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# IFS attractors and Cantor sets

Sylvain Crovisier\* and Michał Rams†

## Abstract

We build a metric space which is homeomorphic to a Cantor set but cannot be realized as the attractor of an iterated function system. We give also an example of a Cantor set  $K$  in  $\mathbb{R}^3$  such that every homeomorphism  $f$  of  $\mathbb{R}^3$  which preserves  $K$  coincides with the identity on  $K$ .

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## 1 Introduction

We are interested in the problem of characterization of compact sets that are limit sets of hyperbolic dynamical systems. In this paper we study iterated function systems that are simplified models for the smooth hyperbolic dynamics. Previous works (see [DH, K]) investigated which compact metric spaces can be attractors of iterated function systems on some euclidean space. We would like to carry on this discussion with the following question:

*Is every Cantor set an attractor of some iterated function system?*

Let us first recall that a continuous mapping  $f$  of a metric space  $(X, d)$  into itself is a *contractive map* if there exists a constant  $\sigma \in (0, 1)$  such that for any points  $x, y$  in  $X$ , the distance  $d(f(x), f(y))$  is less or equal to  $\sigma d(x, y)$ . An *iterated function system* (or *IFS*) is a finite family of contractive maps  $\{f_1, \dots, f_s\}$  acting on a complete metric space. It is well known (see [H]) that such a dynamics possesses a unique compact set  $K \subset X$  which is non-empty and invariant by the IFS, in the following sense:

$$K = \cup_{i=1}^s f_i(K).$$

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One calls this compact set the limit set or the *attractor* of the IFS.

As an example, the middle one third Cantor set is the attractor of the IFS  $\{f_1, f_2\}$  on  $\mathbb{R}$  defined by

$$f_1(x) = x/3, \quad f_2(x) = (x + 2)/3.$$

Our first result shows that for some other metrics, the Cantor set is no more the attractor of an IFS.

**Theorem 1.1.** *There exists a Cantor set  $X_1$  and a Borel probability measure  $\mu$  supported on  $X_1$  such that for every contractive map  $f: X_1 \rightarrow X_1$ , we have  $\mu(f(X_1)) = 0$ .*

This set  $X_1$  cannot be an attractor of an iterated function system  $\{f_1, \dots, f_s\}$  (even if one allows countably many maps in the definition of the IFS) since  $X_1$  has full measure by  $\mu$  but  $f_1(X_1) \cup \dots \cup f_s(X_1)$  has zero measure by  $\mu$ . The example may be generalized:

**Corollary 1.2.** *For any iterated function system on a complete metric space  $(X, d)$ , the attractor  $K$  is not isometric to the Cantor set  $X_1$ , defined in Theorem 1.1.*

In the previous case, the obstruction was metrical. If one specifies the Cantor set  $K$  and the ambient space  $X$  there may exist also topological obstructions for  $K$  to be an attractor of an IFS defined on  $X$ , even if  $X$  is a smooth manifold such as  $\mathbb{R}^d$ .

**Theorem 1.3.** *There exists a Cantor set  $X_2 \subset \mathbb{R}^3$  such that if  $f$  is a homeomorphism of  $\mathbb{R}^3$  which satisfies  $f(X_2) \subset X_2$  then  $f|_{X_2} = \text{id}$ .*

In particular a finite set of homeomorphisms of  $\mathbb{R}^3$  can not be an IFS whose attractor is  $X_2$ . The set  $X_2$  here is a variation on Antoine's necklace.

This example can be easily generalized to higher dimensions but in dimension one or two the situation is completely different:

**Proposition 1.4.** *For any Cantor set  $X \in \mathbb{R}$  (or in  $\mathbb{R}^2$ ) and any two points  $x, y \in X$ , there exists a contractive homeomorphism  $f: \mathbb{R} \rightarrow \mathbb{R}$  (or  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ) such that  $f(X) \subset X$  and  $f(x) = y$ .*

## 2 Constructions

Given  $Y$ , subset of a metric space  $X$  we denote its complement by  $(Y)^c$ , its interior  $\text{Int}(Y)$ , its boundary  $\partial(Y)$  and (in the case  $Y$  is bounded) its diameter  $\text{diam}(Y)$ . We will denote by  $B(z, r)$  the open ball centered at  $z \in X$  with radius  $r$ .

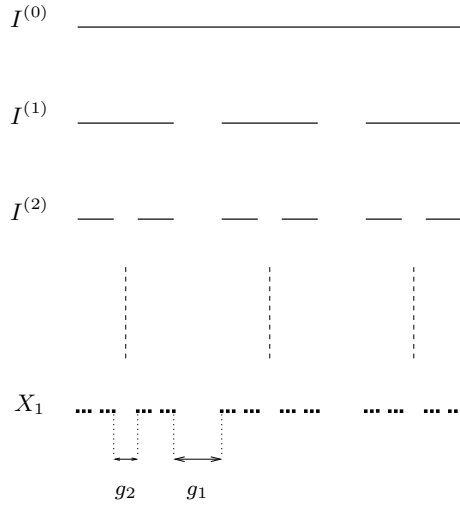


Figure 1: Construction of  $X_1$

## 2.1 Proof of Theorem 1.1

### 2.1.1 Definition of the Cantor set $X_1$ and the measure $\mu$

The Cantor set  $X_1$  is obtained as the intersection of a decreasing sequence  $(I^{(k)})$  of compact sets in  $\mathbb{R}$ . Each set  $I^{(k)}$  is a finite union of pairwise disjoint compact intervals  $I_i^{(k)}$  that have the same length.

We will construct inductively the sequence  $(I^{(k)})$ . The first set  $I^{(0)} = [0, 1]$ .

Given  $I^{(k)} = \bigcup I_j^{(k)}$ , we choose  $n_{k+1}$  closed intervals  $I_i^{(k+1)}$  inside every  $I_j^{(k)}$ . We demand those intervals to be pairwise disjoint and of equal length, that their union contains the endpoints of  $I_j^{(k)}$  and also that the gaps between them are of equal length  $g_{k+1}$ . The set  $I^{(k+1)}$  will be the union of all  $I_i^{(k+1)}$ .

Obviously, the bounded components of  $\mathbb{R} \setminus I^{(k+1)}$  are either gaps created at the previous steps of the construction or new gaps of size  $g_{k+1}$  each.

We take care along the induction that in each consecutive step the number of intervals  $n_k$  increases while the size of gaps  $g_k$  decreases. We define  $X_1 = \bigcap_k I^{(k)}$  which is obviously a Cantor subset of  $\mathbb{R}$ , see the figure 1.

We define the measure  $\mu$  as to be equidistributed: all the intervals  $I_i^{(k)}$  of a given level  $k$  have the same measure, equal to the reciprocal of the total number of intervals of that level, ie.  $\mu(I_i^{(k)}) = \prod_{j=1}^k n_j^{-1}$ . By the Kolmogorov theorem ([B2]) it uniquely defines a probability measure supported on  $X_1$ .

### 2.1.2 Measure of $f(X_1)$

Now, we prove that  $X_1$  satisfies the assertion of Theorem 1.1: let  $f$  be a contraction from  $X_1$  into itself and let  $X_i^{(k)} = X_1 \cap I_i^{(k)}$  be the part of  $X_1$  contained in one of the  $k$ -th level intervals  $I_i^{(k)}$ . We claim that  $f(X_i^{(k)})$  must be contained in some  $(k+1)$ -th level interval.

Assuming it is not the case,  $f(X_i^{(k)})$  intersects at least two  $(k+1)$ -th level intervals. Since  $(g_n)$  is decreasing, these intervals are in distance at least  $g_{k+1}$  from each other. Hence,  $f(X_i^{(k)})$  may be divided onto two subsets  $A, B$  such that the distance between any point from  $A$  and any point from  $B$  is not smaller than  $g_{k+1}$ . As the map  $f$  is contracting, the preimages of  $A$  and  $B$  (covering together whole  $X_i^{(k)}$ ) must lie in distance strictly greater than  $g_{k+1}$ . But if we could divide  $X_i^{(k)}$  into two such subsets, this would imply the existence of a gap inside  $X_i^{(k)}$  of size strictly greater than  $g_{k+1}$ . By construction, such a gap would be created at a step  $\ell$  larger or equal to  $k+1$  and would be of length  $g_\ell$ . The sequence  $(g_n)$  would not be strictly decreasing. This contradiction proves the claim.

Hence, as there are  $\prod_{j=1}^k n_j$  intervals of level  $k$  in  $I^{(k)}$ , the whole set  $f(X_1)$  is contained in the union of at most  $\prod_{j=1}^k n_j$   $(k+1)$ -th level intervals. This implies

$$\mu(f(X_1)) \leq \prod_{i=1}^k n_i \prod_{j=1}^{k+1} n_j^{-1} = n_{k+1}^{-1}$$

and the right hand side tends to zero when  $k$  tends to infinity.  $\square$

## 2.2 Proof of Theorem 1.3

### 2.2.1 Definition of the Cantor set $X_2$

The construction is a variation on Antoine's necklace example built in  $[A_1, A_2]$  (see also for example [M], chapter 18 or [B1], section IV.7 for more details).

We start from a filled compact torus  $T^{(0)} \subset \mathbb{R}^3$  that will also be noted  $T_\emptyset$ . In the first step we find in the interior of  $T^{(0)}$  some number (say,  $n = n_\emptyset > 2$ ) of disjoint compact tori  $T_1, \dots, T_n$  linked together to form a closed chain going around the torus  $T^{(0)}$ , see figure 2. They will be called first level tori. We denote their union by  $T^{(1)}$ . In the second step we take inside each of the first level tori  $T_i$  another chain of smaller tori (called second level tori) going around (of  $n_1$  elements inside  $T_1$ ,  $n_2$  inside  $T_2$  and so on up to  $n_{n_\emptyset}$ ). They will be denoted by  $T_{i,1}, \dots, T_{i,n_i}$  and their union by  $T^{(2)} \subset T^{(1)}$ .

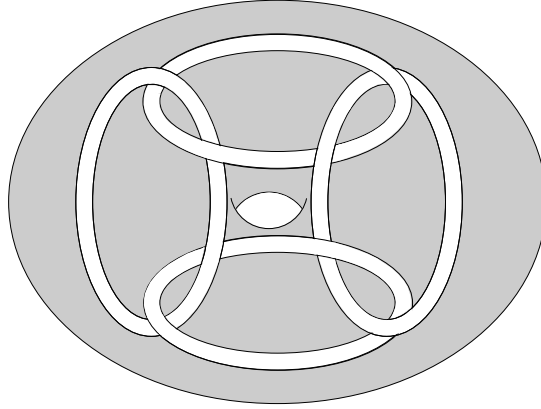


Figure 2: The first sets  $T^{(0)}$  and  $T^{(1)}$  (here we have  $n_\emptyset = 4$ )

We repeat the procedure inductively, taking care that the diameters of the tori we use in the construction go down to 0 and that, at any step, the lengths of the chains are greater than 2, different from each other, and different from the lengths of all other chains introduced at the previous steps (in Antoine's example one takes chains of length four at every step). All the tori of level  $k$  are described by finite sequences  $\omega^k \in \Sigma$  of length  $k$  where:

$$\Sigma = \bigcup_{k \in \mathbb{N}} \{ \omega^k = (\omega_1, \dots, \omega_k) \in \mathbb{N}^k; \forall_{j \leq k} 1 \leq \omega_j \leq n_{\omega^{k-1}} \}.$$

(We denoted by  $\omega^k$  a sequence of length  $k$ , by  $\omega_j$  the  $j$ -th element of the sequence and by  $\omega^j$  the subsequence formed by the  $j$  first terms in  $\omega^k$ .)

We define

$$X_2 = \bigcap_k T^{(k)}.$$

As one can easily check from the construction,  $X_2$  is a Cantor set.

We introduce also the *rings of level  $k$*  in  $X_2$  defined by

$$Y_{\omega^k} = X_2 \cap T_{\omega^k}.$$

Let us remark that the sets  $Y_{\omega^k}$  are open and closed in  $X_2$ . They define a basis for the topology on  $X_2$ .

A *chain  $C$*  in  $X_2$  is a union of rings of same level:

$$C = Y_{\omega^k, i_1} \cup Y_{\omega^k, i_2} \cup \dots \cup Y_{\omega^k, i_r}$$

where  $\omega^k$  belongs to  $\Sigma$  and where  $\{i_1, \dots, i_r\}$  is an interval in  $\mathbb{Z}/n_{\omega^k}\mathbb{Z}$ .

### 2.2.2 Topological properties of $X_2$

Let  $Y$  be a compact set of  $\mathbb{R}^3$ . We will say that  $Y$  *may be cleaved* if there exists a decomposition  $Y = A_1 \cup A_2$  of  $Y$  in two compact disjoint and non empty sets and some isotopy of homeomorphisms  $(h_t)_{t \in [0,1]}$  of such that  $h_0 = \text{id}$  and such that  $h_1(A_1)$  and  $h_1(A_2)$  are contained in two disjoint euclidean balls of  $\mathbb{R}^3$ .

We prove in this section the following topological characterization of rings of  $X_2$ :

**Proposition 2.1.** *A compact subset  $Y$  of  $X_2$  is a ring of  $X_2$  if and only if it satisfies the following properties:*

- a)  $Y$  can not be cleaved.
- b)  $Y$  is “cyclic”: there exists a partition of  $Y$  in three disjoint compact non empty subsets  $Y = A_1 \cup A_2 \cup A_3$  such that each  $A_i$  and each union  $A_i \cup A_j$  can not be cleaved.

We first check that rings satisfy these properties.

**Lemma 2.2.** *No chain  $C$  of  $X_2$  may be cleaved.*

In particular the rings can not be cleaved.

*Proof.* In  $[A_1, A_2]$  (see also [M], problem 18.2), Antoine proved that any 2-sphere in  $\mathbb{R}^3$  that do not intersect the Antoine’s necklace can not separate two points of the necklace from one another. By the same argument, this property is also satisfied by any chain of  $X_2$ . This implies the lemma.  $\square$

We then check that rings are “cyclic” (i.e. satisfy Proposition 2.1 b)): let  $Y_{\omega^k}$  be a ring of  $X_2$ . We set  $A_1 = Y_{\omega^k,1}$ ,  $A_2 = Y_{\omega^k,2}$  and  $A_3 = Y_{\omega^k} \setminus (A_1 \cup A_2)$ . All the sets  $A_i$  or  $A_i \cup A_j$  are chains of  $X_2$  and can not be not cleaved from lemma 2.2, what was to be shown.

The other part of the proof of Proposition 2.1 uses the following lemma.

**Lemma 2.3.** *Let  $Y$  be a compact and proper subset of some ring  $Y_{\omega^k}$  in  $X_2$ . Then, there exists an isotopy  $(h_t)_{t \in [0,1]}$  in the space of homeomorphisms such that  $h_t$  coincides with the identity on  $(\text{Int}(T_\omega))^c$ ,  $h_0 = \text{id}$  and  $h_1$  sends  $Y$  on a small euclidean ball included in  $\text{Int}(T_\omega)$ .*

*Proof.* This lemma is clear when  $Y$  is a chain  $Y_{\omega^k,i_1}, \dots, Y_{\omega^k,i_r}$  of  $T_{\omega^k}$ . In the general case, we note that  $Y_{\omega^k} \setminus Y$  is open in  $Y_{\omega^k}$  and contained in some ring. Hence,  $Y$  is contained in a finite union of pairwise disjoint rings that is strictly included in  $Y_{\omega^k}$ . One observes that it is enough to prove the assertion

for such union of rings, it will imply the result for  $Y$ . Note that such a union of rings is a finite union of pairwise disjoint chains not linked to each other.

The proof is then done by induction on the number of chains : from the remark we did at the beginning of this proof, one starts by shrinking the chains of largest order inside small disjoint balls. This allows us to consider the chains of next largest order and to shrink them. Repeating this procedure, one concludes the proof.  $\square$

We get a counterpart to lemma 2.2

**Lemma 2.4.** *Let  $Y$  be a closed subset of  $X_2$ . If  $Y$  can not be cleaved and contains at least two points, then it is a chain of  $X_2$ .*

*Proof.* Let  $Y$  be a closed subset of  $X_2$  that contains at least two points and can not be cleaved. The proof that  $Y$  is a chain has three steps.

**Step 1:** *Let  $Y_{\omega^k}$  be any ring of  $X_2$ , such that  $Y$  intersects both  $Y_{\omega^k}$  and  $(Y_{\omega^k})^c$ . Then,  $Y_{\omega^k}$  is contained in  $Y$ .*

We prove this fact by contradiction. If  $Y_{\omega^k}$  is not contained in  $Y$ , one can apply Lemma 2.3: the set  $A_1 = Y \cap Y_{\omega^k}$  may be contracted to a small euclidean ball contained in  $\text{Int}(T_{\omega^k})$ . Since  $(Y)^c$  is connected, one can then send this ball outside any bounded neighborhood  $U$  of  $T^{(0)}$ , by an isotopy that fixes the closed set  $Y \setminus A_1$ . The closed set  $A_2 = Y \setminus A_1$  is not empty by assumption and it is possible to contract it in a small euclidean ball through an isotopy whose support is contained in  $U$ . This shows that  $Y$  can be cleaved. This is a contradiction and the fact is proved.

**Step 2:** *There exists a word  $\omega^k$  such that  $Y$  is a union of some rings of the form  $Y_{\omega^k, \ell}$  (with the same level  $k + 1$ ).*

Let  $z_1$  be any point in  $Y$ . By assumption,  $Y$  contains at least an other point  $z_2$ . By construction of  $X_2$ , there exists a ring  $Y_{\omega^k}$  that contains  $z_1$  and not  $z_2$ . By the first step,  $Y_{\omega^k}$  is included in  $Y$ . We hence proved that  $Y$  is a union of rings of  $X$ .

Let us note that, for any two rings  $A_1$  and  $A_2$  of  $X_2$ , either they are disjoint or one is contained in the other. Consequently,  $Y$  is a disjoint union of rings that are maximal in  $Y$  for the inclusion.

Let  $A \subset Y$  be one of these maximal rings and let  $A'$  be the ring of  $X_2$  which contains  $A$  and was built at the previous level. Hence,  $A'$  is not contained in  $Y$ . By the first step of this proof, one deduces that  $Y$  is contained in  $A'$ . Let us denote  $A'$  by  $T_{\omega^k}$ . Repeating this argument with any maximal subring of  $Y$ , one deduces that  $Y$  is a union of rings of the form  $Y_{\omega^k, i}$ . This proves the second step.

**Step 3:** *The set  $Y$  is a chain.*

From the second step,  $Y$  decomposes as an union of chains made of rings of



the form  $Y_{\omega^k, i}$ . If  $Y$  is not a chain, it decomposes as chains  $C_1, \dots, C_r$ , that are not linked together. Hence, one can contract  $C_1$  and  $C_2 \cup \dots \cup C_r$  in two disjoint euclidean balls. This is a contradiction since  $Y$  can not be not cleaved. Hence,  $Y$  is a single chain.  $\square$

*End of the proof of Proposition 2.1.* Let  $Y$  be a closed subset of  $X_2$  that satisfies the properties of Proposition 2.1.

By the second property,  $Y$  decomposes as a union  $A_1 \cup A_2 \cup A_3$ . By Lemma 2.4, all the sets  $A_i$  are chains of  $X_2$ . By the same lemma, the unions  $A_i \cup A_j$  are chains as well, hence  $A_1, A_2$  and  $A_3$  are unions of rings of the form  $Y_{\omega^k, i}$  contained in a same ring  $Y_{\omega^k}$ . But since they are disjoint the only way that  $A_1 \cup A_2, A_2 \cup A_3$  and  $A_3 \cup A_1$  are all chains of  $X_2$  is that their union is the ring  $Y_{\omega^k}$ . This shows that  $Y$  is a ring and concludes the proof.  $\square$

### 2.2.3 Rigidity of $X_2$ under homeomorphisms

We now end the proof of Theorem 1.3.

Let  $f$  be any homeomorphism from  $\mathbb{R}^3$  onto itself such that  $f(X_2) \subset X_2$ . From the Proposition 2.1, the image of any ring  $Y_{\omega^k}$  by  $f$  is a ring  $Y_{\tau^j}$ .

**Lemma 2.5.** *Let  $Y_{\omega^k}$  be a ring of  $X_2$  and  $Y_{\tau^j}$  its image by  $f$ . The images of the subrings  $Y_{\omega^k, i}$  of level  $k + 1$  of  $Y_{\omega^k}$  are subrings  $Y_{\tau^j, \ell}$  of level  $j + 1$  of  $Y_{\tau^j}$ .*

*Proof.* Let us assume that it is not the case. There would exists a ring  $Y$  of level  $j + 1$  in  $Y_{\tau^j}$  which contains strictly the image of some subring  $Y_{\omega^k, i}$  of  $Y_{\omega^k}$ . In other words, the preimage  $f^{-1}(Y)$  is a proper subset of  $Y_{\omega^k}$  but is larger than the subrings of level  $k + 1$  of  $Y_{\omega^k}$ . However, by Proposition 2.1, the set  $f^{-1}(Y)$  is a ring of  $X_2$ . This is a contradiction.  $\square$

By this lemma, the rings  $Y_{\omega^k}$  and  $Y_{\tau^j}$  decompose in the same number of subrings of next level so that  $n_{\omega^k} = n_{\tau^j}$ . By the construction, the lengths of the closed chains of tori in  $X_2$  are all different. This implies that  $Y_{\omega^k} = Y_{\tau^j}$ . Since any point  $z \in X_2$  is the intersection of some family of rings  $Y_{\omega^k}$  (with  $k \nearrow \infty$ ), it is fixed by the map  $f$ . Hence, the restriction of  $f$  to  $X_2$  is the identity. This concludes the proof of Theorem 1.3.

## 2.3 Proof of Proposition 1.4

Let  $X$  be a Cantor set on the plane (the proof for the Cantor set on the line is similar but simpler and will be left for the reader as an exercise). Let  $\Sigma = \bigcup_{n=0}^{\infty} \{0, 1\}^n$  be the space of all finite zero-one sequences.

### 2.3.1 Preliminary constructions

We start by the construction of the covering of  $X$  with a family of topological closed balls  $B_{\omega^n}$  with  $\omega^n \in \Sigma$ , satisfying the following properties:

- i)  $\forall_{i \in \{0,1\}} B_{\omega^n, i} \subset \text{Int } B_{\omega^n}$ ,
- ii)  $\forall_{\omega^n \in \Sigma} \forall_{i \in \{0,1\}} B_{\omega^n, 0} \cap B_{\omega^n, 1} = \emptyset$ ,
- iii)  $\forall_{n \geq 0} X \subset \bigcup_{\omega^n \in \{0,1\}^n} \text{Int } B_{\omega^n}$ ,
- iv)  $\forall_{\omega^n \in \Sigma} B_{\omega^n} \cap X \neq \emptyset$ ,
- v)  $\lim_{n \rightarrow \infty} \sup_{\omega^n \in \{0,1\}^n} \text{diam } B_{\omega^n} = 0$ .

We can easily do this construction for some special Cantor sets (like the usual middle one third Cantor set on a line, contained in the plane). As any two embeddings of Cantor sets in the plane are equivalent with respect to plane homeomorphisms (see [M], chap. 13, theorem 6, p. 93), we get the construction for  $X$ . As the distance from  $X$  to the boundary of any  $B_{\omega^n}$  and the distance between boundaries of  $B_{\omega^n}$  and  $B_{\omega^{n_i}}$  is non-zero, we may freely assume that the boundary of all balls  $B_{\omega^n}$  is  $C^1$ .

Let  $B^{(n)} = \bigcup_{\omega^n \in \{0,1\}^n} B_{\omega^n}$ . By i), they form a decreasing sequence, by iii), iv) and v), we have  $X = \bigcap_n B^{(n)}$ .

In the course of the proof, we will construct inductively another family of topological closed balls (even euclidean balls, but it is not really necessary)  $C_{\omega^n}$  and define the contractive map  $F$  in such a way that  $F((B_{\omega^n})^c) = (C_{\omega^n})^c$ . The sets  $C_{\omega^n}$  will satisfy the properties i), ii), iv) and v) above.

We define the sets  $C^{(n)}$  similarly to  $B^{(n)}$ : it is a decreasing family as well but its limit is only some Cantor subset of  $X$ .

Let us note that the construction of the family  $B_{\omega^n}$  is precisely the point where the proof fails in dimension higher than two. In fact for the Cantor sets in  $\mathbb{R}^3$  for which such a construction is still possible, the rest of the proof follows with minor modifications. These Cantor sets can be easily seen as images of a Cantor set on the line  $\mathbb{R}^1 \subset \mathbb{R}^3$  under homeomorphisms of  $\mathbb{R}^3$ .

### 2.3.2 A geometric lemma

We will use the following lemma:

**Lemma 2.6.** *Let  $A, A_0, A_1$  be closed topological balls with  $C^1$  boundary, such that  $A_0$  and  $A_1$  are disjoint and contained in  $\text{Int}(A)$ . Let  $r$  be a positive real number. Let  $z, z_0, z_1$  be three points such that  $z_0, z_1 \in U = B(z, r)$ .*

Then, there exists a constant  $k(A, A_0, A_1, U, z_0, z_1)$  with the following property: for any Lipschitz homeomorphism  $g : \partial A \rightarrow \partial B(z, r)$  with Lipschitz constant  $L$  and for all sufficiently small  $r_0, r_1$  one can find a map  $h : A \setminus \text{Int}(A_0 \cup A_1) \rightarrow B(z, r) \setminus \text{Int}(B(z_0, r_0) \cup B(z_1, r_1))$  such that:

- a)  $h|_{\partial A} = g$ ,
- b)  $h$  is a homeomorphism,  $h(\partial A_i) = \partial B(z_i, r_i)$ ,
- c)  $h$  is Lipschitz and its Lipschitz constant is not greater than

$$L' = L \cdot k(A, A_0, A_1, U, z_0, z_1),$$

- d) the Lipschitz constant of  $h|_{\partial A_i}$  is not greater than

$$L'' = L \cdot r_i \cdot k(A, A_0, A_1, U, z_0, z_1).$$

Note that the constant  $k(A, A_0, A_1, U, z_0, z_1)$  will not change if we rescale the triple  $\{U, z_0, z_1\}$  by a similitude or exchange  $z_0$  with  $z_1$ .

*Proof.* As all the possible triples  $(A, A_0, A_1)$  are bi-Lipschitz equivalent, it is enough to prove the lemma for  $A = B(z, r)$ ,  $A_0 = B(z_0, \rho)$ ,  $A_1 = B(z_1, \rho)$  where  $\rho$  is the greatest such number that  $B(z_0, 4\rho)$  and  $B(z_1, 4\rho)$  are disjoint and contained in  $B(z, r)$ . Similarly, we may assume that  $r = 1$ ,  $z = (0, 0)$  (the rescaling changes  $L, L'$  and  $L''$  by the same multiplicative constant) and that  $g$  is orientation preserving, ie.

$$g(\cos \phi, \sin \phi) = (\cos g_0(\phi), \sin g_0(\phi))$$

for some orientation-preserving homeomorphism  $g_0$  of  $S^1$ . Note that now  $L \geq 1$ . We assume  $r_0, r_1 < \rho$ .

We define  $h$  as follows:

- on the annulus  $1 \geq |x| \geq 1 - r$  by the formula

$$h(a \cos \phi, a \sin \phi) = \left( a \cos \left( \frac{1-a}{r} \phi + \frac{a+r-1}{r} g_0(\phi) \right), a \sin \left( \frac{1-a}{r} \phi + \frac{a+r-1}{r} g_0(\phi) \right) \right).$$

- on the set  $B((0, 0), 1 - r) \setminus (B(z_0, 2r) \cup B(z_1, 2r))$  as the identity,
- on  $B(z_i, 2r)$  by the formula

$$h(z_i + a \cdot (\cos \phi, \sin \phi)) = z_i + \left( \frac{2r - r_i}{r} a + 2r_i - 2r \right) \cdot (\cos \phi, \sin \phi).$$

This map satisfies a) and b), its Lipschitz constant is not greater than  $L$  in the first part, 1 in the second one and 2 in the third one (this gives c)) and its Lipschitz constant on  $\partial A_i$  is not greater than  $r_i/r$  (we get d)).  $\square$

### 2.3.3 End of the constructions

We are now prepared to construct inductively the sets  $C_{\omega^n}$  and the contractive homeomorphism  $F$  on each set  $\text{Int}(C^{(n)}) \setminus \text{Int}(C^{(n+1)})$ : we consider the Cantor set  $X$ , two points  $x$  and  $y$  belonging to  $X$  and the covering  $\{B_{\omega^n}\}$  constructed in the claim above. Without weakening the assumptions, we may choose zeros and ones in our symbolic notation in such a way that  $\{x\} = \bigcap B_{0^n}$ .

We begin with the definitions of  $C_\emptyset = C^{(0)}$  and  $F$  on  $\mathbb{R}^2 \setminus \text{Int}(C_\emptyset)$ . As the Cantor set is perfect,  $y$  is not an isolated point, hence we have a sequence  $(y_\ell)$  in  $X$  converging to  $y$ . For any  $\ell$ , the triple  $\{B(y, 2|y_\ell - y|), y, y_\ell\}$  is geometrically identical up to a similitude. We take any Lipschitz homeomorphism  $f_0$  from  $B_\emptyset^c$  onto  $B^c(y, 2|y_0 - y|)$ . We can construct a family of Lipschitz homeomorphisms  $f_\ell$  from  $B_\emptyset^c$  onto  $B^c(y, 2|y_\ell - y|)$  for all  $\ell$  by  $f_\ell(x) = S_\ell \circ f_0(x)$ , where  $S_\ell$  is the similitude moving  $\{B(y, 2|y_0 - y|), y, y_0\}$  onto  $\{B(y, 2|y_\ell - y|), y, y_\ell\}$ . The Lipschitz constants of those homeomorphisms are decreasing to zero together with  $|y_\ell - y|$ , hence for some  $\ell$  they will be smaller than  $1/k(B_\emptyset, B_0, B_1, B(y, 2|y_\ell - y|), y, y_\ell) = 1/k(B_\emptyset, B_0, B_1, B(y, 2|y_0 - y|), y, y_0) = c < 1$ . Let us denote  $C_\emptyset = B(y, 2|y_\ell - y|)$  and define  $F = f_\ell$  on  $\mathbb{R}^2 \setminus \text{Int}(C_\emptyset)$ .

Now we explain how to choose  $C_0$  and  $C_1$  and extend  $F$  on  $\text{Int}(C_\emptyset) \setminus \text{Int}(C_0 \cup C_1)$ . At this step,  $F$  is a contraction from the complement of  $\text{Int}(B_\emptyset)$  onto the complement of  $\text{Int}(C_\emptyset)$  and (by Lemma 2.6) it can be extended to an homeomorphism from the complement of  $\text{Int}(B_0 \cup B_1)$  onto the complement of  $\text{Int}(B(y, r_0) \cup B(y_\ell, r_1))$  for some small  $r_0$  and  $r_1$ . By Lemma 2.6 c) and by our choice of the Lipschitz constant of  $F$  in restriction to the boundary  $\partial B_\emptyset$ , the map  $F$  is a contraction.

Moreover by Lemma 2.6 d), for sufficiently small  $r_0$  and  $r_1$  the Lipschitz constant on  $\partial B_0$  and  $\partial B_1$  is arbitrarily small. We may thus choose a point from  $X$  very close to  $y$  and repeat the procedure, obtaining a contraction on  $\text{Int}(B_0) \setminus \text{Int}(B_{00} \cup B_{01})$ . Similarly choosing a point sufficiently close to  $y_n$  lets us extend our function on  $\text{Int}(B_1) \setminus \text{Int}(B_{10} \cup B_{11})$  and so on.

In this inductive procedure we will get a contracting map from the complement of  $X$  onto the complement of some Cantor set  $Y$  on the plane (the closure of the set of all the points we have chosen on all the stages of the construction). As the points chosen were always belonging to  $X$ , the set  $Y$  is included in  $X$ . We then extend the function  $F$  on  $X$  by continuity and we are done.

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