in contradistinction with the dimer problem, there is no “Pfaffian” non-planar graph for the Ising model.

Furthermore, the approach presented here allows for some new representations of the partition function when G is a non-planar graph. For all positive integer n , let \mathbb{C}_n be the multicomplex algebra of order n [29] and Re the linear operator $\mathbb{C}_n \rightarrow \mathbb{R}$ which associates to any element its real part. Our main result is the following:

Theorem 3.2. *Let $G = (V, E)$ be a graph of nonorientable genus \tilde{g} . There exist an incidence matrix A^G on $\mathcal{D}(G)$ (in the sense of 2.22) with coefficients in $\mathbb{C}_{\tilde{g}}$, a constant $\Lambda \in \mathbb{C}_{\tilde{g}}$ and a reference perfect matching M_0 such that for all edge weight functions on G , the partition function on G can be written as*

$$(3.3) \quad Z_G(w) = w(M_0 \cap E_{\mathcal{D}}^E(G)) \text{Re}[\Lambda \text{Pf}(A^{G, M_0}(w))]$$

Note that the above expression depend on the non orientable genus [30], that is the minimal number of crosscaps a surface should have to embed G without edge crossings.

In the present work, we limit ourselves to the simplest application of Equation (3.3), that is the derivation of an expression of the partition function of a nonplanar graph as a sum of Pfaffians. The first one is in terms of the non orientable genus and involves matrices with complex coefficients:

Corollary 3.4. *Let $G = (V, E)$ be a graph of nonorientable genus \tilde{g} . There exist a family of incidence matrices $(A_j^G)_{j \in \{1, \dots, 2^{\tilde{g}}\}}$ on $\mathcal{D}(G)$ (in the sense of 2.22) with coefficients in \mathbb{C} , constants $(\Lambda_j)_{j \in \{1, \dots, 2^{\tilde{g}}\}}$ in \mathbb{C} and a reference perfect matching M_0 such that for all edge weight functions on G , the partition function on G can be written as*

$$(3.5) \quad Z_G(w) = w(M_0 \cap E_{\mathcal{D}}^E(G)) \text{Re}\left[\sum_{j=1}^{2^{\tilde{g}}} \Lambda_j \text{Pf}(A_j^{G, M_0}(w))\right]$$

A similar expression has been derived by Tessler [11], building on Kasteleyn’s approach. The nonorientable genus \tilde{g} of a graph G is related to its orientable genus g through the inequality [31]

$$(3.6) \quad \tilde{g} \leq 2g + 1$$

When the graph is non orientably simple (e.g. projective grids), the non-orientable genus can be much smaller than this bound, so that the expansion 3.5 is a real improvement over an expansion in terms of the orientable genus.

A proof of inequality (3.6) comes from the fact that starting from any orientable embedding in a surface of genus g , one can construct a nonorientable embedding of nonorientable genus $2g+1$. We give an explicit construction on a surface with $2g+1$ crosscaps, showing that every edge passes through them an even number of times. As a consequence, both entries of the incidence matrix and constant that appear in Theorem 3.2 can be taken in the subalgebra of \mathbb{C}_{2g+1} generated by even products of generators. This leads to an expansion in terms of 2^{2g} Pfaffians of matrices with real coefficients, equivalent to the one stated first by Kasteleyn.

Corollary 3.7. *Let $G = (V, E)$ be a graph of orientable genus g . There exist a family of incidence matrices $(A_j^G)_{j \in \{1, \dots, 4g\}}$ on $\mathcal{D}(G)$ (in the sense of 2.22) with coefficients in \mathbb{R} , constants $(\Lambda_j)_{j \in \{1, \dots, 4g\}}$ in \mathbb{R} and a reference perfect matching M_0 such that for all edge weight functions on G , the partition function on G can be written as*

$$(3.8) \quad Z_G(w) = w(M_0 \cap E_D^E(G)) \sum_{j=1}^{4g} \Lambda_j \text{Pf}(A_j^{G, M_0}(w))$$

In the rest of the section, we give a sketch of the proof of Theorem 3.1, explain the connection with McLane’s criterion and prepare for the derivation of Theorem 3.2. The first step is built on Proposition 2.43 and is the following

Theorem 3.9. *Let G_1 and G_2 be two simple graphs such that $G_1 \preceq G_2$ and A^{G_2} an incidence matrix on $\mathcal{D}(G_2)$. There exists an incidence matrix A^{G_1} on $\mathcal{D}(G_1)$, a constant $\Lambda_{1,2}$ and two reference perfect matchings, M_1 on $\mathcal{D}(G_1)$ and M_2 on $\mathcal{D}(G_2)$, such that for every closed curve C in $\mathcal{C}(G_1)$,*

$$(3.10) \quad F_{A^{G_1}}(M_1, C) = \Lambda_{1,2} F_{A^{G_2}}(M_2, \Pi(C))$$

where $\Pi : \mathcal{C}(G_1) \rightarrow \mathcal{C}(G_2)$ is the canonical mapping induced by the transformation from G_2 to its minor G_1 .

In particular, if the partition function on G_2 admits a Pfaffian representation then so does the partition function on G_1 .

Theorem 3.9 allows here to reduce the class of graphs one has to consider to prove both necessity and sufficiency of the planarity condition. Incidentally, it also clarifies a point raised by Fisher long ago[26], related to the fact that a planar graph has a non planar dart graph if some of its vertices are incident with more than three edges. In such a case, the related dimer problem is no longer solvable and two approaches have been proposed: In Kasteleyn approach [8], one keeps the original graph, and has to argue that the multiplicities encountered are exactly compensated by some sign rule; In Fisher’s approach, one deals with a larger, 3–regular graph so that the dart graph is also planar, and reduces later to the original graph. A consequence of Theorem 3.9 is that both approaches are not only always simultaneously possible, but that Fisher’s planar dimer construction implies Kasteleyn’s sign rule.

In the course of the proof of Theorem 3.1, we use Theorem 3.9 twice. On one hand, the proof that planarity is a necessary condition is reduced to an application of Kuratowski’s planarity criterion [27], starting from the following constatation:

Proposition 3.11. *The partition functions on the complete bipartite graph $K_{3,3}$ and on the complete graph K_5 have no Pfaffian representation.*

On the other hand, Theorem 3.9 and the following lemma allows to arbitrarily restrict the proof of sufficiency to 4-regular graphs,

Lemma 3.12. *Given a 2-connected simple graph G embedded in some surface \mathcal{S} , there exists a 2-connected 4-regular simple graph \tilde{G} embeddable in the same surface such that $G \preceq \tilde{G}$.*

Note that 4-regular graphs are just chosen for convenience. We are left with the following

Proposition 3.13. *Let $G = (V, E)$ be a 4-regular, 2-connected simple graph with no loops. If G is planar, then its partition function admits a Pfaffian representation.*

Proof of Proposition 3.13 consists in two parts. First, reduction to 4-regular graphs helps us to write Lemma 2.34 as a set of algebraic equations in the incidence matrix entries, depending on the choice of the cycle basis. We prove next that McLane criterion (Theorem 1.1) is just what is needed to pick a cycle basis so that these algebraic equations do have a consistent solution. First, we introduce some notations.

Let $G = (V, E)$ be as in Proposition 3.13. The vertex set of its dart graph $\mathcal{D}(G)$ is

$$(3.14) \quad V_{\mathcal{D}}(G) = \{(v, e), v \in V, e \in E(v)\}$$

where $E(v) \subset E$ is the set of four edges at vertex $v \in V$. We suppose that the sets V and $E(v)$ for all $v \in V$ have been ordered in some arbitrary way and for all $v \in V$ we denote by e_v^k the k^{th} element of $E(v)$. On $V_{\mathcal{D}}(G)$, we consider the induced lexicographic order.

$$(3.15) \quad (v, e) \leq (v', e') \iff \begin{cases} v < v' \\ \text{or} \\ v = v' \text{ and } e \leq e' \end{cases} \quad \forall ((v, e), (v', e')) \in V_{\mathcal{D}}(G) \times V_{\mathcal{D}}(G)$$

Since there is an even number of edge at each vertex, there exists a perfect matching M_0 on $\mathcal{D}(G)$ such that $M_0 \cap E = \emptyset$. Actually there are $3^{|V|}$ such matchings and we take arbitrarily the following one as our reference perfect matching,

$$(3.16) \quad M_0 = \bigcup_{v \in V} \{ \{(v, e_v^1), (v, e_v^2)\}, \{(v, e_v^3), (v, e_v^4)\} \}$$

We construct an incidence matrix A^G of order $|V_{\mathcal{D}}(G)| = 4|V|$.

$$(3.17) \quad \mathbf{A}^G = (A_{(v,e),(w,f)}^G)_{\substack{(e,f) \in E(v) \times E(w) \\ (v,w) \in V \times V}}$$

By Definition 2.22, entries of A^G are zero except those corresponding to an edge in $\mathcal{D}(G)$. If the edge is in $E_{\mathcal{D}}^E(G)$ (respectively in $E_{\mathcal{D}}^V(G)$), $e = f$ (respectively $v = w$), the coefficient $A_{(v,e),(w,f)}^G$ is termed an ‘‘edge entry’’ (respectively a ‘‘site entry’’).

Recalling that A^G is an antisymmetric matrix, we rename all its edge and site entries as follows. For every edge e in E with endvertices v and w such that $v < w$, we write

$$(3.18) \quad b_e = A_{(v,e),(w,e)}^G$$

while for every vertex $v \in V$ and every permutation σ on $E(v)$, we define

$$(3.19) \quad \begin{aligned} S_v(\sigma) &= A_{(v,\sigma(e_v^1)),(v,\sigma(e_v^2))}^G & \bar{S}_v(\sigma) &= A_{(v,\sigma(e_v^3)),(v,\sigma(e_v^4))}^G \\ T_v(\sigma) &= A_{(v,\sigma(e_v^1)),(v,\sigma(e_v^3))}^G & \bar{T}_v(\sigma) &= A_{(v,\sigma(e_v^2)),(v,\sigma(e_v^4))}^G \\ U_v(\sigma) &= A_{(v,\sigma(e_v^1)),(v,\sigma(e_v^4))}^G & \bar{U}_v(\sigma) &= A_{(v,\sigma(e_v^2)),(v,\sigma(e_v^3))}^G \end{aligned}$$

When σ is the identity, we simplify the notations further as

$$(3.20) \quad \begin{aligned} s_v &= S_v(\underline{1}) & t_v &= T_v(\underline{1}) & u_v &= U_v(\underline{1}) \\ \bar{s}_v &= \bar{S}_v(\underline{1}) & \bar{t}_v &= \bar{T}_v(\underline{1}) & \bar{u}_v &= \bar{U}_v(\underline{1}) \end{aligned}$$

Now let $\gamma = (V_\gamma, E_\gamma)$ be a cycle of length r_γ on G . We consider a cyclic order on both its edge and vertex sets,

$$(3.21) \quad V_\gamma = (v_i^\gamma)_{i \in \{1, \dots, r_\gamma\}}$$

$$(3.22) \quad E_\gamma = (e_i^\gamma)_{i \in \{1, \dots, r_\gamma\}}$$

where the indices are defined modulo r_γ so that v_i^γ is incident with e_{i-1}^γ and e_i^γ for all $i \in \{1, \dots, r_\gamma\}$.

We want to relate both the order on $V_{\mathcal{D}}(G)$ and the order along γ and thus introduce γ -related notations.

For all i in $\{1, \dots, r_\gamma\}$, we rename the edge entry in \mathbf{A}^G associated to e_i^γ as,

$$(3.23) \quad B_i^\gamma = A_{(v_i^\gamma, e_i^\gamma), (v_{i+1}^\gamma, e_i^\gamma)}$$

Note that B_i^γ is equal to either $+b_{e_i^\gamma}$ or $-b_{e_i^\gamma}$ according to whether γ goes through edge e_i^γ compatibly or not with the order on $V_{\mathcal{D}}(G)$. In order to rename the site entries along γ , we consider the following

Lemma 3.24. *Let γ a cycle of length r_γ ordered as above. For every $i \in \{1, \dots, r_\gamma\}$, there exists a unique permutation σ_i^γ on $E(v_i^\gamma)$ with positive signature such that,*

$$(3.25) \quad e_{i-1}^\gamma = \sigma_i^\gamma(e_{v_i^\gamma}^1)$$

$$(3.26) \quad e_i^\gamma = \sigma_i^\gamma(e_{v_i^\gamma}^4)$$

The effect of the σ_i^γ 's is to reorder the edges on each site visited by the cycle, so that it enters a site by the first edge and leave by the forth. When considering a given cycle γ , we also simplify notations (3.19) as

$$(3.27) \quad \begin{aligned} S_i^\gamma &= S_{v_i^\gamma}(\sigma_i^\gamma) & T_i^\gamma &= T_{v_i^\gamma}(\sigma_i^\gamma) & U_i^\gamma &= U_{v_i^\gamma}(\sigma_i^\gamma) \\ \bar{S}_i^\gamma &= \bar{S}_{v_i^\gamma}(\sigma_i^\gamma) & \bar{T}_i^\gamma &= \bar{T}_{v_i^\gamma}(\sigma_i^\gamma) & \bar{U}_i^\gamma &= \bar{U}_{v_i^\gamma}(\sigma_i^\gamma) \end{aligned}$$

The relations between notations (3.20) and (3.27) depend on the actual value of σ_i^γ among twelve possible realizations and are listed in Table 1.

Using these notations, we can state the following proposition:

Proposition 3.28. *Let γ a cycle on G of length r_γ and notations as above. If the entries of the incidence matrix \mathbf{A}^G associated to elements of γ have non zero values and verify the following set of equations:*

$$(3.29) \quad S_i^\gamma \bar{S}_i^\gamma + T_i^\gamma \bar{T}_i^\gamma = 0 \quad \forall i \in \{1, \dots, r_\gamma\}$$

$$(3.30) \quad (B_i^\gamma)^2 = -\frac{U_i^\gamma \bar{T}_i^\gamma}{S_i^\gamma} \frac{U_{i+1}^\gamma T_{i+1}^\gamma}{\bar{S}_{i+1}^\gamma} \quad \forall i \in \{1, \dots, r_\gamma\}$$

$$(3.31) \quad \prod_{i=1}^{r_\gamma} B_i^\gamma = -\prod_{i=1}^{r_\gamma} U_i^\gamma$$

Then, for any perfect matching M_0 such that $M_0 \cap E = \emptyset$, the contribution of every closed curve in $\mathcal{C}(G)$ is invariant under addition of γ ,

$$(3.32) \quad F_{AG}(M_0, C) = F_{AG}(M_0, C \Delta \gamma) \quad \forall C \in \mathcal{C}(G).$$

$\sigma_i^\gamma(e_v^1)$	$\sigma_i^\gamma(e_v^2)$	$\sigma_i^\gamma(e_v^3)$	$\sigma_i^\gamma(e_v^4)$	S_i^γ	\bar{S}_i^γ	T_i^γ	\bar{T}_i^γ	U_i^γ	\bar{U}_i^γ
e_v^1	e_v^2	e_v^3	e_v^4	s_v	\bar{s}_v	t_v	\bar{t}_v	u_v	\bar{u}_v
e_v^2	e_v^1	e_v^4	e_v^3	$-s_v$	$-\bar{s}_v$	$-\bar{t}_v$	$-t_v$	\bar{u}_v	u_v
e_v^3	e_v^4	e_v^1	e_v^2	\bar{s}_v	s_v	$-t_v$	$-\bar{t}_v$	$-\bar{u}_v$	$-u_v$
e_v^4	e_v^3	e_v^2	e_v^1	$-\bar{s}_v$	$-s_v$	\bar{t}_v	t_v	$-u_v$	$-\bar{u}_v$
e_v^1	e_v^3	e_v^4	e_v^2	t_v	\bar{t}_v	u_v	\bar{u}_v	s_v	\bar{s}_v
e_v^3	e_v^1	e_v^2	e_v^4	$-t_v$	$-\bar{t}_v$	$-\bar{u}_v$	$-u_v$	\bar{s}_v	s_v
e_v^4	e_v^2	e_v^1	e_v^3	\bar{t}_v	t_v	$-u_v$	$-\bar{u}_v$	$-\bar{s}_v$	$-s_v$
e_v^2	e_v^4	e_v^3	e_v^1	$-\bar{t}_v$	$-t_v$	\bar{u}_v	u_v	$-s_v$	$-\bar{s}_v$
e_v^1	e_v^4	e_v^2	e_v^3	u_v	\bar{u}_v	s_v	\bar{s}_v	t_v	\bar{t}_v
e_v^4	e_v^1	e_v^3	e_v^2	$-u_v$	$-\bar{u}_v$	$-\bar{s}_v$	$-s_v$	\bar{t}_v	t_v
e_v^2	e_v^3	e_v^1	e_v^4	\bar{u}_v	u_v	$-s_v$	$-\bar{s}_v$	$-\bar{t}_v$	$-t_v$
e_v^3	e_v^2	e_v^4	e_v^1	$-\bar{u}_v$	$-u_v$	\bar{s}_v	s_v	$-t_v$	$-\bar{t}_v$

TABLE 1. Correspondence between the 6 site entries in the upper triangular part of the incidence matrix for a given vertex v (Equation (3.20)), and the new names (Equation (3.27)) after reordering of $E(v) = \{e_v^1, e_v^2, e_v^3, e_v^4\}$ according to a cycle γ passing through v ($= v_i^\gamma$ for some index i). σ_i^γ is the even permutation on $E(v)$ such that γ enters v through edge $\sigma_i^\gamma(e_v^1)$, and leaves it through $\sigma_i^\gamma(e_v^4)$ (Lemma 3.24).

Proposition 3.28 will be proven in Section 5, where we introduce a representation in terms of a Grassmann algebra. Equations (3.29)–(3.31) can be obtained by considering all local configurations around γ with an even number of edges at each site. Some of these configurations may not correspond to an actual element $C \in \mathcal{C}(G)$, for instance when a cut along γ splits the graph G into two pieces, so that some of these equations are possibly not necessary. The set of equations (3.29)–(3.31) always admits a nowhere zero solution, when considering a single cycle. However, compatibility of these conditions for an arbitrary collection of cycles is not granted, even for planar graphs.

Here, MacLane’s planarity criterion comes into play as it provides us with a cycle basis \mathcal{B}_0 with specific properties. In particular, elements of such a basis cannot form a cluster (a double cover of a proper subset of $E(v)$, at some vertex $v \in V$) [20]. Then the absence of clusters with 3 edges implies compatibility of equations (3.29) for all $\gamma \in \mathcal{B}_0$ and all $v \in V$. Similarly, supposing that site entries have been chosen so that equations (3.29) hold for all $\gamma \in \mathcal{B}_0$, the absence of clusters with 2 edges implies that edge entries can be chosen so that equations (3.30) also hold for all $\gamma \in \mathcal{B}_0$.

The remaining equations (3.31) are strongly reminding of Kasteleyn’s orientation prescription. Note that equations (3.29)–(3.30) are independent on the sign of the edge entries B_i^γ ,

and that once they are solved, one has

$$(3.33) \quad \prod_{i=1}^r (B_i^\gamma)^2 = \prod_{i=1}^r (U_i^\gamma)^2$$

so that equation (3.31) just amounts to choose independently the sign of the product of edge entries along γ . Existence of a solution to the equations (3.31) for all $\gamma \in \mathcal{B}_0$ just derives independence of the elements in that basis.

Proposition 3.28 implies that conditions of Lemma 2.34 are verified for any planar 4-regular graph, and Proposition 3.13 follows.

We now generalize these results to non planar graphs.

By Theorem 3.1, we already know that a non planar graph G cannot have a cycle basis such that Equations (3.29)–(3.31) can be solved simultaneously for all its elements. The strategy we adopt here consists in finding the largest free family of cycles on which these equations may hold simultaneously and looking at what happens when completing it to a cycle basis.

Following the generalization to non planar graphs of McLane’s Theorem [20], we say that a family of cycles on a graph is “sparse” if it contains no cluster, where a cluster is a subfamily of cycles covering twice a proper subset of $E(v)$, for some vertex v . Just as in the planar case, Equations (3.29)–(3.31) can be proven to be simultaneously solvable for all cycles forming a sparse family.

Intuitively, a maximal sparse family of cycles has to be related to the embedding properties of the graph, but this relation is rather intricate: in a non planar 2-cell embedding, face boundaries are closed walks but not necessarily cycles and another definition for sparseness is required [20]. The following lemma together with Theorem 3.9 allows us to bypass this point:

Lemma 3.34. *Let G_1 be a 2-connected simple graph embedded in some surface \mathcal{S} . There exists a 4-regular 2-connected graph $G_2 \succcurlyeq G_1$ embeddable in the same surface \mathcal{S} so that all its face boundaries are cycles.*

If the genus of the embedding in \mathcal{S} is minimal for G_1 , it is also minimal for G_2 since $G_2 \succcurlyeq G_1$. When considering such an embedding for G_2 in \mathcal{S} , the family of face boundaries is a collection of cycles and a sparse family of smallest codimension in $\mathcal{C}(G_2)$. Theorem 3.9 allows then to transfer related results back to $G_1 \preccurlyeq G_2$, even if G_1 has no strong embedding of its own genus.

Hence, we consider a 4-regular graph $G = (V, E)$ with a strong 2-cell embedding in some surface \mathcal{S} . We denote by $\mathcal{F}_0(G)$ the set of cycles which are face boundaries in that embedding and by $\mathcal{F}_0^*(G)$ a maximal free subfamily, obtained by dropping out one element of $\mathcal{F}_0(G)$.

Clearly $\mathcal{F}_0^*(G)$ is a sparse family and as in the planar case, this property leads to the proof that the algebraic equations (3.29)–(3.30) can be simultaneously solved for all $\gamma \in \mathcal{F}_0^*(G)$.

A non orientable embedding can be described by an embedding scheme, that is a pair (Π, λ) where $\Pi = (\pi_v)_{v \in V}$ defines a cyclic ordering of edges at every vertex, and $\lambda : E \rightarrow \{-1, 1\}$ is a signature on the edge set [30]. Note that the correspondence between embeddings and embedding schemes is not one to one, and for a graph with $|V|$ vertices there can be as much as $2^{|V|}$ embedding schemes describing the same embedding. We prove the following Lemma.

Lemma 3.35. *Let G a 4-regular graph with a strong 2-cell embedding in some surface \mathcal{S} , defined up to homeomorphism by the embedding scheme (Π, λ) . There exists an incidence matrix A^G with coefficients*

$$(3.36) \quad A_{(v,e),(w,f)}^G \in \begin{cases} i\mathbb{R} & \text{if } e = f \text{ and } \lambda(e) = -1 \\ \mathbb{R} & \text{otherwise.} \end{cases} \quad \text{for all } (v,e), (w,f) \in V_{\mathcal{D}}(G)$$

and a reference perfect matching M_0 as in (3.16) such that

$$(3.37) \quad F_{A^G}(M_0, C) = F_{A^G}(M_0, C \Delta \gamma) \quad \forall C \in \mathcal{C}(G), \forall \gamma \in \mathcal{F}_0^*(G)$$

In particular the coefficients of A^G can be chosen in \mathbb{R} if and only if the embedding is orientable.

When the incidence matrix is chosen as in Lemma 3.35, each closed curve C has a weight in the Pfaffian expansion which depends only on its homology class on \mathcal{S} . Proof of Theorem 3.2 uses the fact that for a sufficiently large class of graphs, these weights can be easily computed. However, the coefficients have to be chosen in a suitable multicomplex algebra \mathbb{C}_n and Lemma 3.35 stated in this larger algebraic context.

The more classical expansions (3.5) and (3.8) are then obtained directly by an algebraic reduction from \mathbb{C}_n to \mathbb{C} and \mathbb{R} , respectively. This suggests that expression (3.3) does contain more information than classical expansions, and we believe that it will prove useful to get new results on non planar Ising model.

4. PROOFS

In this section, we present proofs of all results, with the exception of Propositions 3.28 and 2.41 which are considered in the next section.

Proof of Proposition 2.18.

If $G = (V, E)$ is the dart graph of some other graph, says $G = \mathcal{D}(G')$ and $G' = (V', E')$, its edge set is the union of two distinct parts, $E = E_{\mathcal{D}}^{V'} \cup E_{\mathcal{D}}^{E'}$. Its subgraph $(V, E_{\mathcal{D}}^{E'})$ forms a perfect matching, since by definition, each vertex in V is a dart on G' and, as such, is incident with exactly one edge of $E_{\mathcal{D}}^{E'}$. ■

Proof of Proposition 2.19.

Let M be a perfect matching on $\mathcal{D}(G)$. For each vertex $v \in V$, denote by $E(v)$ the set of edges at v , and by $V_{\mathcal{D}}(v)$ the set of darts at v

$$(4.1) \quad V_{\mathcal{D}}(v) = \{(v, e), e \in E(v)\}$$

An edge of M is either in $E_{\mathcal{D}}^V$ and have thus its two endpoints in the same cluster $V_{\mathcal{D}}(v)$ for some $v \in V$, or is in $E_{\mathcal{D}}^E$ and hits two distinct clusters. Since M is a perfect matching on $\mathcal{D}(G)$, it hits each of its vertices exactly once. Thus for all $v \in V$, the number of edges in $M \cap E_{\mathcal{D}}^E$ which hit $V_{\mathcal{D}}(v)$ has the same parity as $|V_{\mathcal{D}}(v)|$. This number, modulo 2 is thus independent on the matching. In particular, if M_1, M_2 are two perfect matchings on $\mathcal{D}(G)$, we have

$$(4.2) \quad C = (M_1 \Delta M_2) \cap E_{\mathcal{D}}^E = (M_1 \cap E_{\mathcal{D}}^E) \Delta (M_2 \cap E_{\mathcal{D}}^E)$$

and for all $v \in V$, C hits $V_{\mathcal{D}}(v)$ an even number of times. ■

Proof of Lemma 2.34

Substituting both Expressions (2.6) and (2.32) in the definition (2.27) and owing to the algebraic independence of the weights, it is clear that the partition function on G admits a Pfaffian representation in the sense of definition 2.26 if and only if there exists an incidence matrix A^G a reference perfect matching M_0 and a constant $\Lambda \neq 0$ such that

$$(4.3) \quad F_{A^G}(M_0, C) = \Lambda \quad \forall C \in \mathcal{C}(G)$$

The actual value of constant Λ is irrelevant, so we only need to verify that for some reference perfect matching, $F_{A^G}(M_0, \cdot)$ is non zero and is equal on any two closed curves in $\mathcal{C}(G)$,

$$(4.4) \quad F_{A^G}(M_0, C) = F_{A^G}(M_0, C') \quad \text{for all } C, C' \in \mathcal{C}(G)$$

Now $(\mathcal{C}(G), \Delta)$ is a vector space and admits a cycle basis, say \mathcal{B}_G . Therefore the above set of equations can be reduced to an invariance property under addition of any cycle in that basis

$$(4.5) \quad F_{A^G}(M_0, C) = F_{A^G}(M_0, C \Delta \gamma) \quad \text{for all } C \in \mathcal{C}(G) \text{ and all } \gamma \in \mathcal{B}_G$$

Equations (4.5) are clearly a subset of Equations (4.4); by chain rule, it is easy to show that they generate all of them, so that both sets are equivalent. \blacksquare

Proof of Proposition 2.43.

Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $G_1 \preceq G_2$. Consider a given transformation which send G_2 on G_1 , and denote by E_2^c (respectively E_2^d) the set of contracted (respectively deleted) edges of E_2 in that transformation. By construction, each connected component of (V_2, E_2^c) is a tree T_v , which is mapped under contraction to a given vertex v in G_1 . Furthermore, the edges in $E_2 \setminus (E_2^c \cup E_2^d)$ are in one to one correspondence with those of E_1 , which implies that there is also a one to one correspondence between the set of darts $V_{\mathcal{D}}(G_1)$ and the subset of $V_{\mathcal{D}}(G_2)$ defined as

$$(4.6) \quad \overline{K} = \{(v, e) \in V_{\mathcal{D}}(G_2), e \in E_2 \setminus (E_2^c \cup E_2^d)\}$$

For all $d \in V_{\mathcal{D}}(G_1)$, we denote by \tilde{d} its image in \overline{K} and assume that the order on $V_{\mathcal{D}}(G_1)$ is induced from the order on $V_{\mathcal{D}}(G_2)$ through this correspondence.

Let A^{G_2} be an incidence matrix on $\mathcal{D}(G_2)$ such that $A_{d_1, d_2}^{G_2} = 0$ if either d_1 or d_2 is incident with an edge in E_2^d ; Then, by (2.40), the matrix $[A^{G_2}]^{\overline{K}}$ is an antisymmetric matrix of order $|\overline{K}| = |V_{\mathcal{D}}(G_1)|$, and for every pair of darts (d_1, d_2) in $V_{\mathcal{D}}(G_1)$ with $d_1 < d_2$, its entries are

$$(4.7) \quad A_{\tilde{d}_1, \tilde{d}_2}^{\overline{K}} = \text{Pf}(A_{(K \cup \{\tilde{d}_1, \tilde{d}_2\})})$$

where

$$(4.8) \quad K = \{(v, e) \in V_{\mathcal{D}}(G_2), e \in E_2^c \cup E_2^d\}$$

In order to prove that $A^{\overline{K}}$ is an incidence matrix on $\mathcal{D}(G_1)$, we have to check that Equation (2.23) holds for all pairs $(d_1, d_2) \in V_{\mathcal{D}}(G_1)$. Let us consider a pair of darts (d_1, d_2) such that $A_{\tilde{d}_1, \tilde{d}_2}^{\overline{K}} \neq 0$. Then in the expansion of the right hand side of (4.7), there is at least one non zero term which necessarily contains as a factor the term $A_{\tilde{d}_1, \tilde{d}_2}$, or a product of terms $A_{\tilde{d}_1, \tilde{g}_1} A_{\tilde{g}_1, \tilde{g}_2} \cdots A_{\tilde{g}_k, \tilde{d}_2}$ for some $k > 0$ where $(\tilde{g}_i, \tilde{g}'_i)$, $1 \leq i \leq k$, are pairs of darts associated to the same edge in E_2^c (by condition (2.44)).

In the first case, $A_{\tilde{d}_1, \tilde{d}_2} \neq 0$ implies $(\tilde{d}_1, \tilde{d}_2) \in E_{\mathcal{D}}(G_2)$ since A is an antisymmetric incidence matrix. Hence \tilde{d}_1 and \tilde{d}_2 contain the same edge in $E_2 \setminus (E_2^c \cap E_2^d)$, which implies that d_1 and d_2 also contain the same edge in E_1 .

In the second case, $A_{\tilde{d}_1, \tilde{g}_1} \neq 0$, $A_{\tilde{g}'_1, \tilde{g}_2} \neq 0$, \dots , $A_{\tilde{g}'_k, \tilde{d}_2} \neq 0$ imply that each pair of darts, \tilde{d}_1 and \tilde{g}_1 , \tilde{g}'_1 and \tilde{g}_2 , \dots , \tilde{g}'_k and \tilde{d}_2 , share the same vertex, respectively. Since $(\tilde{g}_i, \tilde{g}'_i) \in E_2^c$ for all $1 \leq i \leq k$, these vertices are in the same connected components of (V_2, E_2^c) and thus shrink to the same vertex of G_1 under contraction. Thus d_1 and d_2 contain the same vertex in V_1 .

In both cases, $(d_1, d_2) \in E_{\mathcal{D}}(G_1)$. $A^{\bar{K}}$ is thus an incidence matrix on $\mathcal{D}(G_1)$. ■

Proof of Theorem 3.1.

Let G be a planar, 2-connected simple graph. By Lemma 3.12, there exists a planar, 2-connected, 4-regular graph $\tilde{G} \succ G$, which partition function admits a Pfaffian representation by Proposition 3.13. By Theorem 3.9, this property extends to all its minors and in particular the partition function on G admits a Pfaffian representation.

Let G be a non planar graph. Kuratowski's planarity criterion [27], one of its minors is homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$. By Proposition 3.11, the partition function on either of these two graphs has no Pfaffian representation and by Theorem 3.9, there is also no such Pfaffian representation for the partition function on G . ■

Proof of Theorem 3.2.

Let $G = (V, E)$ be a graph of nonorientable genus \tilde{g} . We first suppose that G is 4-regular and construct another 4-regular graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{G} \preceq G$ in the following way: we consider a surface S with \tilde{g} crosscaps and decompose it into $\tilde{g} + 1$ domains $D_0, D_1, \dots, D_{\tilde{g}}$ so that D_0 is homeomorphic to a sphere with \tilde{g} holes and each of the D_k , $k > 0$ is homeomorphic to a Moebius strip. We draw G on S so that all its vertices are in D_0 . On every domain D_k , $k > 0$, we draw two nested non intersecting simple curves around the crosscap. We call \tilde{G} the 4-regular graph which representation on S results from the superposition of the representation of G and the $2\tilde{g}$ closed curves, adding each intersection point to the vertex set, and associating each line segment to an edge.

Finally, we possibly use Lemma 3.34 and modify \tilde{G} accordingly so that the all face boundaries of \tilde{G} on S are cycles. For every $k \in \{1, \dots, \tilde{g}\}$, we call $\tilde{G}_k = (\tilde{V}_k, \tilde{E}_k)$ the subgraph of \tilde{G} with vertex set the subset of \tilde{V} represented in D_k and edge set the set of edges in \tilde{E} with both endvertices in \tilde{V}_k .

Note that in this embedding every edge of \tilde{G} goes through at most one crosscap. We thus consider an embedding scheme (Π, λ) which describes the embedding of \tilde{G} on S so that $\lambda(e) = -1$ for every edge e going through a crosscap and $\lambda(e) = +1$ otherwise. By Lemma 3.35, there exists an incidence matrix $A^{\tilde{G}}$ with coefficients as in (3.36), such that equations (3.29)–(3.31) hold for all cycles $\gamma \in \mathcal{F}_0^*(\tilde{G})$.

We introduce a multicomplex algebra $\mathbb{C}_{\tilde{g}}$ with generators $i_1, \dots, i_{\tilde{g}}$ such that $i_\alpha^2 = -1$ for all $\alpha \in \{1, \dots, \tilde{g}\}$ and $i_\alpha i_\beta = i_\beta i_\alpha$ for all $\alpha, \beta \in \{1, \dots, \tilde{g}\}$. We construct a new matrix $\tilde{A}^{\tilde{G}}$

with coefficients in $\mathbb{C}_{\tilde{g}}$, so that for every pair of darts $(v, e), (w, f)$ in $V_{\mathcal{D}}(\tilde{G})$

$$(4.9) \quad \tilde{A}_{(v,e),(w,f)}^{\tilde{G}} = \begin{cases} i_k \Im(A_{(v,e),(w,f)}^{\tilde{G}}) & \text{if } e = f \in \tilde{G}_k \text{ and } \lambda(e) = -1 \\ A_{(v,e),(w,f)}^{\tilde{G}} & \text{otherwise.} \end{cases}$$

where $\Im(\cdot)$ denotes the imaginary part. By construction, every cycle in $\mathcal{F}_0^*(\tilde{G})$ has its support in one of the subgraphs \tilde{G}_k , or has all edges with +1 signature.

Thus for every cycle $\gamma \in \mathcal{F}_0^*(\tilde{G})$, the equations (3.29)–(3.31) contain at most one generator i_k , so that they hold for coefficients of $\tilde{A}^{\tilde{G}}$ as soon as they hold for those of $A^{\tilde{G}}$. Thus $\tilde{A}^{\tilde{G}}$ is also an incidence matrix (with coefficients in $\mathbb{C}_{\tilde{g}}$) such that equations (3.29)–(3.31) hold for all cycles $\gamma \in \mathcal{F}_0^*(\tilde{G})$.

The set of closed curves on graph \tilde{G} has $2^{\tilde{g}}$ homology classes and any two elements in the same homology class are given the same contribution in the Pfaffian expansion of $\tilde{A}^{\tilde{G}}$. We now construct a particular element in each homology class which contribution can be computed.

Let us denote by $\mathcal{C}_R(\tilde{G}) \subset \mathcal{C}(\tilde{G})$ the set of closed curves with support in $\cup_{1 \leq k \leq \tilde{g}} \tilde{E}_k$. Given a reference perfect matching M_0 as in (3.16), we also denote by $\mathcal{M}_R(\tilde{G})$ the set of perfect matchings in $\mathcal{M}(\tilde{G})$ which have their image in $\mathcal{C}_R(\tilde{G})$.

$$(4.10) \quad \mathcal{M}_R(\tilde{G}) = \{M \in \mathcal{M}(\tilde{G}), \Phi_{M_0}(M) \in \mathcal{C}_R(\tilde{G})\}$$

We also define the following subsets of darts

$$(4.11) \quad \mathcal{D}_0 = \{(v, e) \in V_{\mathcal{D}}(\tilde{G}), v \in \tilde{V} \setminus \cup_{1 \leq k \leq \tilde{g}} \tilde{V}_k\}$$

and for all $k \in \{1, \dots, \tilde{k}\}$

$$(4.12) \quad \mathcal{D}_k = \{(v, e) \in V_{\mathcal{D}}(\tilde{G}), v \in \tilde{V}_k\}$$

Obviously the chosen reference perfect matching has its support in $\cup_{0 \leq k \leq \tilde{g}} \mathcal{D}_k \times \mathcal{D}_k$ and thus also every perfect matching in $\mathcal{M}_R(\tilde{G})$. Now the sets of dimers $\mathcal{D}_k \times \mathcal{D}_k$, $0 \leq k \leq \tilde{g}$ have disjoint support, so that $\mathcal{M}_R(\tilde{G})$ is a direct product of sets of perfect matchings on each component.

Now define

$$\begin{aligned} f_0 &= F_{\tilde{A}^{\tilde{G}}}(M_0, \gamma_0) \text{ for some } \gamma_0 \in \mathcal{F}_0^*(\tilde{G}) \\ f_k &= F_{\tilde{A}^{\tilde{G}}}(M_0, \gamma_1) \text{ for some } \gamma_1^k \in \mathcal{C}(\tilde{G}_k) \setminus \mathcal{F}_0^*(\tilde{G}) \end{aligned}$$

Note that the value of f_0 is independent of the choice of $\gamma_0 \in \mathcal{F}_0^*(\tilde{G})$, and $\mathcal{C}(\tilde{G}_k) \setminus \mathcal{F}_0^*(\tilde{G})$ is not empty (\tilde{G}_k is not planar) and the value of f_1 is independent on the choice of γ_1^k in that set.

The subgraph \tilde{G}_k contains K_5 as a minor (which can be formed from the two closed lines around the crosscap and two edges of G passing through the crosscap and their 8 crossing points). Thus by Theorem 3.9, there exists an incidence matrix on K_5 which Pfaffian expansion has weights proportional to f_0 and f_k . In particular they verify Equation (4.62) which is homogeneous. Thus weights f_0 and f_k verify the same equation, namely,

$$(4.13) \quad f_k f_0^3 = -f_k^3 f_0$$

Since γ_1^k passes through the crosscap an odd number of times, f_k is necessarily proportional to I_k , which implies then

$$(4.14) \quad f_k = \pm i_k f_0$$

For conveniency, we possibly change the sign of the coefficients of the matrix $\tilde{A}^{\tilde{G}}$ which are proportional to i_k (Indeed, for every cycle $\gamma \in \mathcal{F}_0^*(\tilde{G})$, equations (3.29)–(3.31) keep unchanged in that transformation), so that we can fix those signs to be

$$(4.15) \quad f_k = i_k f_0$$

Now consider a closed curve $C \in \mathcal{C}_R(\tilde{G})$. Denote by C_k the closed curve in $\mathcal{C}(\tilde{G})$ with support in \tilde{G}_k which coincide with C on \tilde{G}_k and by C_0 the null curve. By construction, we have

$$(4.16) \quad F_{\tilde{A}^{\tilde{G}}}(M_0, C) (F_{\tilde{A}^{\tilde{G}}}(M_0, C_0))^{\tilde{g}-1} = \prod_{k=1}^{\tilde{g}} F_{\tilde{A}^{\tilde{G}}}(M_0, C_k)$$

and thus

$$(4.17) \quad F_{\tilde{A}^{\tilde{G}}}(M_0, C) = f_0 \prod_{k=1}^{\tilde{g}} (i_k)^{\epsilon_k}$$

where

$$(4.18) \quad \epsilon_k = \begin{cases} +1 & \text{if } C|_{\tilde{G}_k} \text{ has nonorientable genus 1} \\ 0 & \text{otherwise} \end{cases}$$

where $C|_{\tilde{G}_k}$ is the restriction of C on \tilde{G}_k .

Setting

$$(4.19) \quad \Lambda = \frac{1}{f_0} \prod_{k=1}^{\tilde{g}} (1 - i_k)$$

We get, for every curve $C \in \mathcal{C}(\tilde{G})$,

$$(4.20) \quad \text{Re}[\Lambda F_{\tilde{A}^{\tilde{G}}}(M_0, C)] = 1$$

The Theorem is proven. ■

Proof of Corollary 3.4.

Let $G = (V, E)$ be a graph of nonorientable genus \tilde{g} . By Theorem 3.2, there is an incidence matrix A^G on $\mathcal{D}(G)$ with coefficients in $\mathbb{C}_{\tilde{g}}$, a constant $\Lambda \in \mathbb{C}_{\tilde{g}}$ and a reference perfect matching M_0 such that

$$(4.21) \quad Z_G(w) = w(M_0 \cap E_D^E(G)) \text{Re}[\Lambda \text{Pf}(A^{G, M_0}(w))]$$

Now there are $2^{\tilde{g}}$ distinct algebra homomorphisms $\tilde{H}_k : \mathbb{C}_{\tilde{g}} \rightarrow \mathbb{C}$, $k \in \{1, \dots, 2^{\tilde{g}}\}$ such that for all $j \in \{1, \dots, \tilde{g}\}$, $\tilde{H}_k(i_j) \in \{-i, +i\}$, and we have for every element $w \in \mathbb{C}_{\tilde{g}}$,

$$(4.22) \quad \operatorname{Re}(w) = \frac{1}{2^{\tilde{g}}} \sum_{k=1}^{2^{\tilde{g}}} \tilde{H}_k(w)$$

Here we set

$$(4.23) \quad \Lambda_k = \frac{1}{2^{\tilde{g}}} \tilde{H}_k(\Lambda)$$

and for all $(d_1, d_2) \in V_{\mathcal{D}}(G) \times V_{\mathcal{D}}(G)$

$$(4.24) \quad (A_k^G)_{d_1, d_2} = \tilde{H}_k(A_{d_1, d_2}^G)$$

and we get

$$(4.25) \quad Z_G(w) = w(M_0 \cap E_{\mathcal{D}}^E(G)) \sum_{k=1}^{2^{\tilde{g}}} \Lambda_k \operatorname{Pf}(A_k^{G, M_0}(w))$$

■

Proof of Corollary 3.7.

Let $G = (V, E)$ a graph of orientable genus g . The surface in which it can be minimally embedded can be represented as a (fundamental) polygon with $4g$ sides to be identified pairwise. Conventionally, the polygon can be represented as a succession of translations along the sides, round the polygon, as

$$(4.26) \quad P_g = A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1}$$

where $A_1, B_1, \dots, B_g^{-1}$ are the labels of the sides, and the index $^{-1}$ indicates an orientation opposite to the previous one.

On the other hand, a non orientable surface with genus \tilde{g} can be represented as a fundamental polygon with $2\tilde{g}$ sides, in the form

$$(4.27) \quad \tilde{P}_{\tilde{g}} = C_0 C_0 C_1 C_1 \cdots C_{\tilde{g}-1} C_{\tilde{g}-1}$$

where each side C_k is followed twice in the same direction.

Now we consider (4.26) as an element of the free group \mathbb{F}_{2g} generated by the $2g$ generators $A_1, B_1, \dots, A_g, B_g$. and we show hereafter that it can be set in the form (4.27) with $\tilde{g} = 2g + 1$.

We define $S_0 = T_0 = 1_{\mathbb{F}_{2g}}$ and recursively for all $1 \leq k \leq g$,

$$(4.28) \quad \begin{aligned} S_k &= S_{k-1} A_k B_k A_k^{-1} B_k^{-1} \\ T_k &= B_k A_k T_{k-1} \end{aligned}$$

For every $1 \leq k \leq g$, we write

$$(4.29) \quad \begin{aligned} U_k &= S_{k-1} A_k B_k T_{k-1} \\ V_k &= T_{k-1}^{-1} B_k^{-1} A_k^{-1} S_{k-1}^{-1} T_{k-1}^{-1} B_k T_{k-1} \\ W_k &= T_{k-1}^{-1} B_k^{-1} T_{k-1} S_{k-1} T_{k-1} \end{aligned}$$

From these definitions, it follows that ,

$$(4.30) \quad U_1(V_1V_1)(W_1W_1) = A_1^{-1}B_1^{-1} = T_0A_1^{-1}B_1^{-1}$$

and for all $k \geq 2$,

$$(4.31) \quad U_k(V_kV_k)(W_kW_k)U_{k-1}^{-1} = T_{k-1}^{-1}(A_k^{-1}B_k^{-1})(B_{k-1}A_{k-1})T_{k-2}$$

Using these relations, we compute

$$(4.32) \quad \begin{aligned} & (U_gU_g)(V_gV_g)(W_gW_g) \cdots (V_kV_k)(W_kW_k) \cdots (V_1V_1)(W_1W_1) \\ &= (S_{g-1}A_gB_gT_{g-1})(U_gV_gV_gW_gW_gU_{g-1}^{-1}) \cdots \\ & \quad \times \cdots (U_kV_kV_kW_kW_kU_{k-1}^{-1}) \cdots (U_1V_1V_1W_1W_1) \\ &= (S_{g-1}A_gB_gT_{g-1})(T_{g-1}^{-1}A_g^{-1}B_g^{-1}A_{g-1}B_{g-1}T_{g-2}) \cdots \\ & \quad \times \cdots (T_{k-1}^{-1}A_k^{-1}B_k^{-1}A_{k-1}B_{k-1}T_{k-2}) \cdots (A_1^{-1}B_1^{-1}) \\ &= S_{g-1}A_gB_gA_g^{-1}B_g^{-1} \end{aligned}$$

Which gives the identity

$$(4.33) \quad \begin{aligned} & (U_gU_g)(V_gV_g)(W_gW_g) \cdots (V_kV_k)(W_kW_k) \cdots (V_1V_1)(W_1W_1) \\ &= (A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1}) \cdots (A_gB_gA_g^{-1}B_g^{-1}) \end{aligned}$$

In terms of elements of the free group \mathbb{F}_{2g} , we have thus proven an identity between P_g and $\tilde{P}_{\tilde{g}}$, with $\tilde{g} = 2g + 1$, with the following choice for the factors C_j , $0 \leq j \leq 2g$,

$$(4.34) \quad C_0 = U_g$$

and for all $0 \leq k \leq g - 1$,

$$(4.35) \quad C_{2k+1} = V_{g-k}$$

$$(4.36) \quad C_{2k+2} = W_{g-k}$$

This decomposition proves directly that a graph of orientable genus g can be embedded in a nonorientable surface with $2g + 1$ crosscaps and incidentally leads to a proof of inequality (3.6), longer but distinct from the original one [31]. In Figure 3, we give an example of transformation of a fundamental domain for a surface of orientable genus 1 into that of a surface of nonorientable genus 3, according to Identity (4.33). In Figure 4, we show the embedding of a graph of orientable genus 1 in a nonorientable surface with three crosscaps which results from decomposition (4.34)–(4.35), starting from an orientable embedding.

Here we are interested in the explicit realization of such a nonorientable embedding. We introduce a multicomplex algebra \mathbb{C}_{2g+1} with generators i_0, i_1, i_{2g} , so that $i_k^2 = -1$ and $i_k i_{k'} = i_{k'} i_k$. We now rely on the construction given in the proof of Theorem 3.2. We attach generator i_k to the crosscap $[C_k C_k]$. Considering again a graph $\tilde{G} \succcurlyeq G$ such that an edge crosses at most one crosscap, we recall that the edge entries in the incidence matrix $A_{(v,e),(w,e)}^{\tilde{G}}$ are linear in i_k if and only if edge e crosses the k^{th} crosscap. By Theorem 3.9, the induced incidence matrix for the original graph G gets a similar property: an edge on G corresponds to a simple path in \tilde{G} and its edge entry in A^G is proportional to the product of all edge entries in $A^{\tilde{G}}$ defined on this path. Accordingly, an edge entry in A^G gets a factor i_k each times the edge crosses the k^{th} crosscap, and thus is linear in i_k if and only if edge e crosses the k^{th} crosscap an odd number of times.

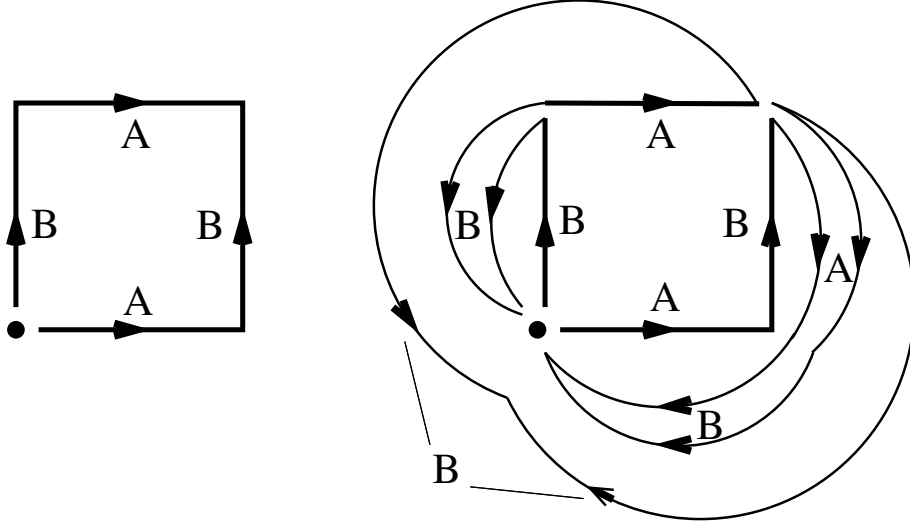


FIGURE 3. Left: Fundamental polygon for a surface of genus one, described as a single curve closing at the point marked with a \bullet . Right: transformation of the same curve using decomposition (4.35). The resulting curve now passes three times at the marked point; After identification of these three points, the curve splits into three lobes, each having the structure of a crosscap.

Now starting from an orientable embedding of G , we have to read off from expressions (4.34)–(4.35), which crosscaps are crossed oddwise by an edge in G which was originally crossing (once) the boundary of the fundamental domain through side A_k (respectively B_k), for all $1 \leq k \leq g$. For this purpose, we have thus to determine for each C_j , $0 \leq j \leq 2g$, the set of generators $\mathcal{O}(C_j)$ which occur an odd number of times. We have

$$(4.37) \quad \mathcal{O}(C_0) = \{A_j, B_j; 1 \leq j \leq g\}$$

and for all $k \in \{0, \dots, g-1\}$,

$$\begin{aligned} \mathcal{O}(C_{2k+1}) &= \{A_j, B_j; 1 \leq j \leq g-k-1\} \cup \{A_{g-k}\} \\ \mathcal{O}(C_{2k+2}) &= \{A_j, B_j; 1 \leq j \leq g-k-1\} \cup \{B_{g-k}\} \end{aligned}$$

It is easy to see that each generator appears in an even number of sets. Thus, for any edge crossing the boundary of the fundamental domain, the corresponding edge entry gets proportional to the product of an even number of generators in \mathbb{C}_{2g+1} . Each matrix entry can be thus considered as an element of a subalgebra with generators $e_k = i_0 i_k$, with $e_k e_{k'} = e_{k'} e_k$ and $e_k^2 = 1$.

Simultaneously, the constant Λ defined in Equation (4.19) can be replaced by an element of the subalgebra without changing the real part of $\Lambda \text{Pf}(A^{G, M_0})(w)$. Namely, we set

$$(4.38) \quad \Lambda = \frac{1}{f_0} \left(\sum_{\substack{S \subset \{1, \dots, 2g\} \\ |S| \text{ even}}} (-1)^{\frac{|S|}{2}} \prod_{j \in S} e_j + \sum_{\substack{S \subset \{1, \dots, 2g\} \\ |S| \text{ odd}}} (-1)^{\frac{|S|-1}{2}} \prod_{j \in S} e_j \right)$$

We follow the same way as in the proof of Corollary 3.7. We denote by \mathbb{R}_{2g} the real ring generated by $\{e_j\}_{1 \leq j \leq 2g}$. There are 2^{2g} distinct algebra homomorphisms $H_k : \mathbb{R}_{2g} \rightarrow \mathbb{R}$,

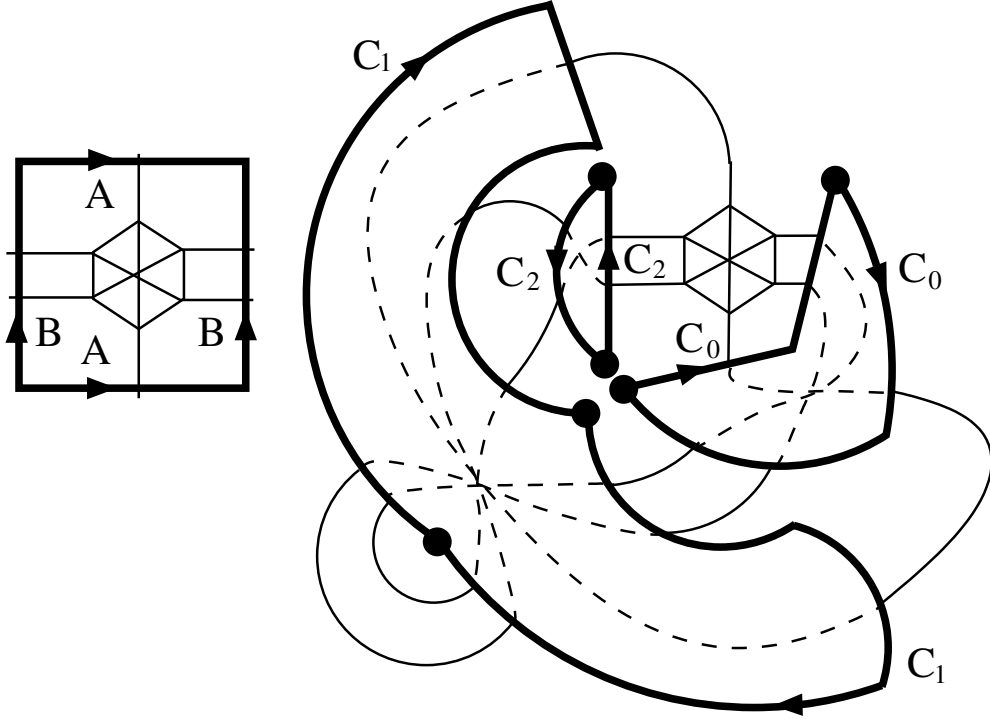


FIGURE 4. Left: Embedding on the torus of a graph of orientable genus 1. Right: Derived embedding in a surface with tree crosscaps, with $C_0 = AB$, $C_1 = B^{-1}A^{-1}B$, $C_2 = B^{-1}$. Multiple occurrences of the same boundary are identified in two ways: inside each crosscap, opposite boundaries are identified (dashed lines); outside crosscaps, the paired new boundaries are joined by solid lines. By construction, edges do not cross outside crosscaps. Note that every edge goes through a crosscap an even number of times.

$k \in \{1, \dots, 2^{2g}\}$ such that for all $j \in \{1, \dots, 2g\}$, $H_k(e_j) \in \{-1, +1\}$, and we have for every element $w \in \mathbb{C}$,

$$(4.39) \quad \operatorname{Re}(w) = \frac{1}{2^{2g}} \sum_{k=1}^{2^{2g}} H_k(w)$$

Here we set

$$(4.40) \quad \Lambda_k = \frac{1}{2^g} H_k(\Lambda)$$

where Λ is defined by Equation (4.38) and for all $(d_1, d_2) \in V_{\mathcal{D}}(G) \times V_{\mathcal{D}}(G)$

$$(4.41) \quad (A_k^G)_{d_1, d_2} = H_k(A_{d_1, d_2}^G)$$

Finally, we get

$$(4.42) \quad Z_G(w) = w(M_0 \cap E_{\mathcal{D}}^E(G)) \sum_{k=1}^{2^{2g}} \Lambda_k \operatorname{Pf}(A_k^{G, M_0}(w))$$

■

Proof of Theorem 3.9.

We use the same notations as in the proof of Proposition 2.43. We first prove that if $G_1 \preceq G_2$, there is an isomorphism between $\mathcal{C}(G_1)$ and the subset of closed curves on $\mathcal{C}(G_2)$ not containing deleted edges,

$$(4.43) \quad \check{\mathcal{C}}(G_2) = \{C \in \mathcal{C}(G_2), C \cap E_2^d = \emptyset\}$$

Under contraction, vertices connected through contracted edges are identified to a single vertex, which degree is thus equal modulo 2 to the sum of the degrees of the initial vertices. In particular, closed curves are sent to closed curves, and this allows to define a mapping π_0 from $\check{\mathcal{C}}(G_2)$ to $\mathcal{C}(G_1)$, which is surjective by hypothesis since $G_1 \preceq G_2$. The mapping is also injective: if two elements in $\check{\mathcal{C}}$ have the same image in $\mathcal{C}(G_1)$, they have the same intersection with $E_2 \setminus (E_2^c \cup E_2^d)$, so that their symmetric difference has its support in E_2^c and thus vanish, since E_2^c contains no cycle. This defines a canonical injective mapping $\Pi : \mathcal{C}(G_1) \rightarrow \mathcal{C}(G_2)$, such that for all $C \in \mathcal{C}(G_1)$,

$$(4.44) \quad \Pi(C) = \pi_0^{-1}(C)$$

Let A^{G_2} be an incidence matrix on $\mathcal{D}(G_2)$. We first construct a modified incidence matrix \check{A}^{G_2} as follows. For every pair of darts d_1, d_2 in $V_{\mathcal{D}}(G_2)$, let $d_1 = (v_1, e_1)$ and $d_2 = (v_2, e_2)$; we set

$$(4.45) \quad \check{A}_{d_1, d_2}^{G_2} = \begin{cases} 0 & \text{if } v_1 = v_2 \text{ and } \{e_1, e_2\} \cap E_2^d \neq \emptyset \\ A_{d_1, d_2}^{G_2} & \text{otherwise} \end{cases}$$

Hereafter we take $M_2 = E_{\mathcal{D}}^E(G_2) \cong E_2$ as the reference perfect matching on $\mathcal{D}(G_2)$ and consider a closed curve $C \in \check{\mathcal{C}}(G_2)$. Thus $C \cap E_2^d = \emptyset$ and Equation (2.20) implies that every perfect matching M in $\phi_{M_2}^{-1}(C)$ contains all pairs of darts $\{(v, e), (v', e)\}$ in $E_{\mathcal{D}}(G_2)$ such that $e \in E_2^d$. In particular, \check{A}^{G_2} and A^{G_2} coincide on every element of M and from Definition (2.33), we get

$$(4.46) \quad F_{\check{A}^{G_2}}(M_2, C) = F_{A^{G_2}}(M_2, C)$$

for all $C \in \check{\mathcal{C}}(G_2)$.

Conversely, consider a closed curve $C \in \mathcal{C}(G_2) \setminus \check{\mathcal{C}}(G_2)$. By definition, there exists a pair of darts d, d' in $V_{\mathcal{D}}(G_2)$ with $d = (v, e)$, $d' = (v', e)$ and $e \in C \cap E_2^d$. Since $\{d, d'\} \in M_2$, Equation (2.20), implies that for every perfect matching M in $\phi_{M_2}^{-1}(C)$, $\{d, d'\} \notin M$. Hence, for every such M , there is $d'' = (v, e'')$ with $\{d, d''\} \in M$ and by (4.45), $\check{A}_{d, d''}^{G_2} = 0$. The contribution of every perfect matching M in $\phi_{M_2}^{-1}(C)$ is thus zero and

$$(4.47) \quad F_{\check{A}^{G_2}}(M_2, C) = 0$$

for all $C \in \mathcal{C}(G_2) \setminus \check{\mathcal{C}}(G_2)$.

We now relate the terms in the Pfaffian expansion of \check{A}^{G_2} to the corresponding terms in the Pfaffian expansion of the incidence matrix on G_1 constructed using Proposition 2.43. First, for every weight function $w : E_1 \rightarrow \mathbb{R}^+$, we define its extension \tilde{w} on E_2 , as

$$(4.48) \quad \tilde{w}(e) = \begin{cases} w(\tilde{e}) & \text{if } e \in E_2 \setminus (E_2^c \cup E_2^d) \\ 1 & \text{if } e \in E_2^c \cup E_2^d \end{cases}$$

where \tilde{e} is the image in E_1 of edge e in $E_2 \setminus (E_2^c \cup E_2^d)$.

Let $\check{G} = (\check{V}, \check{E})$ be the subgraph of G_2 with edge set $\check{E} = E_2^c \cup E_2^d$, and vertex set \check{V} the set of vertices of G_2 incident with some edge in \check{E} . The set of darts on \check{G} is the set K defined in Equation (4.8). By construction there is no closed curve on \check{G} except the null one and those with a non empty intersection with E_2^d . Since the arguments used to get equations (4.46)–(4.47) apply equally to \check{G} , the nonzero terms in the expansion of $\text{Pf}([\check{A}^{G_2, M_0}(\tilde{w})]_K)$ are associated to closed curves with no edge in E^d . Thus only the set of matchings associated to the null curve contributes, and it has only one element, $M_2|_{\check{E}}$. More precisely, for every weight function w on E_1 , we have

$$(4.49) \quad \text{Pf}([\check{A}^{G_2, M_0}(\tilde{w})]_K) = \pm 1 \neq 0$$

which is thus independent on w .

The Pfaffian reduction formula 2.41 reads here

$$(4.50) \quad Pf([\check{A}^{G_2, M_0}(\tilde{w})]_{\check{K}}) = \left(\text{Pf}([\check{A}^{G_2, M_0}(\tilde{w})]_K) \right)^{(n-p-1)} \times \text{Pf}(\check{A}^{G_2, M_0}(\tilde{w}))$$

where $2n = |V_{\mathcal{D}}(G_2)|$ and $p = |E_2^c \cup E_2^d|$.

By Proposition 2.43, we already know that $[\check{A}^{G_2}]_{\check{K}}$ is an incidence matrix on $\mathcal{D}(G_1)$. We claim that the matrix $[\check{A}^{G_2, M_0}(\tilde{w})]_{\check{K}}$ is that the weighted incidence matrix associated to it on $\mathcal{D}(G_1)$ for the weight function w and reference perfect matching $M_1 = E_{\mathcal{D}}^E(G_1) \cong E_1$. Therefore expanding both sides in equation (4.50), for a generic weight function w on E_1 , and identifying terms on both sides using Equation (2.33), one gets

$$(4.51) \quad \begin{aligned} F_{[\check{A}^{G_2}]_{\check{K}}}(M_1, C) &= \Lambda_{1,2} F_{\check{A}^{G_2}}(M_2, \Pi(C)) \\ &= \Lambda_{1,2} F_{A^{G_2}}(M_2, \Pi(C)) \end{aligned}$$

for all $C \in \mathcal{C}(G_1)$, with $\Lambda_{1,2} = \pm 1$ by Equation (4.49) and Π is the canonical mapping from $\mathcal{C}(G_1)$ onto $\check{\mathcal{C}}(G_2)$.

Suppose now that the partition function on G_2 admits a Pfaffian representation. Thus by Lemma 2.34, there exists an incidence matrix A^{G_2} on $\mathcal{D}(G_2)$ such that $F_{\check{A}^{G_2}}(M_2, \cdot)$ is constant and different from zero on $\check{\mathcal{C}}(G_2)$. By (4.51) and (4.49), $F_{[\check{A}^{G_2}]_{\check{K}}}(M_1, \cdot)$ is also constant and different from zero on $\mathcal{C}(G_1)$. Using again Lemma 2.34, this implies that the partition function on G_1 also admits a Pfaffian representation.

In order to conclude the proof, we have to prove the claim.

Given our choice of reference perfect matching, the characterization (2.25) reduces to the identity

$$(4.52) \quad [\check{A}^{G_2, M_0}(\tilde{w})]_{d_1, d_2}^{\check{K}} = w^{-1}(\{d_1, d_2\}) [\check{A}^{G_2}]_{d_1, d_2}^{\check{K}}$$

which has to be checked for every pair of darts d_1, d_2 with a common edge in $E_2 \setminus (E_2^c \cup E_2^d)$.

Let $d_1 = (v_1, e), d_2 = (v_2, e)$ be such a pair of darts. The graph $\check{G}_e = (\check{V} \cup \{v_1, v_2\}, \check{E} \cup \{e\})$ is again a subgraph of G_2 , and every closed curve on \check{G}_e has either one edge in E_2^c or has no edge. We have

$$(4.53) \quad [\check{A}^{G_2, M_0}(\tilde{w})]_{d_1, d_2}^{\check{K}} = \text{Pf}([\check{A}^{G_2, M_0}(\tilde{w})]_{K \cup \{d_1, d_2\}})$$

and the expansion of the Pfaffian on the right hand side thus contains only one term, corresponding to the restriction of the reference perfect matching M_0 on $\mathcal{D}(\check{G}_e)$. In particular, we

have

$$\begin{aligned}
 \text{Pf}([\check{A}^{G_2, M_0}(\tilde{w})]_{K \cup \{d_1, d_2\}}) &= \left(\prod_{e' \in \check{E}_e} \tilde{w}^{-1}(e') \right) \text{Pf}([\check{A}^{G_2}]_{K \cup \{d_1, d_2\}}) \\
 &= w^{-1}(\tilde{e}) \text{Pf}([\check{A}^{G_2}]_{K \cup \{d_1, d_2\}}) \\
 (4.54) \qquad \qquad \qquad &= w^{-1}(\tilde{e}) [\check{A}^{G_2}]_{d_1, d_2}^{\overline{K}}
 \end{aligned}$$

Thus Equation (4.52) holds. The theorem is proven. ■

Proof of Proposition 3.11.

Consider first the complete bipartite graph $K_{3,3}$. Since it is 3-regular, its dart graph $\mathcal{D}(K_{3,3})$ is also 3-regular and there is as many perfect matchings on $\mathcal{D}(K_{3,3})$ as there are closed curves on $K_{3,3}$. Thus given a reference perfect matching M_0 , the mapping ϕ_{M_0} defined in Equation (2.20) is one to one.

We label the vertices and edges of $K_{3,3}$ by letters in $\{a, b, c, d, e, f\}$ and numbers in $\{1, \dots, 9\}$, respectively, as in Figure 5 and if vertex v is incident with edge k , we use the short hand notation v_k to denote the dart (v, k) .

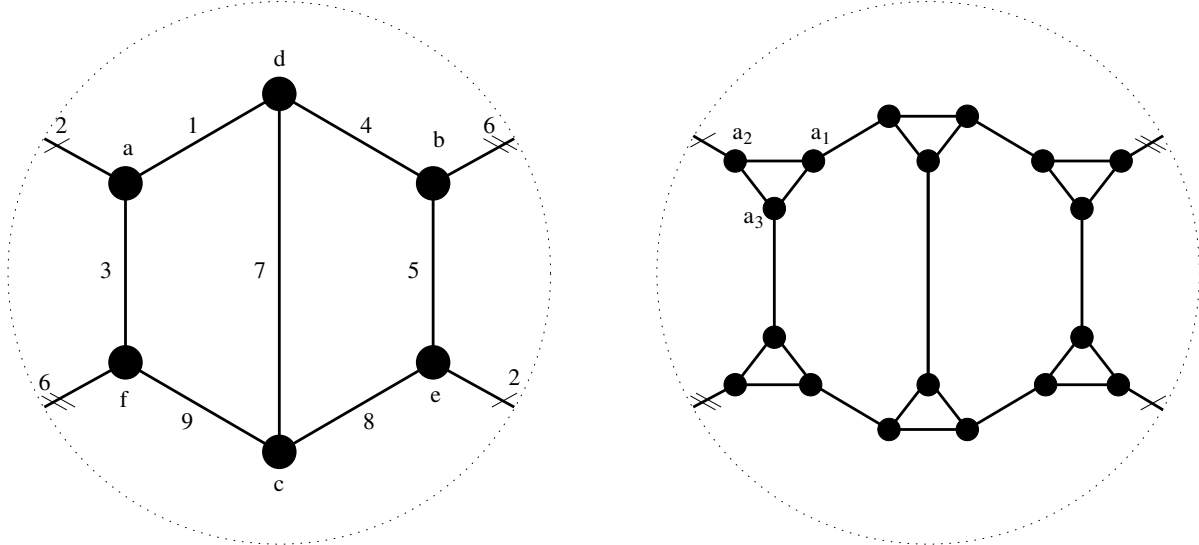


FIGURE 5. Representation of graph $K_{3,3}$ in the projective plane (Left) and its dart graph (Right). Vertices and edges are labelled with letters and numbers, as used in text; darts are named accordingly.

We consider two sets of cycles on $K_{3,3}$, $S = \{\gamma_1, \gamma_2, \gamma_3\}$ and $S' = \{\gamma'_1, \gamma'_2, \gamma'_3\}$, where

$$\begin{aligned}
 (4.55) \qquad \gamma_1 &= \{a, 1, d, 4, b, 6, f, 9, c, 8, e, 2\} \\
 \gamma_2 &= \{a, 1, d, 7, c, 9, f, 3\} \\
 \gamma_3 &= \{b, 4, d, 7, c, 8, e, 5\}
 \end{aligned}$$

and

$$(4.56) \quad \begin{aligned} \gamma'_1 &= \{a, 1, d, 4, b, 5, e, 8, c, 9, f, 3\} \\ \gamma'_2 &= \{a, 1, d, 7, c, 8, e, 2\} \\ \gamma'_3 &= \{b, 4, d, 7, c, 9, f, 6\} \end{aligned}$$

The main property of these two sets is that any subchain of length 4 (a succession of two vertices and two edges in a cycle) appear in both sets with the same multiplicity. For instance the subchain $(d, 4, b, 6)$ appears both in γ_1 and γ'_3 . Now, drawing simultaneously all cycles in a set on the same surface, slightly shifting each drawing, results in a figure as depicted in Figure 6 with possibly some crossings between the lines of different cycles, and the number of crossings depends on the surface and the way the cycles are drawn. However, when drawing the two sets S and S' on two copies of the same surface, the parity of the number of crossings always differs in both drawings (Figure 6).

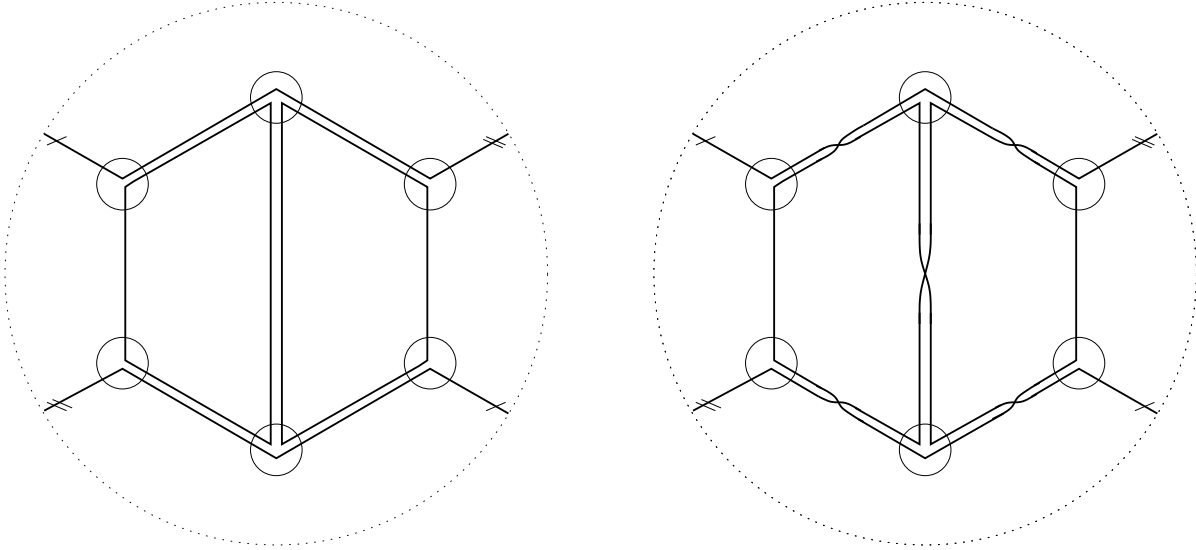


FIGURE 6. The two sets of cycles S (Left) and S' (Right) on $K_{3,3}$ are drawn using the representation of Figure 5. Both sets have locally the same configurations but parity of the numbers of crossings differ.

This property transfers to the perfect matchings on $\mathcal{D}(K_{3,3})$ in the following way: We take for instance $E_{\mathcal{D}}^E(K_{3,3})$ as reference perfect matching, that is, we set

$$(4.57) \quad \begin{aligned} M_0 &= \{\{a_1, d_1\}, \{a_2, e_2\}, \{a_3, f_3\}, \{b_4, d_4\}, \\ &\quad \{b_5, e_5\}, \{b_6, f_6\}, \{c_7, d_7\}, \{c_8, e_8\}, \{c_9, f_9\}\} \end{aligned}$$

so that the two sets of cycles S and S' are associated to two sets of perfect matchings, $S_M = \{M_1, M_2, M_3\}$ and $S'_M = \{M'_1, M'_2, M'_3\}$, respectively, where

$$\begin{aligned} M_1 &= \{\{a_3, f_3\}, \{b_5, e_5\}, \{c_7, d_7\}, \{a_1, a_2\}, \{b_4, b_6\}, \{c_8, c_9\}, \{d_1, d_4\}, \{e_2, e_8\}, \{f_6, f_9\}\} \\ M_2 &= \{\{a_2, e_2\}, \{b_4, d_4\}, \{b_5, e_5\}, \{b_6, f_6\}, \{c_8, e_8\}, \{a_1, a_3\}, \{c_7, c_8\}, \{d_1, d_7\}, \{f_3, f_9\}\} \\ M_3 &= \{\{a_1, d_1\}, \{a_2, e_2\}, \{a_3, f_3\}, \{b_6, f_6\}, \{c_9, f_9\}, \{b_4, b_5\}, \{c_7, c_9\}, \{d_4, d_7\}, \{e_5, e_8\}\} \end{aligned}$$

and

$$\begin{aligned}
 M'_1 &= \{\{a_2, e_2\}, \{b_6, f_6\}, \{c_7, d_7\}, \{a_1, a_3\}, \{b_4, b_5\}, \{c_8, c_9\}, \{d_1, d_4\}, \{e_5, e_8\}, \{f_3, f_9\}\} \\
 M'_2 &= \{\{a_3, f_3\}, \{b_4, d_4\}, \{b_5, e_5\}, \{b_6, f_6\}, \{c_9, f_9\}, \{a_1, a_2\}, \{c_7, c_8\}, \{d_1, d_7\}, \{e_2, e_8\}\} \\
 M'_3 &= \{\{a_1, d_1\}, \{a_2, e_2\}, \{a_3, f_3\}, \{b_5, e_5\}, \{c_8, e_8\}, \{b_4, b_6\}, \{c_7, c_9\}, \{d_4, d_7\}, \{f_6, f_9\}\}
 \end{aligned}$$

As a consequence of the properties of sets S and S' , here each dimer appears in both sets S_M and S'_M with the same multiplicity.

Let A be an arbitrary antisymmetric incidence matrix on $\mathcal{D}(K_{3,3})$. The contributions of the cycles in S and S' to the Pfaffian expansion of A (equation (2.33)) are related by

$$(4.58) \quad \prod_{i=1}^3 F_A(M_0, \gamma_i) = - \prod_{i=1}^3 F_A(M_0, \gamma'_i)$$

This relation derives from the properties of the two sets S and S' . It can also be seen also as a property of the perfect matchings in S_M and S'_M alone and any other choice of reference perfect matching would define two equivalent sets of cycles through equation (2.37). Thus for every antisymmetric incidence matrix on $\mathcal{D}(K_{3,3})$ and every reference perfect matching there exist two sets of closed curves such that relation (4.58) holds. By Lemma 2.34, the partition function on $K_{3,3}$ has no Pfaffian representation.

The proof for the complete graph K_5 is similar.

We label the vertices and edges of K_5 by letters in $\{a, b, c, d, e\}$ and numbers in $\{0, \dots, 9\}$, respectively, as in Figure 7 and we denote the dart (v, k) by v_k . K_5 is 4-regular so that the mapping (2.20) is not one to one: in fact K_5 has 64 closed curves and $\mathcal{D}(K_5)$ has 416 dimer coverings.

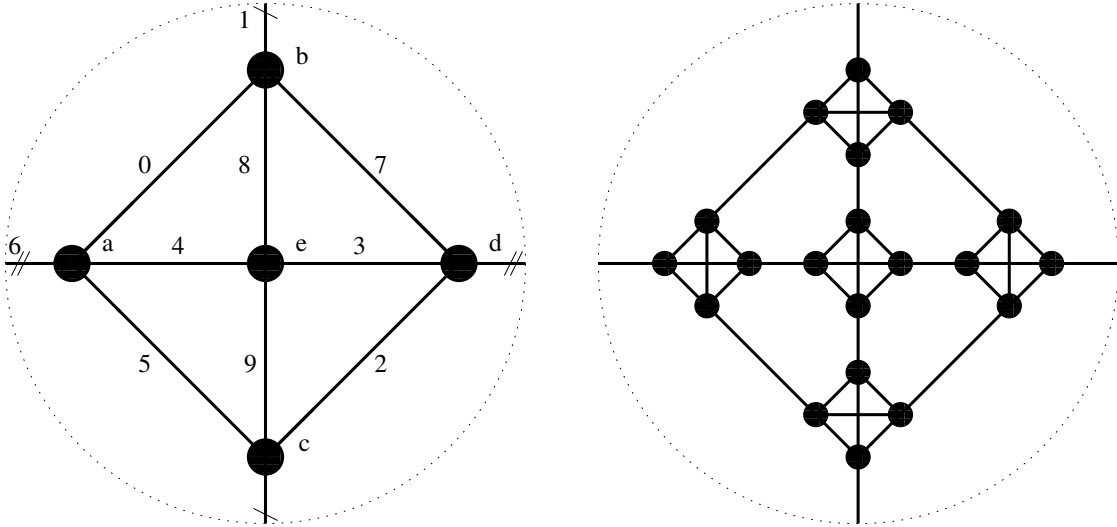


FIGURE 7. Representation of graph K_5 in the projective plane (Left) and its dart graph in the same representation (Right). Vertices and edges are labelled with letters and numbers, as used in text.

We consider two sets of cycles on K_5 , $S = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $S' = \{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4\}$, where

$$(4.59) \quad \begin{aligned} \gamma_1 &= \{a, 4, e, 3, d, 6\} \\ \gamma_2 &= \{a, 0, b, 1, c, 2, d, 6\} \\ \gamma_3 &= \{a, 0, b, 7, d, 2, c, 9, e, 4\} \\ \gamma_4 &= \{a, 0, b, 8, e, 3, d, 2, c, 5\} \end{aligned}$$

and

$$(4.60) \quad \begin{aligned} \gamma'_1 &= \{a, 4, e, 9, c, 2, d, 6\} \\ \gamma'_2 &= \{a, 0, b, 8, e, 3, d, 6\} \\ \gamma'_3 &= \{a, 0, b, 7, d, 2, c, 5\} \\ \gamma'_4 &= \{a, 0, b, 1, c, 2, d, 3, e, 4\} \end{aligned}$$

Here again, any subchain of length 4 (two vertices and two edges) appear in both sets with the same multiplicity. Now, drawing simultaneously all cycles in a set on the same surface, slightly shifting each drawing, results in a figure with some crossings between the lines of different cycles. Again, when drawing the two sets S and S' on two copies of the same surface, the parity of the number of crossings always differs in both drawings (Figure 8).

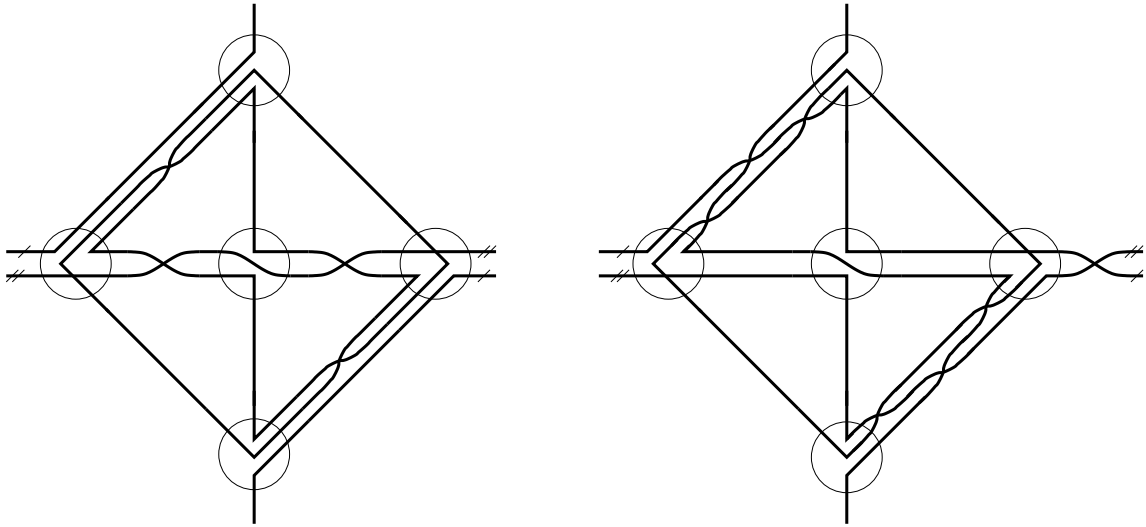


FIGURE 8. The two sets of cycles S (Left) and S' (Right) on K_5 are drawn using the representation of Figure 7. Both sets have locally the same configurations but parity of the numbers of crossings differ.

We take $E_{\mathcal{D}}^E(K_5)$ as reference perfect matching, that is

$$(4.61) \quad \begin{aligned} M_0 &= \{\{a_0, b_0\}, \{b_1, c_1\}, \{c_2, d_2\}, \{d_3, e_3\}, \{a_4, e_4\}, \\ &\quad \{a_5, c_5\}, \{a_6, d_6\}, \{b_7, d_7\}, \{b_8, e_8\}, \{c_9, e_9\}\} \end{aligned}$$

so that each cycle in the two sets S and S' are associated to exactly one perfect matchings in $S_M = \{M_1, M_2, M_3, M_4\}$ and $S'_M = \{M'_1, M'_2, M'_3, M'_4\}$, respectively, where

$$\begin{aligned} M_1 &= \{\{a_0, b_0\}, \{b_1, c_1\}, \{c_2, d_2\}, \{a_5, c_5\}, \{b_7, d_7\}, \{b_8, e_8\}, \{c_9, e_9\}, \{a_4, a_6\}, \{d_3, d_6\}, \{e_3, e_4\}\} \\ M_2 &= \{\{d_3, e_3\}, \{a_4, e_4\}, \{a_5, c_5\}, \{b_7, d_7\}, \{b_8, e_8\}, \{c_9, e_9\}, \{a_0, a_6\}, \{b_0, b_1\}, \{c_1, c_2\}, \{d_2, d_6\}\} \\ M_3 &= \{\{b_1, c_1\}, \{d_3, e_3\}, \{a_5, c_5\}, \{a_6, d_6\}, \{b_8, e_8\}, \{a_0, a_4\}, \{b_0, b_7\}, \{c_2, c_9\}, \{d_2, d_7\}, \{e_4, e_9\}\} \\ M_4 &= \{\{b_1, c_1\}, \{a_4, e_4\}, \{a_6, d_6\}, \{b_7, d_7\}, \{c_9, e_9\}, \{a_0, a_5\}, \{b_0, b_8\}, \{c_2, c_5\}, \{d_2, d_3\}, \{e_3, e_8\}\} \end{aligned}$$

and

$$\begin{aligned} M'_1 &= \{\{a_0, b_0\}, \{b_1, c_1\}, \{d_3, e_3\}, \{a_5, c_5\}, \{b_7, d_7\}, \{b_8, e_8\}, \{a_4, a_6\}, \{c_2, c_9\}, \{d_2, d_6\}, \{e_4, e_9\}\} \\ M'_2 &= \{\{b_1, c_1\}, \{c_2, d_2\}, \{a_4, e_4\}, \{a_5, c_5\}, \{b_7, d_7\}, \{c_9, e_9\}, \{a_0, a_6\}, \{b_0, b_8\}, \{d_3, d_6\}, \{e_3, e_8\}\} \\ M'_3 &= \{\{b_1, c_1\}, \{d_3, e_3\}, \{a_4, e_4\}, \{a_6, d_6\}, \{b_8, e_8\}, \{c_9, e_9\}, \{a_0, a_5\}, \{b_0, b_7\}, \{c_2, c_5\}, \{d_2, d_7\}\} \\ M'_4 &= \{\{a_5, c_5\}, \{a_6, d_6\}, \{b_7, d_7\}, \{b_8, e_8\}, \{c_9, e_9\}, \{a_0, a_4\}, \{b_0, b_1\}, \{c_1, c_2\}, \{d_2, d_3\}, \{e_3, e_4\}\} \end{aligned}$$

As a consequence of the properties of sets S and S' , each dimer appears in both sets S_M and S'_M with the same multiplicity.

Let A be an arbitrary antisymmetric incidence matrix on $\mathcal{D}(K_5)$. The contributions of the cycles in S and S' to the Pfaffian expansion of A (equation (2.33)) are related by

$$(4.62) \quad \prod_{i=1}^4 F_A(M_0, \gamma_i) = - \prod_{i=1}^4 F_A(M_0, \gamma'_i)$$

This relation derives from the properties of the two sets S and S' . It can also be seen also as a property of the perfect matchings in S_M and S'_M alone and any other choice of reference perfect matching would define two equivalent sets of cycles through equation (2.37). Thus for every antisymmetric incidence matrix on $\mathcal{D}(K_5)$ and every reference perfect matching there exist two sets of closed curves such that relation (4.62) holds. By Lemma 2.34, the partition function on K_5 has no Pfaffian representation. ■

Proof of Lemma 3.12.

Let $G = (V, E)$ be a finite 2-connected simple graph 2-cell embedded in some smooth surface Σ . We construct 4-regular graph $\tilde{G} \succcurlyeq G$ by a succession of elementary transformations such as vertex splitting, edge addition and edge subdivision [30], taking care that at each step, the resulting graph is still a 2-connected simple graph embeddable in the same surface.

We first state the following three claims, which we prove for any 2-connected simple graph G 2-cell embedded in some smooth surface Σ :

- Claim 1: If some vertex in graph G has odd valency, there exists a 2-connected simple graph $G' \succcurlyeq G$ embeddable in the same surface, with a strictly smaller number of vertices of odd valency.
- Claim 2: If all vertices in G have even valency and some vertex in G has valency larger than 4, there exists a 2-connected simple graph $G' \succcurlyeq G$ embeddable in the same surface, with all vertices of even valency and a strictly smaller number of vertices of valency larger than 4.

- **Claim 3:** If some vertex in G has valency 2 and all others of valency 4, there exists a 2-connected simple graph $G' \succcurlyeq G$ embeddable in the same surface, with a strictly smaller number of vertices of valency 2, and all others of valency 4.

Clearly, the proof of Lemma 3.12 follows easily from the above three claims and transitivity of $\cdot \succcurlyeq \cdot$: a finite iteration of Claim 1 leads to a graph $G' \succcurlyeq G$ with all vertices of even valency. Claim 2 allows then for the construction of a graph $G'' \succcurlyeq G'$ with all vertices of valency 2 or 4. Finally a repeated use of Claim 3 gives a 4-regular graph $G''' \succcurlyeq G''$.

We now prove the above three claims.

Proof of Claim 1. Suppose that G is a 2-connected simple graph 2-cell embedded in some smooth surface Σ and that some vertex has odd valency. Since the number of such vertices is necessarily even, one can pick a pair of them, says v_1, v_2 . If they belong to the same face boundary, we construct a graph G' by adding an edge $\{v_1, v_2\}$ (or a new vertex v and two edges $\{v_1, v\}, \{v, v_2\}$ if v_1 and v_2 are adjacent on G). If v_1, v_2 don't belong to the same face boundary, one considers a finite sequence of faces $\{F_j\}_{j=0,k}$ such that $v_1 \in \overline{F_0}$, $v_2 \in \overline{F_k}$ and the boundaries of F_{j-1} and F_j share at least one edge, says e_j , for all $1 < j \leq k$. We chose this sequence to be of minimal length and construct G' by replacing each edge e_j by a new vertex w_j and two new edges, each with endvertices w_j and a distinct endvertex of e_j (namely, subdividing each edge e_j), and adding $k+1$ edges $\{v_1, w_1\}, \{w_1, w_2\}, \dots, \{w_k, v_2\}$. In both cases, v_1 and v_2 have even valency in G' , while all new vertices have valency 2 or 4. In addition G' is a 2-connected simple graph since G is and the endvertices of new edges are not neighbors in G . Finally G' can be embedded in the same surface Σ since the transformation consists in splitting one or more faces of the embedding of G .

Proof of Claim 2. Suppose now that all vertices of G have even valency and let v a vertex of valency $r > 4$. We construct a new graph G' by replacing this vertex in G and its r edges by a 4-regular tree T_v with $\frac{r-2}{2}$ points, r external edges and $\frac{r-4}{2}$ internal edges. Since a sufficiently small neighborhood of x_v is homeomorphic to a disk and T_v of genus zero, one can identify the external edges of this tree with the edges of v in G in such a way that G' can still be embedded on Σ . By construction G' is still a simple graph and it is 2-connected since no new vertex is separating.

Proof of Claim 3. Suppose that some vertices of G have valency 2 and all others valency 4. Let v be a vertex of valency 2. If G is a simple graph distinct from K_3 , one of the faces of the embedding which contains v has length at least 4, so one can pick two distinct edges on the boundary of this face with endvertices distinct from v . Now one construct G' from G by adding one subdivision vertex on these two edges, says w_1 and w_2 and the three edges $\{v, w_1\}, \{v, w_2\}$ and $\{w_1, w_2\}$. the three points v, w_1 and w_2 have valency 4 in G' so G' has one point of valency 2 less than G . Furthermore the transformation consists in splitting a face so that G' can still be embedded in Σ . Furthermore G' is again a 2-connected simple graph. If G identifies to the graph K_3 , one may consider $K_{2,2} \succcurlyeq K_3$, and proceed as above. ■

Proof of Proposition 3.13.

Let $G = (V, E)$ be a planar 2-connected 4-regular graph and \mathcal{B}_G a cycle basis on G as in McLane's planarity criterion 1.1. An important property of such a basis is that none of its subsets can form a double cover of a proper subset of $E(v)$, for any vertex $v \in V$. For completeness, we give here a short proof of this property [20]:

Suppose that this property is false at some vertex v , and call a cluster the proper subset of $E(v)$ covered twice. Since \mathcal{B}_G verifies McLane's criterion, every element of \mathcal{B}_G either belong to the covering and has two edges in the cluster, or does not belong to it, and has thus 0 edges in it. The intersection with the cluster thus contains an even number of edges and this holds true also for all elements generated from \mathcal{B}_G by finite difference. However, by hypothesis, there is at least one edge at v outside the cluster, and since G is 2-connected, there exists a cycle in $\mathcal{C}(G)$ containing whichever pair of edges in $E(v)$; and in particular exactly that edge and one edge in the cluster. This is in contradiction with the fact that \mathcal{B}_G is a cycle basis for $\mathcal{C}(G)$, which proves the above property.

We now consider an incidence matrix A^G and notations as in (3.18)–(3.27). We prove that conditions of Proposition 3.28 can hold simultaneously for all cycles in \mathcal{B}_G .

We first consider the site equations (3.29). Let $v \in V$ and $\gamma \in \mathcal{B}_G$ such that $\gamma \ni v$. When rewritten in the original variables (see Table 1), the related site equation may appear in three different forms, depending on which edges in $E(v)$ belong to γ :

$$(4.63) \quad \begin{aligned} s_v \bar{s}_v + t_v \bar{t}_v &= 0 && \text{if } \gamma \cap E(v) = \{e_1, e_4\} \text{ or } \{e_2, e_3\} \\ t_v \bar{t}_v + u_v \bar{u}_v &= 0 && \text{if } \gamma \cap E(v) = \{e_1, e_2\} \text{ or } \{e_3, e_4\} \\ u_v \bar{u}_v + s_v \bar{s}_v &= 0 && \text{if } \gamma \cap E(v) = \{e_1, e_3\} \text{ or } \{e_2, e_4\} \end{aligned}$$

Equations at different vertices are obviously independent. Suppose that there are two cycles in \mathcal{B}_G passing through v and associated to a different equation in (4.63). They share necessarily exactly one edge in $E(v)$. Now take any other cycle in \mathcal{B}_G passing through v . By McLane's criterion, it can neither share the common edge (already appearing twice), nor share a distinct edge with each (which would form a cluster of 3). Thus it is necessarily disjoint in $E(v)$ from one of the other cycle, and the associated equation is the same. Thus at every vertex v , the site equations associated to all elements of \mathcal{B}_G are at most two, and admit a nowhere zero solution.

Note that all three equations together have no nowhere zero solution as it would give $s_v \bar{s}_v = t_v \bar{t}_v = u_v \bar{u}_v = 0$.

We suppose from now on that site entries of the incidence matrix have been given non zero values so that all site equations hold for all $\gamma \in \mathcal{B}_G$, and we consider the set of edge equations.

We consider a cycle $\gamma = (V_\gamma, E_\gamma)$ in \mathcal{B}_G with its elements ordered as in Equations (3.21)–(3.22). For every pair $(v, e) \in V_\gamma \times E_\gamma$ with v incident on e , we define the following ratio

$$(4.64) \quad R_{v,e}^\gamma = \begin{cases} \frac{U_i^\gamma \bar{T}_i^\gamma}{S_i^\gamma} & \text{if } \exists i \in \{1, \dots, r_\gamma\} \text{ such that } e = e_i^\gamma \text{ and } v = v_i^\gamma \\ \frac{U_{i+1}^\gamma T_{i+1}^\gamma}{\bar{S}_{i+1}^\gamma} & \text{if } \exists i \in \{1, \dots, r_\gamma\} \text{ such that } e = e_i^\gamma \text{ and } v = v_{i+1}^\gamma \end{cases}$$

The ratio $R_{v,e}^\gamma$ depends on the intersection of γ with $E(v)$, and is independent on its orientation. On each vertex, this ratio may take twelve different expressions in terms of the edge entries, which can be computed from Table 1 and are reported here in Table 2.

$e \ e'$	e_v^1	e_v^2	e_v^3	e_v^4
e_v^1		$\frac{s_v u_v}{\bar{t}_v}$	$\frac{t_v s_v}{\bar{u}_v}$	$\frac{u_v t_v}{\bar{s}_v}$
e_v^2	$\frac{s_v \bar{u}_v}{t_v}$		$\frac{\bar{u}_v \bar{t}_v}{\bar{s}_v}$	$\frac{\bar{t}_v s_v}{u_v}$
e_v^3	$\frac{t_v \bar{s}_v}{u_v}$	$\frac{\bar{u}_v t_v}{s_v}$		$\frac{\bar{s}_v \bar{u}_v}{\bar{t}_v}$
e_v^4	$\frac{u_v \bar{t}_v}{s_v}$	$\frac{\bar{t}_v \bar{s}_v}{\bar{u}_v}$	$\frac{\bar{s}_v u_v}{t_v}$	

TABLE 2. Expression of the ratio $R_{v,e}^\gamma$ for a vertex v , an edge e and a cycle γ such that $\gamma \cap E(v) = \{e, e'\}$. This expression is written as a function of the location of e, e' in the ordered set $E(v) = \{e_v^1, e_v^2, e_v^3, e_v^4\}$.

Let e be an edge on graph G with endvertices v and w , and γ be a cycle in \mathcal{B}_G containing e . The associated edge equation reads

$$(4.65) \quad b_e^2 = -R_{v,e}^\gamma R_{w,e}^\gamma$$

Now consider two distinct cycles γ and γ' in \mathcal{B}_G passing through a same edge e with endvertex v . By McLane's criterion, they cannot have two successive edges in common, since it would form a cluster of two edges. Thus, up to a permutation, and possibly, exchanging the roles of γ and γ' , one can suppose that $\gamma \cap E(v) = \{e_v^1, e_v^4\}$ and $\gamma' \cap E(v) = \{e_v^1, e_v^2\}$, so that e_v^1 is their common edge at v . From Table 2, the associated ratios read

$$R_{v,e_v^1}^\gamma = \frac{u_v t_v}{\bar{s}_v}$$

$$R_{v,e_v^1}^{\gamma'} = \frac{s_v u_v}{\bar{t}_v}$$

The site equation associated to γ at v reads

$$s_v \bar{s}_v + t_v \bar{t}_v = 0$$

which implies

$$R_{v,e_v^1}^\gamma = -R_{v,e_v^1}^{\gamma'}$$

This relation is invariant under any even permutation on $E(v)$, so we get the following result. Two distinct cycles γ and γ' in \mathcal{B}_G passing through a same edge e with endvertex v have opposite ratio:

$$(4.66) \quad R_{v,e}^\gamma = -R_{v,e}^{\gamma'}$$

Thus, since the left hand side of Equation (4.65) is quadratic in the ratios, it is invariant when considering different cycles in \mathcal{B}_G passing through the same edge.

Thus edge equations are identical for different cycles at the same edge, and are also algebraically independent for different edges. Hence, the whole set of edge equation can be solved simultaneously for all edges and all cycles in \mathcal{B}_G .

We now assume that the coefficients of the incidence matrix have been chosen so that both site and edge equations are fulfilled, and consider the cycle equations (3.31), for all cycles in \mathcal{B}_G . We first note that these equations are not fully independent from the edge and site equations. Multiplying all edge equations associated to a given cycle $\gamma \in \mathcal{B}_G$ gives

$$(4.67) \quad \prod_{i=1}^{r_\gamma} (B_i^\gamma)^2 = (-1)^{r_\gamma} \prod_{i=1}^{r_\gamma} \frac{T_i^\gamma \bar{T}_i^\gamma (\bar{U}_i^\gamma)^2}{S_i^\gamma \bar{S}_i^\gamma} = \prod_{i=1}^{r_\gamma} (\bar{U}_i^\gamma)^2$$

where the last equality comes from the use of site equations (3.29). Thus for every cycle $\gamma \in \mathcal{B}_G$, we have

$$(4.68) \quad \prod_{i=1}^{r_\gamma} B_i^\gamma = -\epsilon_\gamma \prod_{i=1}^{r_\gamma} \bar{U}_i^\gamma$$

with $\epsilon_\gamma \in \{-1, 1\}$.

We thus define a new set of variables $\{\epsilon_e\}_{e \in E}$ with value in $\{-1, 1\}$, and consider the set of equations

$$(4.69) \quad \prod_{e \in \gamma} \epsilon_e = \epsilon_\gamma$$

for all γ in \mathcal{B}_G .

Now the independence of the cycles as elements of the (vector) basis \mathcal{B}_G implies the independence of equations in (4.69) and thus insures the existence of a solution with $\{\epsilon_e\}_{e \in E}$ in $\{-1, 1\}$. Now for every edge e such that $\epsilon_e = -1$, we change the sign of the two associated edge entries in the incidence matrix. The new edge and site entries are all nonzero and such that Equations (3.29)–(3.31) hold for all cycles in \mathcal{B}_G . Thus, by Proposition 3.28, there exists a reference perfect matching M_0 such that Equation (3.32) holds for all cycle $\gamma \in \mathcal{B}_G$. By Lemma 2.34, the partition function on G admits a Pfaffian representation. ■

Proof of Lemma 3.24.

For every vertex v , the set of permutations on $E(v)$ is homeomorphic to \mathcal{S}_4 . There are exactly two permutations σ in \mathcal{S}_4 , with a prescribed value of two images, which differ by a transposition. Thus they have a different signature and only one has a positive one. The twelve permutations are listed explicitly in the first column of Table 1. ■

Proof of Lemma 3.34.

Suppose G_1 is a 2-regular simple graph embedded in some surface \mathcal{S} . Using Lemma 3.12, there exists a 4-regular, 2-connected simple graph $G'_0 \succcurlyeq G_1$ embedded in the same surface. Let K be the number of faces in this embedding which closure is not homeomorphic to a closed disk. If $k \neq 0$, we construct recursively a sequence of graphs G'_1, \dots, G'_K such that $G'_k \succcurlyeq G'_{k-1}$ for all $k \in \{1, \dots, K\}$ so that G'_K has a strong embedding in \mathcal{S} .

For k in $\{0, \dots, K-1\}$, suppose that G'_k is embedded in \mathcal{S} . We pick one face which closure is not homeomorphic to a closed disk, and denote by γ the closed path around this face, which can be described as a succession of vertices and edges as

$$(4.70) \quad \gamma = \{v_1, e_1, \dots, v_i, e_i, \dots, v_r, e_r\}$$

where r is the length of the path and $\{v_i\}_{i \in \{1, \dots, r\}}$ (respectively $\{e_i\}_{i \in \{1, \dots, r\}}$) are vertices (respectively edges) in G'_k such that v_i is incident with e_{i-1} and e_i for all $i \in \{1, \dots, r\}$ (we assume that indices are defined modulo r). Since γ is a path round a face, every edge in γ appears at most twice and successive edges have to be distinct (since G'_k has no vertex incident with only one edge). Denote by $I_1 \subset \{1, \dots, r\}$ the indices of edges appearing only once in γ and by I_2 the set of pairs of indices associated to edges appearing twice in γ

$$(4.71) \quad I_2 = \{(i, j), 1 \leq i < j \leq r, e_i = e_j\}$$

We construct $G'_{k'+1}$ starting from G'_k as follows: on each edge on the path, we add one (respectively two) subdivision points according to its number of occurrences in γ . In other words, we add r vertices $\{w_i\}_{i \in \{1, \dots, r\}}$ to G'_k , delete all edges in γ and replace e_i by two edges (v_i, w_i) and (w_i, v_{i+1}) if $i \in I_1$, or by three edges (v_i, w_i) (w_i, w_j) and (w_j, v_{i+1}) if $(i, j) \in I_2$. Finally we add r edges between subdivision points, (w_1, w_2) , (w_2, w_3) , \dots , (w_r, w_1) . Note that each new edge has its endvertices on successive edges on the boundary of the face, they can be drawn on it without crossing, so that the resulting graph $G'_{k'+1}$ can be embedded on the same surface \mathcal{S} . Furthermore, it is 4-regular, since new vertices have four edges incident on it, and 2-connected since G'_k is. $G'_{k'+1}$ has r more vertices and $2r$ more edges than G_k . On the embedding on \mathcal{S} , the $r+1$ new faces have length 3, 4, 5 or r , and are all bounded by cycles since all their vertices are distinct. Other faces have not changed except by possibly adding one subdivision point on some edges, and there is one face less which boundary is not a cycle. Finally $G'_{k'+1} \succcurlyeq G_k$ since it is constructed from it by adding subdivision points and edges.

Thus, starting from G'_0 and iterating K times this construction, we obtain a graph G'_K embedded in \mathcal{S} with all faces bounded by cycles. Graph G'_K is 4-regular and 2-connected and by transitivity, we also have $G'_K \succcurlyeq G_1$ ■

Proof of Lemma 3.35.

The proof of this lemma is partly similar to that of Proposition 3.13. In fact, as in the planar case, no subset of $\mathcal{F}_0^*(G)$ can form a cluster, that is a double cover of a proper subset of $E(v)$ at some vertex $v \in V$. This implies that all results in the proof of Proposition 3.13 hold true in the non planar orientable case, for the restricted set of cycles \mathcal{F}_0^* which generates the face system and not for the whole basis.

For every v in V , the associated vertex equations consist in a system of two equations out of the three possible ones written in Equation (4.63), which has clearly solutions in \mathbb{R} . Therefore we suppose from now on that all vertex entries of the incidence matrix have been given a real non zero value so that the set of vertex equations for all vertices and all cycles in $\mathcal{F}_0^*(G)$, are fulfilled.

We now consider the ratios $R_{v,e}^\gamma$ defined in 4.64. We have already noted that their value does not depend on the orientation of the associated cycle γ , and that if two cycles pass through the same edge, the ratios have opposite value. Here we show that the two ratios associated to the same cycle γ at a given vertex, says v_i^γ , have values with opposite signs.

Indeed, from the definition (4.64), we have

$$(4.72) \quad R_{v_i^\gamma, e_i^\gamma}^\gamma = \frac{U_i^\gamma \bar{T}_i^\gamma}{S_i^\gamma}$$

$$(4.73) \quad R_{v_i^\gamma, e_{i-1}^\gamma}^\gamma = \frac{U_i^\gamma T_i^\gamma}{\bar{S}_i^\gamma}$$

while the vertex equation reads

$$(4.74) \quad S_i^\gamma \bar{S}_i^\gamma + T_i^\gamma \bar{T}_i^\gamma = 0$$

and thus

$$(4.75) \quad R_{v_i^\gamma, e_i^\gamma}^\gamma R_{v_i^\gamma, e_{i-1}^\gamma}^\gamma = -(U_i^\gamma)^2 < 0$$

Therefore the ratios at every vertex are alternating signs; at a given vertex v , either $R_{v,e}^\gamma > 0$ and $R_{v,\pi_v(e)}^\gamma < 0$ for all edges $e \in E(v)$ and all cycles γ such that $\gamma \cap E(v) = \{e, \pi_v(e)\}$, or all signs are simultaneously reversed. We also note that given a solution $(s_v, t_v, u_v, \bar{s}_v, \bar{t}_v, \bar{u}_v)$, one gets another solution by reversing the signs of u_v and \bar{u}_v , for which all ratio at v change sign. Thus given a rotation system $\Pi = (\pi_v)_{v \in V}$, one can choose the solutions of the site equations so that the ratios $R_{v,e}^\gamma > 0$ simultaneously for all $v \in V$, all $e \in E(v)$ and all $\gamma \in \mathcal{F}_0^*(G)$ such that $\gamma \cap E(v) = \{e, \pi_v(e)\}$.

Now consider an edge $e \in E$ with endvertices v and w . If its signature $\lambda(e) = +1$, a cycle γ passing through e behaves either according to π_v on v and π_w on w , or according to π_v^{-1} on v and π_w^{-1} on w . Thus either $\gamma \cap E(v) = \{\pi_v^{-1}(e), e\}$ and $\gamma \cap E(w) = \{e, \pi_w(e)\}$, or $\gamma \cap E(w) = \{\pi_w^{-1}(e), e\}$ and $\gamma \cap E(v) = \{e, \pi_v(e)\}$. Therefore the two ratios $R_{v,e}^\gamma$ and $R_{w,e}^\gamma$ have opposite sign and the associated edge equation

$$(4.76) \quad b_e^2 = -R_{v,e}^\gamma R_{w,e}^\gamma > 0$$

has a solution in \mathbb{R} . In the other case, $\lambda(e) = -1$, a cycle γ passing through e behaves either according to π_v on v and π_w^{-1} on w , or according to π_v^{-1} on v and π_w on w . Thus either $\gamma \cap E(v) = \{\pi_v^{-1}(e), e\}$ and $\gamma \cap E(w) = \{\pi_w^{-1}(e), e\}$, or $\gamma \cap E(w) = \{e, \pi_w(e)\}$ and $\gamma \cap E(v) = \{e, \pi_v(e)\}$. In that case, the two ratios $R_{v,e}^\gamma$ and $R_{w,e}^\gamma$ have the same sign and the associated edge equation

$$(4.77) \quad b_e^2 = -R_{v,e}^\gamma R_{w,e}^\gamma < 0$$

has only purely imaginary solutions.

If the surface is orientable, there exists an embedding scheme with $\lambda(\cdot) = +1$, which implies that there exists an incidence matrix A^G with real coefficients such that Equation (3.37) holds.

Suppose now that there exists an incidence matrix with real coefficients such that Equations (3.29)–(3.31) hold for all cycles in $\mathcal{F}_0^*(G)$. Starting from an embedding scheme (Π, λ) , we define a new rotation system $\tilde{\Pi} = (\tilde{\pi}_v)_{v \in V}$ such that on each vertex $v \in V$,

$$(4.78) \quad \tilde{\pi}_v = \begin{cases} \pi_v & \text{if } \exists \gamma \in \mathcal{F}_0^*(G) \text{ such that } \gamma \cap E(v) = \{e, \pi_v(e)\} \text{ and } R_{v,e}^\gamma < 0 \\ \pi_v^{-1} & \text{otherwise} \end{cases}$$

Due to the alternating sign properties of the ratios studied previously, one gets that for all $v \in V$ and all $\gamma \in \mathcal{F}_0^*(G)$ such that $\gamma \cap E(v) = \{e, \tilde{\pi}_v(e)\}$, one has $R_{v,e}^\gamma < 0$. Since the right hand side of Equation (4.65) has to be positive for all edges $e \in E$, all cycles in $\mathcal{F}_0^*(G)$ can

be oriented so that on each vertex the ratio on the incoming edge is negative, which in turn implies that they behave everywhere according to the rotation system $\tilde{\Pi}$. The surface is thus orientable. \blacksquare

5. GRASSMANN ALGEBRA AND REPRESENTATION OF PFAFFIANS.

In this section, we introduce a Grassmann Algebra and use it to represent Pfaffians as Grassmann integrals. We refer to references [24] and [25] for a thorough introduction on this matter. We first recall the basic properties which are useful here, and use this representation to give a proof of Proposition 3.28. We end this section by a short proof of Proposition 2.42 using the same representation.

A Grassmann algebra over \mathbb{R} (or \mathbb{C}) is an associative algebra with a set of generators $\{\xi_i\}_{i \in \{1, \dots, n\}}$ obeying the relations

$$(5.1) \quad \xi_i \xi_j = -\xi_j \xi_i \quad \forall i, j \in \{1, \dots, n\}$$

and in particular

$$(5.2) \quad \xi_i^2 = 0 \quad \forall i \in \{1, \dots, n\}$$

Integration on the Grassmann algebra [24] is defined through the following formulas,

$$(5.3) \quad \int d\xi_i = 0 \quad \int d\xi_i \xi_i = 1$$

and multiple integrals are defined as iterates

$$(5.4) \quad \int d\xi_{i_r} d\xi_{i_{r-1}} \cdots d\xi_{i_1} \xi_{i_1} \cdots \xi_{i_{r-1}} \xi_{i_r} = 1 \quad \text{for all } 1 \leq i_1 < \cdots < i_{r-1} < i_r \leq n$$

With this prescription, integrals of products of factors depending on disjoint subsets of Grassmann variables is just the product of the integrals,

$$(5.5) \quad \int d\xi_{i_{r+s}} \cdots d\xi_{i_{r+1}} d\xi_{i_r} \cdots d\xi_{i_1} \phi_1(\xi_{i_1}, \dots, \xi_{i_r}) \phi_2(\xi_{i_{r+1}}, \dots, \xi_{i_{r+s}}) = \left[\int d\xi_{i_r} \cdots d\xi_{i_1} \phi_1(\xi_{i_1}, \dots, \xi_{i_r}) \right] \left[\int d\xi_{i_{r+s}} \cdots d\xi_{i_{r+1}} \phi_2(\xi_{i_{r+1}}, \dots, \xi_{i_{r+s}}) \right]$$

Here, our main interest in the introduction of a Grassmann algebra is the following. Let A be an antisymmetric matrix of even order n , then its Pfaffian can be expressed as the following Grassmann integral [25]:

$$(5.6) \quad Pf(A) = \int d\xi_n \cdots d\xi_1 \exp\left\{\frac{1}{2} \sum_{i,j=1}^n \xi_i A_{i,j} \xi_j\right\}$$

Proof of Proposition 3.28

For the purpose of using identity (5.6) in the context of Proposition 3.28, we introduce a Grassmann algebra of order $n = |V_{\mathcal{D}}(G)|$, with generators indexed by the darts on G ,

$\{\xi_d\}_{\{d \in V_{\mathcal{D}}(G)\}}$ and write the Pfaffian of the antisymmetric incidence matrix A defined in (3.17) as

$$(5.7) \quad Pf(A) = \int \overleftarrow{\prod}_{d \in V_{\mathcal{D}}(G)} d\xi_d \exp\{\mathcal{L}_G(\underline{\xi})\}$$

with

$$(5.8) \quad \mathcal{L}_G(\underline{\xi}) = \sum_{\{d, d'\} \in E_{\mathcal{D}}(G)} \xi_d A_{d, d'} \xi_{d'}$$

In Equation (5.7), we have also used the shorthand notation $\overleftarrow{\prod}$ to indicate that the differential elements in the product have to be ordered from left to right in decreasing order with respect to the (lexicographic) order defined on $V_{\mathcal{D}}(G)$. Note also that in (5.8), the sum is over all edges in $E_{\mathcal{D}}(G)$, which are defined as unordered pairs of elements in $V_{\mathcal{D}}(G)$; since A is antisymmetric and Grassmann generators anticommute, $\xi_d A_{d, d'} \xi_{d'} = \xi_{d'} A_{d', d} \xi_d$, that is, the value of each summand is independent on the choice of which representation, (d, d') or (d', d) is chosen for the unordered pair $\{d, d'\}$, the right hand side of (5.8) is well defined and Equation (5.7) identifies with (5.6). Up to a permutation of positive signature, we may write

$$(5.9) \quad \overleftarrow{\prod}_{d \in V_{\mathcal{D}}(G)} d\xi_d = \prod_{v \in V} d\xi_{(v, e_v^4)} d\xi_{(v, e_v^3)} d\xi_{(v, e_v^2)} d\xi_{(v, e_v^1)}$$

where for all $v \in V$, $e_v^1 < e_v^2 < e_v^3 < e_v^4$ denote the four ordered edges at v .

Given a weight function w on E , the Pfaffian of the related weighted incidence matrix can be expanded as a sum over the set of closed curves on G

$$(5.10) \quad Pf(A(w)) = \sum_{C \in \mathcal{C}(G)} \prod_{e \in C} w(e) F_A(C)$$

For any closed curve $C \in \mathcal{C}(G)$, the expression of $F_A(C)$ in terms of a Grassmann integral can be obtained through an expansion of the exponential in (5.7) over all subsets of $E_{\mathcal{D}}^E(G)$. Non zero terms necessarily involve an even number of generator at each vertex $v \in V$, and thus correspond to closed curves on G . The contribution to $F_A(C)$ can then be readily identified as,

$$(5.11) \quad F_A(C) = \int \overleftarrow{\prod}_{d \in V_{\mathcal{D}}(G)} d\xi_d \left[\prod_{\{d, d'\} \in E_{\mathcal{D}}^E(C)} A_{d, d'} \xi_d \xi_{d'} \right] \exp\left\{ \sum_{\{d, d'\} \in E_{\mathcal{D}}^V(G)} \xi_d A_{d, d'} \xi_{d'} \right\}$$

where for clarity, we have denoted by $E_{\mathcal{D}}^E(C)$ the (canonical) image of the closed curve C on the dart graph,

$$(5.12) \quad E_{\mathcal{D}}^E(C) = \{\{(v, e), (v', e)\} \in E_{\mathcal{D}}^E(G) : e \in C\}$$

Now, let $\Gamma = (V_{\Gamma}, E_{\Gamma})$ be a cycle of length r on G . We note $V_{\mathcal{D}}(\Gamma)$ the set of darts with vertex in V_{Γ}

$$(5.13) \quad V_{\mathcal{D}}(\Gamma) = \{(v, e) \in V_{\mathcal{D}}(G), v \in V_{\Gamma}, e \in E(v)\}$$

and we partition the elements of $E_{\mathcal{D}}^E(C)$ and $E_{\mathcal{D}}^V(G)$ according to their intersection with $V_{\mathcal{D}}(\Gamma)$,

$$(5.14) \quad E_k^E(C) = \{ \{(v, e), (v', e)\} \in E_{\mathcal{D}}^E(C) : |V_{\mathcal{D}}(\Gamma) \cap \{(v, e), (v', e)\}| = k \}, \quad k = 0, 1, 2$$

$$(5.15) \quad E_k^V = \{ \{(v, e), (v, e')\} \in E_{\mathcal{D}}^V(G) : |V_{\mathcal{D}}(\Gamma) \cap \{(v, e), (v, e')\}| = k \}, \quad k = 0, 2$$

These sets are pairwise disjoint and we have

$$(5.16) \quad E_{\mathcal{D}}^E(C) = E_0^E(C) \cup E_1^E(C) \cup E_2^E(C)$$

$$(5.17) \quad E_{\mathcal{D}}^V(G) = E_0^V \cup E_2^V$$

We thus rewrite the expression of $F_A(C)$ using this decomposition,

$$\begin{aligned} F_A(C) = & \int \overleftarrow{\prod}_{d \in V_{\mathcal{D}}(G)} d\xi_d \left[\prod_{\{d, d'\} \in E_0^E(C)} A_{d, d'} \xi_d \xi_{d'} \right] \exp \left\{ \sum_{\{d, d'\} \in E_0^V} \xi_d A_{d, d'} \xi_{d'} \right\} \\ & \times \left[\prod_{\{d, d'\} \in E_1^E(C)} A_{d, d'} \xi_d \xi_{d'} \right] \\ & \times \left[\prod_{\{d, d'\} \in E_2^E(C)} A_{d, d'} \xi_d \xi_{d'} \right] \exp \left\{ \sum_{\{d, d'\} \in E_2^V} \xi_d A_{d, d'} \xi_{d'} \right\} \end{aligned}$$

In order to factorize this expression, we rewrite the product over E_1^E as

$$(5.18) \quad \prod_{\{d, d'\} \in E_1^E(C)} A_{d, d'} \xi_d \xi_{d'} = \left[\overleftarrow{\prod}_{\substack{\{d, d'\} \in E_1^E(C) \\ d' \in V_{\mathcal{D}}(\Gamma)}}^{(\Gamma)} A_{d, d'} \xi_d \right] \left[\overrightarrow{\prod}_{\substack{\{d, d'\} \in E_1^E(C) \\ d' \in V_{\mathcal{D}}(\Gamma)}}^{(\Gamma)} \xi_{d'} \right]$$

where the notation $\overleftarrow{\prod}^{(\Gamma)}$ (respectively $\overrightarrow{\prod}^{(\Gamma)}$) indicates that the factors have to be written in decreasing (respectively increasing) order with respect to an order on $V_{\mathcal{D}}(\Gamma)$ to be precised later. The above identity just states that the reordering involves an even number of transpositions, whatever the order chosen on $V_{\mathcal{D}}(\Gamma)$.

Using identity (5.5), we can now write $F_A(C)$ as a product of two factors

$$(5.19) \quad F_A(C) = \overline{F}_A(C, \Gamma) F_A(C, \Gamma)$$

where

$$\begin{aligned} \overline{F}_A(C, \Gamma) = & \int \overleftarrow{\prod}_{d \in V_{\mathcal{D}}(G) \setminus V_{\mathcal{D}}(\Gamma)} d\xi_d \left[\prod_{\{d, d'\} \in E_0^E(C)} A_{d, d'} \xi_d \xi_{d'} \right] \left[\overleftarrow{\prod}_{\substack{\{d, d'\} \in E_1^E(C) \\ d' \in V_{\mathcal{D}}(\Gamma)}}^{(\Gamma)} A_{d, d'} \xi_d \right] \\ (5.20) \quad & \times \exp \left\{ \sum_{\{d, d'\} \in E_0^V} \xi_d A_{d, d'} \xi_{d'} \right\} \end{aligned}$$

and

$$(5.21) \quad F_A(C, \Gamma) = \int \overleftarrow{\prod}_{d \in V_{\mathcal{D}}(\Gamma)} d\xi_d \quad \left[\prod_{\{d, d'\} \in E_2^E(C)} A_{d, d'} \xi_d \xi_{d'} \right] \left[\overrightarrow{\prod}_{\substack{\{d, d'\} \in E_1^E(C) \\ d' \in V_{\mathcal{D}}(\Gamma)}} \xi_{d'} \right] \\ \times \exp \left\{ \sum_{\{d, d'\} \in E_2^V(C)} \xi_d A_{d, d'} \xi_{d'} \right\}$$

Since by construction, C and $C \Delta \Gamma$ coincide on $E \setminus \Gamma$, one has necessarily

$$(5.22) \quad \begin{aligned} E_0^E(C \Delta \Gamma) &= E_0^E(C) \\ E_0^V(C \Delta \Gamma) &= E_0^V(C) \end{aligned}$$

Hence

$$(5.23) \quad \overline{F}_A(C \Delta \Gamma, \Gamma) = \overline{F}_A(C, \Gamma)$$

We now prove that $F_A(C \Delta \Gamma, \Gamma)$ and $F_A(C, \Gamma)$ are equal. We first define the following notation for the generators indexed in $V_{\mathcal{D}}(\Gamma)$: for all $i \in \{1, \dots, r\}$ and all $k \in \{1, 2, 3, 4\}$, we write

$$(5.24) \quad \eta_i^k = \xi_{(v_i, \sigma_i(e_{v_i}^k))}$$

where σ_i is the permutation defined in Lemma 3.24. We also define

$$(5.25) \quad \psi_i^k(C) = \begin{cases} \eta_i^k & \text{if } \sigma_i(e_{v_i}^k) \in C \\ 1 & \text{otherwise.} \end{cases}$$

and given $\underline{\eta}_i = \{\eta_i^1, \eta_i^2, \eta_i^3, \eta_i^4\}$,

$$(5.26) \quad \mathcal{L}_i(\underline{\eta}_i) = S_i \eta_i^1 \eta_i^2 + \overline{S}_i \eta_i^3 \eta_i^4 + T_i \eta_i^1 \eta_i^3 + \overline{T}_i \eta_i^4 \eta_i^2 + U_i \eta_i^1 \eta_i^4 + \overline{U}_i \eta_i^2 \eta_i^3$$

Using these and notations (3.27), (3.23), we rewrite the expression of $F_A(C, \Gamma)$ as

$$(5.27) \quad \begin{aligned} F_A(C, \Gamma) &= \int \overleftarrow{\prod}_{(i,k)=(1,1)}^{(r,4)} d\eta_i^k \quad \left[\prod_{i: e_i \in C} B_i \eta_{i-1}^4 \eta_i^1 \right] \\ &\quad \times \left[\overrightarrow{\prod}_{i=1}^r (\psi_i^2(C) \psi_i^3(C)) \right] \exp \left\{ \sum_i \mathcal{L}_i(\underline{\eta}_i) \right\} \\ &= \int \overleftarrow{\prod}_{(i,k)=(1,1)}^{(r,4)} d\eta_i^k \quad \left[\prod_{i: e_i \in C} B_i \right] \left[\overrightarrow{\prod}_{i=1}^r \psi_{i-1}^4(C) \psi_i^1(C) \right] \\ &\quad \times \left[\overrightarrow{\prod}_{i=1}^r (\psi_i^2(C) \psi_i^3(C)) \right] \exp \left\{ \sum_i \mathcal{L}_i(\underline{\eta}_i) \right\} \end{aligned}$$

where the order of the ‘‘differentials’’ is induced by the lexicographic order on $\{1, \dots, r\} \times \{1, \dots, 4\}$. Each factor in the first product contains an even number of generators (2 or 0 depending on whether $e_i \in C$ and thus commute with any other term. Thus we can write

$$\begin{aligned}
F_A(C, \Gamma) &= \int \prod_{(i,k)=(1,1)}^{\overleftarrow{(r,4)}} d\eta_i^k \left[\prod_{i:e_i \in C} B_i \right] \\
&\quad \times \psi_0^4(C) \left[\prod_{i=1}^{r-1} (\psi_i^1(C) \psi_i^2(C) \psi_i^3(C) \psi_i^4(C)) \right] \left[(\psi_r^1(C) \psi_r^2(C) \psi_r^3(C)) \right] \exp \left\{ \sum_i \mathcal{L}_i(\underline{\eta}_i) \right\} \\
&= (-1)^{|\{e_0\} \cap C|} \int \prod_{(i,k)=(1,1)}^{\overleftarrow{(r,4)}} d\eta_i^k \left[\prod_{i:e_i \in C} B_i \right] \left[\prod_{i=1}^r (\psi_i^1(C) \psi_i^2(C) \psi_i^3(C) \psi_i^4(C)) \exp \{ \mathcal{L}_i(\underline{\eta}_i) \} \right]
\end{aligned}$$

where in the last line, the term $\psi_0^4(C) = \psi_r^4(C)$ has been shifted on the right of all other terms, taking into account that the total number of generators is even, leading to a possible minus sign by anticommutation. Factorization and shifts of the exponential terms takes into account that the arguments are quadratic in the generators and thus commute with other terms.

$F_A(C, \Gamma)$ can then be expressed as a product of r Grassmann integrals,

$$(5.28) \quad F_A(C, \Gamma) = (-1)^{|\{e_0\} \cap C|} \left[\prod_{i:e_i \in C} B_i \right] \prod_1^r W_i(C, \Gamma)$$

with

$$(5.29) \quad W_i(C, \Gamma) = \int d\eta_i^4 d\eta_i^3 d\eta_i^2 d\eta_i^1 \left[\psi_i^1(C) \psi_i^2(C) \psi_i^3(C) \psi_i^4(C) \right] \exp \{ \mathcal{L}_i(\underline{\eta}_i) \}$$

The expression of $W_i(C, \Gamma)$ can be computed for all possible values of the $\{\psi_i^k(C)\}_{1 \leq k \leq 4}$. Since C is a closed curve, v_i is incident with an even number of edges in C , and there are thus eight possible configurations, giving rise to eight possible expressions for $W_i(C, \Gamma)$, listed in the table below as a function of which edges (indexed relatively to Γ) are in C

$\{k \sigma_i(e_i^k) \in C\}$	$W_i(C)$	$\{k \sigma_i(e_i^k) \in C\}$	$W_i(C)$
\emptyset	$S_i \bar{S}_i + T_i \bar{T}_i + U_i \bar{U}_i$	$\{1, 4\}$	\bar{U}_i
$\{1, 2, 3, 4\}$	1	$\{2, 3\}$	U_i
$\{1, 2\}$	\bar{S}_i	$\{2, 4\}$	$-T_i$
$\{3, 4\}$	S_i	$\{1, 3\}$	\bar{T}_i

TABLE 3. Expression of the Grassmann integral $W_i(C)$ defined in Equation (5.29) for the eight possible configurations for C on $E(v_i)$, as a function of their ordering with respect to cycle Γ (Lemma 3.24). Note that first and third columns are exchanged under the transformation $C \rightarrow C \Delta \Gamma$.

Now define the following two functions

$$(5.30) \quad \varphi_i(C) = \begin{cases} \frac{U_i \bar{T}_i}{S_i} & \text{if } \sigma_i(e_i^4) \in C \\ 1 & \text{otherwise.} \end{cases}$$

$$(5.31) \quad \bar{\varphi}_i(C) = \begin{cases} \frac{U_i S_i}{\bar{T}_i} & \text{if } \sigma_i(e_i^1) \in C \\ 1 & \text{otherwise.} \end{cases}$$

Under the hypothesis of Proposition 3.28, Equations (3.29) are fulfilled. In Table 4, we check by direct inspection that for all $i \in \{1, \dots, r\}$, the following relation holds between $W_i(C)$ and $W_i(C\Delta\Gamma)$:

$$(5.32) \quad \frac{1}{U_i} \bar{\varphi}_i(C) W_i(C) \varphi_i(C) = W_i(C\Delta\Gamma)$$

$\{k \sigma_i(e_i^k) \in C\}$	$\bar{\varphi}_i(C)$	$W_i(C)$	$\varphi_i(C)$	$\frac{\bar{\varphi}_i(C) W_i(C) \varphi_i(C)}{U_i}$	$W_i(C\Delta\Gamma)$
\emptyset	1	$U_i \bar{U}_i$	1	\bar{U}_i	\bar{U}_i
$\{1, 4\}$	$\frac{U_i S_i}{\bar{T}_i}$	\bar{U}_i	$\frac{U_i \bar{T}_i}{S_i}$	$U_i \bar{U}_i$	$U_i \bar{U}_i$
$\{1, 2, 3, 4\}$	$\frac{U_i S_i}{\bar{T}_i}$	1	$\frac{U_i \bar{T}_i}{S_i}$	U_i	U_i
$\{2, 3\}$	1	U_i	1	1	1
$\{1, 2\}$	$\frac{U_i S_i}{\bar{T}_i}$	\bar{S}_i	1	$-T_i$	$-T_i$
$\{2, 4\}$	1	$-T_i$	$\frac{U_i \bar{T}_i}{S_i}$	\bar{S}_i	\bar{S}_i
$\{3, 4\}$	1	S_i	$\frac{U_i \bar{T}_i}{S_i}$	\bar{T}_i	\bar{T}_i
$\{1, 3\}$	$\frac{U_i S_i}{\bar{T}_i}$	\bar{T}_i	1	S_i	S_i

TABLE 4. Relation between $W_i(C)$ and $W_i(C\Delta\Gamma)$ under the hypothesis that $S_i \bar{S}_i + T_i \bar{T}_i = 0$ (Equation (3.29)). The table shows that relation (5.32) holds for the eight possible local configurations

We suppose now that Equations (3.29), (3.29) hold for all $i \in \{1, \dots, r\}$ together with (3.31). We have

$$\begin{aligned}
F_A(C, \Gamma) &= (-1)^{|\{e_0\} \cap C|} \left[\prod_{i:e_i \in C} B_i \right] \left[\prod_{i=1}^r W_i(C, \Gamma) \right] \\
&= -(-1)^{|\{e_0\} \cap C|} \left[\prod_{i:e_i \in C} B_i \right] \left[\prod_{i=1}^r B_i \right] \left[\prod_{i=1}^r \frac{1}{U_i} \right] \left[\prod_{i=1}^r W_i(C, \Gamma) \right] \\
&= -(-1)^{|\{e_0\} \cap C|} \left[\prod_{i:e_i \in C} B_i^2 \right] \left[\prod_{i:e_i \notin C} B_i \right] \left[\prod_{i=1}^r \frac{1}{U_i} \right] \left[\prod_{i=1}^r W_i(C, \Gamma) \right] \\
&= -(-1)^{|\{e_0\} \cap C|} \left[\prod_{i:e_i \notin C} B_i \right] \left[\prod_{i:e_i \in C} \frac{U_i \bar{T}_i U_{i+1} S_{i+1}}{S_i \bar{T}_{i+1}} \right] \left[\prod_{i=1}^r \frac{1}{U_i} \right] \left[\prod_{i=1}^r W_i(C, \Gamma) \right] \\
&= -(-1)^{|\{e_0\} \cap C|} \left[\prod_{i:e_i \notin C} B_i \right] \left[\prod_{i=1}^r \varphi_i(C) \bar{\varphi}_{i+1}(C) \right] \left[\prod_{i=1}^r \frac{1}{U_i} \right] \left[\prod_{i=1}^r W_i(C, \Gamma) \right] \\
&= -(-1)^{|\{e_0\} \cap C|} \left[\prod_{i:e_i \notin C} B_i \right] \left[\prod_{i=1}^r \frac{\bar{\varphi}_i(C) W_i(C, \Gamma) \varphi_i(C)}{U_i} \right] \\
&= (-1)^{|\{e_0\} \cap (C \Delta \Gamma)|} \left[\prod_{i:e_i \in C \Delta \Gamma} B_i \right] \left[\prod_{i=1}^r W_i(C \Delta \Gamma, \Gamma) \right] \\
(5.33) \quad &= F_A(C \Delta \Gamma, \Gamma)
\end{aligned}$$

To get the second line we have introduced (3.31) as an identity

$$(5.34) \quad \left[\prod_{i=1}^r B_i \right] \left[\prod_{i=1}^r \frac{1}{U_i} \right] = -1$$

For the fourth line of (5.33), we have used Equations (3.30) for all i such that $e_i \in C$; To get the fifth line, we used the definitions (5.30)–(5.31); In the sixth line appears the left hand side of (5.32) and the seventh is just expression (5.28) for $C \Delta \Gamma$.

Now inserting Equation (5.23) and the result of Equation (5.33) in Equation (5.19), one gets

$$\begin{aligned}
F_A(C) &= \bar{F}_A(C, \Gamma) F_A(C, \Gamma) \\
&= \bar{F}_A(C \Delta \Gamma, \Gamma) F_A(C \Delta \Gamma, \Gamma) \\
(5.35) \quad &= F_A(C \Delta \Gamma)
\end{aligned}$$

Proposition 3.28 is proven. ■

Proof of Proposition 2.42.

We conclude this section by a proof of Proposition 2.42. The Pfaffian reduction formula is obviously not new but we found no clear recent textbook reference. In addition, one of the few available references [23] appears to have some misprints. We report here an easy but rather lengthy proof.

We prove Proposition 2.41 by induction on p . Let us first prove Equation (2.42) for $p = 1$. We set $K = \{\alpha, \beta\}$, where $\alpha < \beta$ is a pair of indices in $\{1, \dots, 2n\}$ such that $A_{\alpha, \beta} \neq 0$ and write the Pfaffian of A as an integral over a Grassmann algebra of dimension $2n$ (Equation (5.6)). Expand the terms in the exponential containing elements indexed by K gives,

$$\begin{aligned}
 Pf(A) &= \int \prod_{k=1}^{\overleftarrow{2n}} d\xi_k \exp\left\{\frac{1}{2} \sum_{i,j=1}^{2n} A_{i,j} \xi_i \xi_j\right\} \\
 &= \int \prod_{k=1}^{\overleftarrow{2n}} d\xi_k \exp\left\{\frac{1}{2} \sum_{\substack{i,j=1 \\ |\{i,j\} \cap \{\alpha, \beta\}|=0}}^{2n} A_{i,j} \xi_i \xi_j + \sum_{i \neq \alpha} A_{i,\beta} \xi_i \xi_\beta + \sum_{j \neq \beta} A_\alpha \xi_\alpha \xi_j + A_{\alpha,\beta} \xi_\alpha \xi_\beta\right\} \\
 &= \int \prod_{k=1}^{\overleftarrow{2n}} d\xi_k \exp\left\{\frac{1}{2} \sum_{\substack{i,j=1 \\ |\{i,j\} \cap \{\alpha, \beta\}|=0}}^{2n} A_{i,j} \xi_i \xi_j\right\} \\
 &\quad \times \left(1 + \sum_{i \neq \alpha} A_{i,\beta} \xi_i \xi_\beta\right) \left(1 + \sum_{j \neq \beta} A_\alpha \xi_\alpha \xi_j\right) \left(1 + A_{\alpha,\beta} \xi_\alpha \xi_\beta\right) \\
 (5.36) \quad &= \int \prod_{k=1}^{\overleftarrow{2n}} d\xi_k \exp\left\{\frac{1}{2} \sum_{\substack{i,j=1 \\ |\{i,j\} \cap \{\alpha, \beta\}|=0}}^{2n} A_{i,j} \xi_i \xi_j\right\} \left(A_{\alpha,\beta} - \sum_{\substack{i \neq \alpha \\ j \neq \beta}} A_{i,\beta} A_{\alpha,j} \eta_i \eta_j\right) \eta_\alpha \eta_\beta
 \end{aligned}$$

After integration over $(\eta_\alpha, \eta_\beta)$, the expanded part reduces to the sum of a non zero constant ($A_{\alpha,\beta}$), and a product of two linear combinations of Grassmann generators. Thus successive powers of this second term are zero and one can re-exponentiate this part. We have,

$$\begin{aligned}
 Pf(A) &= (-1)^{\beta-\alpha+1} \int \prod_{\substack{k=1 \\ k \neq \alpha, \beta}}^{\overleftarrow{2n}} d\xi_k \exp\left\{\frac{1}{2} \sum_{i,j \neq \alpha, \beta} A_{i,j} \eta_i \eta_j\right\} \times \left(A_{\alpha,\beta} - \sum_{p,q \neq \alpha, \beta} A_{p,\alpha} A_{q,\beta} \eta_p \eta_q\right) \\
 &= (-1)^{\beta-\alpha+1} \int \prod_{k=1}^{\overleftarrow{2n}} d\xi_k \exp\left\{\frac{1}{2} \sum_{i,j \neq \alpha, \beta} A_{i,j} \eta_i \eta_j - A_{\alpha,\beta}^{-1} \sum_{p,q \neq \alpha, \beta} A_{p,\alpha} A_{q,\beta} \eta_p \eta_q\right\} \\
 (5.37) \quad &= (-1)^{\beta-\alpha+1} \int \prod_{k=1}^{\overleftarrow{2n}} d\xi_k \exp\left\{\frac{1}{2A_{\alpha,\beta}} \sum_{i,j \neq \alpha, \beta} (A_{i,j} A_{\alpha,\beta} - A_{i,\alpha} A_{j,\beta} + A_{j,\alpha} A_{i,\beta}) \eta_i \eta_j\right\}
 \end{aligned}$$

In order to get the correct signs in all terms of the sum, we introduce the following change of variables

$$(5.38) \quad \tilde{\eta}_i = \begin{cases} -\eta_i & \text{if } \alpha < i < \beta \\ \eta_i & \text{otherwise} \end{cases}$$

The Jacobian of the transformation is $(-1)^{\alpha+\beta-1}$ and we get, using homogeneity

$$\begin{aligned}
\text{Pf}(A) &= \left(\frac{1}{A_{\alpha,\beta}}\right)^{n-2} \int \left(\prod_{k \neq \alpha,\beta} d\tilde{\eta}_k\right) \exp\left\{\sum_{\substack{i < j \\ i,j \neq \alpha,\beta}} \text{Pf}(A_{\{i,j,\alpha,\beta\}}) \tilde{\eta}_i \tilde{\eta}_j\right\} \\
&= \left(\frac{1}{A_{\alpha,\beta}}\right)^{n-2} \int \left(\prod_{k \neq \alpha,\beta} d\eta_k\right) \exp\left\{\frac{1}{2} \sum_{i,j \neq \alpha,\beta} A_{i,j}^{\overline{\{\alpha,\beta\}}} \eta_i \eta_j\right\} \\
(5.39) \quad &= \text{Pf}(A_{\{\alpha,\beta\}})^{-(n-2)} \text{Pf}(A^{\overline{\{\alpha,\beta\}}})
\end{aligned}$$

Equation (2.42) is proven for $p = 1$. We now fix p with $1 < p < n$ and suppose that equation (2.42) holds for all $p' < p$. We consider K a subset of indices of order $2p$ in $\{1, \dots, 2n\}$ such that $\text{Pf}(A_K) \neq 0$. Thus there exists at least one proper submatrix of A_K , says A_{K_1} with $|K_1| = 2p_1$, $0 < p_1 < p$, such that $\text{Pf}(A_{K_1}) \neq 0$. Let $K_2 = K \setminus K_1$ and $p_2 = p - p_1$. We first have the two following identities which are a direct consequence of the definitions (2.39) and (2.40):

$$(5.40) \quad [A_{K_1 \cup K_2}]_{K_1} = A_{K_1}$$

$$(5.41) \quad [A^{\overline{K_1}}]_{K_2} = [A_{K_1 \cup K_2}]^{\overline{K_1}}$$

Now applying the recurrence hypothesis for p_1 for the matrix $A_{K_1 \cup K_2}$ and both (5.40) and (5.41), we first get,

$$\begin{aligned}
\text{Pf}(A_{K_1 \cup K_2}) &= \text{Pf}([A_{K_1 \cup K_2}]_{K_1})^{-(p_2-1)} \text{Pf}([A_{K_1 \cup K_2}]^{\overline{K_1}}) \\
(5.42) \quad &= \text{Pf}(A_{K_1})^{-(p_2-1)} \text{Pf}([A^{\overline{K_1}}]_{K_2})
\end{aligned}$$

We also need to prove that both matrices $A^{\overline{K_1 \cup K_2}}$ and $[A^{\overline{K_1}}]^{\overline{K_2}}$ are proportional. For every pair of indices $i, j \in K \setminus (K_1 \cup K_2)$, we have

$$\begin{aligned}
(A^{\overline{K_1 \cup K_2}})_{i,j} &= \text{Pf}(A_{K_1 \cup (K_2 \cup \{i,j\})}) \\
&= \text{Pf}(A_{K_1})^{-p_2} \text{Pf}([A^{\overline{K_1}}]_{(K_2 \cup \{i,j\})}) \\
(5.43) \quad &= \text{Pf}(A_{K_1})^{-p_2} ([A^{\overline{K_1}}]^{\overline{K_2}})_{i,j}
\end{aligned}$$

Since this relation holds for every entry of both matrices, we have

$$(5.44) \quad A^{\overline{K_1 \cup K_2}} = \text{Pf}(A_{K_1})^{-p_2} [A^{\overline{K_1}}]^{\overline{K_2}}$$

Now we can write

$$\begin{aligned}
\text{Pf}(A) &= \text{Pf}(A_{K_1})^{-(n-p_1-1)} \text{Pf}(A^{K_1}) \\
&= \text{Pf}(A_{K_1})^{-(n-p_1-1)} \text{Pf}([A^{\overline{K_1}}]_{K_2})^{-(n-p_1-p_2-1)} \text{Pf}([A^{\overline{K_1}}]^{\overline{K_2}}) \\
&= \text{Pf}(A_{K_1})^{-(n-p_1-1)+p_2(n-p_1-p_2)} \text{Pf}([A^{\overline{K_1}}]_{K_2})^{-(n-p_1-p_2-1)} \text{Pf}(A^{\overline{K_1 \cup K_2}}) \\
&= \text{Pf}(A_{K_1})^{-(n-p_1-1)+p_2(n-p_1-p_2)-(p_2-1)(n-p_1-p_2-1)} \text{Pf}(A_{K_1 \cup K_2})^{-(n-p_1-p_2-1)} \text{Pf}
\end{aligned}$$

we have used the hypothesis in the first two lines and equations (5.41), (5.42), (5.44) in the last two lines, respectively. The exponent of $Pf(A_{K_1})$ being zero in the last line, we finally get

$$(5.45) \quad Pf(A) = Pf(A_{K_1 \cup K_2})^{-(n-(p_1+p_2)-1)} Pf(A^{\overline{K_1 \cup K_2}})$$

which is (2.42) for $p = p_1 + p_2$.

Proposition 2.41 is proven. ■

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