



**HAL**  
open science

## Junction of ferromagnetic thin films

Rejeb Hadiji, Antonio Gaudiello

► **To cite this version:**

Rejeb Hadiji, Antonio Gaudiello. Junction of ferromagnetic thin films. *Calculus of Variations and Partial Differential Equations*, 2010, 39 (no 3-4), p. 593-619. hal-00795492

**HAL Id: hal-00795492**

**<https://hal.science/hal-00795492v1>**

Submitted on 28 Feb 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Junction of ferromagnetic thin films

Antonio Gaudiello\* and Rejeb Hadiji†

## Abstract

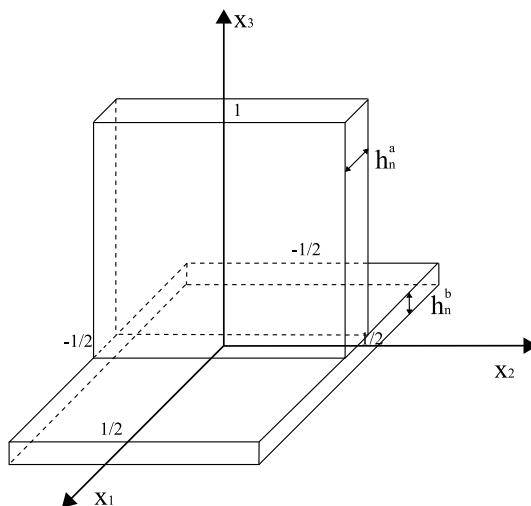
In this paper, starting from the classical 3D micromagnetic energy, we determine, via an asymptotic analysis, the free energy of two joined ferromagnetic thin films. We distinguish different regimes depending on the limit of the ratio between the small thicknesses of the two films.

Keywords: micromagnetics, variational problem, thin films, junctions.

2000 AMS subject classifications: 78A25, 49S05, 78M35

## 1 Introduction

In this paper, starting from the classical 3D micromagnetic energy (see W. F. Brown [5] and L. D. Landau and E. M. Lifshitz [18]), we determine, via an asymptotic analysis, the free energy of two joined ferromagnetic thin films. Precisely, let



---

\*DAEIMI, Università degli Studi di Cassino, via G. Di Biasio 43, 03043 Cassino (FR), Italia. e-mail: gaudiell@unina.it

†Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, UFR des Sciences et Technologie, 61, Avenue du Général de Gaulle Bât. P3, 4e étage, 94010 Créteil Cedex, France. e-mail: hadiji@univ-paris12.fr

$$\Omega_n = \left( \left[ -\frac{h_n^a}{2}, \frac{h_n^a}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times [0, 1[ \right) \cup \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \times ]-h_n^b, 0[ \right), \quad n \in N,$$

be a 3D ferromagnetic multidomain consisting of two orthogonal joined films, as in figure, with small thicknesses  $h_n^a$  and  $h_n^b$ , respectively, where  $h_n^a$  and  $h_n^b$  are two positive parameters tending to zero, as  $n$  diverges. For instance, such a structure appears as a component of a rotor of a permanent magnetic synchronous micro-machine (see S. S. Irudayaraj and A. Emadi [15]). In general, magnetic thin-film elements are used in many applications: inductive thin films heads in magnetic recording, megnetoresistive sensors, thin films memories, etc. About this subject we refer to A. Hubert and R. Schafer [14] and the references quoted therein. The aim of our paper is to study the asymptotic behavior, as  $n$  diverges, of the following non-convex, nonlocal variational problem:

$$\left\{ \begin{array}{l} J_n = \min \left\{ \int_{\Omega_n} \left( \alpha |DM|^2 + \varphi(M) + \frac{1}{2} DU_M M - 2F_n M \right) dx : M \in H^1(\Omega_n, S^2) \right\}, \\ \operatorname{div}(-DU_M + M) = 0 \text{ in } \mathbb{R}^3, \end{array} \right.$$

where  $\alpha$  is a positive constant,  $\varphi : S^2 \rightarrow [0, +\infty[$  is a continuous and even function,  $F_n \in L^2(\Omega_n, \mathbb{R}^3)$ , and  $S^2$  denotes the unit sphere of  $\mathbb{R}^3$ . Moreover, it is understood that  $M = 0$  in  $\mathbb{R}^3 \setminus \Omega_n$ . As we shall prove, the limit problem depends on the limit of the ratio between the thicknesses  $h_n^b$  and  $h_n^a$ .

In classical theory of micromagnetics,  $M : \Omega_n \rightarrow \mathbb{R}^3$  denotes the magnetization and the body is always locally magnetized to a saturation magnetization  $|M(x)| = m(T) > 0$  unless the local temperature  $T$  is greater or equal to Curie temperature depending on the body, in the latter case  $m(T) = 0$ , and the material ceases to behave ferromagnetically. In this paper we suppose constant temperature lower than Curie temperature and, without loss of generality, we assume that  $m = 1$ , that is  $M(x) \in S^2$ . The exchange energy  $\int_{\Omega_n} |DM|^2 dx$  penalizes the spatial variation of  $M$ , driving the body to have large regions of uniform magnetization separated by thin transition layers. The scalar function  $U_M : \mathbb{R}^3 \rightarrow \mathbb{R}$  is the so-called magnetostatic potential. The magnetostatic energy  $\int_{\Omega_n} DU_M M dx = \int_{\mathbb{R}^3} |DU_M|^2 dx$  favors  $\operatorname{div} M = 0$  in  $\Omega_n$  and  $M \cdot \nu = 0$  on  $\partial\Omega_n$ , where  $\nu$  is the exterior unit normal to  $\partial\Omega_n$ . The constant  $\alpha$  is typically on order of 100 nanometers and measures the relative strength of exchange energy with respect to the magnetostatic energy. The anisotropy energy  $\int_{\Omega_n} \varphi(M) dx$  favors magnetization along special crystallographic directions, while the external (Zeeman) energy  $\int_{\Omega_n} F_n M dx$  favors magnetization parallel to an externally applied field.

After having reformulated the problem on a fixed domain through appropriate rescalings of the kind proposed by P.G. Ciarlet and P. Destuynder [7] and having imposed appropriate convergence assumptions on the rescaled exterior fields, using the main ideas of  $\Gamma$ -convergence method introduced by E. De Giorgi [8], we derive the limit problem which depends on the limit  $\lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty[$ . Precisely, when  $q \in ]0, +\infty[$  (i.e.  $h_n^b \simeq h_n^a$ ), in Theorem 4.1 we

prove that

$$\begin{aligned} \lim_n \frac{J_n}{h_n^a + h_n^b} &= \frac{1}{1+q} \\ \min \left\{ \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \alpha |(D_{x_2} \mu^a | D_{x_3} \mu^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\mu_1^a|^2 - 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \mu^a \right) d(x_2, x_3) \right. \\ &+ q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \alpha |(D_{x_1} \mu^b | D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 - 2 \int_{-1}^0 f^b(x_1, x_2, x_3) dx_3 \mu^b \right) d(x_1, x_2), \\ &\left. (\mu^a, \mu^b) \in H^1 \left( ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2 \right) \times H^1 \left( ]-\frac{1}{2}, \frac{1}{2}[^2, S^2 \right), \quad \mu^a(x_2, 0) = \mu^b(0, x_2) \text{ in } ]-\frac{1}{2}, \frac{1}{2}[ \right\}, \end{aligned}$$

where  $f^a$  and  $f^b$  (see (3.6) and (3.9)) are the  $L^2$ -weak limits of the rescaled exterior fields in the vertical domain and in the horizontal domain, respectively, while  $\mu_1^a$  and  $\mu_3^b$  are the first and the third component of  $\mu^a$  and  $\mu^b$ , respectively. Remark that  $h_n^a + h_n^b$  is the measure of  $\Omega_n$ . We obtain two  $2D$  problems coupled by the junction condition  $\mu^a(x_2, 0) = \mu^b(0, x_2)$  in  $]-\frac{1}{2}, \frac{1}{2}[$ . Moreover, the magnetostatic energy transforms into  $\frac{1}{2} \left( \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\mu_1^a|^2 d(x_2, x_3) + q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |\mu_3^b|^2 d(x_1, x_2) \right)$ , so that the limit problem is completely local. It is easy to see that, if  $\varphi = 0$ ,  $f^a = 0$  and  $f^b = 0$ , then the minimum in the limit problem is zero and it is attained by  $((0, 1, 0), (0, 1, 0))$  and  $((0, -1, 0), (0, -1, 0))$ , i.e. the limit magnetization is parallel to the two orthogonal thin films.

In the other two cases, the structure behaves like a single thin film. Precisely, when  $q = 0$  (i.e.  $h_n^b \ll h_n^a$ ), in Theorem 4.2 we prove that the limit problem reduces to a  $2D$  problem in the vertical thin film losing the junction condition:

$$\begin{aligned} \lim_n \frac{J_n}{h_n^a + h_n^b} &= \min \left\{ \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \alpha |(D_{x_2} \mu^a | D_{x_3} \mu^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\mu_1^a|^2 \right) d(x_2, x_3) + \right. \\ &\left. \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( -2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \mu^a \right) d(x_2, x_3) : \quad \mu^a \in H^1 \left( ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2 \right) \right\}. \end{aligned}$$

In this case, if  $\varphi = 0$  and  $f^a = 0$ , then the minimum in the limit problem is zero and it is attained by constant functions  $(0, c_2, c_3) \in S^2$ , i.e. the limit magnetization is contained in the vertical plane, but its orientation is undetermined. Analogously, when  $q = +\infty$  (i.e.  $h_n^b \gg h_n^a$ ), in Theorem 4.3 we prove that the limit problem reduces to a  $2D$  problem in the horizontal thin film:

$$\begin{aligned} \lim_n \frac{J_n}{h_n^a + h_n^b} &= \min \left\{ \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \alpha |(D_{x_1} \mu^b | D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 \right) d(x_1, x_2) + \right. \\ &\left. \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( -2 \int_{-1}^0 f^b(x_1, x_2, x_3) dx_3 \mu^b \right) d(x_1, x_2) : \quad \mu^b \in H^1 \left( ]-\frac{1}{2}, \frac{1}{2}[^2, S^2 \right) \right\}. \end{aligned}$$

In all three cases, strong convergences in  $H^1$ -norm are obtained for the rescaled magnetization and in  $L^2$ -norm for the gradient of the rescaled magnetostatic potential.

The proofs of these results are developed in several steps. In the case  $q \in ]0, +\infty[$ , we begin by proving, in Subsection 5.1, a general convergence result for the magnetostatic energy. The proof of this result has the framework of the proof of Proposition 4.1 in G. Gioia and R. D. James [12], but in our case we have to build couples of test functions satisfying suitable junction conditions between the two films. In Subsection 5.2, we obtain classical *a priori* estimates on the magnetization providing the convergence of the magnetization to a couple  $(\widehat{\mu}^a, \widehat{\mu}^b) \in H^1(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2) \times H^1(]-\frac{1}{2}, \frac{1}{2}[^2, S^2)$ . A real difficulty is to recover the junction condition  $\widehat{\mu}^a(x_2, 0) = \widehat{\mu}^b(0, x_2)$  in  $]-\frac{1}{2}, \frac{1}{2}[$ . It is obtained in Subsection 5.3 through a suitable splitting of the trace and deducing sharp estimates for this. The crucial point of this paper is the density result in Subsection 5.4, where we approximate couples of  $H^1$  maps, with values in  $S^2$ , defined on the two  $2D$  limit domains and with the same trace on the line joining the two orthogonal thin films, by couples of regular maps, with values in  $S^2$ , satisfying the same junction condition. This result is not trivial since our limit domain  $(\{0\} \times [-\frac{1}{2}, \frac{1}{2}] \times [0, 1]) \cup \left( [-\frac{1}{2}, \frac{1}{2}]^2 \times \{0\} \right)$  is not a manifold and its elaborate proof is based on the combination of an approximation result proved by F. Bethuel and X. Zheng [4] with splitting techniques introduced by H. Le Dret [19] and with a projection technique from  $\mathbb{R}^3$  into  $S^2$  as in R. Hardt, D. Kinderlehrer and F. H. Lin [13]. In Subsection 5.5, combining convex arguments with projection techniques as we used in [10] and with the convergence of the magnetostatic energy, we build a recovery sequence for a generic regular couple in the limit space and, by virtue of the density result, we conclude the proof in the case  $q \in ]0, +\infty[$ . Section 6 is devoted to the cases  $q = 0$  and  $q = +\infty$ .

In what concerns the study of a single ferromagnetic thin film, several results are present in literature. The fact that the magnetostatic energy behaves, at the limit, like an anisotropic local term which forces the magnetization to be tangent to the thin film was proved, for the first time, by G. Gioia and R. D. James [12]. This result was extended by C. Leone and R. Alicandro [1] to the case with non-convex exchange energy, and by M. Baía and E. Zappale [3] to a thin film with nonhomogeneous profile. The time-dependent case was treated by H. Ammari, L. Halpern and K. Hamdache [2], and by G. Carbou [6]. Very different regimes were considered by A. Desimone, R.V. Kohn, S. Muller and F. Otto [9], and by R.V. Kohn and V.V. Slastikov in [17], where  $\frac{h}{l}$  and  $\frac{\alpha}{l}$  vanish,  $h$  being the film thickness,  $l$  the in-plane diameter and  $\alpha$  the exchange length of the ferromagnetic material.

Our paper is, to our knowledge, the first work on the junction of ferromagnetic thin bodies, unless we consider papers [10] and [11] where we developed an asymptotic analysis of minimizing maps with values in  $S^2$  for the energy  $\int_{C_n} (|DM|^2 - 2F_n M) dx$ , neglecting the term with the nonlocal magnetostatic energy. The geometry of  $C_n$  consists of two cylinders attached together that shrink respectively to a segment and to a  $2D$  disc in the first paper, while the two cylinders transform into a  $T$ -shaped domain in the second one. The limit problem is uncoupled in the former, while it is coupled in the latter.

## 2 The setting of the problem

In the sequel,  $x = (x_1, x_2, x_3)$  denotes the generic point of  $\mathbb{R}^3$ . If  $a, b, c \in \mathbb{R}^3$ , then  $(a|b|c)$  denotes the  $3 \times 3$  real matrix having  $a^T$  as first column,  $b^T$  as second column, and  $c^T$  as third column. In according to this notation, if  $v : A \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , then  $Dv$  denotes the  $3 \times 3$  real matrix  $(D_{x_1}v|D_{x_2}v|D_{x_3}v)$ , where  $D_{x_i}v \in \mathbb{R}^3$ ,  $i=1,2,3$ , stands for the derivative of  $v$  with respect to  $x_i$ . Moreover,  $\tilde{v}$  denotes the zero-extension of  $v$  to  $\mathbb{R}^3$ .

Let  $\{h_n^a\}_{n \in \mathbb{N}}, \{h_n^b\}_{n \in \mathbb{N}} \subset ]0, 1[$  be two sequences such that

$$\begin{cases} \lim_n h_n^a = 0 = \lim_n h_n^b, \\ \lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty], \end{cases} \quad (2.1)$$

and, for every  $n \in \mathbb{N}$ , let  $\Omega_n^a = ]-\frac{h_n^a}{2}, \frac{h_n^a}{2}[ \times ]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[$ ,  $\Omega_n^b = ]-\frac{1}{2}, \frac{1}{2}[^2 \times ]-\frac{h_n^b}{2}, \frac{h_n^b}{2}[$  and  $\Omega_n = \Omega_n^a \cup \Omega_n^b$ , as in figure.

Let  $B$  be an interval containing  $\overline{\Omega_n}$  for every  $n \in \mathbb{N}$ , for instance let  $B = ]-1, 1[^2 \times ]-2, 2[$ , and for every  $n \in \mathbb{N}$ , set

$$\mathcal{U} = \left\{ U \in L_{loc}^1(\mathbb{R}^3) : U \in L^2(B), DU \in (L^2(\mathbb{R}^3))^3, \int_B U dx = 0 \right\}.$$

It is easy to prove that  $\mathcal{U}$  is contained in  $L_{loc}^2(\mathbb{R}^3)$  and it is a Hilbert space with the inner product:  $(U, V) = \int_{\mathbb{R}^3} DUDV dx + \int_B UV dx$ . Moreover, from the Poincaré-Wirtinger inequality it follows that a norm on  $\mathcal{U}$  equivalent to  $(U, U)^{\frac{1}{2}}$  is given by  $(\int_{\mathbb{R}^3} |DU|^2 dx)^{\frac{1}{2}}$ . Then, Lax-Milgram Theorem provides that, for  $M \in L^2(\Omega_n, \mathbb{R}^3)$ , the following equation:

$$\begin{cases} U_{M,n} \in \mathcal{U}, \\ \int_{\mathbb{R}^3} (-DU_{M,n} + \tilde{M})DU dx = 0, \quad \forall U \in \mathcal{U}, \end{cases}$$

admits a unique solution and  $U_{M,n}$  is characterized as the unique minimizer of the following problem:

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |DU - \tilde{M}|^2 dx : U \in \mathcal{U} \right\}.$$

Moreover,  $U_M$  belongs to  $H^1(\mathbb{R}^3)$  (see [16]).

Let  $\alpha$  be a positive constant,  $\varphi : S^2 \rightarrow [0, +\infty[$  be a continuous, even function and, for every  $n \in \mathbb{N}$ ,  $F_n \in L^2(\Omega_n, \mathbb{R}^3)$ . The following problem:

$$\min \left\{ \int_{\Omega_n} \left( \alpha |DM|^2 + \varphi(M) + \frac{1}{2} DU_{M,n} M - 2F_n M \right) dx : M \in H^1(\Omega_n, S^2) \right\}. \quad (2.2)$$

has at least one solution (see [20]). In general, one can not expect a unique solution, because of the non-convexity of the constraint  $M(x) \in S^2$ . The aim of this paper is to study the asymptotic behavior, as  $n$  diverges, of problem (2.2). As we shall show, its asymptotic behavior depends on the limit  $q$  given in (2.1) and on some assumptions on  $F_n$ .

### 3 The rescaled problem

As it is usual (see [7]), problem (2.2) will be reformulated on a fixed domain through the following rescalings:

$$\left\{ \begin{array}{l} (x_1, x_2, x_3) \in \Omega^a = \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] 0, 1 \left[ \longrightarrow (h_n^a x_1, x_2, x_3) \in \text{Int}(\Omega_n^a), \\ (x_1, x_2, x_3) \in \Omega^b = \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] -1, 0 \left[ \longrightarrow (x_1, x_2, h_n^b x_3) \in \Omega_n^b, \end{array} \right.$$

where  $\text{Int}(\Omega_n^a)$  denotes the interior of  $\Omega_n^a$ . Moreover, the energy will be multiplied by  $\frac{1}{h_n^a}$  when  $q \neq +\infty$ , by  $\frac{1}{h_n^b}$  when  $q = +\infty$ . Namely, let  $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ ,  $\mathbb{R}_-^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$  and, for every  $n \in \mathbb{N}$ ,  $B_n^a = \left] -\frac{1}{h_n^a}, \frac{1}{h_n^a} \left[ \times \left] -1, 1 \left[ \times \right] 0, 2 \left[$ ,  $B_n^b = \left] -1, 1 \left[ \times \left] -\frac{2}{h_n^b}, 0 \left[$  and

$$\begin{aligned} \mathcal{U}_n = \{ & u = (u^a, u^b) \in L_{loc}^1(\overline{\mathbb{R}_+^3}) \times L_{loc}^1(\overline{\mathbb{R}_-^3}) : \\ & (u^a|_{B_n^a}, u^b|_{B_n^b}) \in L^2(B_n^a) \times L^2(B_n^b), (Du^a, Du^b) \in (L^2(\mathbb{R}_+^3))^3 \times (L^2(\mathbb{R}_-^3))^3, \\ & \int_{B_n^a} u^a dx + \frac{h_n^b}{h_n^a} \int_{B_n^b} u^b dx = 0, \\ & u^a(x_1, x_2, 0) = u^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } \mathbb{R}^2 \}. \end{aligned} \quad (3.1)$$

Then, for every  $\underline{m} = (\underline{m}^a, \underline{m}^b) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$ , the following equation:

$$\left\{ \begin{array}{l} u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^b) \in \mathcal{U}_n, \\ \int_{\mathbb{R}_+^3} \left( - \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m},n}^a, D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) + \widetilde{m}^a \right) \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) dx + \\ \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( - \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m},n}^b \right) + \widetilde{m}^b \right) \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^b} D_{x_3} u^b \right) dx = 0, \\ \forall u = (u^a, u^b) \in \mathcal{U}_n. \end{array} \right. \quad (3.2)$$

admits a unique solution and  $u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^b) \in \mathcal{U}_n$  is characterized as the the unique solution of the following problem:

$$j_{\underline{m},n}(u_{\underline{m},n}) = \min \{ j_{\underline{m},n}(u) : u \in \mathcal{U}_n \}, \quad (3.3)$$

where

$$\begin{aligned}
j_{\underline{m},n} : u = (u^a, u^b) \in \mathcal{U}_n &\longrightarrow \frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u^a, D_{x_2} u^a, D_{x_3} u^a \right) - \widetilde{\underline{m}}^a \right|^2 dx + \\
&\frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^b} D_{x_3} u^b \right) - \widetilde{\underline{m}}^b \right|^2 dx.
\end{aligned} \tag{3.4}$$

Remark that  $u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^b)$  belongs to  $H^1(\mathbb{R}_+^3) \times H^1(\mathbb{R}_-^3)$ .

For every  $n \in \mathbb{N}$ , let

$$\begin{aligned}
\mathcal{M}_n = \{ \underline{m} = (\underline{m}^a, \underline{m}^b) \in H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2) : \\
\underline{m}^a(x_1, x_2, 0) = \underline{m}^b(h_n^a x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ] - \frac{1}{2}, \frac{1}{2}[^2 \},
\end{aligned} \tag{3.5}$$

$$f_n : x \in \Omega^a \cup \Omega^b \rightarrow f_n(x) = \begin{cases} f_n^a(x) = F_n(h_n^a x_1, x_2, x_3), & \text{for } x \text{ a.e. in } \Omega^a, \\ f_n^b(x) = F_n(x_1, x_2, h_n^b x_3), & \text{for } x \text{ a.e. in } \Omega^b, \end{cases} \tag{3.6}$$

and

$$\begin{aligned}
E_n : \underline{m} = (\underline{m}^a, \underline{m}^b) \in \mathcal{M}_n &\longrightarrow \\
&\int_{\Omega^a} \left( \alpha \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}^a, D_{x_2} \underline{m}^a, D_{x_3} \underline{m}^a \right) \right|^2 + \varphi(\underline{m}^a) - 2f_n^a \underline{m}^a \right) dx + \\
&\frac{1}{2} \int_{\Omega^a} \left( \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m},n}^a, D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \underline{m}^a \right) dx + \\
&\frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \alpha \left| \left( D_{x_1} \underline{m}^b, D_{x_2} \underline{m}^b, \frac{1}{h_n^b} D_{x_3} \underline{m}^b \right) \right|^2 + \varphi(\underline{m}^b) - 2f_n^b \underline{m}^b \right) dx + \\
&\frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m},n}^b \right) \underline{m}^b \right) dx.
\end{aligned} \tag{3.7}$$

Then, the function defined by

$$\begin{cases} M_n(h_n^a x_1, x_2, x_3), & \text{for } x \text{ a.e. in } \Omega^a, \\ M_n(x_1, x_2, h_n^b x_3), & \text{for } x \text{ a.e. in } \Omega^b, \end{cases}$$

with  $M_n$  solution of problem (2.2), is a solution of the following problem:

$$\min \{ E_n(\underline{m}) : \underline{m} \in \mathcal{M}_n \}. \tag{3.8}$$

Actually, the goal of this paper becomes to study the asymptotic behavior, as  $n$  diverges, of problem (3.8). To this aim, it will be assumed that

$$f_n^a \rightharpoonup f^a \text{ weakly in } L^2(\Omega^a, \mathbb{R}^3), \quad f_n^b \rightharpoonup f^b \text{ weakly in } L^2(\Omega^b, \mathbb{R}^3). \tag{3.9}$$



Remark that, setting for every  $n \in \mathbb{N}$

$$\begin{aligned}
E_n^{mag} : \underline{m} = (\underline{m}^a, \underline{m}^b) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3) \longrightarrow \\
\frac{1}{2} \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m},n}^a, D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \right|^2 dx + \\
\frac{1}{2} \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m},n}^b \right) \right|^2 dx,
\end{aligned} \tag{3.10}$$

from (3.2) it follows that

$$\begin{aligned}
E_n(\underline{m}) &= \int_{\Omega^a} \left( \alpha \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}^a, D_{x_2} \underline{m}^a, D_{x_3} \underline{m}^a \right) \right|^2 + \varphi(\underline{m}^a) - 2f_n^a \underline{m}^a \right) dx + \\
\frac{h_n^b}{h_n^a} \int_{\Omega^b} \left( \alpha \left| \left( D_{x_1} \underline{m}^b, D_{x_2} \underline{m}^b, \frac{1}{h_n^b} D_{x_3} \underline{m}^b \right) \right|^2 + \varphi(\underline{m}^b) - 2f_n^b \underline{m}^b \right) dx + \\
E_n^{mag}(\underline{m}), \quad \forall \underline{m} = (\underline{m}^a, \underline{m}^b) \in \mathcal{M}_n, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.11}$$

## 4 The main results

Let

$$\begin{aligned}
\mathcal{M} &= \left\{ \mu = (\mu^a, \mu^b) \in H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2) : \right. \\
\mu^a &\text{ is independent of } x_1, \quad \mu^b \text{ is independent of } x_3, \\
\mu^a(0, x_2, 0) &= \mu^b(0, x_2, 0), \text{ for } x_2 \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[ \simeq \\
\left. \left\{ \mu = (\mu^a, \mu^b) \in H^1\left(]-\frac{1}{2}, \frac{1}{2}[\times]0, 1[, S^2\right) \times H^1\left(]-\frac{1}{2}, \frac{1}{2}[^2, S^2\right) : \right. \right. \\
\left. \left. \mu^a(x_2, 0) = \mu^b(0, x_2), \text{ for } x_2 \text{ a.e. in } ]-\frac{1}{2}, \frac{1}{2}[ \right\} \right.
\end{aligned} \tag{4.1}$$

and, for  $q \in ]0, +\infty[$ , let

$$\begin{aligned}
E_q : \mu = (\mu^a, \mu^b) \in \mathcal{M} \longrightarrow \\
\int_{]-\frac{1}{2}, \frac{1}{2}[\times]0, 1[} \left( \alpha |(D_{x_2} \mu^a, D_{x_3} \mu^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\mu_1^a|^2 - 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \mu^a \right) d(x_2, x_3) + \\
q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \alpha |(D_{x_1} \mu^b, D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 - 2 \int_{-1}^0 f^b(x_1, x_2, x_3) dx_3 \mu^b \right) d(x_1, x_2),
\end{aligned} \tag{4.2}$$

then this paper is essentially devoted to prove the following result:

**Theorem 4.1.** *Assume (2.1) with  $q \in ]0, +\infty[$  and (3.9). For every  $n \in \mathbb{N}$  let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of (3.8) and  $u_{\underline{m}_n, n} = (u_{\underline{m}_n, n}^a, u_{\underline{m}_n, n}^b)$  be the unique solution of (3.3) corresponding to  $\underline{m}_n$ . Moreover, let  $\mathcal{M}$  and  $E_q$  be defined by (4.1) and (4.2), respectively. Then, there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\widehat{\mu} = (\widehat{\mu}^a, \widehat{\mu}^b) \in \mathcal{M}$ , depending on the selected subsequence, such that*

$$\begin{cases} \underline{m}_{n_i}^a \rightarrow \widehat{\mu}^a \text{ strongly in } H^1(\Omega^a, S^2), \\ \underline{m}_{n_i}^b \rightarrow \widehat{\mu}^b \text{ strongly in } H^1(\Omega^b, S^2), \end{cases} \quad (4.3)$$

$$\begin{cases} \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases} \quad (4.4)$$

$$\begin{cases} \frac{1}{h_{n_i}^a} D_{x_1} u_{\underline{m}_{n_i}, n_i}^a \rightarrow \widetilde{\mu}_1^a, & D_{x_2} u_{\underline{m}_n, n}^a \rightarrow 0, & D_{x_3} u_{\underline{m}_n, n}^a \rightarrow 0 & \text{strongly in } L^2(\mathbb{R}_+^3), \\ D_{x_1} u_{\underline{m}_n, n}^b \rightarrow 0, & D_{x_2} u_{\underline{m}_n, n}^b \rightarrow 0, & \frac{1}{h_{n_i}^b} D_{x_3} u_{\underline{m}_{n_i}, n_i}^b \rightarrow \widetilde{\mu}_3^b & \text{strongly in } L^2(\mathbb{R}_-^3), \end{cases} \quad (4.5)$$

as  $i$  and  $n$  diverge, and  $\widehat{\mu}$  is a solution of the following problem:

$$E_q(\widehat{\mu}) = \min \{E_q(\mu) : \mu \in \mathcal{M}\}, \quad (4.6)$$

where  $\widetilde{\mu}_1^a$  and  $\widetilde{\mu}_3^b$  denote the zero-extension of  $\widehat{\mu}_1^a$  and  $\widehat{\mu}_3^b$  to  $\mathbb{R}^3$ , respectively. Moreover, it results that

$$\lim_n E_n(\underline{m}_n) = E_q(\widehat{\mu}). \quad (4.7)$$

In this paper we also study the cases  $q = 0$  and  $q = +\infty$ . Precisely, in the case  $q = 0$ , let

$$\mathcal{M}_0 = \{\mu^a \in H^1(\Omega^a, S^2) : \mu^a \text{ is independent of } x_1, \} \simeq H^1(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2) \quad (4.8)$$

and

$$E_0 : \mu^a \in \mathcal{M}_0 \rightarrow$$

$$\int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \alpha |(D_{x_2} \mu^a | D_{x_3} \mu^a)|^2 + \varphi(\mu^a) + \frac{1}{2} |\mu_1^a|^2 - 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \mu^a \right) d(x_2, x_3), \quad (4.9)$$

then the following result will be proved:

**Theorem 4.2.** *Assume (2.1) with  $q = 0$  and (3.9). For every  $n \in \mathbb{N}$  let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of (3.8) and  $u_{\underline{m}_n, n} = (u_{\underline{m}_n, n}^a, u_{\underline{m}_n, n}^b)$  be the unique solution of (3.3) corresponding to  $\underline{m}_n$ . Moreover, let  $\mathcal{M}_0$  and  $E_0$  be defined by (4.8) and (4.9), respectively. Then, there exist*

an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\widehat{\mu}^a \in \mathcal{M}_0$ , depending on the selected subsequence, such that

$$\begin{cases} \underline{m}_{n_i}^a \rightarrow \widehat{\mu}^a \text{ strongly in } H^1(\Omega^a, S^2), \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} \underline{m}_n^b \rightarrow 0 \text{ strongly in } H^1(\Omega^b, S^2), \end{cases} \quad (4.10)$$

$$\begin{cases} \frac{1}{h_n^a} D_{x_1} \underline{m}_n^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n^b} \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_3} \underline{m}_n^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases} \quad (4.11)$$

$$\begin{cases} \frac{1}{h_{n_i}^a} D_{x_1} u_{\underline{m}_{n_i}, n_i}^a \rightarrow \widetilde{\mu}_1^a, \quad D_{x_2} u_{\underline{m}_n, n}^a \rightarrow 0, \quad D_{x_3} u_{\underline{m}_n, n}^a \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}_+^3), \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_1} u_{\underline{m}_n, n}^b \rightarrow 0, \quad \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_2} u_{\underline{m}_n, n}^b \rightarrow 0, \quad \frac{1}{h_n^b} \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_3} u_{\underline{m}_n, n}^b \rightarrow 0 \\ \text{strongly in } L^2(\mathbb{R}_-^3), \end{cases} \quad (4.12)$$

as  $i$  and  $n$  diverge, and  $\widehat{\mu}^a$  is a solution of the following problem:

$$E_0(\widehat{\mu}^a) = \min \{E_0(\mu^a) : \mu^a \in \mathcal{M}_0\}, \quad (4.13)$$

where  $\widetilde{\mu}_1^a$  denotes the zero-extension of  $\widehat{\mu}_1^a$ . Moreover, it results that

$$\lim_n E_n(\underline{m}_n) = E_0(\widehat{\mu}^a). \quad (4.14)$$

In the case  $q = +\infty$ , let

$$\mathcal{M}_\infty = \{\mu^b \in H^1(\Omega^b, S^2) : \mu^b \text{ is independent of } x_3, \} \simeq H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right]^2, S^2\right) \quad (4.15)$$

and

$$E_\infty : \mu^b \in \mathcal{M}_\infty \rightarrow$$

$$\int_{\left]-\frac{1}{2}, \frac{1}{2}\right]^2} \left( \alpha |(D_{x_1} \mu^b | D_{x_2} \mu^b)|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 - 2 \int_{-1}^0 f^b(x_1, x_2, x_3) dx_3 \mu^b \right) d(x_1, x_2), \quad (4.16)$$

then the following result will be proved:

**Theorem 4.3.** *Assume (2.1) with  $q = +\infty$ , and (3.9). For every  $n \in \mathbb{N}$  let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of (3.8) and  $u_{\underline{m}_n, n} = (u_{\underline{m}_n, n}^a, u_{\underline{m}_n, n}^b)$  be the unique solution of (3.3) corresponding to  $\underline{m}_n$ . Moreover, let  $\mathcal{M}_\infty$  and  $E_\infty$  be defined by (4.15) and (4.16), respectively. Then, there*

exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\widehat{\mu}^b \in \mathcal{M}_\infty$ , depending on the selected subsequence, such that

$$\begin{cases} \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} \underline{m}_n^a \rightarrow 0 \text{ strongly in } H^1(\Omega^a, S^2), \\ \underline{m}_{n_i}^b \rightarrow \widehat{\mu}^b \text{ strongly in } H^1(\Omega^b, S^2), \end{cases}$$

$$\begin{cases} \frac{1}{h_n^a} \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} D_{x_1} \underline{m}_n^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n^b} D_{x_3} \underline{m}_n^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{cases}$$

$$\begin{cases} \frac{1}{h_n^a} \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} D_{x_1} u_{\underline{m}_n, n}^a \rightarrow 0, \quad \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} D_{x_2} u_{\underline{m}_n, n}^a \rightarrow 0, \quad \left( \frac{h_n^a}{h_n^b} \right)^{\frac{1}{2}} D_{x_3} u_{\underline{m}_n, n}^a \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}_+^3), \\ D_{x_1} u_{\underline{m}_n, n}^b \rightarrow 0, \quad D_{x_2} u_{\underline{m}_n, n}^b \rightarrow 0, \quad \frac{1}{h_{n_i}^b} D_{x_3} u_{\underline{m}_{n_i}, n_i}^b \rightarrow \widetilde{\mu}_3^b \quad \text{strongly in } L^2(\mathbb{R}_-^3), \end{cases}$$

as  $i$  and  $n$  diverge, and  $\widehat{\mu}^b$  is a solution of the following problem:

$$E_\infty(\widehat{\mu}^b) = \min \{ E_\infty(\mu^b) : \mu^b \in \mathcal{M}_\infty \}, \quad (4.17)$$

where  $\widetilde{\mu}_3^b$  denotes the zero-extension of  $\widehat{\mu}_3^b$ . Moreover, it results that

$$\lim_n \left( \frac{h_n^a}{h_n^b} E_n(\underline{m}_n) \right) = E_\infty(\widehat{\mu}^b).$$

## 5 The case $q \in ]0, +\infty[$

The proof of Theorem 4.1 will be developed in several steps. We begin by proving a general convergence result for the magnetostatic energy. Its proof is inspired by the proof of Proposition 4.1 in [12], but we have to build couples of test functions satisfying the junction condition in (3.1).

### 5.1 A Convergence result for the magnetostatic energy

**Proposition 5.1.** *Assume (2.1) with  $q \in ]0, +\infty[$ . Let  $\{\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)\}_{n \in \mathbb{N}} \subset L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  and  $\mu = (\mu^a, \mu^b) = ((\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^b, \mu_2^b, \mu_3^b)) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  be such that*

$$(\underline{m}_n^a, \underline{m}_n^b) \rightarrow (\mu^a, \mu^b) \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3), \quad (5.1)$$

as  $n$  diverges. Moreover, for every  $n \in \mathbb{N}$  let  $u_{\underline{m}_n, n} = (u_{\underline{m}_n, n}^a, u_{\underline{m}_n, n}^b)$  be the unique solution of (3.3) corresponding to  $\underline{m}_n$ , and let  $E_n^{mag}$  be defined by (3.10). Then, it results that

$$\begin{cases} \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a \rightarrow \widetilde{\mu}_1^a, & D_{x_2} u_{\underline{m}_n, n}^a \rightarrow 0, & D_{x_3} u_{\underline{m}_n, n}^a \rightarrow 0 & \text{strongly in } L^2(\mathbb{R}_+^3), \\ D_{x_1} u_{\underline{m}_n, n}^b \rightarrow 0, & D_{x_2} u_{\underline{m}_n, n}^b \rightarrow 0, & \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b \rightarrow \widetilde{\mu}_3^b & \text{strongly in } L^2(\mathbb{R}_-^3), \end{cases} \quad (5.2)$$

as  $n$  diverges, and

$$\lim_n E_n^{mag}(\underline{m}_n) = \frac{1}{2} \left( \int_{\Omega^a} |\mu_1^a|^2 dx + q \int_{\Omega^b} |\mu_3^b|^2 dx \right). \quad (5.3)$$

*Proof.* By choosing  $u = (0, 0)$  as test function in (3.3) corresponding to  $\underline{m}_n$ , by virtue of (2.1) with  $q \neq +\infty$  and (5.1) it results that

$$\begin{aligned} \exists c \in ]0, +\infty[ : & \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a, D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a \right) - \widetilde{m}_n^a \right|^2 dx + \\ & \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b \right) - \widetilde{m}_n^b \right|^2 dx \leq c, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Consequently, applying the triangle inequality and using again (2.1) with  $q \neq 0$  and (5.1), it follows that

$$\begin{cases} \exists c \in ]0, +\infty[ : \\ \left\| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, & \|D_{x_2} u_{\underline{m}_n, n}^a\|_{L^2(\mathbb{R}_+^3)} \leq c, & \|D_{x_3} u_{\underline{m}_n, n}^a\|_{L^2(\mathbb{R}_+^3)} \leq c, \\ \|D_{x_1} u_{\underline{m}_n, n}^b\|_{L^2(\mathbb{R}_-^3)} \leq c, & \|D_{x_2} u_{\underline{m}_n, n}^b\|_{L^2(\mathbb{R}_-^3)} \leq c, & \left\| \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \end{cases} \quad (5.4)$$

$\forall n \in \mathbb{N}.$

Since  $(u_{\underline{m}_n}^a, u_{\underline{m}_n}^b)$  belongs to  $H^1(\mathbb{R}_+^3) \times H^1(\mathbb{R}_-^3)$ , the Sobolev-Gagliardo-Nirenberg Inequality and (5.4) provide that

$$\exists c \in ]0, +\infty[ : \|u_{\underline{m}_n, n}^a\|_{L^6(\mathbb{R}_+^3)} \leq c, \quad \|u_{\underline{m}_n, n}^b\|_{L^6(\mathbb{R}_-^3)} \leq c, \quad \forall n \in \mathbb{N}. \quad (5.5)$$

Estimates (5.4) and (5.5) guarantee the existence of a function  $u = (u^a, u^b) \in L^6(\mathbb{R}_+^3) \times L^6(\mathbb{R}_-^3)$ , with  $Du = (Du^a, Du^b) \in (L^2(\mathbb{R}_+^3))^3 \times (L^2(\mathbb{R}_-^3))^3$ ,  $u^a$  independent of  $x_1$  and  $u^b$  independent of  $x_3$ , such that on extraction of a suitable subsequence (not relabelled)

$$\begin{cases} u_{\underline{m}_n, n}^a \rightharpoonup u^a \text{ weakly in } L^6(\mathbb{R}_+^3), & Du_{\underline{m}_n, n}^a \rightharpoonup Du^a \text{ weakly in } (L^2(\mathbb{R}_+^3))^3, \\ u_{\underline{m}_n, n}^b \rightharpoonup u^b \text{ weakly in } L^6(\mathbb{R}_-^3), & Du_{\underline{m}_n, n}^b \rightharpoonup Du^b \text{ weakly in } (L^2(\mathbb{R}_-^3))^3, \end{cases} \quad (5.6)$$

as  $n$  diverges. Moreover, the fact that  $u^a$  is independent of  $x_1$  and  $Du^a \in (L^2(\mathbb{R}_+^3))^3$  provides that

$$+\infty > \int_{\mathbb{R}_+^3} |Du^a|^2 dx \geq \int_a^b \left( \int_{\mathbb{R}_+^2} (|D_{x_2} u^a|^2 + |D_{x_3} u^a|^2) d(x_2, x_3) \right) dx_1 =$$

$$(b-a) \int_{\mathbb{R}_+^2} (|D_{x_2} u^a|^2 + |D_{x_3} u^a|^2) d(x_2, x_3) \quad \forall a, b \in \mathbb{R} \text{ with } a < b,$$

which gives that  $u^a$  is independent of  $x_2$  and  $x_3$ , too. Then  $u^a = 0$ , since zero is the only constant function belonging to  $L^6(\mathbb{R}_+^3)$ . Similarly, one proves that  $u^b = 0$ . Then, from (5.6) one concludes that

$$\begin{cases} Du_{\underline{m}_n, n}^a \rightharpoonup 0 \text{ weakly in } (L^2(\mathbb{R}_+^3))^3, \\ Du_{\underline{m}_n, n}^b \rightharpoonup 0 \text{ weakly in } (L^2(\mathbb{R}_-^3))^3, \end{cases} \quad (5.7)$$

as  $n$  diverges, and these convergences hold true for the whole sequence.

The first and the last estimate in (5.4) guarantee the existence of  $\xi^a \in L^2(\mathbb{R}_+^3)$  and  $\xi^b \in L^2(\mathbb{R}_-^3)$  such that on extraction of a suitable subsequence (not relabelled)

$$\frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a \rightharpoonup \xi^a \text{ weakly in } L^2(\mathbb{R}_+^3), \quad \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b \rightharpoonup \xi^b \text{ weakly in } L^2(\mathbb{R}_-^3), \quad (5.8)$$

as  $n$  diverges.

For proving that convergences (5.7) are (5.8) are strong and for identifying  $\xi^a$  and  $\xi^b$ , introduce a sequence  $\{\varepsilon\}$  of positive numbers converging to zero, a diverging sequence  $\{\lambda\}$  of positive numbers, and two sequences  $\{\mu_\varepsilon^a\}_\varepsilon \subset C_0^\infty(\Omega^a)$  and  $\{\mu_\varepsilon^b\}_\varepsilon \subset C_0^\infty(\Omega^b)$  such that

$$\mu_\varepsilon^a \rightarrow \mu_1^a \text{ strongly in } L^2(\Omega^a), \quad \mu_\varepsilon^b \rightarrow \mu_3^b \text{ strongly in } L^2(\Omega^b), \quad (5.9)$$

as  $\varepsilon$  tends to zero. Moreover, for every  $\varepsilon$ ,  $\lambda$  and  $n \in \mathbb{N}$  set

$$u_{\varepsilon, \lambda, n} = \begin{cases} u_{\varepsilon, \lambda, n}^a = h_n^a \int_{-\frac{1}{2}}^{x_1} \mu_\varepsilon^a(s, x_2, x_3) ds - \frac{h_n^a}{\lambda} \int_{\frac{1}{2}}^{x_1} \chi_{[\frac{1}{2}, \frac{1}{2} + \lambda]}(r) dr \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds + c_{\varepsilon, \lambda, n}, \\ \hspace{15em} \text{a.e. in } \mathbb{R}_+^3, \\ u_{\varepsilon, \lambda, n}^b = -h_n^b \int_{x_3}^0 \mu_\varepsilon^b(x_1, x_2, s) ds + \frac{h_n^b}{\lambda} \int_{x_3}^{-1} \chi_{[-1 - \lambda, -1]}(r) dr \int_{-1}^0 \mu_\varepsilon^b(x_1, x_2, s) ds + c_{\varepsilon, \lambda, n}, \\ \hspace{15em} \text{a.e. in } \mathbb{R}_-^3, \end{cases}$$

where the constant  $c_{\varepsilon, \lambda, n}$  is chosen in such a way to have  $\int_{B_n^a} u_{\varepsilon, \lambda, n}^a dx + \frac{h_n^b}{h_n^a} \int_{B_n^b} u_{\varepsilon, \lambda, n}^b dx = 0$ . Remark that  $u_{\varepsilon, \lambda, n}^a = c_{\varepsilon, \lambda, n}$  in  $\mathbb{R}_+^3 \setminus ([-\frac{1}{2}, \frac{1}{2} + \lambda] \times [-\frac{1}{2}, \frac{1}{2}] \times [0, 1])$ ,  $u_{\varepsilon, \lambda, n}^b = c_{\varepsilon, \lambda, n}$  in  $\mathbb{R}_-^3 \setminus ([-\frac{1}{2}, \frac{1}{2}] \times [-1 - \lambda, 0])$ . Since  $u_{\varepsilon, \lambda, n} \in \mathcal{U}_n$ , by choosing  $u = u_{\varepsilon, \lambda, n}$  as test function in (3.3) corresponding to  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b) = ((m_{1,n}^a, m_{2,n}^a, m_{3,n}^a), (m_{1,n}^b, m_{2,n}^b, m_{3,n}^b))$ , it results

that

$$\begin{aligned}
& \int_{\mathbb{R}_+^3} \left( |\xi^a - \widetilde{m_{1,n}^a}|^2 + 2 \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a - \xi^a \right) (\xi^a - \widetilde{m_{1,n}^a}) + \left| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a - \xi^a \right|^2 + \right. \\
& \left. |(D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a)|^2 - 2(D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a)(\widetilde{m_{2,n}^a}, \widetilde{m_{3,n}^a}) + |(\widetilde{m_{2,n}^a}, \widetilde{m_{3,n}^a})|^2 \right) dx + \\
& \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( |(D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b)|^2 - 2(D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b)(\widetilde{m_{1,n}^b}, \widetilde{m_{2,n}^b}) + |(\widetilde{m_{1,n}^b}, \widetilde{m_{2,n}^b})|^2 + \right. \\
& \left. |\xi^b - \widetilde{m_{3,n}^b}|^2 + 2 \left( \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b - \xi^b \right) (\xi^b - \widetilde{m_{3,n}^b}) + \left| \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b - \xi^b \right|^2 \right) dx = \\
& 2j_{\underline{m}_n, n}(u_{\underline{m}_n}) \leq 2j_{\underline{m}_n, n}(u_{\varepsilon, \lambda, n}) = \\
& \int_{\mathbb{R}_+^3} \left( \left| \mu_\varepsilon^a - \frac{1}{\lambda} \chi_{[\frac{1}{2}, \frac{1}{2} + \lambda]}(x_1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{m_{1,n}^a} \right|^2 + \right. \\
& \left| h_n^a D_{x_2} \int_{-\frac{1}{2}}^{x_1} \mu_\varepsilon^a(s, x_2, x_3) ds - \frac{h_n^a}{\lambda} \int_{\frac{1}{2}}^{x_1} \chi_{[\frac{1}{2}, \frac{1}{2} + \lambda]}(r) dr D_{x_2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{m_{2,n}^a} \right|^2 + \\
& \left. \left| h_n^a D_{x_3} \int_{-\frac{1}{2}}^{x_1} \mu_\varepsilon^a(s, x_2, x_3) ds - \frac{h_n^a}{\lambda} \int_{\frac{1}{2}}^{x_1} \chi_{[\frac{1}{2}, \frac{1}{2} + \lambda]}(r) dr D_{x_3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{m_{3,n}^a} \right|^2 \right) dx + \\
& \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( \left| -h_n^b D_{x_1} \int_{x_3}^0 \mu_\varepsilon^b(x_1, x_2, s) ds + \frac{h_n^b}{\lambda} \int_{x_3}^{-1} \chi_{[-1-\lambda, -1]}(r) dr D_{x_1} \int_{-1}^0 \mu_\varepsilon^b(x_1, x_2, s) ds - \widetilde{m_{1,n}^b} \right|^2 + \right. \\
& \left| -h_n^b D_{x_2} \int_{x_3}^0 \mu_\varepsilon^b(x_1, x_2, s) ds + \frac{h_n^b}{\lambda} \int_{x_3}^{-1} \chi_{[-1-\lambda, -1]}(r) dr D_{x_2} \int_{-1}^0 \mu_\varepsilon^b(x_1, x_2, s) ds - \widetilde{m_{2,n}^b} \right|^2 + \\
& \left. \left| \mu_\varepsilon^b - \frac{1}{\lambda} \chi_{[-1-\lambda, -1]}(x_3) \int_{-1}^0 \mu_\varepsilon^b(x_1, x_2, s) ds - \widetilde{m_{3,n}^b} \right|^2 \right) dx, \quad \forall \varepsilon, \lambda, n,
\end{aligned}$$

from which, fixing  $\varepsilon$  and  $\lambda$ , passing to the limit as  $n$  diverges and using (2.1) with  $q \neq +\infty$ ,

(5.1), (5.7), (5.8), it follows that

$$\begin{aligned}
& \limsup_n \left[ \int_{\mathbb{R}_+^3} \left( |\xi^a - \widetilde{m}_{1,n}^a|^2 + \left| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a - \xi^a \right|^2 + |(D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a)|^2 \right) dx + \right. \\
& \left. \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( |(D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b)|^2 + |\xi^b - \widetilde{m}_{3,n}^b|^2 + \left| \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b - \xi^b \right|^2 \right) dx \right] \leq \\
& \int_{\mathbb{R}_+^3} \left( \left| \mu_\varepsilon^a - \frac{1}{\lambda} \chi_{[\frac{1}{2}, \frac{1}{2} + \lambda]}(x_1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{\mu}_1^a \right|^2 \right) dx + \\
& q \int_{\mathbb{R}_-^3} \left( \left| \mu_\varepsilon^b - \frac{1}{\lambda} \chi_{[-1-\lambda, -1]}(x_3) \int_{-1}^0 \mu_\varepsilon^b(x_1, x_2, s) ds - \widetilde{\mu}_3^b \right|^2 \right) dx \leq \\
& 2 \int_{\Omega^a} |\mu_\varepsilon^a - \mu_1^a|^2 dx + \frac{2}{\lambda} \int_{\Omega^a} |\mu_\varepsilon^a|^2 dx + \\
& 2q \int_{\Omega^b} |\mu_\varepsilon^b - \mu_3^b|^2 dx + \frac{2q}{\lambda} \int_{\Omega^b} |\mu_\varepsilon^b|^2 dx \quad \forall \varepsilon, \lambda.
\end{aligned} \tag{5.10}$$

Finally, passing to the limit in (5.10), as  $\lambda$  diverges and  $\varepsilon$  tends to zero, and using (5.9), one obtains

$$\begin{aligned}
& \lim_n \left[ \int_{\mathbb{R}_+^3} \left( |\xi^a - \widetilde{m}_{1,n}^a|^2 + \left| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a - \xi^a \right|^2 + |(D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a)|^2 \right) dx + \right. \\
& \left. \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( |(D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b)|^2 + |\xi^b - \widetilde{m}_{3,n}^b|^2 + \left| \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b - \xi^b \right|^2 \right) dx \right] = 0,
\end{aligned}$$

from which, by virtue of (2.1) with  $q \neq 0$  and (5.1), converges (5.2) follow and these converges hold true for the whole sequence. Convergence (5.3) is a consequence of (2.1) with  $q \neq +\infty$  and (5.2).  $\square$

## 5.2 *A priori* estimates on the magnetization

**Proposition 5.2.** *Assume (2.1) with  $q \in [0, +\infty[$  and (3.9). For every  $n \in \mathbb{N}$  let  $\underline{m}_n$  be a solution of (3.8). Then, it results that*

$$\exists c \in ]0, +\infty[ : E_n(\underline{m}_n) \leq c, \quad \forall n \in \mathbb{N}. \tag{5.11}$$

*Proof.* By choosing  $\underline{m} = ((0, 1, 0), (0, 1, 0))$  as test function in (3.8) and by taking into account (3.11), it follows that

$$E_n(\underline{m}_n) \leq \varphi((0, 1, 0)) + 2\|f_n^a\|_{L^2(\Omega^a)} + \frac{h_n^b}{h_n^a} (\varphi((0, 1, 0)) + 2\|f_n^b\|_{L^2(\Omega^b)}) +$$

$$E_n^{mag}(((0, 1, 0), (0, 1, 0))), \quad \forall n \in \mathbb{N}.$$



Consequently, taking into account (2.1) with  $q \neq +\infty$  and (3.9), for obtaining (5.11) it remains to prove that

$$\exists c \in ]0, +\infty[ : E_n^{mag}(((0, 1, 0), (0, 1, 0))) \leq c, \quad \forall n \in \mathbb{N}. \quad (5.12)$$

By choosing  $u = \left( u_{((0,1,0),(0,1,0)),n}^a, u_{((0,1,0),(0,1,0)),n}^b \right)$  as test function in (3.2) with  $\underline{m} = ((0, 1, 0)(0, 1, 0))$  and using the Cauchy-Schwarz inequality, one derives that

$$2E_n^{mag}(((0, 1, 0), (0, 1, 0))) \leq \left( 1 + \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \right) (2E_n^{mag}(((0, 1, 0), (0, 1, 0))))^{\frac{1}{2}}, \quad \forall n \in \mathbb{N},$$

which gives (5.12), since  $q \neq +\infty$ .  $\square$

The following result is an immediate consequence of Proposition 5.2:

**Corollary 5.3.** *Assume (2.1) with  $q \in ]0, +\infty[$  and (3.9). For every  $n \in \mathbb{N}$  let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of (3.8). Then, there exists  $c \in ]0, +\infty[$  such that*

$$\begin{aligned} \|D_{x_1} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} &\leq ch_n^a, & \|D_{x_2} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} &\leq c, & \|D_{x_3} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} &\leq c, & \forall n \in \mathbb{N}, \\ \|D_{x_1} \underline{m}_n^b\|_{(L^2(\Omega^b))^3} &\leq c, & \|D_{x_2} \underline{m}_n^b\|_{(L^2(\Omega^b))^3} &\leq c, & \|D_{x_3} \underline{m}_n^b\|_{(L^2(\Omega^b))^3} &\leq ch_n^b, & \forall n \in \mathbb{N}. \end{aligned}$$

### 5.3 Convergence of the magnetization

Previous *a priori* estimates provide that the magnetization converges to a function  $(\hat{\mu}^a, \hat{\mu}^b) \in H^1(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2) \times H^1(]-\frac{1}{2}, \frac{1}{2}[^2, S^2)$ . A real difficulty is to recover the junction condition  $\hat{\mu}^a(x_2, 0) = \hat{\mu}^b(0, x_2)$  in  $]-\frac{1}{2}, \frac{1}{2}[$ . This is proved in the following proposition through a suitable splitting of the trace and deducing sharp estimates for this.

**Proposition 5.4.** *Assume (2.1) with  $q \in ]0, +\infty[$  and (3.9). For every  $n \in \mathbb{N}$  let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of (3.8) and let  $\mathcal{M}$  be defined by (4.1). Then, there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\hat{\mu} = (\hat{\mu}^a, \hat{\mu}^b) \in \mathcal{M}$ , depending on the selected subsequence, such that*

$$\begin{cases} \underline{m}_{n_i}^a \rightharpoonup \hat{\mu}^a \text{ weakly in } H^1(\Omega^a, \mathbb{R}^3), \\ \underline{m}_{n_i}^b \rightharpoonup \hat{\mu}^b \text{ weakly in } H^1(\Omega^b, \mathbb{R}^3). \end{cases} \quad (5.13)$$

*Proof.* By taking into account that  $|\underline{m}_n| = 1$  a.e. in  $\Omega^a \cup \Omega^b$  for every  $n \in \mathbb{N}$ , Corollary 5.3 and assumption (2.1), there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$ ,  $\hat{\mu}^a \in H^1(\Omega^a, S^2)$  independent of  $x_1$ ,  $\hat{\mu}^b \in H^1(\Omega^b, S^2)$  independent of  $x_3$ , such that convergences (5.13) hold true. For asserting that  $(\hat{\mu}^a, \hat{\mu}^b) \in \mathcal{M}$ , it remains to prove that

$$\hat{\mu}^a(x_2, 0) = \hat{\mu}^b(0, x_2), \text{ for } x_2 \text{ a.e. in } \left] -\frac{1}{2}, \frac{1}{2} \right[. \quad (5.14)$$

The proof of (5.14) will be performed in several steps. At first, remark that

$$\lim_i \int_{]-\frac{1}{2}, \frac{1}{2}[^2} (\underline{m}_{n_i}^a(x_1, x_2, 0) - \hat{\mu}^a(x_2, 0)) \varphi(x_2) d(x_1, x_2) = 0, \quad \forall \varphi \in C_0^\infty \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \right). \quad (5.15)$$

Now, for every  $i \in \mathbb{N}$ , set

$$\rho_i(x_3) = \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( |D_{x_1} \underline{m}_{n_i}^b(x_1, x_2, x_3)|^2 + |D_{x_2} \underline{m}_{n_i}^b(x_1, x_2, x_3)|^2 + |\underline{m}_{n_i}^b(x_1, x_2, x_3)|^2 \right) d(x_1, x_2),$$

for  $x_3$  a.e. in  $] - 1, 0[$ .

From Fatou's Lemma, Corollary (5.3) and the fact that  $\underline{m}_{n_i}^b = 1$  a.e. in  $\Omega^b$ , it follows that

$$\int_{-1}^0 \liminf_i \rho_i(x_3) dx_3 \leq \liminf_i \int_{-1}^0 \rho_i(x_3) dx_3 < +\infty.$$

Consequently, there exist two constants  $c \in ]0, +\infty[$  and  $\bar{x}_3 \in ] - 1, 0[$ , and an increasing sequence of positive integer numbers  $\{i_k\}_{k \in \mathbb{N}}$  such that

$$\rho_{i_k}(\bar{x}_3) < c, \quad \forall k \in \mathbb{N}.$$

Then, by virtue of the second convergence in (5.13), it results that

$$\underline{m}_{n_{i_k}}^b(\cdot, \cdot, \bar{x}_3) \rightharpoonup \widehat{\mu}^b \text{ weakly in } H^1 \left( \left] -\frac{1}{2}, \frac{1}{2} \right]^2, \mathbb{R}^3 \right), \quad (5.16)$$

as  $k \rightarrow +\infty$ .

The next step is devoted to prove that

$$\lim_k \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) - \widehat{\mu}^b(0, x_2) \right) \varphi(x_2) d(x_1, x_2) = 0, \quad (5.17)$$

$\forall \varphi \in C_0^\infty \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \right).$

To this aim, for a fixed  $\varphi \in C_0^\infty \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \right)$ , the integral in (5.17) will be split in the following way:

$$\begin{aligned} & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) - \widehat{\mu}^b(0, x_2) \right) \varphi(x_2) d(x_1, x_2) = \\ & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) - \underline{m}_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) \right) \varphi(x_2) d(x_1, x_2) + \\ & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) - \underline{m}_{n_{i_k}}^b(0, x_2, \bar{x}_3) \right) \varphi(x_2) d(x_1, x_2) + \\ & \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(0, x_2, \bar{x}_3) - \widehat{\mu}^b(0, x_2) \right) \varphi(x_2) d(x_1, x_2) \quad \forall k \in \mathbb{N}, \end{aligned} \quad (5.18)$$

and one will pass to the limit, as  $k$  diverges, in each term of this decomposition.

By virtue of the last inequality in Corollary 5.3 and of (2.1) with  $q \neq +\infty$ , there exists a constant  $c \in ]0, +\infty[$  such that

$$\begin{aligned}
& \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b (h_{n_{i_k}}^a x_1, x_2, 0) - \underline{m}_{n_{i_k}}^b (h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) \right) \varphi(x_2) d(x_1, x_2) \right| = \\
& \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \int_{\bar{x}_3}^0 D_{x_3} \underline{m}_{n_{i_k}}^b (h_{n_{i_k}}^a x_1, x_2, x_3) dx_3 \right) \varphi(x_2) d(x_1, x_2) \right| \leq \\
& \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} |\Omega^b|^{\frac{1}{2}} \limsup_k \left( \int_{\Omega^b} |D_{x_3} \underline{m}_{n_{i_k}}^b (h_{n_{i_k}}^a x_1, x_2, x_3)|^2 dx \right)^{\frac{1}{2}} \leq \tag{5.19} \\
& \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} |\Omega^b|^{\frac{1}{2}} \limsup_k \left( \frac{1}{h_{n_{i_k}}^a} \int_{\Omega^b} |D_{x_3} \underline{m}_{n_{i_k}}^b (x_1, x_2, x_3)|^2 dx \right)^{\frac{1}{2}} \leq \\
& \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} |\Omega^b|^{\frac{1}{2}} c \lim_k \frac{h_{n_{i_k}}^b}{\sqrt{h_{n_{i_k}}^a}} = 0.
\end{aligned}$$

By virtue of (5.16), there exists a constant  $c \in ]0, +\infty[$  such that

$$\begin{aligned}
& \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b (h_{n_{i_k}}^a x_1, x_2, \bar{x}_3) - \underline{m}_{n_{i_k}}^b (0, x_2, \bar{x}_3) \right) \varphi(x_2) d(x_1, x_2) \right| = \\
& \limsup_k \left| \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \int_0^{h_{n_{i_k}}^a x_1} D_t \underline{m}_{n_{i_k}}^b (t, x_2, \bar{x}_3) dt \right) \varphi(x_2) d(x_1, x_2) \right| \leq \\
& \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \limsup_k \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \int_0^{\frac{h_{n_{i_k}}^a}{2}} |D_t \underline{m}_{n_{i_k}}^b (t, x_2, \bar{x}_3)| dt \right) dx_2 + \\
& \frac{1}{2} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \limsup_k \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \int_{-\frac{h_{n_{i_k}}^a}{2}}^0 |D_t \underline{m}_{n_{i_k}}^b (t, x_2, \bar{x}_3)| dt \right) dx_2 \leq \tag{5.20} \\
& \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} \limsup_k \left( \frac{h_{n_{i_k}}^a}{2} \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |D_{x_1} \underline{m}_{n_{i_k}}^b (x_1, x_2, \bar{x}_3)|^2 d(x_1, x_2) \right)^{\frac{1}{2}} \leq \\
& \frac{1}{\sqrt{2}} \|\varphi\|_{L^\infty(]-\frac{1}{2}, \frac{1}{2}[)} c \lim_k \sqrt{h_{n_{i_k}}^a} = 0.
\end{aligned}$$

From (5.16) it follows that

$$\underline{m}_{n_{i_k}}^b (0, \cdot, \bar{x}_3) \rightarrow \widehat{\mu}^b(0, \cdot) \text{ strongly in } L^2 \left( \left[ -\frac{1}{2}, \frac{1}{2} \right], \mathbb{R}^3 \right),$$

as  $k \rightarrow +\infty$ . Consequently, it results that

$$\begin{aligned} \lim_k \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(0, x_2, \bar{x}_3) - \widehat{\mu}^b(0, x_2) \right) \varphi(x_2) d(x_1, x_2) = \\ \lim_k \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \underline{m}_{n_{i_k}}^b(0, x_2, \bar{x}_3) - \widehat{\mu}^b(0, x_2) \right) \varphi(x_2) dx_2 = 0. \end{aligned} \quad (5.21)$$

Then, passing to the limit in (5.18), as  $k$  diverges, and taking into account (5.19)-(5.21), one obtains (5.17).

Finally, junction condition (5.14) is obtained by passing to the limit, as  $k$  diverges, in

$$\begin{aligned} \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \underline{m}_{n_{i_k}}^a(x_1, x_2, 0) \varphi(x_2) d(x_1, x_2) = \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \underline{m}_{n_{i_k}}^b(h_{n_{i_k}}^a x_1, x_2, 0) \varphi(x_2) d(x_1, x_2), \\ \forall k \in \mathbb{N}, \quad \forall \varphi \in C_0^\infty \left( ]-\frac{1}{2}, \frac{1}{2}[ \right), \end{aligned}$$

and using (5.15) and (5.17). □

## 5.4 A density result

Let

$$\begin{aligned} \mathcal{M}_{reg} = \left\{ \begin{aligned} \mu = (\mu^a, \mu^b) \in C^1 \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \times [0, 1], S^2 \right) \times C \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2, S^2 \right) : \\ \mu^b|_{\left[ -\frac{1}{2}, 0 \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right]} \in C^1 \left( \left[ -\frac{1}{2}, 0 \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right], S^2 \right), \\ \mu^b|_{\left[ 0, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right]} \in C^1 \left( \left[ 0, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right], S^2 \right), \\ \mu^a(x_2, 0) = \mu^b(0, x_2), \text{ for } x_2 \in \left] -\frac{1}{2}, \frac{1}{2} \right[ \end{aligned} \right\}. \end{aligned} \quad (5.22)$$

Remark that  $\mathcal{M}_{reg} \subset W^{1,\infty} \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \times ]0, 1[, S^2 \right) \times W^{1,\infty} \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2, S^2 \right)$ .

Next proposition is devoted to prove that  $\mathcal{M}_{reg}$  is dense in  $\mathcal{M}$ . We point out that this result is not evident since the limit domain  $(\{0\} \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times [0, 1]) \cup \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \times \{0\} \right)$  is not a manifold. The proof is based on the combination of an approximation result proved in [4] with splitting techniques introduced in [19] and a projection technique from  $\mathbb{R}^3$  into  $S^2$  as in [13].

**Proposition 5.5.** *Let  $\mathcal{M}$  and  $\mathcal{M}_{reg}$  be the spaces defined in (4.1) and (5.22), respectively. Then,  $\mathcal{M}_{reg}$  is dense in  $\mathcal{M}$ .*

*Proof.* Let  $(\mu^a, \mu^b) \in \mathcal{M}$ . The goal is to find a sequence  $\{(\mu_n^a, \mu_n^b)\}_{n \in \mathbb{N}} \subset \mathcal{M}_{reg}$  such that

$$(\mu_n^a, \mu_n^b) \rightarrow (\mu^a, \mu^b) \text{ strongly in } H^1 \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \times ]0, 1[, S^2 \right) \times H^1 \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2, S^2 \right), \quad (5.23)$$

as  $n$  diverges. The proof of (5.23) will be developed in several steps.

At first, consider a sequence  $\{(v_n^a, v_n^b)\}_{n \in \mathbb{N}} \subset C^\infty([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], S^2) \times C^\infty([-\frac{1}{2}, \frac{1}{2}]^2, S^2)$  such that

$$(v_n^a, v_n^b) \rightarrow (\mu^a, \mu^b) \text{ strongly in } H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times ]0, 1[, S^2\right) \times H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, S^2\right), \quad (5.24)$$

as  $n \rightarrow +\infty$  (see [4]). Next steps are devoted to modify  $v_n^a$  in a neighbourhood of  $x_3 = 0$  and  $v_n^b$  in a neighbourhood of  $x_1 = 0$ , in order to obtain the junction condition required in (5.22). To this aim, we begin by splitting  $\mu^b$  in the even part and in the odd part with respect to  $x_1$ :

$$\mu^e : (x_1, x_2) \in \left] -\frac{1}{2}, \frac{1}{2} \right[ \rightarrow \frac{1}{2}(\mu^b(x_1, x_2) + \mu^b(-x_1, x_2))$$

and

$$\mu^o : (x_1, x_2) \in \left] -\frac{1}{2}, \frac{1}{2} \right[ \rightarrow \frac{1}{2}(\mu^b(x_1, x_2) - \mu^b(-x_1, x_2)).$$

Remark that  $\mu^e, \mu^o \in H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right) \cap L^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right)$ ,  $\mu^e$  is an even function with respect to  $x_1$ ,  $\mu^o$  is an odd function with respect to  $x_1$ ,  $\mu^e(0, \cdot) = \mu^b(0, \cdot) = \mu^a(\cdot, 0)$  and  $\mu^o(0, \cdot) = 0$  a.e. in  $\left]-\frac{1}{2}, \frac{1}{2}\right[$ . Consequently, by convolution, it is easy to build three sequences (see [19])  $\{\zeta_n^a\}_{n \in \mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1], \mathbb{R}^3\right)$ ,  $\{\zeta_n^e\}_{n \in \mathbb{N}} \subset C\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right)$  and  $\{\zeta_n^o\}_{n \in \mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right)$  such that

$$\left\{ \begin{array}{l} \zeta_n^a \rightarrow \mu^a \text{ strongly in } H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times ]0, 1[, \mathbb{R}^3\right), \text{ as } n \rightarrow +\infty, \\ \zeta_n^e \rightarrow \mu^e \text{ strongly in } H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right), \text{ as } n \rightarrow +\infty, \\ \zeta_n^o \rightarrow \mu^o \text{ strongly in } H^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right), \text{ as } n \rightarrow +\infty, \\ \left\{ \zeta_n^e|_{\left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]} \right\}_{n \in \mathbb{N}} \subset C^\infty\left(\left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right], \mathbb{R}^3\right), \\ \left\{ \zeta_n^e|_{\left[0, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]} \right\}_{n \in \mathbb{N}} \subset C^\infty\left(\left[0, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right], \mathbb{R}^3\right), \\ \exists r_1 \in ]0, +\infty[ : \|\zeta_n^a\|_{L^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times ]0, 1[, \mathbb{R}^3\right)} < r_1, \quad \|\zeta_n^e\|_{L^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right)} < r_1, \\ \|\zeta_n^o\|_{L^\infty\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, \mathbb{R}^3\right)} < r_1, \quad \forall n \in \mathbb{N}, \\ \zeta_n^a(\cdot, 0) = \zeta_n^e(0, \cdot) \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[, \quad \forall n \in \mathbb{N}, \\ \zeta_n^o(0, \cdot) = 0 \text{ in } \left]-\frac{1}{2}, \frac{1}{2}\right[, \quad \forall n \in \mathbb{N}, \end{array} \right.$$

from which, by setting  $\zeta_n^b = \zeta_n^e + \zeta_n^o \in C([- \frac{1}{2}, \frac{1}{2}]^2, \mathbb{R}^3)$ , since  $\mu^b = \mu^e + \mu^o$ , one derives that

$$\left\{ \begin{array}{l} \zeta_n^a \rightarrow \mu^a \text{ strongly in } H^1 \left( ] - \frac{1}{2}, \frac{1}{2}[ \times ] 0, 1[, \mathbb{R}^3 \right), \text{ as } n \rightarrow +\infty, \\ \zeta_n^b \rightarrow \mu^b \text{ strongly in } H^1 \left( ] - \frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3 \right), \text{ as } n \rightarrow +\infty, \\ \{\zeta_n^a\}_{n \in \mathbb{N}} \subset C^\infty \left( \left[ - \frac{1}{2}, \frac{1}{2} \right] \times [0, 1], \mathbb{R}^3 \right) \\ \{\zeta_n^b\}_{n \in \mathbb{N}} \subset C \left( \left[ - \frac{1}{2}, \frac{1}{2} \right]^2, \mathbb{R}^3 \right), \\ \left\{ \zeta_n^b \Big|_{\left[ - \frac{1}{2}, 0 \right] \times \left[ - \frac{1}{2}, \frac{1}{2} \right]} \right\}_{n \in \mathbb{N}} \subset C^\infty \left( \left[ - \frac{1}{2}, 0 \right] \times \left[ - \frac{1}{2}, \frac{1}{2} \right], \mathbb{R}^3 \right), \\ \left\{ \zeta_n^b \Big|_{\left[ 0, \frac{1}{2} \right] \times \left[ - \frac{1}{2}, \frac{1}{2} \right]} \right\}_{n \in \mathbb{N}} \subset C^\infty \left( \left[ 0, \frac{1}{2} \right] \times \left[ - \frac{1}{2}, \frac{1}{2} \right], \mathbb{R}^3 \right), \\ \exists r_2 \in ] 0, +\infty[ : \|\zeta_n^a\|_{L^\infty \left( ] - \frac{1}{2}, \frac{1}{2}[ \times ] 0, 1[, \mathbb{R}^3 \right)} < r_2, \quad \|\zeta_n^b\|_{L^\infty \left( ] - \frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3 \right)} < r_2, \quad \forall n \in \mathbb{N}, \\ \zeta_n^a(\cdot, 0) = \zeta_n^b(0, \cdot) \text{ in } ] - \frac{1}{2}, \frac{1}{2}[ , \quad \forall n \in \mathbb{N}. \end{array} \right. \quad (5.25)$$

Let, now,

$$\eta \in C^\infty(\mathbb{R}, [0, 1]) : \eta(t) = 1 \quad \forall t \in [-1, 1], \quad \eta(t) = 0 \quad \forall t \in \mathbb{R} \setminus ] - 2, 2[, \quad (5.26)$$

and, for every  $n \in \mathbb{N}$ , set

$$w_n^a : (x_2, x_3) \in \left] - \frac{1}{2}, \frac{1}{2} \left[ \times ] 0, 1[ \rightarrow \zeta_n^a(x_2, x_3) \eta \left( \frac{x_3}{\varepsilon_n} \right) + v_n^a(x_2, x_3) \left( 1 - \eta \left( \frac{x_3}{\varepsilon_n} \right) \right),$$

$$w_n^b : (x_1, x_2) \in \left] - \frac{1}{2}, \frac{1}{2} \left[ ^2 \rightarrow \zeta_n^b(x_1, x_2) \eta \left( \frac{x_1}{\varepsilon_n} \right) + v_n^b(x_1, x_2) \left( 1 - \eta \left( \frac{x_1}{\varepsilon_n} \right) \right),$$

where

$$\varepsilon_n = \max \left\{ \sqrt{\frac{1}{n} + \|v_n^a - \zeta_n^a\|_{L^2 \left( ] - \frac{1}{2}, \frac{1}{2}[ \times ] 0, 1[, \mathbb{R}^3 \right)}}, \sqrt{\frac{1}{n} + \|v_n^b - \zeta_n^b\|_{L^2 \left( ] - \frac{1}{2}, \frac{1}{2}[^2, \mathbb{R}^3 \right)}} \right\}. \quad (5.27)$$

From (5.24), the first line in (5.25), (5.26) and (5.27), it follows that

$$\begin{aligned}
& \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} |D_{x_2} w_n^a(x_2, x_3)|^2 dx_2 dx_3 = \\
& \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} \left| D_{x_2} \zeta_n^a(x_2, x_3) \eta\left(\frac{x_3}{\varepsilon_n}\right) + D_{x_2} v_n^a(x_2, x_3) \left(1 - \eta\left(\frac{x_3}{\varepsilon_n}\right)\right) \right|^2 dx_2 dx_3 \leq \\
& 2 \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^1 |D_{x_2} \zeta_n^a(x_2, x_3) - D_{x_2} v_n^a(x_2, x_3)|^2 dx_2 dx_3 + \\
& 4 \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^1 |D_{x_2} v_n^a(x_2, x_3) - D_{x_2} \mu^a(x_2, x_3)|^2 dx_2 dx_3 + \\
& 4 \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} |D_{x_2} \mu^a(x_2, x_3)|^2 dx_2 dx_3 = 0
\end{aligned}$$

and

$$\begin{aligned}
& \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} |D_{x_3} w_n^a(x_2, x_3)|^2 dx_2 dx_3 = \\
& \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} \left| D_{x_3} \zeta_n^a(x_2, x_3) \eta\left(\frac{x_3}{\varepsilon_n}\right) + D_{x_3} v_n^a(x_2, x_3) \left(1 - \eta\left(\frac{x_3}{\varepsilon_n}\right)\right) + \right. \\
& \qquad \qquad \qquad \left. (\zeta_n^a(x_2, x_3) - v_n^a(x_2, x_3)) \frac{1}{\varepsilon_n} \eta'\left(\frac{x_3}{\varepsilon_n}\right) \right|^2 dx_2 dx_3 \leq \\
& 2 \lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} \left| D_{x_3} \zeta_n^a(x_2, x_3) \eta\left(\frac{x_3}{\varepsilon_n}\right) + D_{x_3} v_n^a(x_2, x_3) \left(1 - \eta\left(\frac{x_3}{\varepsilon_n}\right)\right) \right|^2 dx_2 dx_3 + \\
& 2 \|\eta'\|_{L^\infty(\mathbb{R})}^2 \lim_n \frac{\|v_n^a - \zeta_n^a\|_{L^2(\cdot)_{-\frac{1}{2}, \frac{1}{2}}[\times]0, 1[\cdot, \mathbb{R}^3])}^2}{\varepsilon_n^2} = 0,
\end{aligned}$$

that is

$$\lim_n \|Dw_n^a\|_{L^2(\cdot)_{-\frac{1}{2}, \frac{1}{2}}[\times]0, 2\varepsilon_n[\cdot, \mathbb{R}^6)} = 0. \tag{5.28}$$

Similarly, one obtains that

$$\lim_n \|Dw_n^b\|_{L^2(\cdot)_{-2\varepsilon_n, 2\varepsilon_n[\times] -\frac{1}{2}, \frac{1}{2}[\cdot, \mathbb{R}^6)} = 0. \tag{5.29}$$

Then, (5.24), (5.25), (5.26), (5.28) and (5.29) provide that

$$\left\{ \begin{array}{l}
w_n^a \rightarrow \mu^a \text{ strongly in } H^1 \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] 0, 1 \left[ , \mathbb{R}^3 \right) \right), \text{ as } n \rightarrow +\infty, \\
w_n^b \rightarrow \mu^b \text{ strongly in } H^1 \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ ^2, \mathbb{R}^3 \right) \right), \text{ as } n \rightarrow +\infty, \\
\{w_n^a\}_{n \in \mathbb{N}} \subset C^\infty \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left[ 0, 1 \right], \mathbb{R}^3 \right), \\
\{w_n^b\}_{n \in \mathbb{N}} \subset C \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^2, \mathbb{R}^3 \right), \\
\left\{ w_n^b \Big|_{\left[ -\frac{1}{2}, 0 \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right]} \right\}_{n \in \mathbb{N}} \subset C^\infty \left( \left[ -\frac{1}{2}, 0 \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right], \mathbb{R}^3 \right), \\
\left\{ w_n^b \Big|_{\left[ 0, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right]} \right\}_{n \in \mathbb{N}} \subset C^\infty \left( \left[ 0, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right], \mathbb{R}^3 \right), \\
\exists r_3 \in ]0, +\infty[ : \|w_n^a\|_{L^\infty \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] 0, 1 \left[ , \mathbb{R}^3 \right)} < r_3, \|w_n^b\|_{L^\infty \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ ^2, \mathbb{R}^3 \right)} < r_3, \forall n \in \mathbb{N}, \\
|w_n^a| = 1 \text{ in } \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] 2\varepsilon_n, 1 \left[ , \quad \forall n \in \mathbb{N}, \\
|w_n^b| = 1 \text{ in } \left( \left] -\frac{1}{2}, -2\varepsilon_n \left[ \cup \right] 2\varepsilon_n, \frac{1}{2} \left[ \right) \times \left] -\frac{1}{2}, \frac{1}{2} \left[ , \quad \forall n \in \mathbb{N}, \\
w_n^a(\cdot, 0) = w_n^b(0, \cdot) \text{ in } \left] -\frac{1}{2}, \frac{1}{2} \left[ , \quad \forall n \in \mathbb{N}.
\end{array} \right. \right. \tag{5.30}$$

For obtaining (5.23), it remains to normalize  $w_n^a$  in a neighbourhood of  $x_3 = 0$  and  $w_n^b$  in a neighbourhood of  $x_1 = 0$ . To this aim, for  $y \in B_{\frac{1}{2}}(0) = \{y \in \mathbb{R}^3 : |y| \leq \frac{1}{2}\}$ , project  $x \in \mathbb{R}^3 \setminus \{y\}$  on  $S^2$  by the intersection of  $S^2$  with the half line  $y + t(x - y) : t \in ]0, +\infty[$  (compare [13] and [10]). Precisely, introduce the projection

$$\pi_y : x \in \mathbb{R}^3 \setminus \{y\} \rightarrow y + \frac{-y(x - y) + \sqrt{(y(x - y))^2 + |x - y|^2(1 - |y|^2)}}{|x - y|^2} (x - y) \in S^2. \tag{5.31}$$

It is easy to see that

$$\pi_y(x) = x, \quad \forall x \in S^2. \tag{5.32}$$

Moreover, it results that (see appendix)

$$\exists c \in ]0, +\infty[ : |D\pi_y(x)|^2 \leq \frac{c}{|x - y|^2}, \quad \forall y \in B_{\frac{1}{2}}(0), \quad \forall x \in \mathbb{R}^3 \setminus \{y\}. \tag{5.33}$$

Then, for every  $n \in \mathbb{N}$  and for every

$$y \in B_{\frac{1}{2}}(0) \setminus \bigcup_{k \in \mathbb{N}} \left( w_k^a \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ \times \right] 0, 1 \left[ , \mathbb{R}^3 \right) \right) \cup w_k^b \left( \left] -\frac{1}{2}, \frac{1}{2} \left[ ^2, \mathbb{R}^3 \right) \right) \right)$$



(remark that the 3D Lebesgue measure of

$$\bigcup_{k \in \mathbb{N}} \left( w_k^a \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]0, 1[, \mathbb{R}^3 \right) \cup w_k^b \left( \left] -\frac{1}{2}, \frac{1}{2} \right[{}^2, \mathbb{R}^3 \right) \right)$$

is zero), the functions  $\pi_y \circ w_n^a$  and  $\pi_y \circ w_n^b$  are well defined and it results that

$$\left\{ \begin{array}{l} \pi_y \circ w_n^a \in C^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1], S^2\right), \\ \pi_y \circ w_n^b \in C\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^2, S^2\right), \\ (\pi_y \circ w_n^b)|_{\left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]} \in C^1\left(\left[-\frac{1}{2}, 0\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right], S^2\right), \\ (\pi_y \circ w_n^b)|_{\left[0, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]} \in C^1\left(\left[0, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right], S^2\right), \\ (\pi_y \circ w_n^a)(\cdot, 0) = (\pi_y \circ w_n^b)(0, \cdot) \text{ in } \left] -\frac{1}{2}, \frac{1}{2} \right[. \end{array} \right. \quad (5.34)$$

Moreover, from (5.33) and the 7th line in (5.30) one derives that

$$\begin{aligned} & \int_{B_{\frac{1}{2}}(0) \setminus \bigcup_{k \in \mathbb{N}} \left( w_k^a \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]0, 1[, \mathbb{R}^3 \right) \cup w_k^b \left( \left] -\frac{1}{2}, \frac{1}{2} \right[{}^2, \mathbb{R}^3 \right) \right)} \\ & \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} |D(\pi_y(w_n^a(x_2, x_3)))|^2 d(x_2, x_3) + \int_{-2\varepsilon_n}^{2\varepsilon_n} \int_{-\frac{1}{2}}^{\frac{1}{2}} |D(\pi_y(w_n^b(x_1, x_2)))|^2 d(x_1, x_2) \right) dy \leq \\ & \int_{B_{\frac{1}{2}}(0) \setminus \bigcup_{k \in \mathbb{N}} \left( w_k^a \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]0, 1[, \mathbb{R}^3 \right) \cup w_k^b \left( \left] -\frac{1}{2}, \frac{1}{2} \right[{}^2, \mathbb{R}^3 \right) \right)} \\ & c \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} \frac{|Dw_n^a(x_2, x_3)|^2}{|w_n^a(x_2, x_3) - y|^2} d(x_2, x_3) + \int_{-2\varepsilon_n}^{2\varepsilon_n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|Dw_n^b(x_1, x_2)|^2}{|w_n^b(x_1, x_2) - y|^2} d(x_1, x_2) \right) dy \leq \\ & c \int_{B_{r_3 + \frac{1}{2}}(0)} \frac{1}{|z|^2} dz \\ & \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\varepsilon_n} |Dw_n^a(x_2, x_3)|^2 d(x_2, x_3) + \int_{-2\varepsilon_n}^{2\varepsilon_n} \int_{-\frac{1}{2}}^{\frac{1}{2}} |Dw_n^b(x_1, x_2)|^2 d(x_1, x_2) \right) \quad \forall n \in \mathbb{N}, \end{aligned}$$

where  $c$  and  $r_3$  are the constants given in (5.33) and in (5.30), respectively, and  $B_{r_3 + \frac{1}{2}}(0) = \{z \in \mathbb{R}^3 : |z| \leq r_3 + \frac{1}{2}\}$ . Consequently, by using the Fatou Lemma and by taking into account (5.28), (5.29) and the fact that  $\int_{B_{r_3 + \frac{1}{2}}(0)} \frac{1}{|z|^2} dz < +\infty$ , there exists a subsequence of  $\{n\}$ , still

denoted by  $\{n\}$ , and  $\bar{y} \in B_{\frac{1}{2}}(0) \setminus \bigcup_{k \in \mathbb{N}} \left( w_k^a \left( \left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]0, 1[, \mathbb{R}^3 \right) \cup w_k^b \left( \left] -\frac{1}{2}, \frac{1}{2} \right[{}^2, \mathbb{R}^3 \right) \right)$   
such that

$$\lim_n \|D(\pi_{\bar{y}} \circ w_n^a)\|_{L^2(\left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]0, 2\varepsilon_n[, \mathbb{R}^6)} = 0 = \lim_n \|D(\pi_{\bar{y}} \circ w_n^b)\|_{L^2(\left] -2\varepsilon_n, 2\varepsilon_n \times \left] -\frac{1}{2}, \frac{1}{2} \right[{}^2, \mathbb{R}^6)}. \quad (5.35)$$

Finally, by setting  $\mu_n^a = \pi_{\bar{y}} \circ w_n^a$  and  $\mu_n^b = \pi_{\bar{y}} \circ w_n^b$ , (5.34) assures that  $\{(\mu_n^a, \mu_n^b)\}_{n \in \mathbb{N}} \subset \mathcal{M}_{reg}$ . Moreover, by taking into account that  $\mu_n^a = w_n^a$  in  $\left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]2\varepsilon_n, 1[, \mu_n^b = w_n^b$  in  $\left] -\frac{1}{2}, -2\varepsilon_n \cup ]2\varepsilon_n, \frac{1}{2} \right[ \times \left] -\frac{1}{2}, \frac{1}{2} \right[$ , the first two lines in (5.30) and (5.35) provide (5.23).  $\square$

## 5.5 Proof of Theorem 4.1

Now, combining convex arguments with projection techniques we used in [10] and with the convergence of the magnetostatic energy of Proposition 5.1, we build a recovery sequence for a generic couple in the space  $\mathcal{M}_{reg}$ . Then, using the density result of Proposition 5.5, we conclude the proof of Theorem 4.1.

*Proof.* Proposition 5.4 assures the existence of an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\hat{\mu} = (\hat{\mu}^a, \hat{\mu}^b) \in \mathcal{M}$ , depending on the selected subsequence, such that

$$\begin{cases} \underline{m}_{n_i}^a \rightharpoonup \hat{\mu}^a \text{ weakly in } H^1(\Omega^a, \mathbb{R}^3), \\ \underline{m}_{n_i}^b \rightharpoonup \hat{\mu}^b \text{ weakly in } H^1(\Omega^b, \mathbb{R}^3), \end{cases} \quad (5.36)$$

as  $i$  diverges. Consequently, by virtue of Proposition 5.1, convergences (4.5) hold true, and it results that

$$\lim_i E_{n_i}^{mag}(\underline{m}_{n_i}) = \frac{1}{2} \left( \int_{\left] -\frac{1}{2}, \frac{1}{2} \right[ \times ]0, 1[} |\hat{\mu}_1^a(x_2, x_3)|^2 d(x_2, x_3) + q \int_{\left] -\frac{1}{2}, \frac{1}{2} \right[{}^2} |\hat{\mu}_3^b(x_1, x_2)|^2 d(x_1, x_2) \right), \quad (5.37)$$

where  $E_{n_i}^{mag}$  is defined in (3.10).

Now, for  $\mu = (\mu^a, \mu^b) \in \mathcal{M}_{reg}$ , where  $\mathcal{M}_{reg}$  is defined in (5.22), we shall build a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset H^1(\Omega^a, S^2)$  such that

$$\left\{ \begin{array}{l} g_n \rightarrow \mu^a \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \text{ as } n \rightarrow +\infty, \\ \left( \frac{1}{h_n^a} D_{x_1} g_n |D_{x_2} g_n |D_{x_3} g_n \right) \rightarrow (0 |D_{x_2} \mu^a |D_{x_3} \mu^a) \text{ strongly in } L^2(\Omega^a, \mathbb{R}^9), \\ \hspace{15em} \text{as } n \rightarrow +\infty, \\ g_n(x_1, x_2, 0) = \mu^b(h_n^a x_1, x_2), \text{ for } (x_1, x_2) \in \left] -\frac{1}{2}, \frac{1}{2} \right[{}^2, \quad \forall n \in \mathbb{N}, \end{array} \right. \quad (5.38)$$

and we shall choose  $(g_n, \mu^b)$  as test function in (3.8). To this aim, for every  $n \in \mathbb{N}$ , set

$$g_n(x) = \begin{cases} \mu^a(x_2, x_3), & \text{if } x = (x_1, x_2, x_3) \in ]-\frac{1}{2}, \frac{1}{2}[^2 \times ]h_n^a, 1[, \\ \mu^a(x_2, h_n^a) \frac{x_3}{h_n^a} + \mu^b(h_n^a x_1, x_2) \frac{h_n^a - x_3}{h_n^a}, & \text{if } x = (x_1, x_2, x_3) \in ]-\frac{1}{2}, \frac{1}{2}[^2 \times [0, h_n^a]. \end{cases}$$

Obviously,  $\{g_n\}_{n \in \mathbb{N}} \subset H^1(\Omega^a, \mathbb{R}^3)$  and the last line of (5.38) is satisfied. Moreover, by using the properties of  $\mathcal{M}_{reg}$ , it is easy to see that

$$\begin{aligned} \lim_n \int_{]-\frac{1}{2}, \frac{1}{2}[^2 \times ]0, h_n^a[} \left| \frac{1}{h_n^a} D_{x_1} g_n(x) \right|^2 dx &\leq \|\mu^b\|_{W^{1,\infty}(]-\frac{1}{2}, \frac{1}{2}[^2, S^2))}^2 \lim_n h_n^a = 0, \\ \lim_n \int_{]-\frac{1}{2}, \frac{1}{2}[^2 \times ]0, h_n^a[} |D_{x_2} g_n(x)|^2 dx &\leq \\ 2(\|\mu^a\|_{W^{1,\infty}(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2))}^2 + \|\mu^b\|_{W^{1,\infty}(]-\frac{1}{2}, \frac{1}{2}[^2, S^2))}^2) \lim_n h_n^a &= 0, \end{aligned}$$

and

$$\begin{aligned} \lim_n \int_{]-\frac{1}{2}, \frac{1}{2}[^2 \times ]0, h_n^a[} |D_{x_3} g_n(x)|^2 dx &= \\ \lim_n \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \frac{1}{h_n^a} |\mu^a(x_2, h_n^a) - \mu^b(h_n^a x_1, x_2)|^2 d(x_1, x_2) &= \\ \lim_n \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \frac{1}{h_n^a} |\mu^a(x_2, h_n^a) - \mu^a(x_2, 0) + \mu^b(0, x_2) - \mu^b(h_n^a x_1, x_2)|^2 d(x_1, x_2) &\leq \\ 2(\|\mu^a\|_{W^{1,\infty}(]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[, S^2))}^2 + \|\mu^b\|_{W^{1,\infty}(]-\frac{1}{2}, \frac{1}{2}[^2, S^2))}^2) \lim_n h_n^a &= 0, \end{aligned}$$

that is

$$\lim_n \left\| \left( \frac{1}{h_n^a} D_{x_1} g_n |D_{x_2} g_n |D_{x_3} g_n \right) \right\|_{L^2(]-\frac{1}{2}, \frac{1}{2}[^2 \times ]0, h_n^a[, \mathbb{R}^9)} = 0. \quad (5.39)$$

Consequently, convergence (5.38) holds true. Unfortunately,  $|g_n(x)| \leq 1$  for every  $x \in ]-\frac{1}{2}, \frac{1}{2}[^2 \times ]0, h_n^a[$ . To overcome this difficulty, using projection (5.31) and arguing as in the proof of Theorem 2.1 in [10], it is possible to project  $g_n$  on  $S^2$  such that this projection, still denoted by  $g_n$ , satisfies (5.38). Remark that the first line in (5.38) and Proposition 5.1 provide that

$$\begin{aligned} \lim_n E_n^{mag}((g_n, \mu^b)) &= \\ \frac{1}{2} \left( \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\mu_1^a(x_2, x_3)|^2 d(x_2, x_3) + q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} |\mu_3^b(x_1, x_2)|^2 d(x_1, x_2) \right). & \quad (5.40) \end{aligned}$$

Finally, choosing  $(g_n, \mu^b)$  as test function in (3.8) and taking into account (3.11), one has that

$$\begin{aligned}
& \int_{\Omega^a} \left( \alpha \left| \left( \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a | D_{x_2} \underline{m}_{n_i}^a | D_{x_3} \underline{m}_{n_i}^a \right) \right|^2 + \varphi(\underline{m}_{n_i}^a) - 2f_{n_i}^a \underline{m}_{n_i}^a \right) dx + \\
& \frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} \left( \alpha \left| \left( D_{x_1} \underline{m}_{n_i}^b | D_{x_2} \underline{m}_{n_i}^b | \frac{1}{h_{n_i}^b} D_{x_3} \underline{m}_{n_i}^b \right) \right|^2 + \varphi(\underline{m}_{n_i}^b) - 2f_{n_i}^b \underline{m}_{n_i}^b \right) dx + E_{n_i}^{mag}(\underline{m}_{n_i}) \leq \\
& \int_{\Omega^a} \left( \alpha \left| \left( \frac{1}{h_{n_i}^a} D_{x_1} g_{n_i} | D_{x_2} g_{n_i} | D_{x_3} g_{n_i} \right) \right|^2 + \varphi(g_{n_i}) - 2f_{n_i}^a g_{n_i} \right) dx + \\
& \frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} \left( \alpha \left| (D_{x_1} \mu^b | D_{x_2} \mu^b | 0) \right|^2 + \varphi(\mu^b) - 2f_{n_i}^b \mu^b \right) dx + E_{n_i}^{mag}((g_{n_i}, \mu^b)), \quad \forall i \in \mathbb{N},
\end{aligned}$$

from which, splitting  $D_{x_2} \underline{m}_{n_i}^a$ ,  $D_{x_3} \underline{m}_{n_i}^a$ ,  $D_{x_1} \underline{m}_{n_i}^b$  and  $D_{x_2} \underline{m}_{n_i}^b$  in the following way:

$$\begin{aligned}
D_{x_2} \underline{m}_{n_i}^a &= D_{x_2} \widehat{\mu}^a + (D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a), & D_{x_3} \underline{m}_{n_i}^a &= D_{x_3} \widehat{\mu}^a + (D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a), \\
D_{x_1} \underline{m}_{n_i}^b &= D_{x_1} \widehat{\mu}^b + (D_{x_1} \underline{m}_{n_i}^b - D_{x_1} \widehat{\mu}^b), & D_{x_2} \underline{m}_{n_i}^b &= D_{x_2} \widehat{\mu}^b + (D_{x_2} \underline{m}_{n_i}^b - D_{x_2} \widehat{\mu}^b),
\end{aligned}$$

it follows that

$$\begin{aligned}
& \alpha \int_{\Omega^a} \left( \left| \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a \right|^2 + |D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a|^2 \right) dx + \\
& \frac{h_{n_i}^b}{h_{n_i}^a} \alpha \int_{\Omega^b} \left( |D_{x_1} \underline{m}_{n_i}^b - D_{x_1} \widehat{\mu}^b|^2 + |D_{x_2} \underline{m}_{n_i}^b - D_{x_2} \widehat{\mu}^b|^2 + \left| \frac{1}{h_{n_i}^b} D_{x_3} \underline{m}_{n_i}^b \right|^2 \right) dx \leq \\
& -\alpha \int_{\Omega^a} (|D_{x_2} \widehat{\mu}^a|^2 + 2D_{x_2} \widehat{\mu}^a (D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a)) dx + \\
& -\alpha \int_{\Omega^a} (|D_{x_3} \widehat{\mu}^a|^2 + 2D_{x_3} \widehat{\mu}^a (D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a)) dx - \int_{\Omega^a} (\varphi(\underline{m}_{n_i}^a) - 2f_{n_i}^a \underline{m}_{n_i}^a) dx + \\
& -\frac{h_{n_i}^b}{h_{n_i}^a} \alpha \int_{\Omega^b} (|D_{x_1} \widehat{\mu}^b|^2 + 2D_{x_1} \widehat{\mu}^b (D_{x_1} \underline{m}_{n_i}^b - D_{x_1} \widehat{\mu}^b)) dx + \tag{5.41} \\
& -\frac{h_{n_i}^b}{h_{n_i}^a} \alpha \int_{\Omega^b} (|D_{x_2} \widehat{\mu}^b|^2 + 2D_{x_2} \widehat{\mu}^b (D_{x_2} \underline{m}_{n_i}^b - D_{x_2} \widehat{\mu}^b)) dx + \\
& -\frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} (\varphi(\underline{m}_{n_i}^b) - 2f_{n_i}^b \underline{m}_{n_i}^b) dx - E_{n_i}^{mag}(\underline{m}_{n_i}) + \\
& \int_{\Omega^a} \left( \alpha \left| \left( \frac{1}{h_{n_i}^a} D_{x_1} g_{n_i} |D_{x_2} g_{n_i} |D_{x_3} g_{n_i} \right) \right|^2 + \varphi(g_{n_i}) - 2f_{n_i}^a g_{n_i} \right) dx + \\
& \frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} \left( \alpha |(D_{x_1} \mu^b |D_{x_2} \mu^b |0)|^2 + \varphi(\mu^b) - 2f_{n_i}^b \mu^b \right) dx + E_{n_i}^{mag}((g_{n_i}, \mu^b)), \quad \forall i \in \mathbb{N}.
\end{aligned}$$

By passing to the limit, in (5.41), as  $i$  diverges, and using (2.1) (recall that  $q \neq +\infty$ ), (3.9),

(5.36), (5.37), (5.38) and (5.40), one obtains that

$$\begin{aligned}
& \limsup_i \left[ \alpha \int_{\Omega^a} \left( \left| \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a \right|^2 + |D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a|^2 \right) dx + \right. \\
& \left. \frac{h_{n_i}^b}{h_{n_i}^a} \alpha \int_{\Omega^b} \left( |D_{x_1} \underline{m}_{n_i}^b - D_{x_1} \widehat{\mu}^b|^2 + |D_{x_2} \underline{m}_{n_i}^b - D_{x_2} \widehat{\mu}^b|^2 + \left| \frac{1}{h_{n_i}^b} D_{x_3} \underline{m}_{n_i}^b \right|^2 \right) dx \right] \leq \\
& - \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \alpha (|D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \widehat{\mu}^a|^2) + \varphi(\widehat{\mu}^a) - 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \widehat{\mu}^a + \frac{1}{2} |\widehat{\mu}_1^a|^2 \right) d(x_2, x_3) + \\
& - q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \alpha (|D_{x_1} \widehat{\mu}^b|^2 + |D_{x_2} \widehat{\mu}^b|^2) + \varphi(\widehat{\mu}^b) - 2 \int_{-1}^0 f^b(x_1, x_2, x_3) dx_3 \widehat{\mu}^b + \frac{1}{2} |\widehat{\mu}_3^b|^2 \right) d(x_1, x_2) + \\
& \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \alpha (|D_{x_2} \mu^a|^2 + |D_{x_3} \mu^a|^2) + \varphi(\mu^a) - 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \mu^a + \frac{1}{2} |\mu_1^a|^2 \right) d(x_2, x_3) + \\
& q \int_{]-\frac{1}{2}, \frac{1}{2}[^2} \left( \alpha (|D_{x_1} \mu^b|^2 + |D_{x_2} \mu^b|^2) + \varphi(\mu^b) - 2 \int_{-1}^0 f^b(x_1, x_2, x_3) dx_3 \mu^b + \frac{1}{2} |\mu_3^b|^2 \right) d(x_1, x_2).
\end{aligned} \tag{5.42}$$

Since  $\mu = (\mu^a, \mu^b)$  is a generic element of  $\mathcal{M}_{reg}$ , by virtue of the density result stated in Proposition 5.5, this inequality holds true also with  $\mu = \widehat{\mu}$ . Consequently, it results that

$$\begin{aligned}
& \limsup_i \left[ \alpha \int_{\Omega^a} \left( \left| \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a \right|^2 + |D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a|^2 \right) dx + \right. \\
& \left. \frac{h_n^b}{h_n^a} \alpha \int_{\Omega^b} \left( |D_{x_1} \underline{m}_{n_i}^b - D_{x_1} \widehat{\mu}^b|^2 + |D_{x_2} \underline{m}_{n_i}^b - D_{x_2} \widehat{\mu}^b|^2 + \left| \frac{1}{h_{n_i}^b} D_{x_3} \underline{m}_{n_i}^b \right|^2 \right) dx \right] = 0.
\end{aligned} \tag{5.43}$$

By combining this limit with (2.1) (recall that  $q \neq 0$ ) and (5.36), one derives (4.3) and (4.4). Moreover, (5.42), (5.43) and Proposition 5.5 provide that  $\widehat{\mu}$  is a solution of (4.6). Furthermore, convergence of the energies (4.7) is a consequence of (2.1) with  $q \in ]0, +\infty[$ , (4.3), (4.4) and (4.5).  $\square$

## 6 The cases $q = 0$ and $q = +\infty$

### 6.1 A Convergence result for the magnetostatic energy

**Proposition 6.1.** *Assume (2.1) with  $q \in [0, +\infty]$ . Let  $\{\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)\}_{n \in \mathbb{N}} \subset L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  and  $\mu^a = (\mu_1^a, \mu_2^a, \mu_3^a) \in L^2(\Omega^a, \mathbb{R}^3)$  be such that*

$$\left( \underline{m}_n^a, \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \underline{m}_n^b \right) \rightarrow (\mu^a, 0) \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3), \tag{6.1}$$

as  $n$  diverges. Moreover, for every  $n \in \mathbb{N}$  let  $u_{\underline{m}_n, n} = (u_{\underline{m}_n, n}^a, u_{\underline{m}_n, n}^b)$  be the unique solution of (3.3) corresponding to  $\underline{m}_n$ , and let  $E_n^{mag}$  be defined by (3.10). Then, it results that

$$\left\{ \begin{array}{l} \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a \rightarrow \widetilde{\mu}_1^a, \quad D_{x_2} u_{\underline{m}_n, n}^a \rightarrow 0, \quad D_{x_3} u_{\underline{m}_n, n}^a \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}_+^3), \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_1} u_{\underline{m}_n, n}^b \rightarrow 0, \quad \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_2} u_{\underline{m}_n, n}^b \rightarrow 0, \quad \frac{1}{h_n^b} \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_3} u_{\underline{m}_n, n}^b \rightarrow 0 \end{array} \right. \quad (6.2)$$

strongly in  $L^2(\mathbb{R}_-^3)$ ,

as  $n$  diverges and

$$\lim_n E_n^{mag}(\underline{m}_n) = \frac{1}{2} \int_{\Omega^a} |\mu_1^a|^2 dx. \quad (6.3)$$

*Proof.* By choosing  $u = (0, 0)$  as test function in (3.3) corresponding to  $\underline{m}_n$ , by virtue of (6.1) it results that

$$\begin{aligned} \exists c \in ]0, +\infty[ : \int_{\mathbb{R}_+^3} \left| \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a, D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a \right) - \widetilde{m}_n^a \right|^2 dx + \\ \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left| \left( D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n, n}^b \right) - \widetilde{m}_n^b \right|^2 dx \leq c, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Consequently, applying the triangle inequality and using again (6.1), it follows that

$$\left\{ \begin{array}{l} \exists c \in ]0, +\infty[ : \\ \left\| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a \right\|_{L^2(\mathbb{R}_+^3)} \leq c, \quad \|D_{x_2} u_{\underline{m}_n, n}^a\|_{L^2(\mathbb{R}_+^3)} \leq c, \quad \|D_{x_3} u_{\underline{m}_n, n}^a\|_{L^2(\mathbb{R}_+^3)} \leq c, \\ \left\| \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_1} u_{\underline{m}_n, n}^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \quad \left\| \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_2} u_{\underline{m}_n, n}^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \\ \left\| \frac{1}{h_n^b} \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_3} u_{\underline{m}_n, n}^b \right\|_{L^2(\mathbb{R}_-^3)} \leq c, \quad \forall n \in \mathbb{N}. \end{array} \right. \quad (6.4)$$

Since  $(u_{\underline{m}_n}^a, u_{\underline{m}_n}^b)$  belongs to  $H^1(\mathbb{R}_+^3) \times H^1(\mathbb{R}_-^3)$ , the Sobolev imbedding theorem and (6.4) provide that

$$\exists c \in ]0, +\infty[ : \|u_{\underline{m}_n, n}^a\|_{L^6(\mathbb{R}_+^3)} \leq c, \quad \left\| \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} u_{\underline{m}_n, n}^b \right\|_{L^6(\mathbb{R}_-^3)} \leq c, \quad \forall n \in \mathbb{N}. \quad (6.5)$$

Estimates (6.4) and (6.5) guarantee the existence of a function  $u = (u^a, u^b) \in L^6(\mathbb{R}_+^3) \times L^6(\mathbb{R}_-^3)$ , with  $Du = (Du^a, Du^b) \in (L^2(\mathbb{R}_+^3))^3 \times (L^2(\mathbb{R}_-^3))^3$ ,  $u^a$  independent of  $x_1$  and  $u^b$  independent of  $x_3$ , such that on extraction of a suitable subsequence (not relabelled)

$$\left\{ \begin{array}{l} u_{\underline{m}_n, n}^a \rightharpoonup u^a \text{ weakly in } L^6(\mathbb{R}_+^3), \quad Du_{\underline{m}_n, n}^a \rightharpoonup Du^a \text{ weakly in } (L^2(\mathbb{R}_+^3))^3, \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} u_{\underline{m}_n, n}^b \rightharpoonup u^b \text{ weakly in } L^6(\mathbb{R}_-^3), \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} Du_{\underline{m}_n, n}^b \rightharpoonup Du^b \text{ weakly in } (L^2(\mathbb{R}_-^3))^3, \end{array} \right. \quad (6.6)$$

as  $n$  diverges. By arguing as in the proof of Proposition 5.1, one obtains that  $u^a = 0$  and  $u^b = 0$ . Then, from (6.6) one concludes that

$$\left\{ \begin{array}{l} Du_{\underline{m}_n, n}^a \rightharpoonup 0 \text{ weakly in } (L^2(\mathbb{R}_+^3))^3, \\ \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} Du_{\underline{m}_n, n}^b \rightharpoonup 0 \text{ weakly in } (L^2(\mathbb{R}_-^3))^3, \end{array} \right. \quad (6.7)$$

as  $n$  diverges, and these convergences hold true for the whole sequence.

The first and the last estimate in (6.4) guarantee the existence of  $\xi^a \in L^2(\mathbb{R}_+^3)$  and  $\xi^b \in L^2(\mathbb{R}_-^3)$  such that on extraction of a suitable subsequence (not relabelled)

$$\frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a \rightharpoonup \xi^a \text{ weakly in } L^2(\mathbb{R}_+^3), \quad \frac{1}{h_n^b} \left(\frac{h_n^b}{h_n^a}\right)^{\frac{1}{2}} D_{x_3} u_{\underline{m}_n, n}^b \rightharpoonup \xi^b \text{ weakly in } L^2(\mathbb{R}_-^3), \quad (6.8)$$

as  $n$  diverges.

For proving that convergences (6.7) are (6.8) are strong and for identifying  $\xi^a$  and  $\xi^b$ , arguing as in the proof of Proposition 5.1, introduce a sequence  $\{\varepsilon\}$  of positive numbers converging to zero, a diverging sequence  $\{\lambda\}$  of positive numbers, and a sequence  $\{\mu_\varepsilon^a\}_\varepsilon \subset C_0^\infty(\Omega^a)$  such that

$$\mu_\varepsilon^a \rightarrow \mu_1^a \text{ strongly in } L^2(\Omega^a), \quad (6.9)$$

as  $\varepsilon$  tends to zero. Moreover, for every  $\varepsilon, \lambda$  and  $n \in \mathbb{N}$  set

$$u_{\varepsilon, \lambda, n} = \left\{ \begin{array}{l} u_{\varepsilon, \lambda, n}^a = h_n^a \int_{-\frac{1}{2}}^{x_1} \mu_\varepsilon^a(s, x_2, x_3) ds - \frac{h_n^a}{\lambda} \int_{\frac{1}{2}}^{x_1} \chi_{[\frac{1}{2}, \frac{1}{2} + \lambda]}(r) dr \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds + c_{\varepsilon, \lambda, n}, \\ u_{\varepsilon, \lambda, n}^b = c_{\varepsilon, \lambda, n}, \text{ a.e. in } \mathbb{R}_-^3, \end{array} \right. \quad \text{a.e. in } \mathbb{R}_+^3,$$

where the constant  $c_{\varepsilon, \lambda, n}$  is chosen in such a way to have  $\int_{B_n^a} u_{\varepsilon, \lambda, n}^a dx + \frac{h_n^b}{h_n^a} \int_{B_n^b} u_{\varepsilon, \lambda, n}^b dx = 0$ . Since  $u_{\varepsilon, \lambda, n} \in \mathcal{U}_n$ , choosing  $u = u_{\varepsilon, \lambda, n}$  as test function in (3.3) corresponding to  $\underline{m}_n =$



$(\underline{m}_n^a, \underline{m}_n^b) = ((\underline{m}_{1,n}^a, \underline{m}_{2,n}^a, \underline{m}_{3,n}^a), (\underline{m}_{1,n}^b, \underline{m}_{2,n}^b, \underline{m}_{3,n}^b))$ , it results that

$$\begin{aligned}
& \int_{\mathbb{R}_+^3} \left( \left| \xi^a - \widetilde{\underline{m}}_{1,n}^a \right|^2 + 2 \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n^a}^a - \xi^a \right) \left( \xi^a - \widetilde{\underline{m}}_{1,n}^a \right) + \left| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n^a}^a - \xi^a \right|^2 + \right. \\
& \left. \left| (D_{x_2} u_{\underline{m}_n^a}^a, D_{x_3} u_{\underline{m}_n^a}^a) \right|^2 - 2(D_{x_2} u_{\underline{m}_n^a}^a, D_{x_3} u_{\underline{m}_n^a}^a)(\widetilde{\underline{m}}_{2,n}^a, \widetilde{\underline{m}}_{3,n}^a) + |(\widetilde{\underline{m}}_{2,n}^a, \widetilde{\underline{m}}_{3,n}^a)|^2 \right) dx + \\
& \frac{h_n^b}{h_n^a} \int_{\mathbb{R}_-^3} \left( \left| (D_{x_1} u_{\underline{m}_n^b}^b, D_{x_2} u_{\underline{m}_n^b}^b) \right|^2 - 2(D_{x_1} u_{\underline{m}_n^b}^b, D_{x_2} u_{\underline{m}_n^b}^b)(\widetilde{\underline{m}}_{1,n}^b, \widetilde{\underline{m}}_{2,n}^b) + |(\widetilde{\underline{m}}_{1,n}^b, \widetilde{\underline{m}}_{2,n}^b)|^2 + \right. \\
& \left. \left| \widetilde{\underline{m}}_{3,n}^b \right|^2 - 2 \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n^b}^b \widetilde{\underline{m}}_{3,n}^b + \left| \frac{1}{h_n^b} D_{x_3} u_{\underline{m}_n^b}^b \right|^2 \right) dx = \\
& 2j_{\underline{m}_n,n}(u_{\underline{m}_n}) \leq 2j_{\underline{m}_n,n}(u_{\varepsilon,\lambda,n}) = \\
& \int_{\mathbb{R}_+^3} \left( \left| \mu_\varepsilon^a - \frac{1}{\lambda} \chi_{[\frac{1}{2}, \frac{1}{2}+\lambda]}(x_1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{\underline{m}}_{1,n}^a \right|^2 + \right. \\
& \left| h_n^a D_{x_2} \int_{-\frac{1}{2}}^{x_1} \mu_\varepsilon^a(s, x_2, x_3) ds - \frac{h_n^a}{\lambda} \int_{\frac{1}{2}}^{x_1} \chi_{[\frac{1}{2}, \frac{1}{2}+\lambda]}(r) dr D_{x_2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{\underline{m}}_{2,n}^a \right|^2 + \\
& \left. \left| h_n^a D_{x_3} \int_{-\frac{1}{2}}^{x_1} \mu_\varepsilon^a(s, x_2, x_3) ds - \frac{h_n^a}{\lambda} \int_{\frac{1}{2}}^{x_1} \chi_{[\frac{1}{2}, \frac{1}{2}+\lambda]}(r) dr D_{x_3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{\underline{m}}_{3,n}^a \right|^2 \right) dx + \\
& \int_{\mathbb{R}_-^3} \left| \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} \widetilde{\underline{m}}_n^b \right|^2 dx, \quad \forall \varepsilon, \lambda, n,
\end{aligned}$$

from which, fixing  $\varepsilon$  and  $\lambda$ , passing to the limit as  $n$  diverges and using (6.1), (6.7), (6.8), it

follows that

$$\begin{aligned}
& \limsup_n \left[ \int_{\mathbb{R}_+^3} \left( |\xi^a - \widetilde{m}_{1,n}^a|^2 + \left| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a - \xi^a \right|^2 + |(D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a)|^2 \right) dx + \right. \\
& \left. \int_{\mathbb{R}_-^3} \left( \left| \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} (D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b) \right|^2 + \left| \frac{1}{h_n^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_3} u_{\underline{m}_n, n}^b \right|^2 \right) dx \right] \leq \\
& \int_{\mathbb{R}_+^3} \left( \left| \mu_\varepsilon^a - \frac{1}{\lambda} \chi_{[\frac{1}{2}, \frac{1}{2} + \lambda]}(x_1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu_\varepsilon^a(s, x_2, x_3) ds - \widetilde{\mu}_1^a \right|^2 \right) dx \leq \\
& 2 \int_{\Omega^a} |\mu_\varepsilon^a - \mu_1^a|^2 dx + \frac{2}{\lambda} \int_{\Omega^a} |\mu_\varepsilon^a|^2 dx \quad \forall \varepsilon, \lambda.
\end{aligned} \tag{6.10}$$

Finally, passing to the limit in (6.10), as  $\lambda$  diverges and  $\varepsilon$  tends to zero, and using (6.9), one obtains

$$\begin{aligned}
& \lim_n \left[ \int_{\mathbb{R}_+^3} \left( |\xi^a - \widetilde{m}_{1,n}^a|^2 + \left| \frac{1}{h_n^a} D_{x_1} u_{\underline{m}_n, n}^a - \xi^a \right|^2 + |(D_{x_2} u_{\underline{m}_n, n}^a, D_{x_3} u_{\underline{m}_n, n}^a)|^2 \right) dx + \right. \\
& \left. \int_{\mathbb{R}_-^3} \left( \left| \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} (D_{x_1} u_{\underline{m}_n, n}^b, D_{x_2} u_{\underline{m}_n, n}^b) \right|^2 + \left| \frac{1}{h_n^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_3} u_{\underline{m}_n, n}^b \right|^2 \right) dx \right] = 0,
\end{aligned}$$

from which, by virtue of (6.1), converges (6.2) follow and the converges hold true for the whole sequence. Convergence (6.3) is a consequence of (6.2).  $\square$

**Remark 6.2.** Proposition 6.1 holds true for every  $q \in [0, +\infty]$ . Really, in the sequel, this result will be applied to a sequence  $\{\underline{m}_n\}_{n \in \mathbb{N}}$ , where  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b) \in \mathcal{M}_n$  is a solution of (3.8). Then, for obtain that such a sequence satisfies (6.1), the restrictive assumption  $q = 0$  in (2.1) will need (see below).

## 6.2 Proof of Theorem 4.2

*Proof.* When in (2.1)  $q = 0$ , from Proposition 5.2 it is not possible to derive Corollary 5.3, but only the existence of a constant  $c \in ]0, +\infty[$  such that

$$\begin{aligned}
& \|D_{x_1} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq c h_n^a, \quad \|D_{x_2} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq c, \quad \|D_{x_3} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq c, \\
& \left\| \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_1} \underline{m}_n^b \right\|_{(L^2(\Omega^b))^3} \leq c, \quad \left\| \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_2} \underline{m}_n^b \right\|_{(L^2(\Omega^b))^3} \leq c, \quad \left\| \frac{1}{h_n^b} \left( \frac{h_n^b}{h_n^a} \right)^{\frac{1}{2}} D_{x_3} \underline{m}_n^b \right\|_{(L^2(\Omega^b))^3} \leq c,
\end{aligned}$$

for every  $n \in \mathbb{N}$ . Consequently, taking also into account (2.1) with  $q = 0$ , and the fact that  $|\underline{m}_n^a| = 1$  a.e. in  $\Omega^a$  and  $|\underline{m}_n^b| = 1$  a.e. in  $\Omega^b$ , there exist an increasing sequence

of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\widehat{\mu}^a \in H^1(\Omega^a, S^2)$ , independent of  $x_1$ ,  $(\xi^a, \xi^b) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  depending on the selected subsequence, such that

$$\left( \underline{m}_{n_i}^a, \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} \underline{m}_{n_i}^b \right) \rightharpoonup (\widehat{\mu}^a, 0) \text{ weakly in } H^1(\Omega^a, \mathbb{R}^3) \times H^1(\Omega^b, \mathbb{R}^3), \quad (6.11)$$

$$\left( \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a, \frac{1}{h_{n_i}^b} \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_3} \underline{m}_{n_i}^b \right) \rightharpoonup (\xi^a, \xi^b) \text{ weakly in } L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3),$$

as  $i$  diverges. Consequently, by virtue of Proposition 6.1, convergences (4.12) hold true, and it results that

$$\lim_i E_{n_i}^{mag}(\underline{m}_{n_i}) = \frac{1}{2} \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\widehat{\mu}_1^a(x_2, x_3)|^2 d(x_2, x_3), \quad (6.12)$$

where  $E_{n_i}^{mag}$  is defined in (3.10).

For  $\mu^a \in C^\infty(]-\frac{1}{2}, \frac{1}{2}[ \times [0, 1], S^2)$ , set

$$g(x) = \begin{cases} \mu^a(x_2, x_3), & \text{if } x = (x_1, x_2, x_3) \in \Omega^a, \\ \mu^a(x_2, 0), & \text{if } x = (x_1, x_2, x_3) \in \Omega^b. \end{cases}$$

Then, Proposition 6.1 provides that

$$\lim_n E_n^{mag}(g) = \frac{1}{2} \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} |\mu_1^a(x_2, x_3)|^2 d(x_2, x_3). \quad (6.13)$$

Finally, choosing  $g$  as test function in (3.8) and taking into account (3.11), one has that

$$\begin{aligned} & \int_{\Omega^a} \left( \alpha \left| \left( \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a \mid D_{x_2} \underline{m}_{n_i}^a \mid D_{x_3} \underline{m}_{n_i}^a \right) \right|^2 + \varphi(\underline{m}_{n_i}^a) - 2f_{n_i}^a \underline{m}_{n_i}^a \right) dx + \\ & \frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} \left( \alpha \left| \left( D_{x_1} \underline{m}_{n_i}^b \mid D_{x_2} \underline{m}_{n_i}^b \mid \frac{1}{h_{n_i}^b} D_{x_3} \underline{m}_{n_i}^b \right) \right|^2 + \varphi(\underline{m}_{n_i}^b) - 2f_{n_i}^b \underline{m}_{n_i}^b \right) dx + E_{n_i}^{mag}(\underline{m}_{n_i}) \leq \\ & \int_{\Omega^a} (\alpha |(0 \mid D_{x_2} \mu^a \mid D_{x_3} \mu^a)|^2 + \varphi(\mu^a) - 2f_{n_i}^a \mu^a) dx + \\ & \frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} (\alpha |(0 \mid D_{x_2} \mu^a(x_2, 0) \mid 0)|^2 + \varphi(\mu^a(x_2, 0)) - 2f_{n_i}^b \mu^a(x_2, 0)) dx + E_{n_i}^{mag}(g), \quad \forall i \in \mathbb{N}, \end{aligned}$$

from which, splitting  $D_{x_2} \underline{m}_{n_i}^a$  and  $D_{x_3} \underline{m}_{n_i}^a$  in the following way:

$$D_{x_2} \underline{m}_{n_i}^a = D_{x_2} \widehat{\mu}^a + (D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a), \quad D_{x_3} \underline{m}_{n_i}^a = D_{x_3} \widehat{\mu}^a + (D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a),$$

it follows that

$$\begin{aligned}
& \alpha \int_{\Omega^a} \left( \left| \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a \right|^2 + |D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a|^2 \right) dx + \\
& \alpha \int_{\Omega^b} \left( \left| \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_1} \underline{m}_{n_i}^b \right|^2 + \left| \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_2} \underline{m}_{n_i}^b \right|^2 + \left| \frac{1}{h_{n_i}^b} \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_3} \underline{m}_{n_i}^b \right|^2 \right) dx \leq \\
& -\alpha \int_{\Omega^a} (|D_{x_2} \widehat{\mu}^a|^2 + 2D_{x_2} \widehat{\mu}^a (D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a)) dx + \\
& -\alpha \int_{\Omega^a} (|D_{x_3} \widehat{\mu}^a|^2 + 2D_{x_3} \widehat{\mu}^a (D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a)) dx - \int_{\Omega^a} (\varphi(\underline{m}_{n_i}^a) - 2f_{n_i}^a \underline{m}_{n_i}^a) dx + \tag{6.14} \\
& -\frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} (\varphi(\underline{m}_{n_i}^b) - 2f_{n_i}^b \underline{m}_{n_i}^b) dx - \frac{1}{2} E_{n_i}^{mag}(\underline{m}_{n_i}) + \\
& \int_{\Omega^a} (\alpha |(0|D_{x_2} \mu^a|D_{x_3} \mu^a)|^2 + \varphi(\mu^a) - 2f_{n_i}^a \mu^a) dx + \\
& \frac{h_{n_i}^b}{h_{n_i}^a} \int_{\Omega^b} (\alpha |(0|D_{x_2} \mu^a(x_2, 0)|0)|^2 + \varphi(\mu^a(x_2, 0)) - 2f_{n_i}^b \mu^a(x_2, 0)) dx + E_{n_i}^{mag}(g),
\end{aligned}$$

for every  $i \in \mathbb{N}$ . By passing to the limit, in (6.14), as  $i$  diverges, and using (2.1) with  $q = 0$ , (3.9), (6.11), (6.12) and (6.13), one obtains that

$$\begin{aligned}
& \limsup_i \left[ \alpha \int_{\Omega^a} \left( \left| \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a \right|^2 + |D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a|^2 \right) dx + \right. \\
& \left. \alpha \int_{\Omega^b} \left( \left| \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_1} \underline{m}_{n_i}^b \right|^2 + \left| \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_2} \underline{m}_{n_i}^b \right|^2 + \left| \frac{1}{h_{n_i}^b} \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_3} \underline{m}_{n_i}^b \right|^2 \right) dx \right] \leq \\
& - \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \alpha (|D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \widehat{\mu}^a|^2) + \varphi(\widehat{\mu}^a) - 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \widehat{\mu}^a + \frac{1}{2} |\widehat{\mu}_1^a|^2 \right) d(x_2, x_3) + \\
& \int_{]-\frac{1}{2}, \frac{1}{2}[ \times ]0, 1[} \left( \alpha (|D_{x_2} \mu^a|^2 + |D_{x_3} \mu^a|^2) + \varphi(\mu^a) - 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^a(x_1, x_2, x_3) dx_1 \mu^a + \frac{1}{2} |\mu_1^a|^2 \right) d(x_2, x_3). \tag{6.15}
\end{aligned}$$

Since  $C^\infty([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], S^2)$  is dense in  $H^1([-\frac{1}{2}, \frac{1}{2}] \times ]0, 1[, S^2)$  (see [4]), this inequality holds

true also with  $\mu^a = \widehat{\mu}^a$ . Consequently, it results that

$$\begin{aligned} & \lim_i \left[ \alpha \int_{\Omega^a} \left( \left| \frac{1}{h_{n_i}^a} D_{x_1} \underline{m}_{n_i}^a \right|^2 + |D_{x_2} \underline{m}_{n_i}^a - D_{x_2} \widehat{\mu}^a|^2 + |D_{x_3} \underline{m}_{n_i}^a - D_{x_3} \widehat{\mu}^a|^2 \right) dx + \right. \\ & \left. \alpha \int_{\Omega^b} \left( \left| \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_1} \underline{m}_{n_i}^b \right|^2 + \left| \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_2} \underline{m}_{n_i}^b \right|^2 + \left| \frac{1}{h_{n_i}^b} \left( \frac{h_{n_i}^b}{h_{n_i}^a} \right)^{\frac{1}{2}} D_{x_3} \underline{m}_{n_i}^b \right|^2 \right) dx \right] \quad (6.16) \\ & = 0. \end{aligned}$$

By combining this limit with (6.11), one derives (4.10) and (4.11). Moreover, (6.15), (6.16) and the density of  $C^1([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], S^2)$  in  $H^1([-\frac{1}{2}, \frac{1}{2}] \times [0, 1], S^2)$  provide that  $\widehat{\mu}^a$  is a solution of (4.13). Furthermore, convergence of the energies (4.14) is a consequence of (2.1) with  $q = 0$ , (4.10), (4.11) and (4.12).  $\square$

### 6.3 Proof of Theorem 4.3

The proof of Theorem 4.3 is very similar to that of Theorem 4.3. So we omit it. We only suggest of working with the functional  $\frac{h_n^a}{h_n^b} E_n$ , i.e.

$$\begin{aligned} \underline{m} &= (\underline{m}^a, \underline{m}^b) \in \mathcal{M}_n \longrightarrow \\ & \frac{h_n^a}{h_n^b} \int_{\Omega^a} \left( \alpha \left| \left( \frac{1}{h_n^a} D_{x_1} \underline{m}^a | D_{x_2} \underline{m}^a | D_{x_3} \underline{m}^a \right) \right|^2 + \varphi(\underline{m}^a) - 2f_n^a \underline{m}^a \right) dx + \\ & \frac{1}{2} \frac{h_n^a}{h_n^b} \int_{\Omega^a} \left( \left( \frac{1}{h_n^a} D_{x_1} u_{\underline{m},n}^a, D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \underline{m}^a \right) dx + \\ & \int_{\Omega^b} \left( \alpha \left| \left( D_{x_1} \underline{m}^b | D_{x_2} \underline{m}^b | \frac{1}{h_n^b} D_{x_3} \underline{m}^b \right) \right|^2 + \varphi(\underline{m}^b) - 2f_n^b \underline{m}^b \right) dx + \\ & \frac{1}{2} \int_{\Omega^b} \left( \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^b} D_{x_3} u_{\underline{m},n}^b \right) \underline{m}^b \right) dx \end{aligned}$$

and of remarking that  $\lim_n \frac{h_n^a}{h_n^b} = 0$ , since  $q = +\infty$ .

## 7 Appendix

For sake of completeness we give the proof of the following known result:

**Proposition 7.1.** *For  $y \in B_{\frac{1}{2}}(0) = \{x \in \mathbb{R}^3 : |x| \leq \frac{1}{2}\}$ , let  $\pi_y$  be the projection defined by*

$$\pi_y : x \in \mathbb{R}^3 \setminus \{y\} \rightarrow y + \frac{-y(x-y) + \sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}}{|x-y|^2} (x-y) \in S^2.$$

Then,

$$\exists c \in ]0, +\infty[ \quad : \quad |D\pi_y(x)|^2 \leq \frac{c}{|x-y|^2}, \quad \forall y \in B_{\frac{1}{2}}(0), \quad \forall x \in \mathbb{R}^3 \setminus \{y\}. \quad (7.1)$$

*Proof.* For  $y \in B_{\frac{1}{2}}(0)$  and  $j = 1, 2, 3$ , let

$$\pi_y^j : x \in \mathbb{R}^3 \setminus \{y\} \rightarrow y_j + \frac{-y(x-y) + \sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}}{|x-y|^2}(x_j - y_j).$$

Then, for  $i, j = 1, 2, 3$ , it results that

$$\begin{aligned} \partial_{x_i} \pi_y^j(x) &= \frac{-y(x-y) + \sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}}{|x-y|^2} \delta_{i,j} + \\ &\frac{\left(-y_i + \frac{y(x-y)y_i + (x_i - y_i)(1-|y|^2)}{\sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}}\right)}{|x-y|^2} (x_j - y_j) + \\ &\frac{\left(-y(x-y) + \sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}\right) 2(x_i - y_i)}{|x-y|^4} (x_j - y_j), \end{aligned} \quad (7.2)$$

for every  $y \in B_{\frac{1}{2}}(0)$  and every  $x \in \mathbb{R}^3 \setminus \{y\}$ .

As regard as the first and last term in the right-hand side of (7.2), it is easily seen that, for  $j = 1, 2, 3$ ,

$$\left| \frac{-y(x-y) + \sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}}{|x-y|^2} \delta_{i,j} \right| \leq \frac{|y|+1}{|x-y|} \leq \frac{3}{2} \frac{1}{|x-y|}, \quad (7.3)$$

$$\left| \frac{\left(-y(x-y) + \sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}\right) 2(x_i - y_i)}{|x-y|^4} (x_j - y_j) \right| \leq \quad (7.4)$$

$$2 \frac{|y|+1}{|x-y|} \leq \frac{3}{|x-y|},$$

for every  $y \in B_{\frac{1}{2}}(0)$  and every  $x \in \mathbb{R}^3 \setminus \{y\}$ . As regard as the second term in the right-hand side of (7.2), since  $|y| < 1$ , it results that, for  $j = 1, 2, 3$ ,

$$\begin{aligned} (y(x-y)y_i + (x_i - y_i)(1-|y|^2))^2 &\leq 2 \left( (y(x-y))^2 y_i^2 + (x_i - y_i)^2 (1-|y|^2)^2 \right) \\ &\leq 2 \left( (y(x-y))^2 + |x-y|^2 (1-|y|^2) \right), \end{aligned}$$

and consequently

$$\frac{\left(-y_i + \frac{y(x-y)y_i + (x_i - y_i)(1-|y|^2)}{\sqrt{(y(x-y))^2 + |x-y|^2(1-|y|^2)}}\right)}{|x-y|^2} (x_j - y_j) \leq \frac{\frac{1}{2} + \sqrt{2}}{|x-y|}, \quad (7.5)$$

for every  $y \in B_{\frac{1}{2}}(0)$  and every  $x \in \mathbb{R}^3 \setminus \{y\}$ . Finally, by combining estimate (7.2) with (7.3)-(7.4), one obtains (7.1).  $\square$

## Acknowledgments

This paper is part of the project: "Strutture sottili" of the program 2004-2006: "Collaborazioni interuniversitarie internazionali" of the Italian Ministry of Education, University and Research, and also of the project GNAMPA 2009 "Problemi di giunzione in multi-strutture"

The authors wish to thank Professor Hervé Le Dret for the helpful discussions they had on the subject.

## References

- [1] R. ALICANDRO, C. LEONE, 3D-2D Asymptotic Analysis for Micromagnetic Thin Films, *ESAIM Control Optim. Calc. Var.* **6** (2001), 489-498.
- [2] H. AMMARI, L. HALPERN, K. HAMDACHE, Asymptotic behavior of thin ferromagnetic films, *Asymptot. Anal.* **24** (2000), 277-294.
- [3] M. BAÍA, E. ZAPPALE, A Note on the 3D-2D Dimensional Reduction of a Micromagnetic Thin Film with Nonhomogeneous Profile, *Appl. Anal.* **86** (2007), n. 5, 555-575.
- [4] F. BETHUEL, X. ZHENG, Density of Smooth Functions between Two Manifolds in Sobolev Spaces, *J. Functional Anal.* **80** (1988), n.2, 60-75.
- [5] W. F. BROWN, Micromagnetics, *John Willey and Sons, New York*, (1963).
- [6] G. CARBOU, Thin Layers in micromagnetism, *M<sup>3</sup>AS: Math. Models Methods Appl. Sci.* **11** (2001), n.9, 1529-1546.
- [7] P.G. CIARLET, P. DESTUYNDER, A Justification of the Two-Dimensional Linear Plate Model, *J. Mécanique* **18** (1979), n.2, 315-344.
- [8] E. DE GIORGI, T. FRANZONI, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **58** (8) (1975), n.6, 842-850.
- [9] A. DESIMONE, R.V. KOHN, S. MULLER, F. OTTO, A Reduced Theory for Thin-Film Micromagnetics, *Commun. Pure Appl. Math.* **55** (2002), 11, 1408-1460.
- [10] A. GAUDIELLO, R. HADIJI, Asymptotic Analysis, in a Thin Multidomain, of Minimizing Maps with values in  $S^2$ , *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **26** (2009), 1, 59-80.
- [11] A. GAUDIELLO, R. HADIJI, Junction of One-Dimensional Minimization Problems Involving  $S^2$  Valued Maps, *Adv. Differ. Equ.* **13** (2008), 9-10, 935-958.
- [12] G. GIOIA, R.D. JAMES, Micromagnetism of Very Thin Films, *Proc. R. Lond. A* **453** (1997), 213-223.
- [13] R. HARDT, D. KINDERLEHRER, F. H. LIN, Existence and Partial Regularity of Static Liquid Crystal Configurations *Commun. Math. Phys.* **105** (1986), 547-570.

- [14] A. HUBERT, R. SCHAFER, Magnetic Domains: The Analysis of Magnetic Microstructures, *Springer-Verlag*, (1998).
- [15] S.S. IRUDAYARAJ, A. EMADI, Micromachines: Principles of Operation, Dynamics, and Control, *Electric Machines and Drives, 2005 IEEE International Conference* (2005), 1108-1115.
- [16] R.D. JAMES, D. KINDERLEHRER, Frustration in Ferromagnetic Materials, *Continuum Mech. Thermodyn.* **2** (1990), 215-239.
- [17] R.V. KOHN, V.V. SLASTIKOV, Another Thin-Film Limit of Micromagnetics, *Arch. Rational Mech. Anal.* **178** (2005), 227-245.
- [18] L. D. LANDAU, E.M. LIFSHITZ, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, *Phy. Z. Sowjetunion* **8** 153 (1935), ter Haar, D. (eds.). Reproduced in *Collected Papers of L. D. Landau*, pp.101-114, New York: Pergamon Press (1965)
- [19] H. LE DRET, Problèmes variationnels dans le multi-domaines: modélisation des jonctions et applications. Research in Applied Mathematics, **19**. *Masson, Paris*, (1991).
- [20] A. VISINTIN, On Landau-Lifschitz' Equations for Ferromagnetism, *Jap. J. Appl. Math.* **2** (1985), 69-84.