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# Homotopy commutative algebra and 2-nilpotent Lie algebra

Michel Dubois-Violette and Todor Popov

**Abstract** The homotopy transfer theorem due to Tornike Kadeishvili induces the structure of a homotopy commutative algebra, or  $C_\infty$ -algebra, on the cohomology of the free 2-nilpotent Lie algebra. The latter  $C_\infty$ -algebra is shown to be generated in degree one by the binary and the ternary operations.

## 1 Introduction

Every Universal Enveloping Algebra (UEA)  $U\mathfrak{g}$  of a finite dimensional positively graded Lie algebra  $\mathfrak{g}$  belongs to the class of Artin-Schelter regular algebras (see e.g. [4]). As every finitely generated graded connected algebra,  $U\mathfrak{g}$  has a free minimal resolution which is canonically built from the data of its Yoneda algebra  $\mathcal{E} := \text{Ext}_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K})$ . By construction the Yoneda algebra  $\mathcal{E}$  is isomorphic (as algebra) to the cohomology of the Lie algebra (with coefficients in the trivial representation provided by the ground field  $\mathbb{K}$ )

$$\mathcal{E} = \text{Ext}_{U\mathfrak{g}}^\bullet(\mathbb{K}, \mathbb{K}) \cong H^\bullet(\mathfrak{g}, \mathbb{K}) \quad (1)$$

equipped with wedge product between cohomological classes in  $H^\bullet(\mathfrak{g}, \mathbb{K})$ .

The homotopy transfer theorem of Tornike Kadeishvili [7] implies that the Yoneda algebra  $\mathcal{E} = \text{Ext}_{U\mathfrak{g}}^\bullet(\mathbb{K}, \mathbb{K})$  has the structure of homotopy associative algebra, or  $A_\infty$ -algebra. Since  $\mathcal{E} \cong H^\bullet(\mathfrak{g}, \mathbb{K})$  is the cohomology of the exterior algebra

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$\wedge \mathfrak{g}^*$  which is graded-commutative, it has the structure of homotopy commutative and associative algebra, or  $C_\infty$ -algebra.

Throughout the text  $\mathfrak{g}$  will be the free 2-nilpotent graded Lie algebra, with degree one generators in the finite dimensional vector space  $V$  over a field  $\mathbb{K}$  of characteristic 0,

$$\mathfrak{g} = V \oplus \wedge^2 V .$$

The UEA  $U(V \oplus \wedge^2 V)$  arises naturally in physics in the universal Fock-like space of the parastatistics algebra introduced by H.S. Green [5](see also [3]). Here we will concentrate on the case when  $V$  is an ordinary (even) vector space  $V$ , when the algebra  $U\mathfrak{g}$  is the parafermionic algebra.

The aim of this note is to describe the Yoneda algebra  $\mathcal{E}$  of the UEA  $U\mathfrak{g}$ , i.e., the cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K})$  with its  $C_\infty$ -structure induced by the isomorphism (1) through the homotopy transfer.

The cohomology space  $H^\bullet(\mathfrak{g}, \mathbb{K})$  has a natural  $GL(V)$ -action. The decomposition of the  $GL(V)$ -module  $H^\bullet(\mathfrak{g}, \mathbb{K})$  into irreducible Schur modules  $V_\lambda$  is known since the work of Józefiak and Weyman [6]; it contains all  $GL(V)$ -modules with self-conjugated Young diagrams  $\lambda = \lambda'$  once and exactly once. The decomposition of  $\mathcal{E} = H^\bullet(\mathfrak{g}, \mathbb{K})$  into Schur modules provides a powerful tool to handle its  $C_\infty$ -algebra structure.

## 2 Artin-Schelter regularity

Let  $\mathfrak{g}$  be the 2-nilpotent graded Lie algebra  $\mathfrak{g} = V \oplus \wedge^2 V$  generated by the finite dimensional vector space  $V$  having Lie bracket

$$[x, y] := \begin{cases} x \wedge y & x, y \in V \\ 0 & \text{otherwise} \end{cases} . \quad (2)$$

We denote the Universal Enveloping Algebra  $U\mathfrak{g}$  by  $PS$  and will refer to it as *parastatistics algebra* (by some abuse<sup>1</sup>). The parastatistics algebra  $PS(V)$  generated in  $V$  is graded

$$PS(V) := U\mathfrak{g} = U(V \oplus \wedge^2 V) = T(V)/([V, V], V) .$$

We shall write simply  $PS$  when the space of generators  $V$  is clear from the context.

Artin and Schelter [1] introduced a class of regular algebras sharing some “good” homological properties with the polynomial algebra  $\mathbb{K}[V]$ . These algebras were dubbed Artin-Schelter regular algebras (AS-regular algebra for short).

**Definition 1.** (AS-regular algebras) A connected graded algebra  $\mathcal{A} = \mathbb{K} \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots$  is called Artin-Schelter regular of dimension  $d$  if

- (i)  $\mathcal{A}$  has finite global dimension  $d$ ,
- (ii)  $\mathcal{A}$  has finite Gelfand-Kirillov dimension,

<sup>1</sup> Strictly speaking  $PS(V)$  is the creation parastatistics algebra, closed by creation operators alone.

(iii)  $\mathcal{A}$  is Gorenstein, i.e.,  $\text{Ext}_{\mathcal{A}}^i(\mathbb{K}, \mathcal{A}) = \delta^{i,d} \mathbb{K}$ .

A general theorem claims that the UEA of a finite dimensional positively graded Lie algebra is an AS-regular algebra of global dimension equal to the dimension of the Lie algebra [4]. Hence the parastatistics algebra  $PS$  is AS-regular of global dimension  $d = \frac{\dim V(\dim V + 1)}{2}$ . In particular the finite global dimension of  $PS$  implies that the ground field  $\mathbb{K}$  has a minimal resolution  $P_\bullet$  by projective left  $PS$ -modules  $P_n$

$$P_\bullet : \quad 0 \rightarrow P_d \rightarrow \cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\varepsilon} \mathbb{K} \rightarrow 0 . \quad (3)$$

Here  $\mathbb{K}$  is a trivial left  $PS$ -module, the action being defined by the projection  $\varepsilon$  onto  $PS_0 = \mathbb{K}$ . Since  $PS$  is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [2], we have  $P_n \cong PS \otimes E_n$  where  $E_n$  are finite dimensional vector spaces.

The minimal projective resolution is unique (up to an isomorphism). Minimality implies that the complex  $\mathbb{K} \otimes_{PS} P_\bullet$  has “zero differentials” hence

$$H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet = E_n .$$

One can calculate the derived functor  $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$  using the resolution  $P_\bullet$ , it yields

$$\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) = E_n . \quad (4)$$

The data of a minimal resolution of  $\mathbb{K}$  by free  $PS$ -modules provides an easy way to find  $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ . Conversely if the spaces  $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$  are known, then one can construct a minimal free resolution of  $\mathbb{K}$ .

The Gorenstein property guarantees that when applying the functor  $\text{Hom}_{PS}(-, PS)$  to the minimal free resolution  $P_\bullet$  we get another minimal free resolution  $P^\bullet := \text{Hom}_{PS}(P_\bullet, PS)$  of  $\mathbb{K}$  by right  $PS$ -modules

$$P^\bullet : \quad 0 \leftarrow \mathbb{K} \leftarrow P_d' \leftarrow \cdots \leftarrow P_n' \leftarrow \cdots \leftarrow P_2' \leftarrow P_1' \leftarrow P_0' \leftarrow 0 \quad (5)$$

with  $P_n' \cong E_n^* \otimes PS$ . Note that by construction  $E_n^* = \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})$ , thus one has vector space isomorphisms [2]

$$E_n \cong E_n^* \cong \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K}) . \quad (6)$$

The Gorenstein property is an analog of the Poincaré duality, it implies  $E_{d-n}^* \cong E_n$ . The finite global dimension  $d$  of  $PS$  and the Gorenstein condition imply that its Yoneda algebra

$$\mathcal{E}^\bullet := \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K}) \cong \bigoplus_{n=0}^d E_n^*$$

is Frobenius [10]. More on Gorenstein property you can find in the first autor's lecture “Poincaré duality for Koszul algebras” in the present volume.

### 3 Homology and cohomology of $\mathfrak{g}$

A non-minimal projective(in fact free) resolution of  $\mathbb{K}$ ,  $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$  is given by the standard Chevalley-Eilenberg chain complex  $C_\bullet(\mathfrak{g}) = (U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$  with differential maps

$$\begin{aligned} d_p(u \otimes x_1 \wedge \dots \wedge x_p) &= \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_p \\ &+ \sum_{i < j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_p \end{aligned} \quad (7)$$

The resolution  $C_\bullet(\mathfrak{g})$  calculates the homologies of the derived complex  $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$

$$E_n = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})) = H_n(\mathfrak{g}, \mathbb{K}),$$

coinciding with the homologies  $H_n(\mathfrak{g}, \mathbb{K})$  of the Lie algebra  $\mathfrak{g}$  with trivial coefficients. The derived complex  $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$  is the chain complex with degrees  $\wedge^\bullet \mathfrak{g} = \mathbb{K} \otimes_{PS} PS \otimes \wedge^\bullet \mathfrak{g}$  and differentials  $\partial_p := id \otimes_{PS} d_p : \wedge^p \mathfrak{g} \rightarrow \wedge^{p-1} \mathfrak{g}$ . One has

$$\wedge^p \mathfrak{g} = \wedge^p (V \oplus \wedge^2 V) = \bigoplus_{s+r=p} \wedge^s (\wedge^2 V) \otimes \wedge^r (V) \quad (8)$$

and differentials  $\partial_{p=r+s} : \wedge^s (\wedge^2 V) \otimes \wedge^r (V) \rightarrow \wedge^{s+1} (\wedge^2 V) \otimes \wedge^{r-2} (V)$  are given by

$$\begin{aligned} \partial_p : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge e_r \mapsto \\ \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_r. \end{aligned}$$

The differential  $\partial$  is induced by the Lie bracket  $[\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ , it identifies a pair of degree 1 generators  $e_i, e_j \in$  with one degree 2 generator  $e_{ij} := (e_i \wedge e_j) = [e_i, e_j]$ . The differential  $\partial_p$  is the extension of  $\partial_2 := -[\cdot, \cdot]$  as coderivation on  $\wedge^p \mathfrak{g}$ .

The dual cochain complex  $\text{Hom}_{PS}(C(\mathfrak{g}), \mathbb{K}) = (\wedge^\bullet \mathfrak{g}^*, \delta)$  calculates cohomology<sup>2</sup>

$$E_n^* = \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K}) \cong H^n(\text{Hom}_{PS}(C(\mathfrak{g}), \mathbb{K})) = H^n(\mathfrak{g}, \mathbb{K}). \quad (9)$$

The coboundary map  $\delta^p : \wedge^p \mathfrak{g}^* \rightarrow \wedge^{p+1} \mathfrak{g}^*$  is transposed to the differential  $\partial_{p+1}$

$$\begin{aligned} \delta^p : e_{i_1 j_1}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{i_1}^* \wedge \dots \wedge e_{i_r}^* \mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1}^* \wedge \dots \wedge \hat{e}_{i_k j_k}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{i_k}^* \wedge e_{j_k}^* \wedge e_{i_1}^* \wedge \dots \wedge \dots \wedge e_{i_r}^*, \end{aligned} \quad (10)$$

it is (up to a conventional sign) the extension as derivation of the dualization of the Lie bracket  $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$ . Thus the algebra  $(\wedge^\bullet \mathfrak{g}^*, \delta)$  equipped with  $\delta$  is a (graded-)commutative DGA.

<sup>2</sup> In the presence of metric one has  $\delta := \partial^*$  (see below)

#### 4 Homology of $\mathfrak{g}$ as a $GL(V)$ -module

An irreducible polynomial  $GL(V)$ -module  $V_\lambda$  is called Schur module, it has a basis labelled by semistandard Young tableaux which are fillings of the Young diagram  $\lambda$  with the numbers of the set  $\{1, \dots, \dim V\}$ . The action of the linear group  $GL(V)$  on the space  $V$  of the generators of the Lie algebra  $\mathfrak{g}$  induces a  $GL(V)$ -action on the UEA  $PS = U\mathfrak{g} \cong S(V \oplus \wedge^2 V)$  and on the space  $\wedge^\bullet \mathfrak{g} \cong \wedge^\bullet(V \oplus \wedge^2 V)$ .

In the presence of metric  $g$  one has an identification  $V \xrightarrow{g} V^*$ , and  $\wedge^\bullet \mathfrak{g} \xrightarrow{g} \wedge^\bullet \mathfrak{g}^*$ . The adjoint operator  $\partial_p^* : \wedge^p \mathfrak{g} \rightarrow \wedge^{p+1} \mathfrak{g}$  is defined by  $g(\partial_p^* v, w) = g(v, \partial_{p+1} w)$ . It turns out that the action of  $\partial_p^*$  always takes the form (similar to the action of  $\delta^p$ )

$$\begin{aligned} \partial_p^* : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \dots \wedge e_{l_r} \mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \dots \wedge \hat{e}_{i_k j_k} \wedge \dots \wedge e_{i_s j_s} \otimes e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \dots \wedge e_{l_r}, \end{aligned} \quad (11)$$

It is obvious that the maps  $\partial$  and  $\partial^*$  both commute with the  $GL(V)$ -action. The Laplacian  $\Delta = \bigoplus_{p \geq 0} \Delta_p$  of the pair  $(\mathfrak{g}, g)$  is defined to be the self-adjoint operator

$$\Delta_p = \partial_{p+1} \partial_{p+1}^* + \partial_p^* \partial_p \in \text{End}(\wedge^p \mathfrak{g}).$$

Its kernel is a complete set of representatives for the homology classes in  $H_p(\mathfrak{g}, \mathbb{K})$

$$\ker \Delta_p \cong H_p(\mathfrak{g}, \mathbb{K}).$$

The decomposition of the  $GL(V)$ -module  $H_n(\mathfrak{g}, \mathbb{K})$  into irreducible polynomial representations  $V_\lambda$  is given by the following theorem;

**Theorem 1 (Józefiak and Weyman [6], Sigg [11]).** *The homology  $H_\bullet(\mathfrak{g}, \mathbb{K})$  of the 2-nilpotent Lie algebra  $\mathfrak{g} = V \oplus \wedge^2 V$  decomposes into irreducible  $GL(V)$ -modules*

$$H_n(\mathfrak{g}, \mathbb{K}) = H_n(\wedge^\bullet \mathfrak{g}, \partial) \cong \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda \quad (12)$$

where the sum is over self-conjugate Young diagrams  $\lambda$  such that  $n = \frac{1}{2}(|\lambda| + r(\lambda))$ .

The data  $H_n(\mathfrak{g}, \mathbb{K}) = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$  encodes the minimal free resolution  $P_\bullet$  (cf. (3)).

The Euler characteristics of  $P_\bullet$  implies an identity about the  $GL(V)$ -characters

$$\text{ch} PS(V) \cdot \text{ch} \left( \bigoplus_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} V_\lambda \right) = 1.$$

The character of a Schur module  $V_\lambda$  is the Schur function,  $\text{ch} V_\lambda = s_\lambda(x)$ . Due to the Poincaré-Birkhoff-Witt theorem  $\text{ch} PS(V) = \text{ch} S(V \oplus \wedge^2 V)$  thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i < j} \frac{1}{(1-x_i x_j)} \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) = 1. \quad (13)$$

But the latter identity is nothing but rewriting of the Littlewood identity [6]. The moral is that the Littlewood identity reflects a homological property of the algebra  $PS$ , namely the above particular structure of the minimal projective (free) resolution of  $\mathbb{K}$  by  $PS$ -modules.

## 5 Homotopy algebras $A_\infty$ and $C_\infty$

**Definition 2.** ( $A_\infty$ -algebra) A homotopy associative algebra, or  $A_\infty$ -algebra, over  $\mathbb{K}$  is a  $\mathbb{Z}$ -graded vector space  $A = \bigoplus_{i \in \mathbb{Z}} A^i$  endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \rightarrow A, \quad \deg(m_n) = 2 - n \quad n \geq 1$$

satisfying the Stasheff identities **SI**( $\mathbf{n}$ ) for  $n \geq 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad \mathbf{SI}(\mathbf{n})$$

where the sum runs over all decompositions  $n = r + s + t$ .

Here we assume the Koszul sign convention  $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$ . We define the shuffle product  $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \rightarrow A^{\otimes p+q}$  throughout the expression

$$(a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} \text{sgn}(\sigma) a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all  $(p, q)$ -shuffles  $Sh_{p,q}$ , i.e., over all permutations  $\sigma \in S_{p+q}$  such that  $\sigma(1) < \sigma(2) < \dots < \sigma(p)$  and  $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$ .

**Definition 3.** ( $C_\infty$ -algebra [7]) A homotopy commutative algebra, or  $C_\infty$ -algebra, is an  $A_\infty$ -algebra  $\{A, m_n\}$  such that each operation  $m_n$  vanishes on non-trivial shuffles

$$m_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, \quad 1 \leq p \leq n-1. \quad (14)$$

In particular for  $m_2$  we have  $m_2(a \otimes b \pm b \otimes a) = 0$ , so a  $C_\infty$ -algebra such that  $m_n = 0$  for  $n \geq 3$  is a (super-)commutative DGA.

A morphism of two  $A_\infty$ -algebras  $A$  and  $B$  is a family of graded maps  $f_n : A^{\otimes n} \rightarrow B$  for  $n \geq 1$  with  $\deg f_n = 1 - n$  such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = \sum_{1 \leq r \leq n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r})$$

where the sum is on all decompositions  $i_1 + \dots + i_r = n$  and the sign on RHS is determined by  $S = \sum_{k=1}^{r-1} (r-k)(i_k - 1)$ . The morphism  $f$  is a *quasi-isomorphism* of  $A_\infty$ -algebras if  $f_1$  is a quasi-isomorphism. It is strict if  $f_i = 0$  for all  $i \neq 1$ . The identity morphism of  $A$  is the strict morphism  $f$  such that  $f_1$  is the identity of  $A$ .

A morphism of  $C_\infty$ -algebras is a morphism of  $A_\infty$ -algebras vanishing on non-trivial shuffles  $f_n((a_1 \otimes \dots \otimes a_p) \sqcup (a_{p+1} \otimes \dots \otimes a_n)) = 0, 1 \leq p \leq n-1$ .

## 6 Homotopy Transfer Theorem

**Lemma 1.** *Every cochain complex  $(A, d)$  of vector spaces over a field  $\mathbb{K}$  has its cohomology  $H^\bullet(A)$  as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex  $(A, d)$  such that  $A^n \cong B^n \oplus H^n \oplus B^{n+1}$  where  $H^n$  is the cohomology and  $B^n$  is the space of coboundaries,  $B^n = dA^{n-1}$ . We choose a homotopy  $h : A^n \rightarrow A^{n-1}$  which identifies  $B^n$  with its copy in  $A^{n-1}$  and is 0 on  $H^n \oplus B^{n+1}$ . The projection  $p$  to the cohomology and the cocycle-choosing inclusion  $i$  given by  $A^n \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} H^n$  are chain homomorphisms (satisfying the additional conditions  $hh = 0$ ,  $hi = 0$  and  $ph = 0$ ). With these choices done the complex  $(H^\bullet(A), 0)$  is a deformation retract of  $(A, d)$

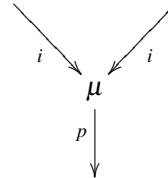
$$h \circlearrowleft (A, d) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H^\bullet(A), 0) , \quad pi = Id_{H^\bullet(A)} , \quad ip - Id_A = dh + hd .$$

Let now  $(A, d, \mu)$  be a DGA, i.e.,  $A$  is endowed with an associative product  $\mu$  compatible with  $d$ . The cochain complexes  $(A, d)$  and its contraction  $H^\bullet(A)$  are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on  $A$  can be transferred to an  $A_\infty$ -structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract  $H^\bullet(A)$ . For a friendly introduction to homotopy transfer theorems in much boarder context we send the reader to the textbook [9], see chapter 9.

**Theorem 2 (Kadeishvili [7]).** *Let  $(A, d, \mu)$  be a (commutative) DGA over a field  $\mathbb{K}$ . There exists a  $A_\infty$ -algebra ( $C_\infty$ -algebra) structure on the cohomology  $H^\bullet(A)$  and a  $A_\infty(C_\infty)$ -quasi-isomorphism  $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \rightarrow (A, \{d, \mu, 0, 0, \dots\})$  such that the inclusion  $f_1 = i : H^\bullet(A) \rightarrow A$  is a cocycle-choosing homomorphism of cochain complexes. The differential  $m_1$  on  $H^\bullet(A)$  is zero ( $m_1 = 0$ ) and  $m_2$  is strictly associative operation induced by the multiplication on  $A$ . The resulting structure is unique up to quasi-isomorphism.*

Kontsevich and Soibelman [12] gave an explicit expressions for the higher operations of the induced  $A_\infty$ -structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion  $i$ , the root by the projection  $p$  the vertices by the product  $\mu$  of the (commutative) DGA  $(A, d, \mu)$  and the internal edges by the homotopy  $h$ . The  $C_\infty$ -structure implies additional symmetries on trees. We will make use of the graphic representation for the binary operation on  $H^\bullet(A)$

$$m_2(x, y) := p\mu(i(x), i(y)) \quad \text{or} \quad m_2 =$$





and the ternary one  $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$  being the sum of two planar binary trees with three leaves

$$m_3 = \begin{array}{c} \begin{array}{ccc} & & \\ & i & \\ & \swarrow & \searrow \\ & \mu & \\ & \swarrow & \searrow \\ i & & i \\ \swarrow & & \searrow \\ & \mu & \\ & \swarrow & \searrow \\ & h & \\ & \mu & \\ & \swarrow & \searrow \\ & p & \\ & \downarrow & \end{array} & - & \begin{array}{ccc} & & \\ & i & \\ & \swarrow & \searrow \\ & \mu & \\ & \swarrow & \searrow \\ i & & i \\ \swarrow & & \searrow \\ & \mu & \\ & \swarrow & \searrow \\ & h & \\ & \mu & \\ & \swarrow & \searrow \\ & p & \\ & \downarrow & \end{array} \end{array}$$

**Theorem 3.** *The cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$  of the 2-nilpotent graded Lie algebra  $\mathfrak{g} = V \otimes \wedge^2 V$  is a homotopy commutative algebra which is generated in degree 1 (i.e., in  $H^1(\mathfrak{g}, \mathbb{K})$ ) by the operations  $m_2$  and  $m_3$ .*

**Sketch of the proof.** Let us choose a metric  $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$  on the vector space  $V$  and an orthonormal basis  $\langle e_i, e_j \rangle = \delta_{ij}$ . The choice induces a metric on  $\wedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \wedge^\bullet \mathfrak{g}^*$ .

Due to the isomorphisms  $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})$  (see eq. 6) and  $V \cong V^*$  the theorem 1 implies the decomposition of  $H^\bullet(\mathfrak{g}, \mathbb{K})$  into irreducible  $GL(V)$ -modules

$$H^n(\mathfrak{g}, \mathbb{K}) \cong H^n(\wedge \mathfrak{g}^*, \delta) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})(V^*) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams  $\lambda$  such that  $n = \frac{1}{2}(|\lambda| + r(\lambda))$ .

In the presence of metric  $g$  the differential  $\delta$  is identified with the adjoint of  $\partial$ ,  $\delta \stackrel{g}{=} \partial^*$  while  $\partial$  plays the role of a homotopy. In view of lemma 1 we have the cohomology  $H^\bullet(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$  as deformation retract of the complex  $(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ ,

$$pi = Id_{H^\bullet(\wedge^\bullet \mathfrak{g}^*)}, \quad ip - Id_{\wedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta, \quad \delta^* \stackrel{g}{=} \partial.$$

Here the projection  $p$  identifies the subspace  $\ker \delta \cap \ker \delta^*$  with  $H^\bullet(\wedge^\bullet \mathfrak{g}^*)$ , which is the orthogonal complement of the space of the coboundaries  $\text{im} \delta$ . The cocycle-choosing homomorphism  $i$  is  $Id$  on  $H^\bullet(\wedge^\bullet \mathfrak{g}^*)$  and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 2 for the commutative DGA  $(\wedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$  and its deformation retract  $H^\bullet(\wedge^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g}, \mathbb{K})$  and conclude that the cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K})$  is a  $C_\infty$ -algebra.

The Kontsevich and Soibelman tree representations of the operations  $m_n$  provide explicit expressions. Let us take  $\mu$  to be the super-commutative product  $\wedge$  on the DGA  $(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ . The projection  $p$  maps onto the Schur modules  $V_\lambda$  with  $\lambda = \lambda'$ .

The binary operation on the degree 1 generators  $e_i \in H^1(\mathfrak{g}, \mathbb{K})$  is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(12)}) = 0.$$

Hence  $H^\bullet(\mathfrak{g}, \mathbb{K})$  could not be generated in  $H^1(\mathfrak{g}, \mathbb{K})$  as algebra with product  $m_2$ .

The ternary operation  $m_3$  restricted to  $H^1(\mathfrak{g}, \mathbb{K})$  is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i, e_j, e_k) &= p \{ e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k \} = p \{ e_{ij} \wedge e_k - e_i \wedge e_{jk} \} \\ &= p \{ (e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) - e_{ki} \wedge e_j \} = e_{ik} \wedge e_j \in H^2(\mathfrak{g}, \mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets (...) spans the Schur module  $V_{(1^3)}$ ,  $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$  yields a Jacobi-type identity. The monomials  $e_{ij} \wedge e_k$  modulo  $V_{(1^3)}$  span a Schur module  $V_{(2,1)} \in H^2(\mathfrak{g}, \mathbb{K})$  with basis in bijection with the semistandard Young tableaux  $e_{ik} \wedge e_j \leftrightarrow \begin{array}{|c|c|} \hline i & j \\ \hline k & \\ \hline \end{array}$  and  $e_{ij} \wedge e_k \leftrightarrow \begin{array}{|c|c|} \hline i & k \\ \hline j & \\ \hline \end{array}$ .

We check the symmetry condition on ternary operation  $m_3$  in  $C_\infty$ -algebra; indeed  $m_3$  vanishes on the (signed) shuffles  $Sh_{1,2}$  and  $Sh_{2,1}$

$$m_3(e_i \sqcup e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0 = m_3(e_i \otimes e_j \sqcup e_k).$$

It is important that in the complexes  $(\wedge^p \mathfrak{g}, \partial_p)$  and  $(\wedge^p \mathfrak{g}^*, \delta^p)$  two different degrees are involved; one is the homological degree  $p := r + s$  counting the number of  $\mathfrak{g}$ -generators, while the second is the tensor degree  $t := 2s + r$  (also called weight). The differentials  $\partial$  and  $\delta$  preserve the tensor degree  $t$  but the spaces  $H_n(\mathfrak{g}, \mathbb{K})$  and  $H^n(\mathfrak{g}, \mathbb{K})$  are not homogeneous in  $t$ . The operation  $m_n$  is bigraded by homological and tensor gradings of bidegree  $(p, t) = (2 - n, 0)$ . The bi-grading impose the vanishing of many higher products.

On the level of Schur modules the ternary operation glues three fundamental  $GL(V)$ -representations  $V_\square$  into a Schur module  $V_{(2,1)}$ . By iteration of the process of gluing boxes we generate all elementary hooks  $V_k := V_{(k+1, 1^k)}$ ,

$$m_3(V_\square, V_\square, V_\square) = V_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}, \quad m_3 \left( V_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}, V_\square, V_\square \right) = V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}}, \dots, m_3(V_k, V_0, V_0) = V_{k+1}.$$

In our context the more convenient notation for Young diagrams is due to Frobenius:  $\lambda := (a_1, \dots, a_r | b_1, \dots, b_r)$  stands for a diagram  $\lambda$  with  $a_i$  boxes in the  $i$ -th row on the right of the diagonal, and with  $b_i$  boxes in the  $i$ -th column below the diagonal and the rank  $r = r(\lambda)$  is the number of boxes on the diagonal.

For self-dual diagrams  $\lambda = \lambda'$ , i.e.,  $a_i = b_i$  we set  $V_{a_1, \dots, a_r} := V_{(a_1, \dots, a_r | a_1, \dots, a_r)}$  when  $a_1 > a_2 > \dots > a_r \geq 0$  (and set the convention  $V_{a_1, \dots, a_r} := 0$  otherwise). Any two elementary hooks  $V_{a_1}$  and  $V_{a_2}$  can be glued together by the binary operation  $m_2$ , the decomposition of  $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$  is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1, a_2} \oplus \left( \bigoplus_{i=1}^{a_2} V_{a_1+i, a_2-i} \right) \quad a_1 \geq a_2$$

where the ‘‘leading’’ term  $V_{a_1, a_2}$  has the diagram with minimal height. Hence any  $m_2$ -bracketing of the hooks  $V_{a_1}, V_{a_2}, \dots, V_{a_r}$  yields<sup>3</sup> a sum of  $GL(V)$ -modules

<sup>3</sup> The operation  $m_2$  is associative thus the result does not depend on the choice of the bracketing.

$$m_2(\dots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \dots, V_{a_r}) = V_{a_1, \dots, a_r} \oplus \dots$$

whose module with minimal height is precisely  $V_{a_1, \dots, a_r}$ . We conclude that all elements in the  $C_\infty$ -algebra  $H^\bullet(\mathfrak{g}, \mathbb{K})$  can be generated in  $H^1(\mathfrak{g}, \mathbb{K})$  by  $m_2$  and  $m_3$ .  
□

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