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Jean-Marc Bardet, Donatas Surgailis. Nonparametric estimation of the local Hurst function of multifractional processes. 2010. hal-00526294v1

HAL Id: hal-00526294

<https://hal.science/hal-00526294v1>

Preprint submitted on 14 Oct 2010 (v1), last revised 7 Jun 2012 (v2)

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Nonparametric estimation of the local Hurst function of multifractional processes

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Abstract

Consistency, almost sure convergence and central limit theorems are provided for two nonparametric estimators of the local Hurst function of Gaussian multifractional processes. In the case of multifractional Brownian motions, we correct results of Coeurjolly (2005) [11] and Benassi *et al.* (1998a) [8].

Keywords: Nonparametric estimators, Hurst function, tangent process, multifractional Brownian motion, Gaussian process, central limit theorem.

2000 MSC: Primary: 62G05, Secondary 62G20, 60F05, 60G22

1. Introduction

In the sequel, $B_H = \{B_H(t), t \in \mathbb{R}\}$ is a normalized ($EB_H(t) = 0$ and $EB_H(1) = 1$) fractional Brownian motion (FBM) with Hurst parameter $H \in (0, 1)$ defined by the following harmonizable representation (see for instance [15] for details)

$$B_H(t) := (K(2H))^{1/2} \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H+1/2}} W(dx), \quad (1.1)$$

where $W(dx)$ is a complex-valued Gaussian noise with variance dx and

$$K(z) := \frac{z\Gamma(z)\sin(\pi z/2)}{\pi}, \quad z \in (0, 2). \quad (1.2)$$

Note that $K(\cdot)$ is positive and analytic on $(0, 2)$. It is well-known that local roughness of the path of a FBM is identical everywhere and is completely determined by the parameter H . Several generalizations of this process were proposed in the last decade (see for instance [1]), whose trajectories can display varying Hölder exponents and "multifractional behavior". In particular, [14] introduced the multifractional Brownian motion (MBM) $X = \{X(t), t \in \mathbb{R}\}$ by substituting H in (1.1) by a function $H(t), t \in \mathbb{R}$ taking values in $(0, 1)$:

$$X(t) := B_{H(t)}(t) = (K(2H(t)))^{1/2} \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H(t)+1/2}} W(dx), \quad (1.3)$$

Below, we call $H(\cdot)$ the *local Hurst function* of MBM. Under some regularity condition on $H(\cdot)$ (see Assumption $C(\eta)$ below) the MBM is locally asymptotically self-similar at each point $t \in \mathbb{R}$ having a fBm $B_{H(t)}$ as its tangent process at t [10] and its pointwise Hlder exponent coincides with $H(t)$ [3]. Note $EX^2(t) = |t|^{2H(t)}$ for $t \in \mathbb{R}$.

A natural generalization of MBM is

$$Y = \{Y(t), t \in \mathbb{R}\} \quad \text{with} \quad Y(t) := \sigma(t)X(t) \quad \text{for} \quad t \in \mathbb{R} \quad (1.4)$$

and $\sigma(\cdot)$ a function on \mathbb{R} taking values in $\mathbb{R}_+ := (0, \infty)$. In the sequel, we shall assume the following conditions on $H(\cdot)$ and $\sigma(\cdot)$.

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Assumption $C(\eta)$: $H(\cdot)$ and $\sigma(\cdot)$ are \mathcal{C}^η functions with exponent $\eta > 0$, $\inf_{t \in \mathbb{R}} \sigma(t) > 0$ and

$$0 < \inf_{t \in \mathbb{R}} H(t) \leq \sup_{t \in \mathbb{R}} H(t) < \min(1, \eta). \quad (1.5)$$

Note, (1.5) excludes discontinuous function $H(\cdot)$ or $H(\cdot) \notin (0, 1)$. [2] and [7] proposed generalizations of FBM (called Generalized Multifractional Brownian Motion (GMBM) and Step Fractional Brownian Motion (SFBM), respectively) which allow a discontinuous function $H(\cdot)$. [16] introduced another generalization of FBM from nonhomogeneous fractional integration which allows $H(\cdot) \notin (0, 1)$.

Several studies have been devoted to nonparametric estimation of the function $H(\cdot)$ of MBM from a sample $(X(1/n), X(2/n), \dots, X(n/n))$. [8] discussed an estimator based generalized quadratic variations and obtained its a.s. convergence under Assumption $C(1)$, together with some bounds on the convergence rate (but in Remark 3 below we prove that an asymptotic expansion of quadratic variation covariance used in this paper is erroneous). [7] applied another generalized quadratic variations estimator $\hat{h}_{n,\alpha}$ (see Remark 1 below) to estimate parameters of a piecewise constant function $H(\cdot)$ of SFBM. [3] obtained the a.s. convergence of this estimator under weak assumptions allowing discontinuity of $H(\cdot)$ in the case of GFBM. For MBM satisfying Assumption $C(1)$ with $\sup_{t \in [0,1]} H(t) < 1/8$, they also proved asymptotic normality of $\hat{h}_{n,\alpha}$, without specifying the exact convergence rate. For this estimator and a SFBM with a piecewise constant $H(\cdot)$, [4] established a central limit theorem without the previous limitation on $\sup_{t \in [0,1]} H(t)$ but with a $1/\log n$ bias depending on $H(\cdot)$ which does not allow to use this result for applications. Probably, the most comprehensive results on nonparametric estimation of $H(\cdot)$ were obtained by [11], who used a generalized quadratic variations estimator $\hat{H}_{n,\alpha}^{(QV)}$ in (1.9) below and discussed its asymptotic normality under Assumption $C(\eta)$, $\eta \in (0, 1]$ together with a convergence rate $n^{\eta/(1+2\eta)}$. [11] also proposed an adaptive procedure for selecting an optimal localization parameter α for this estimator. However, Lemma 1 of [11] (the key point in the proof of his asymptotic results for the generalized variations estimator) is erroneous, see Remark 2 below, leading to the necessity of a new study of asymptotic properties of the estimator $\hat{H}_{n,\alpha}^{(QV)}$.

The present paper discusses two estimators of $H(\cdot)$ from observed sample $(Y(1/n), Y(2/n), \dots, Y(n/n))$ of Y in (1.4), namely, the generalized quadratic variations estimator $\hat{H}_{n,\alpha}^{(QV)}(t)$ of [11] and the estimator $\hat{H}_{n,\alpha}^{(IR)}(t)$, based on increment ratios and defined in [6]. The last estimator is a localized version of the estimator discussed in [17].

Let us define these estimators. Consider a filter $a := (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ such that there exists $m \in \mathbb{N}^*$ satisfying

$$\sum_{\ell=0}^q \ell^p a_\ell = 0 \quad \text{for } p = 0, \dots, (m-1) \quad \text{and} \quad \sum_{\ell=0}^q \ell^m a_\ell \neq 0. \quad (1.6)$$

For $n \in \mathbb{N}^*$, $t \in [0, 1 - q/n] \rightarrow Z(t)$ (= a random or nonrandom function on $[0, 1)$), define the generalized variations of $Z = \{Z(t), t \in [0, 1)\}$ by

$$V_n^a Z(t) = \sum_{\ell=0}^q a_\ell Z(t + \ell/n). \quad (1.7)$$

The main examples of generalized variations are the usual simple variations corresponding to $a = (1, -1)$, $m = 1$, and the second order variations corresponding to $a = a^* := (1, -2, 1)$, $m = 2$. More generally, for a filter $a = (a_0, \dots, a_q)$, for $p \in \mathbb{N}$, $p \geq 2$, and $j = 1, \dots, p$, define its j th dilatation by

$$a^{(j)} := (a_0^{(j)}, \dots, a_{jq}^{(j)}), \quad \text{with } a_{ij}^{(j)} = a_i \quad \text{and} \quad a_k^{(j)} = 0 \quad \text{if } k \notin j\mathbb{N}.$$

For $\alpha \in (0, 1)$, define a neighborhood of t and its cardinal by:

$$\mathcal{V}_{n,\alpha}(t) := \left\{ k \in \mathbb{N}, |k/n - t| \leq n^{-\alpha} \right\} \quad \text{and} \quad v_{n,\alpha}(t) := \# \mathcal{V}_{n,\alpha}(t).$$

Note that $v_{n,\alpha}(t) = [2n^{1-\alpha}]$ or $[2n^{1-\alpha}] + 1$ depending on the parity of $[2n^{1-\alpha}] =$ the integer part of $2n^{1-\alpha}$. For $H \in (0, 1)$, define

$$\rho_2(H) := \text{Cor}(V_n^{a^*} B_H(0), V_n^{a^*} B_H(1/n)) = \frac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}}, \quad (1.8)$$

$$\Lambda_2(H) := \mathbb{E} \frac{|V_n^{a^*} B_H(0) + V_n^{a^*} B_H(1/n)|}{|V_n^{a^*} B_H(0)| + |V_n^{a^*} B_H(1/n)|} = \frac{1}{\pi} \arccos(-\rho_2(H)) + \frac{1}{\pi} \sqrt{\frac{1 + \rho_2(H)}{1 - \rho_2(H)}} \log \left(\frac{2}{1 + \rho_2(H)} \right).$$

The function Λ_2 does not depend on n by self-similarity of FMB and is monotone increasing on $(0, 1)$, see [6]. The estimators $\widehat{H}_n^{(QV)}$ and $\widehat{H}_n^{(IR)}$ are defined for $t \in (0, 1)$ as follows:

$$\widehat{H}_{n,\alpha}^{(QV)}(t) := \frac{1}{2} \frac{A}{AA^\top} \left(\log \left(\frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} |V_n^{a^{(i)}} Y(k/n)|^2 \right) \right)_{1 \leq i \leq p}^\top, \quad (1.9)$$

$$\widehat{H}_{n,\alpha}^{(IR)}(t) := \Lambda_2^{-1} \left(\frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \frac{|V_n^{a^*} Y(k/n) + V_n^{a^*} Y((k+1)/n)|}{|V_n^{a^*} Y(k/n)| + |V_n^{a^*} Y((k+1)/n)|} \right), \quad (1.10)$$

where $A := (\log i - \frac{1}{p} \sum_{j=1}^p \log j)_{1 \leq i \leq p} \in \mathbb{R}^p$ is a row vector, A^\top is the transposed vector (vector-column), and Λ_2^{-1} is the inverse function of Λ_2 . Thus, $\widehat{H}_{n,\alpha}^{(QV)}(t)$ is defined from a log-regression of generalized quadratic variations, see also [11], whereas $\widehat{H}_{n,\alpha}^{(IR)}(t)$ is defined from a mean of ratios of second order variations.

Remark 1. [3] and [1, 4] studied the estimator $\widehat{h}_{n,\alpha}(t) := \frac{1}{2} \left(1 - \alpha - \frac{\log \sum_{k \in \mathcal{V}_{n,\alpha}(t)} (V_n^{a^*} Y(k/n))^2}{\log n} \right)$. We do not discuss this estimator because it is a simpler and less robust version of $\widehat{H}_{n,\alpha}^{(QV)}(t)$ in (1.9). However, limit theorems from the present paper can be extended to $\widehat{h}_{n,\alpha}$ as well.

Below, strong consistency and asymptotic normality of $\widehat{H}_{n,\alpha}^{(QV)}(t)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t)$ are obtained following conditions on the bandwidth parameter α and the smoothness parameter η in Assumption $C(\eta)$. The conditions on α and η in this paper are more restrictive as compared with those in [8] and [11] because of the reasons explained above (see also Remark 2 and Remark 3). The proofs of our results crucially depend on precise decorrelation inequalities for generalized variations of MBM which correct the previous decorrelation inequalities in the above mentioned papers. The inequalities for covariances of generalized variations are derived from asymptotic expansions in Section 2. Section 3 is devoted to the limit theorems satisfied by $\widehat{H}_{n,\alpha}^{(QV)}(t)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t)$. The empirical performance of the estimators is illustrated in Section 4 (Simulations) under different smoothness assumptions on $H(\cdot)$. Section 5 contains the proofs.

In what follows, C stands for a constant whose precise value is unimportant and which may change from line to line. Also, we write $\xrightarrow[n \rightarrow \infty]{P}$, $\xrightarrow[n \rightarrow \infty]{a.s.}$, $\xrightarrow[n \rightarrow \infty]{D}$ for convergence in probability, almost sure (a.s.) convergence and the (weak) convergence of probability distributions, respectively.

2. Expansions and inequalities for covariances of generalized variations of MBM

In this section we assume that Y is given as in (1.4) and Assumption $C(\eta)$ holds.

Property 1. For any $t_0 \in (0, 1)$, $p \in \mathbb{N}^*$, there exists a constant $C(p) > 0$ such that for all $n \geq 1$,

$$\max_{k \in \mathcal{V}_{n,\alpha}(t_0)} \left| \frac{\mathbb{E} \left(Y\left(\frac{k+p}{n}\right) - Y\left(\frac{k}{n}\right) \right)^2}{\left(\frac{p}{n}\right)^{2H(t_0)}} - \sigma^2(t_0) \right| \leq C(p) \left(\frac{\log n}{n^{(\eta \wedge 1)\alpha}} + \frac{1}{n^{2(\eta - H(t_0))}} \right). \quad (2.1)$$

Property 2. Let $a = (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ satisfy property (1.6) with $m \geq 1$. For any $t_0 \in (0, 1)$, $\alpha \in (0, 1)$ there exist $C > 0$, n_0 such that for any $n \geq n_0$ and any $k, k' \in \mathcal{V}_{n,\alpha}(t_0)$,

$$\left| \mathbb{E} \left[V_n^a Y\left(\frac{k}{n}\right) V_n^a Y\left(\frac{k'}{n}\right) \right] \right| \leq C \frac{1}{n^{2H(t_0)}} \left(\frac{1}{(|k' - k| + 1)^{m - H(t_0)}} + \frac{1}{n^{(\eta \wedge m) - H(t_0)}} \right)^2. \quad (2.2)$$

Property 1 provides the rate of convergence of Y to its tangent process, which is a FBM. Properties 1 and 2 imply that correlation $\text{Cor}(V_n^a X(k/n), V_n^a X(k'/n))$ for $k, k' \in \mathcal{V}_{n,\alpha}(t_0)$ decays as $O(|k' - k|^{2H(t_0)-2m-2})$ when $|k' - k| \leq n^{\frac{\eta \wedge m - H(t_0)}{m - H(t_0)}}$ and as $O(n^{-2(\eta \wedge m - H(t_0))})$ otherwise (see Figure 4.1 for an illustration of this property).

Properties 1 and 2 follow from the asymptotic expansion of the covariance of generalized variations given in Lemma 2 (see Section 5). In (2.7) below we present this expansion for simple variations with $a = (1, -2, 1)$, $m = 2$ but for general $m \geq 0$ the formulation of Lemma 2 is more involved.

Let Y be defined as in (1.4). Note, for $t, t' \in [0, 1]$

$$\mathbb{E}Y(t)Y(t') = Q(\sigma(t), \sigma(t'), H(t), H(t'), t, t'), \quad (2.3)$$

where Q is a function of 6 variables $t, t' \in [0, 1]$, $H, H' \in (0, 1)$, $\sigma, \sigma' \in (0, \infty)$ defined by

$$Q(\sigma, \sigma', H, H', t, t') := \frac{1}{2} \sigma \sigma' L(H, H') \left(t^{H+H'} + t'^{H+H'} - |t - t'|^{H+H'} \right), \quad (2.4)$$

$$\text{and } L(H, H') := \frac{(K(2H)K(2H'))^{1/2}}{K(H+H')} \quad (2.5)$$

with $K(\cdot)$ defined in (1.2).

For $0 < t < t' < 1$, introduce the 3×3 -matrix of the second order partial derivatives:

$$R^{(2)}(t, t') := \begin{pmatrix} R_{H,H'}^{(2)}(t, t') & R_{H,\sigma'}^{(2)}(t, t') & R_{H,t'}^{(2)}(t, t') \\ R_{\sigma,H'}^{(2)}(t, t') & R_{\sigma,\sigma'}^{(2)}(t, t') & R_{\sigma,t'}^{(2)}(t, t') \\ R_{t,H'}^{(2)}(t, t') & R_{t,\sigma'}^{(2)}(t, t') & R_{t,t'}^{(2)}(t, t') \end{pmatrix}, \quad (2.6)$$

whose elements are the corresponding partial derivatives of the function Q computed at $H := H(t)$, $H' := H(t')$, $\sigma := \sigma(t)$, $\sigma' := \sigma(t')$,

$$R_{H,H'}^{(2)}(t, t') := \frac{\partial^2 Q}{\partial H \partial H'}(\sigma(t), \sigma(t'), H(t), H(t'), t, t'), \quad \dots, \quad R_{t,t'}^{(2)}(t, t') := \frac{\partial^2 Q}{\partial t \partial t'}(\sigma(t), \sigma(t'), H(t), H(t'), t, t').$$

In particular,

$$\begin{aligned} R_{t,t'}^{(2)}(t, t') &= \frac{1}{2} \sigma(t) \sigma(t') L(H(t), H(t')) (H(t) + H(t')) (H(t) + H(t') - 1) (t' - t)^{H(t)+H(t')-2}, \\ R_{H,H'}^{(2)}(t, t') &= \frac{\sigma(t) \sigma(t')}{2} \left\{ L(H(t), H(t')) [t^{H(t)+H(t')} \log^2 t + t'^{H(t)+H(t')} \log^2 t' - (t' - t)^{H(t)+H(t')} \log^2(t' - t)] \right. \\ &\quad + 2 \frac{\partial L}{\partial H}(H(t), H(t')) [t^{H(t)+H(t')} \log t + t'^{H(t)+H(t')} \log t' - (t' - t)^{H(t)+H(t')} \log(t' - t)] \\ &\quad \left. + \frac{\partial^2 L}{\partial H \partial H'}(H(t), H(t')) [t^{H(t)+H(t')} + t'^{H(t)+H(t')} - (t' - t)^{H(t)+H(t')}] \right\}. \end{aligned}$$

Then, from Lemma 2, since $a = (1, -2, 1)$, $m = 2$, for $k, k' \in \mathcal{V}_{n,\alpha}(t)$, when $n, k, k' - k \rightarrow \infty$,

$$\begin{aligned} \mathbb{E} \left[V_n^a Y\left(\frac{k}{n}\right) V_n^a Y\left(\frac{k'}{n}\right) \right] &= \left(V_n^a H\left(\frac{k}{n}\right), V_n^a \sigma\left(\frac{k}{n}\right), V_n^a \frac{k}{n} \right) \times R^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) \times \left(V_n^a H\left(\frac{k'}{n}\right), V_n^a \sigma\left(\frac{k'}{n}\right), V_n^a \frac{k'}{n} \right)^\top \\ &\quad + \frac{1}{2} \sigma^2(t) \left(\prod_{i=0}^3 (2H(t) - i) \right) \frac{|k - k'|^{2H(t)-4}}{n^{2H(t)}} + O\left(\frac{1}{n^{(3\eta) \wedge 5}} + \frac{|k' - k|^{2H(t)-5}}{n^{2H(t)}} \right), \quad (2.7) \end{aligned}$$

where $(V_n^a H(\frac{k}{n}), V_n^a \sigma(\frac{k}{n}), V_n^a \frac{k}{n}) = (H(\frac{k+2}{n}) - 2H(\frac{k+1}{n}) + H(\frac{k}{n}), \sigma(\frac{k+2}{n}) - 2\sigma(\frac{k+1}{n}) + \sigma(\frac{k}{n}), 0) \in \mathbb{R}^3$ following the definition in (1.7) with $a = (1, -2, 1)$. Next, let $\sigma(\cdot) \equiv 1$, then $V_n^a \sigma(\frac{k}{n}) = 0$ and the terms involving partial derivatives with respect to σ, σ', t, t' in the quadratic form on the r.h.s. of (2.7) disappear. We see from the above expressions of partial derivatives that $R_{HH'}^{(2)}(\frac{k}{n}, \frac{k'}{n}) = O(1)$ and hence the main term of the expansion on the r.h.s. of (2.7) decays as

$$O\left(\left| V_n^a H\left(\frac{k}{n}\right) V_n^a H\left(\frac{k'}{n}\right) \right| + |k' - k|^{2H(t)-4} n^{-2H(t)} \right) = O\left(n^{-2(\eta \wedge 2)} + |k' - k|^{2H(t)-4} n^{-2H(t)} \right)$$

which agrees with the bound in (2.2) for $m = 2$. The above argument suggests that (2.2) is an optimal or nearly optimal decay rate of covariance of generalized variations of MBM which can be achieved e.g. by taking for $H(\cdot)$ a “typical” trajectory of FBM with Hurst parameter $\eta \in (0, 1)$.

Remark 2. According to [11] (Lemma 1), for any $\alpha, t_0 \in (0, 1)$, $k, k' \in \mathcal{V}_{n,\alpha}(t_0)$ and n large enough,

$$\mathbb{E}\left[V_n^a X\left(\frac{k}{n}\right)V_n^a X\left(\frac{k'}{n}\right)\right] = \mathbb{E}\left[V_n^a B_{H(t_0)}\left(\frac{k}{n}\right)V_n^a B_{H(t_0)}\left(\frac{k'}{n}\right)\right] \left(1 + O\left(n^{-\alpha\eta} \log n\right)\right). \quad (2.8)$$

For $m = 2$, $a = (1, -2, 1)$ we have

$$\begin{aligned} \mathbb{E}\left[V_n^a B_{H(t_0)}\left(\frac{k}{n}\right)V_n^a B_{H(t_0)}\left(\frac{k'}{n}\right)\right] &= -\frac{1}{2n^{2H(t_0)}} \left(|k - k' + 2|^{2H(t_0)} - 4|k - k' + 1|^{2H(t_0)} + 6|k - k'|^{2H(t_0)} \right. \\ &\quad \left. - 4|k - k' - 1|^{2H(t_0)} + |k - k' - 2|^{2H(t_0)} \right) \\ &\sim -\frac{1}{2n^{2H(t_0)}} (2H(t_0))(2H(t_0) - 1)(2H(t_0) - 2)(2H(t_0) - 3) |k - k'|^{2H(t_0) - 4} \end{aligned} \quad (2.9)$$

and hence according to (2.8) the decay rate is

$$\left| \mathbb{E}\left[V_n^a X\left(\frac{k}{n}\right)V_n^a X\left(\frac{k'}{n}\right)\right] \right| \leq C |k' - k|^{2H(t_0) - 4} n^{-2H(t_0)}. \quad (2.10)$$

Note the bound in (2.10) does not depend on $\eta \in (0, 1]$ and therefore (2.10) is much better than our bound (2.2). However, (2.8) and (2.10) are erroneous, which fact easily follows from our expansion in (2.7). Indeed, let $\sigma(\cdot) \equiv 1$ and

$$H(t) := \begin{cases} H_0, & 0 \leq t \leq 1/2, \\ H_0 + (t - \frac{1}{2})^\eta, & 1/2 < t \leq 1 \end{cases} \quad (2.11)$$

for some $0 < H_0 < 1/2 < \eta < 1$. Let $k = [n/2]$, $k' > k$, $k, k' \in \mathcal{V}_{n,\alpha}(1/2)$. Then for even n

$$V_n^a H\left(\frac{k}{n}\right) = n^{-\eta}, \quad V_n^a H\left(\frac{k'}{n}\right) = H\left(\frac{k' + 2}{n}\right) - 2H\left(\frac{k' + 1}{n}\right) + H\left(\frac{k'}{n}\right) \sim \eta(\eta - 1)(k' - k)^{\eta - 2} n^{-\eta}.$$

From (2.7), and since $L(H_0, H_0) = 1$, $\frac{\partial L}{\partial H}(H_0, H_0) = 0$ and one obtains

$$\begin{cases} R_{H,H'}^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) \xrightarrow{n \rightarrow \infty} c_0 := 2^{-2H_0} \left[\log^2 2 - 2 \frac{\partial L}{\partial H}(H_0, H_0) \log 2 + \frac{\partial^2 L}{\partial H \partial H'}(H_0, H_0) \right] \neq 0, \\ R_{t,H'}^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) = O\left(\left(\frac{k' - k}{n}\right)^{2H_0 - 1} \log n\right), \\ R_{t,t'}^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) = O\left(\left(\frac{k' - k}{n}\right)^{2H_0 - 2}\right) \end{cases}.$$

Therefore, for $k' - k = [n^{1-\alpha}]$ and α such as $1 - \alpha > \frac{2(\eta - H_0)}{2 - 2H_0 + \eta}$ i.e. $0 < \alpha < \frac{2 - \eta}{2 + \eta - 2H_0} < 1$, from (2.7) we obtain

$$\begin{aligned} \mathbb{E}\left[V_n^a X\left(\frac{k}{n}\right)V_n^a X\left(\frac{k'}{n}\right)\right] &= (c_0 + o(1)) V_n^a H\left(\frac{k}{n}\right) V_n^a H\left(\frac{k'}{n}\right) + O\left((k' - k)^{2H_0 - 4} n^{-2H_0}\right) \\ &= c_0 \eta (\eta - 1) n^{-(2-\eta)(1-\alpha) - 2\eta} (1 + o(1)) \end{aligned} \quad (2.12)$$

which contradicts (2.10).

Remark 3. [8] consider the case of a MBM where $X_t = a_\infty(t) \int \frac{e^{itx} - 1}{|x|^{H(t)+1/2}} W(dx)$, where $a_\infty(\cdot) \in \mathcal{C}^1$, $H(\cdot) \in \mathcal{C}^1$. Note that with our notations $a_\infty(t) = \sigma(t)(K(2H(t)))^{1/2}$. Lemma 2 in Benassi *et al.* (1998a), based on more general results published in [9], gives an expansion for

$$I(S, S)_{k,k'} = \sum_{i,j=0}^q a_i a_j \mathbb{E} X\left(\frac{k+i}{n}\right) X\left(\frac{k'+j}{n}\right) = \mathbb{E} V_n^a X\left(\frac{k}{n}\right) V_n^a X\left(\frac{k'}{n}\right)$$

for $k, k' \in \mathcal{V}_{n,\alpha}(t)$. This expansion is

$$\mathbb{E} V_n^a X\left(\frac{k}{n}\right) V_n^a X\left(\frac{k'}{n}\right) = a_\infty\left(\frac{k}{n}\right) a_\infty\left(\frac{k'}{n}\right) n^{-H(k/n) - H(k'/n)} F_{H(k/n) + H(k'/n)}(k - k') + O\left(\frac{1}{n^{2\delta+1} |k - k'|^2}\right), \quad (2.13)$$

where $\delta < H(t)$ is arbitrarily close to $H(t)$ and $F_\gamma(k) = 2 \sum_{i,j=0}^q a_i a_j \int e^{ikx} \frac{\sin^2((i-j)x/2)}{|x|^{\gamma+1}} dx$. Remark that the lower bounds $\delta > \max(H(t) - 1/2, 0)$ and $\sup(H(s), s \in \mathcal{V}_t) < \delta + 1/2$ in [8] are not necessary since in (2.13) it makes sense to choose δ as large as possible.

Let us clarify the notation F_γ and (2.13). We have $2 \sin^2(jx/2) = 1 - \cos(jx) = (1/2)|e^{ijx} - 1|^2$ and therefore

$$\begin{aligned} F_{H(k/n)+H(k'/n)}(k-k') &= \frac{1}{2} \sum_{i,j=0}^q a_i a_j \int e^{i(k-k')x} \frac{|e^{i(i-j)x} - 1|^2}{|x|^{H(k/n)+H(k'/n)+1}} dx \\ &= \frac{1}{2} \sum_{i,j=0}^q a_i a_j \int \frac{2(e^{i(k-k')x} - 1) + (1 - e^{i(k-k'+i-j)x}) + (1 - e^{i(k-k'+i-j)x})}{|x|^{H(k/n)+H(k'/n)+1}} dx \\ &= \sum_{i,j=0}^q a_i a_j \int \frac{1 - e^{i(k-k'+i-j)x}}{|x|^{H(k/n)+H(k'/n)+1}} dx \quad \text{using relation (1.6)} \\ &= -\frac{1}{K(H(k/n) + H(k'/n))} \sum_{i,j=0}^q a_i a_j |k + i - k' - j|^{H(k/n)+H(k'/n)} \\ &= \frac{2}{\sqrt{K(H(k/n))K(H(k'/n))}} \text{EV}_1^a B_{H(k/n)}(k) V_1^a B_{H(k'/n)}(k'). \end{aligned}$$

Hence and because of $a_\infty(t) = \sigma(t)(K(2H(t)))^{1/2}$,

$$a_\infty\left(\frac{k}{n}\right) a_\infty\left(\frac{k'}{n}\right) n^{-H(k/n)-H(k'/n)} F_{H(k/n)+H(k'/n)}(k-k') = 2 \sigma\left(\frac{k}{n}\right) \sigma\left(\frac{k'}{n}\right) \text{EV}_n^a B_{H(k/n)}\left(\frac{k}{n}\right) V_n^a B_{H(k'/n)}\left(\frac{k'}{n}\right).$$

In the particular case $q = 2, a_0 = a_2 = 1, a_1 = -2$, with $H := H(k/n), H' := H(k'/n)$, using (2.9) we obtain

$$\text{EV}_n^a B_{H(k/n)}\left(\frac{k}{n}\right) V_n^a B_{H(k'/n)}\left(\frac{k'}{n}\right) \sim -\frac{1}{2n^{H+H'}} (H+H')(H+H'-1)(H+H'-2)(H+H'-3) |k-k'|^{H+H'-4}.$$

Therefore, with C an identifiable quantity, for $k, k' \in \mathcal{V}_{n,\alpha}(t)$, relation (2.13) becomes

$$\text{EV}_n^a X\left(\frac{k}{n}\right) V_n^a X\left(\frac{k'}{n}\right) \sim \frac{C}{n^{H(k/n)+H(k'/n)} |k' - k|^{4-H(k/n)-H(k'/n)}} + O\left(\frac{1}{n^{2H(t)+1-\varepsilon} |k - k'|^2}\right). \quad (2.14)$$

On the other hand from our Lemma 2 under the same conditions (notably $\eta = 1$) we should have

$$\text{EV}_n^a X\left(\frac{k}{n}\right) V_n^a X\left(\frac{k'}{n}\right) \sim \frac{C}{n^{H(k/n)+H(k'/n)} |k' - k|^{4-H(k/n)-H(k'/n)}} + \frac{C_1}{n^2} + O\left(\frac{1}{n^3} + \frac{1}{n^{2H(t)} |k - k'|^{5-2H(t)}}\right). \quad (2.15)$$

To obtain a contradiction between (2.14) and (2.15), it suffices to take $|k' - k|$ large enough so that the main term in (2.15) is $C_1 n^{-2}$, and at the same time to take $|k' - k|$ so that the remainder term in (2.14) is negligible. For concreteness, let $H(t) = 5/6, |k' - k| = n^{1/4}$, then since $H(\cdot) \in \mathcal{C}^1$, so $n^{H(k/n)+H(k'/n)} \sim n^{2H(t)} \sim n^{5/3}$,

$$n^{2H(t)} |k' - k|^{4-2H(t)} = n^{5/3} n^{7/12} = n^{27/12} \gg n^2; \quad n^{2H(t)+1} |k' - k|^2 = n^{8/3} n^{1/2} = n^{19/6} \gg n^{27/12}$$

Therefore and since $C_1 \neq 0$, we see that $\text{EV}_n^a X\left(\frac{k}{n}\right) V_n^a X\left(\frac{k'}{n}\right) \sim C_1 n^{-2}$ and therefore (2.14) [= the expansion in [8] (Lemma 2)] is incorrect.

3. Limit theorems satisfied by the estimators of the Hurst function

For a general Gaussian process $Z = \{Z(t), t \in [0, 1]\}$ and a filter $a = (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ satisfying the property (1.6) with $m \geq 1$ and $\alpha \in (0, 1)$, we introduce the following assumptions adopted from [6].

(A.1) There exist continuous functions $0 < H(t) < 1$ and $c(t) > 0$ for $t \in [0, 1]$ satisfying $H(\cdot) \in \mathcal{C}^\eta([0, 1])$ and $c(\cdot) \in \mathcal{C}^\eta([0, 1])$, $\eta > 0$ such that for any $p \in \mathbb{N}^*$ and $t_0 \in (0, 1)$,

$$\lim_{n \rightarrow \infty} \max_{k \in \mathcal{V}_{n,\alpha}(t_0)} \left| \frac{\mathbb{E}\left(Z\left(\frac{k+p}{n}\right) - Z\left(\frac{k}{n}\right)\right)^2}{\left(\frac{p}{n}\right)^{2H(t_0)}} - c(t_0) \right| = 0. \quad (3.1)$$

(A.1') There exist continuous functions $0 < H(t) < 1$ and $c(t) > 0$ for $t \in [0, 1]$ satisfying $H(\cdot) \in \mathcal{C}^\eta([0, 1])$ and $c(\cdot) \in \mathcal{C}^\eta([0, 1])$ with $\eta > 0$, $\max(0, 1 - 2\eta) < \alpha < 1$ such that for any $p \in \mathbb{N}^*$ and $t_0 \in (0, 1)$,

$$\lim_{n \rightarrow \infty} n^{(1-\alpha)/2} \max_{k \in \mathcal{V}_{n,\alpha}(t_0)} \left| \frac{\mathbb{E}(Z(\frac{k+p}{n}) - Z(\frac{k}{n}))^2}{\binom{p}{n}^{2H(t_0)}} - c(t_0) \right| = 0. \quad (3.2)$$

(A.2) For any $t_0 \in (0, 1)$ there exist $C > 0$, $\gamma > 1/2$ and $0 \leq \theta < \gamma/2$ such that for any $n \in \mathbb{N}^*$ and $k, k' \in \mathcal{V}_{n,\alpha}(t_0)$,

$$\left| \text{Cor}(V_n^a Z(\frac{k}{n}), V_n^a Z(\frac{k'}{n})) \right| \leq C n^{(1-\alpha)\theta} (|k' - k| + 1)^{-\gamma}. \quad (3.3)$$

If condition **(A.1)** or **(A.1')** is satisfied then Z admits a tangent process which is a FBM (with **(A.1')**, the rate of convergence of Z to its tangent process is controlled). Condition **(A.2)** provides a bound for the correlation of the process $\{V_n^a Z(\frac{k}{n}), k \in \mathcal{V}_{n,\alpha}(t_0)\}$. Under such conditions, the strong consistency and a CLT for $\widehat{H}_{n,\alpha}^{(QV)}(t_0)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t_0)$ can be established.

Theorem 1. Let $Z = \{Z(t), t \in [0, 1]\}$ be a Gaussian process, and $a = (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ be such that (1.6) is satisfied with $m \geq 1$.

(i) If Z satisfies condition **(A.1)** then for all $t_0 \in (0, 1)$ and any $\alpha \in (0, 1)$

$$\widehat{H}_{n,\alpha}^{(QV)}(t_0) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} H(t_0) \quad \text{and} \quad \widehat{H}_{n,\alpha}^{(IR)}(t_0) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} H(t_0).$$

(ii) If Z satisfies conditions **(A.1)** and **(A.2)** with $0 < \alpha < \frac{\gamma-2\theta}{2(\gamma-\theta)}$, then for all $t_0 \in (0, 1)$

$$\widehat{H}_{n,\alpha}^{(QV)}(t_0) \xrightarrow[n \rightarrow \infty]{a.s.} H(t_0) \quad \text{and} \quad \widehat{H}_{n,\alpha}^{(IR)}(t_0) \xrightarrow[n \rightarrow \infty]{a.s.} H(t_0).$$

Theorem 2. Let $Z = \{Z(t), t \in [0, 1]\}$ be a Gaussian process satisfying **(A.1')** and **(A.2)** with and $a = (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ be such that (1.6) is satisfied with $m \geq 1$. Then for all $t_0 \in (0, 1)$

$$n^{(1-\alpha)/2} (\widehat{H}_{n,\alpha}^{(QV)}(t_0) - H(t_0)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}\left(0, \frac{A \Gamma(H(t_0)) A^\top}{4(A A^\top)^2}\right) \quad (3.4)$$

$$\text{and} \quad n^{(1-\alpha)/2} (\widehat{H}_{n,\alpha}^{(IR)}(t_0) - H(t_0)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma_2(H(t_0))) \quad (3.5)$$

with A as in (1.9), $\Gamma(H)$ as in (5.34) and $H \rightarrow \Sigma_2(H)$ provided and drawn in [6].

Remark 4. Note that the conditions for the strong consistency or the asymptotic normality of $\widehat{H}_{n,\alpha}^{(QV)}(t_0)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t_0)$ are the same for both estimators. A key point in the proofs of these limit theorems is the fact that these estimators are written as sums of functions of Gaussian vectors and these functions have the same Hermite rank 2 in both cases.

Corollary 1. Let $a = (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ satisfy property (1.6) with $m \geq 1$ and Assumption $C(\eta)$ holds with some $\eta > 0$. Then for all $t_0 \in (0, 1)$,

1. For all $\alpha \in (0, 1)$, $\widehat{H}_{n,\alpha}^{(QV)}(t_0) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} H(t_0)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t_0) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} H(t_0)$.
2. $\widehat{H}_{n,\alpha}^{(QV)}(t_0) \xrightarrow[n \rightarrow \infty]{a.s.} H(t_0)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t_0) \xrightarrow[n \rightarrow \infty]{a.s.} H(t_0)$ when

$$\eta \wedge m > H(t_0) + \frac{1}{8} \quad \text{and} \quad 0 < \alpha < \min \left\{ \frac{1}{2}, 1 - \frac{1}{8(m - H(t_0))} \right\}. \quad (3.6)$$

3. If $\max \left\{ \frac{1}{1 + 2(\eta \wedge 1)}, 1 - 4((\eta \wedge m) - H(t_0)) \right\} < \alpha < 1$, then the CLTs (3.4) and (3.5) hold when $H(t_0) < m - 1/4$.

Remark 5. From the proofs of Properties 2 and 1 and Theorems 1 and 2, it is clear that if Hlderian properties of $H(\cdot)$ and $\sigma(\cdot)$ are provided in a neighborhood of t_0 , then results of Corollary 1 are also valid when η is replaced by $\eta(t_0)$.

Note that if $m \geq 2$ the last inequality in (3.6) simplifies to $0 < \alpha < 1/2$ while condition $H(t_0) < m - 1/4$ for the CLTs is automatically satisfied. If $m = 1$, we find again the condition $H(t_0) < 3/4$ for the statement of the CLTs, condition already obtained for $\widehat{H}_{n,0}^{(QV)}(t_0)$ in [12] or [13].

Since α may depend on η and $H(t_0)$ that are not available, it is a problem to apply the CLTs (3.4) and (3.5) of Corollary 1 in concrete situations. However, when $\eta \geq 1$ and $H(t_0) < 5/6$ it is possible to select any $\alpha > 1/3$ and even to use $n^{1/3} \log^2 n$ instead of n^α and then the convergence rate of the CLT is $n^{1/3}$ up to a logarithm term. Note that [11] proposed a procedure by minimization of the MISE to select an optimal $\widehat{\alpha}$ and then an adaptive estimator of $H(t_0)$ is $\widehat{H}_{n,\widehat{\alpha}}^{(QV)}(t_0)$.

4. Simulations

4.1. Simulations concerning Property 2

This property is essential for proving the CLTs satisfied by the estimators $\widehat{H}_{n,\alpha}^{(QV)}(\cdot)$ and $\widehat{H}_{n,\alpha}^{(IR)}(\cdot)$. As it was said in Remark 2, Property 2 also indicates that Lemma 1 of [11] or Lemma 2 of [8] are erroneous. It is easy to obtain the explicit expression of the covariance $|\mathbb{E}[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n})]|$. Thus it is possible to obtain by deterministic computations the convergence rate of $|\mathbb{E}[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n})]|$ when $n \rightarrow \infty$. The concrete numerical experiment procedure is the following:

- $m = 2$, $a = (1, -2, 1)$;
- α is chosen to be 0.1, 0.11, 0.12, \dots , 0.89, 0.9;
- we have chosen $k = [nt_0] - 0.5n^{1-\alpha}$ and $k' = [nt_0] + n^{1-\alpha}$ with $t_0 = 0.5$;
- for each α , the log-log regression of $|\mathbb{E}[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n})]|$ onto n is realized, where $n = 100 \times i$, $i = 1, 2, \dots, 10^4$. The slope of the log-log-regression line provides an approximation of the exponent of the power law in n (Remark that even if this is a deterministic numerical experiment the convergence rate of $|\mathbb{E}[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n})]|$ when $n \rightarrow \infty$ could only be approximated because the sign of this covariance can change when n increases and a log-log regression is sensitive to such a change).

The purpose of this simulation is to exhibit, with the chosen values $|k' - k| \approx \frac{3}{2}n^{1-\alpha}$, the covariance decay rate $|\mathbb{E}[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n})]| \approx C(n^{-2(\eta \wedge 2)} + n^{-2H(t_0)\alpha - 4(1-\alpha)})$ as indicated by Property 2 (since $m = 2$), or the characteristic exponent in (4.1), below, with a break of slope in α at $\alpha_0 := \frac{2-\eta \wedge 2}{2-H}$.

For the above task, the choice of the function $H(\cdot)$ is crucial. If $H(\cdot)$ is chosen a deterministic function as in (2.11) which satisfies the Hölder condition with exponent η at a single point t_0 , the decay rate $n^{-2(\eta \wedge 2)}$ will be not reached, see (2.12). Hence, to achieve our goal, the function $H(\cdot)$ needs to be exactly a \mathcal{C}^η function in a neighborhood of t_0 . For this reason, we chose $H(\cdot)$ a trajectory of a FBM with Hurst parameter $\eta \in (0, 1)$, and, similarly, a trajectory of integrated FBM with Hurst parameter $\eta - 1 \in (0, 1)$ for $1 < \eta < 2$. It is well-known that such a trajectory is a.s. η' -Hlderian for all $\eta' < \eta$. Figure 1 contains the results of simulations for two different values of η , viz. $\eta = 0.6$ and $\eta = 1.5$. This figure shows that with such a function $H(\cdot)$, the asymptotic behavior

$$\log(|\mathbb{E}[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n})]|) / \log(1/n) \sim \min(2\eta, 2H(t_0)\alpha + 4(1-\alpha)) \quad (4.1)$$

is observed, including the change point at α_0 of the behavior of the characteristic exponent in (4.1).

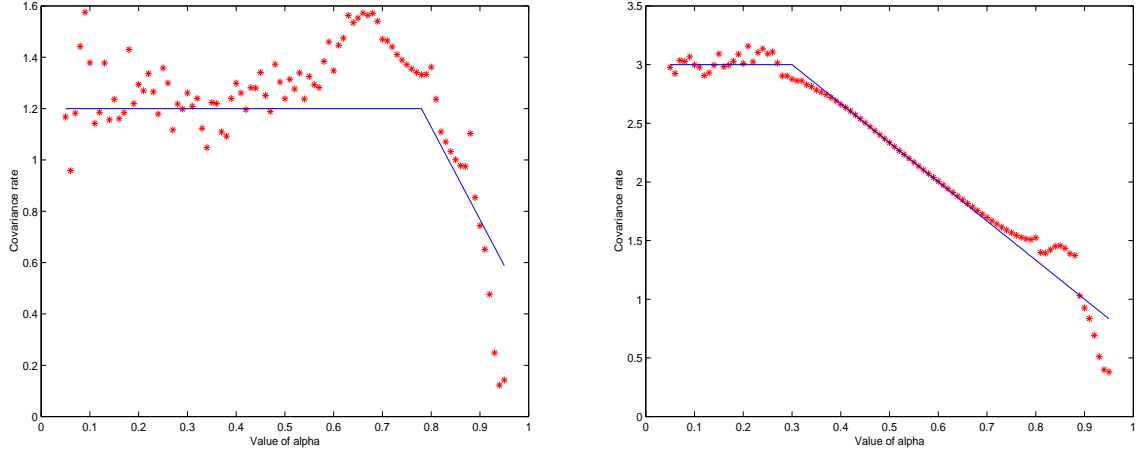


Figure 1: Simulation of $\lim_{n \rightarrow \infty} \log(|E[V_n^\alpha Y(\frac{k}{n})V_n^\alpha Y(\frac{k'}{n})]|)/\log(1/n)$ when $H(\cdot)$ is a trajectory of a FBM with Hurst parameter $\eta = 0.6$ (left) and a trajectory of integrated FBM with Hurst parameter $\eta - 1 = 1.5 - 1$ (right) (in both cases, $H(t_0) = \eta/3$). The theoretical asymptotic rates as given by the r.h.s. of (4.1) are also drawn.

4.2. Simulations on the convergence rate of $\widehat{H}_{n,\alpha}^{(QV)}$ and $\widehat{H}_{n,\alpha}^{(IR)}$

Since trajectories of MBM in our simulation study are generated (for a given Hurst function $H(\cdot)$) using the Choleski decomposition of the covariance matrix, the number of observation points limited to $n = 5000$. Although this data length may appear rather small in the present context, some interesting features can nevertheless be noted. Three cases are considered: Case 1: $H(\cdot)$ is a smooth function; Case 2: $H(\cdot)$ is a trajectory of an integrated FBM with Hurst parameter $0 < H < 1$ and independent of X , therefore $H(\cdot) \in \mathcal{C}^{\eta-}$ with $\eta = 1 + H \in (1, 2)$; and Case 3: $H(\cdot)$ is a trajectory of a FBM with Hurst parameter $0 < \eta < 1$, independent of X .

For each Hurst function $H(\cdot)$, Monte-Carlo experiments are realized from 100 independent replications of MBM trajectories for the following choice of parameters:

- $\alpha = 0.2, 0.3, 0.4$ and 0.5 ;
- $p = 5$ (= the number of dilatations) in all cases;
- $a = (1, -2, 1)$ and therefore $m = 2$ in all cases.

For each estimator, the estimation of the function $H(\cdot)$ is done for $t = \{n^{-\alpha}, n^{-\alpha} + 0.01, \dots, \min(1 - n^{-\alpha}, n^{-\alpha} + 0.99)\}$ and therefore an approximation of $\sqrt{MISE} = (\int_0^1 E(\widehat{H}_n(t) - H(t))^2 dt)^{1/2}$ can be computed. Here there are the results of simulations:

Case 1: $H(\cdot) \in \mathcal{C}^\infty$

We have chosen $H(t) = 0.5(1 + \sqrt{t}\sin(20t))$ in order that $H(\cdot)$ displays ample fluctuations. Figure 2 provides graphs of the two estimators $\widehat{H}_{n,\alpha}^{(QV)}$ and $\widehat{H}_{n,\alpha}^{(IR)}$ for $n = 5000$ and three values of α ($\alpha = 0.3, 0.4$ and 0.5). We can observe that the first estimator is more biased at $\alpha = 0.3$ vs. the second estimator and that both estimators become less stable as α increases (in general, $\widehat{H}_{n,\alpha}^{(IR)}$ appears to be more sensitive to the choice of α than $\widehat{H}_{n,\alpha}^{(QV)}$). The corresponding MISE are given in Table 1.

In agreement with the theory (see the remark at the end of section 3), our simulation suggests to select α close to $1/3$ for optimal results, more precisely, $\alpha \simeq 0.3$ for $\widehat{H}_{n,\alpha}^{(IR)}$, and $\alpha \simeq 0.4$ for $\widehat{H}_{n,\alpha}^{(QV)}$. The above table also confirms that, with the above choice of the bandwidth α , the accuracy of both estimators increases according to the theoretical convergence rate $n^{(1-\alpha)/2}$ as n increases from $n = 1000$ to $n = 5000$. Our general impression from simulations with smooth Hurst functions is that in the case when $H(\cdot)$ does not vary much (in particularly, when $H(\cdot)$ is a constant as in the case of FBM), the estimator $\widehat{H}_{n,\alpha}^{(QV)}$ is more accurate compared to $\widehat{H}_{n,\alpha}^{(IR)}$, while in the opposite case, the second estimator is preferable to the first one.

	α	0.2	0.3	0.4	0.5
$n = 1000$	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(QV)}$	0.29	0.17	0.16	0.17
	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(IR)}$	0.13	0.15	0.17	0.23
$n = 5000$	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(QV)}$	0.21	0.13	0.09	0.10
	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(IR)}$	0.13	0.08	0.11	0.16

Table 1: Values of the (empirical) MISE for estimators $\hat{H}_{n,\alpha}^{(QV)}$ and $\hat{H}_{n,\alpha}^{(IR)}$ of the smooth Hurst function $H(t) = 0.5(1 + \sqrt{t} \sin(20t))$ in Figure 2, for $n \in \{1000, 5000\}$, $\alpha \in \{0.2, 0.3, 0.4, 0.5\}$ and $a = (1, -2, 1)$.

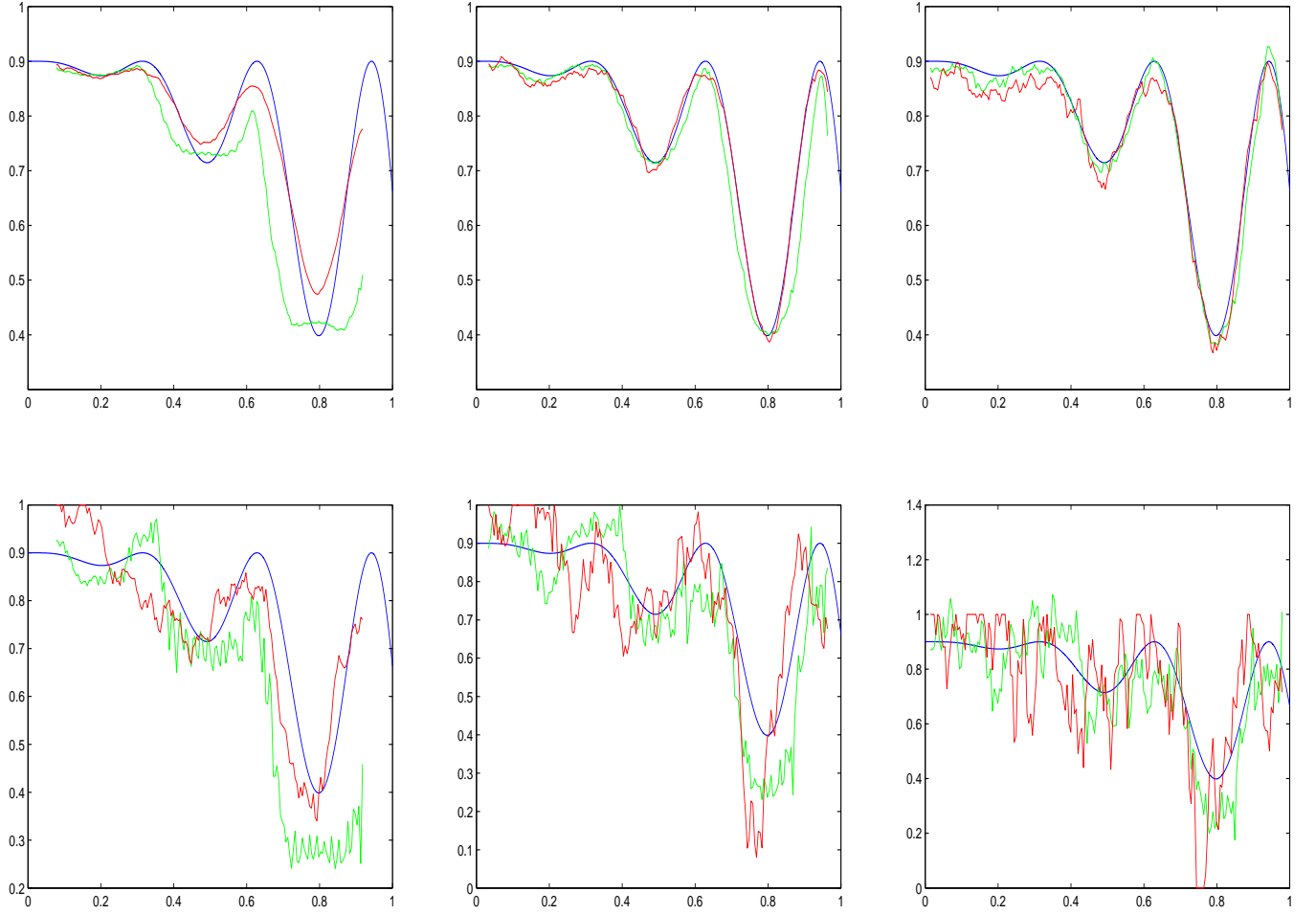


Figure 2: Estimations of the function $H(t) = 0.5(1 + \sqrt{t} \sin(20t))$ with $t \in (0, 1)$ for $n = 5000$ and $\alpha = 0.3, 0.4$ and 0.5 (from left to right). The top row represents the graphs of the sample means of $\hat{H}_{n,\alpha}^{(QV)}(t)$ and $\hat{H}_{n,\alpha}^{(IR)}(t)$ of 100 independent replications of MBM with the above function $H(\cdot)$. The bottom row represents the graphs of $\hat{H}_{n,\alpha}^{(QV)}(t)$ and $\hat{H}_{n,\alpha}^{(IR)}(t)$ from a single trajectory of a MBM with the same function $H(\cdot)$. The graphs of $H(t)$, $\hat{H}_{n,\alpha}^{(QV)}(t)$, and $\hat{H}_{n,\alpha}^{(IR)}(t)$ are in blue, green, and red, respectively.

	α	0.2	0.3	0.4	0.5
$n = 1000$	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(QV)}$	0.17	0.16	0.16	0.18
	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(IR)}$	0.15	0.15	0.22	0.27
$n = 5000$	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(QV)}$	0.16	0.14	0.12	0.15
	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(IR)}$	0.12	0.12	0.20	0.26

Table 2: Values of the (empirical) MISE for estimators $\hat{H}_{n,\alpha}^{(QV)}$ and $\hat{H}_{n,\alpha}^{(IR)}$ of the differentiable Hurst function $H(\cdot) \in \mathcal{C}^{1.5-}$ in Figure 3, for $n \in \{1000, 5000\}$, $\alpha \in \{0.2, 0.3, 0.4, 0.5\}$ and $a = (1, -2, 1)$.

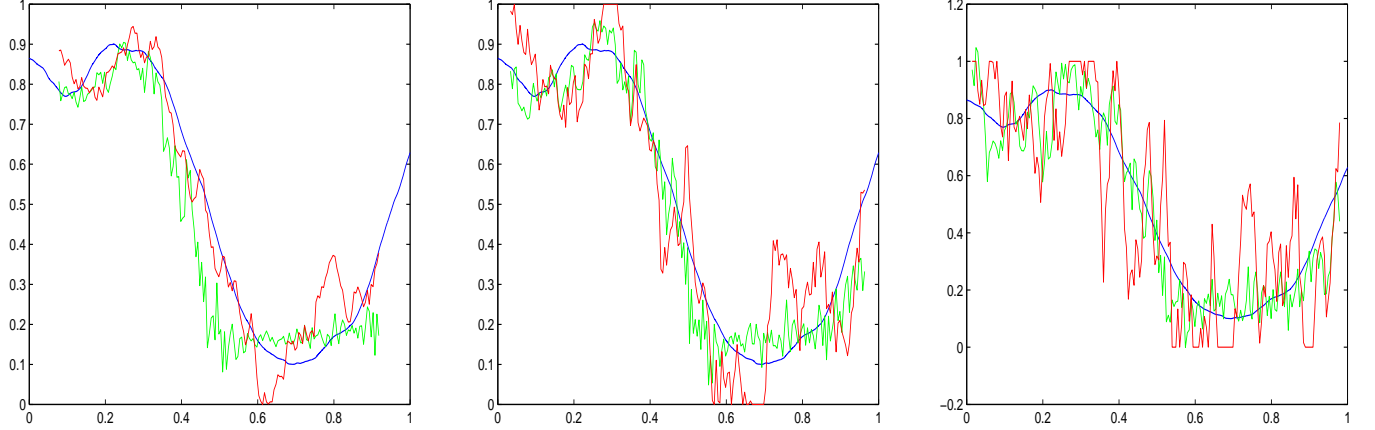


Figure 3: Sample means of 100 independent replications of $\hat{H}_{n,\alpha}^{(QV)}(t)$ and $\hat{H}_{n,\alpha}^{(IR)}(t)$ of differentiable Hurst function $H(\cdot) \in \mathcal{C}^{1.5-}$ for $n = 5000$ and $\alpha = 0.3, 0.4, 0.5$ (from left to right). The graphs of $H(t)$, $\hat{H}_{n,\alpha}^{(QV)}(t)$ and $\hat{H}_{n,\alpha}^{(IR)}(t)$ are respectively in blue, green and red.

Case 2: $H(\cdot) \in \mathcal{C}^{\eta-}$, $1 < \eta < 2$

We have chosen $\eta = 1.5$ but other simulations with different values of η lead to similar conclusions. Figure 3 provides graphs of mean values (over 100 independent simulation) of both estimators for $n = 5000$ and $\alpha \in \{0.3, 0.4, 0.5\}$. We can see from this figure that both estimators are rather unbiased although their variances seem to increase with α , especially for $\hat{H}_{n,\alpha}^{(IR)}$. The last fact can be also observed from Table 2, containing the empirical MISE. Similarly as in Case 1, the choice of bandwidth $\alpha \simeq 0.3$ for $\hat{H}_{n,\alpha}^{(IR)}$, and $\alpha \simeq 0.4$ for $\hat{H}_{n,\alpha}^{(QV)}$ appears to be optimal.

Case 3: $H(\cdot) \in \mathcal{C}^{\eta-}$, $0 < \eta < 1$

We have chosen here $\eta = 0.6$. An example of a graph of a function $H(\cdot) \in \mathcal{C}^{0.6-}$ and both the estimators is drawn in Figure 4. For such a function $H(\cdot)$, the following table provides the computations of \sqrt{MISE} :

The relatively small values of the \sqrt{MISE} in Table 3 with respect to Table 2 can be explained by the fact that the values of $H(\cdot)$ in Table 3 are generally smaller than in Table 2 since the condition (1.5) had to be imposed in the simulations. From Table 3 we can observe that an optimal choice of α is $\simeq 0.2$ for $\hat{H}_{n,\alpha}^{(IR)}(t)$ and $\simeq 0.3$ or 0.4 for $\hat{H}_{n,\alpha}^{(QV)}(t)$. It is not possible to observe that there is a bias if $\alpha \leq (1 + 2\eta)^{-1} = 1/2.2 \simeq 0.46$ as the theory (Corollary 1) seems to indicate; it perhaps requires to consider larger n than 5000. From Figure 4 and Table 3, it appears that in Case 3, $\hat{H}_{n,\alpha}^{(QV)}(t)$ is more accurate from the two estimators.

	α	0.2	0.3	0.4	0.5
$n = 1000$	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(QV)}$	0.15	0.13	0.13	0.17
	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(IR)}$	0.14	0.17	0.21	0.27
$n = 5000$	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(QV)}$	0.09	0.07	0.07	0.10
	\sqrt{MISE} for $\hat{H}_{n,\alpha}^{(IR)}$	0.07	0.10	0.14	0.17

Table 3: Values of the (empirical) MISE for estimators $\hat{H}_{n,\alpha}^{(QV)}$ and $\hat{H}_{n,\alpha}^{(IR)}$ of the nondifferentiable Hurst function $H(\cdot) \in \mathcal{C}^{0.6-}$ in Figure 4, for $n \in \{1000, 5000\}$, $\alpha \in \{0.2, 0.3, 0.4, 0.5\}$ and $a = (1, -2, 1)$.

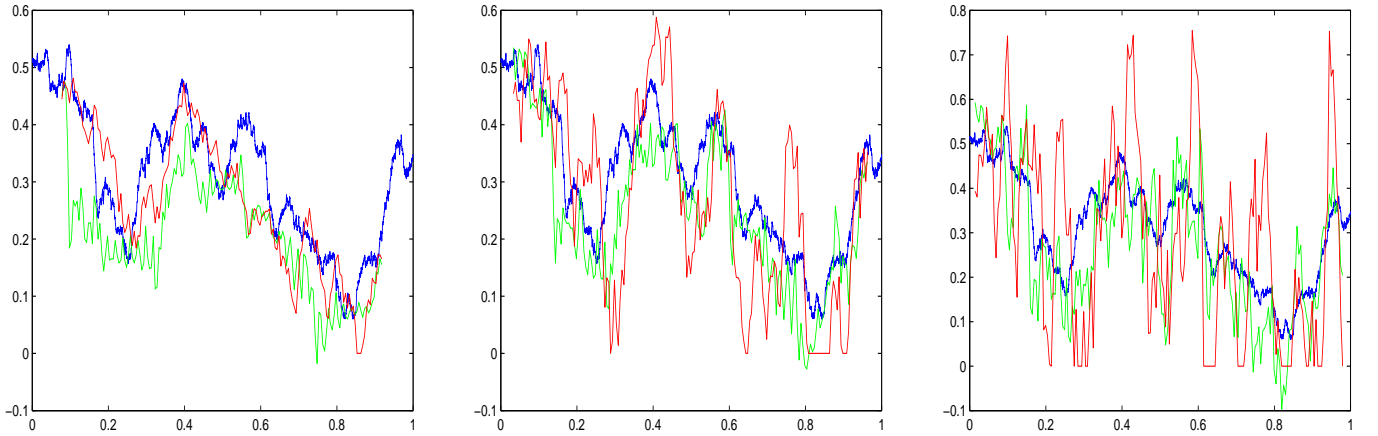


Figure 4: Sample means of 100 independent replications of $\hat{H}_{n,\alpha}^{(QV)}(t)$ and $\hat{H}_{n,\alpha}^{(IR)}(t)$ of nondifferentiable Hurst function $H(\cdot) \in \mathcal{C}^{0.6-}$ for $n = 5000$ and $\alpha = 0.3, 0.4, 0.5$ (from left to right). $H(t)$, $\hat{H}_{n,\alpha}^{(QV)}(t)$ and $\hat{H}_{n,\alpha}^{(IR)}(t)$ are respectively in blue, green and red.

5. Lemmas and proofs

In Lemmas 1 and 2, below, we assume that $a = (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ satisfy property (1.6) and Assumption $C(\eta)$ holds, with some $\eta > 0$, $m \geq 1$.

Lemma 1. *There exists $C_{\sigma, H}(q) \geq 0$ only depending on q , $\sigma(\cdot)$ and $H(\cdot)$ such that for any $k \in \{1, \dots, n - q\}$,*

$$\left| \frac{\text{Var}(V_n^a Y(\frac{k}{n}))}{\sigma^2(\frac{k}{n}) \text{Var}(V_n^a B_{H(\frac{k}{n})}(\frac{k}{n}))} - 1 \right| \leq C_{\sigma, H}(q) \frac{\log n}{n^{\eta \wedge 1}}.$$

Proof of Lemma 1. For any $k \in \{0, 1, \dots, n - q\}$,

$$\begin{aligned} \text{Var}(V_n^a Y(\frac{k}{n})) &= \frac{1}{2} \sum_{i, j=0}^q a_i a_j (\sigma(\frac{k}{n}) + \beta_i) (\sigma(\frac{k}{n}) + \beta_j) L(H(\frac{k}{n}) + \delta_i, H(\frac{k}{n}) + \delta_j) \\ &\quad \times \left(\left| \frac{k+i}{n} \right|^{2H(\frac{k}{n}) + \delta_i + \delta_j} + \left| \frac{k+j}{n} \right|^{2H(\frac{k}{n}) + \delta_i + \delta_j} - \left| \frac{j-i}{n} \right|^{2H(\frac{k}{n}) + \delta_i + \delta_j} \right) \end{aligned}$$

We know from the assumptions that there exists $C \geq 0$ (not depending on k and n) such as $\max\{|\beta_i|, |\beta_j|, |\delta_i|, |\delta_j|\} \leq C n^{-\eta \wedge 1}$. But $\left| \frac{j-i}{n} \right|^{2H(\frac{k}{n}) + \delta_i + \delta_j} = \left| \frac{j-i}{n} \right|^{2H(\frac{k}{n})} \exp\left\{(\delta_i + \delta_j) \log \left| \frac{j-i}{n} \right|\right\} = \left| \frac{j-i}{n} \right|^{2H(\frac{k}{n})} \left(1 + O\left(\frac{\log n}{n^{\eta \wedge 1}}\right)\right)$. The same expansion can also be obtained by replacing $j - i$ by i or by j . Note that here $O(\cdot)$ is only depending on function $H(\cdot)$ and q (and does not depend on k and n). Moreover since $(x, x') \mapsto L(x, x')$ is an analytic function on $(0, 1)^2$, it is clear that $L(H(\frac{k}{n}) + \delta_i, H(\frac{k}{n}) + \delta_j) = L(H(\frac{k}{n}), H(\frac{k}{n})) + O(n^{-\eta \wedge 1})$ and $O(\cdot)$ does not depend on k and n . Therefore one obtains that

$$\begin{aligned} \text{Var}(V_n^a Y(\frac{k}{n})) &= \left\{ \sum_{i, j=0}^q a_i a_j \sigma(\frac{k}{n}) \sigma(\frac{k}{n}) L(H(\frac{k}{n}), H(\frac{k}{n})) \left(\left| \frac{k+i}{n} \right|^{2H(\frac{k}{n})} + \left| \frac{k+j}{n} \right|^{2H(\frac{k}{n})} - \left| \frac{j-i}{n} \right|^{2H(\frac{k}{n})} \right) \right\} \left(1 + O\left(\frac{\log n}{n^{\eta \wedge 1}}\right)\right) \\ &= \sigma^2(\frac{k}{n}) \text{Var}(V_n^a B_{H(\frac{k}{n})}(\frac{k}{n})) \left(1 + O\left(\frac{\log n}{n^{\eta \wedge 1}}\right)\right), \end{aligned}$$

where the bounded function $O(\cdot)$ only depends on q , $H(\cdot)$ and $\sigma(\cdot)$. \square

Proof of Property 1. Consider the filter sequence $(a_0, \dots, a_p) = (-1, 0, \dots, 0, 1)$; therefore $m = 1$ and for all $k \in \{1, \dots, n - p\}$, $\text{Var}(V_n^a Y(\frac{k}{n})) = \mathbb{E}(Y(\frac{k+p}{n}) - Y(\frac{k}{n}))^2$. From Lemma 1, there exists $C_{\sigma, H}(p)$ such that for $k \in \mathcal{V}_{n, \alpha}(t_0)$,

$$\left| \frac{\text{Var}(V_n^a Y(\frac{k}{n}))}{\sigma^2(\frac{k}{n}) \text{Var}(V_n^a B_{H(\frac{k}{n})}(\frac{k}{n}))} - 1 \right| = \left| \frac{\mathbb{E}(Y(\frac{k+p}{n}) - Y(\frac{k}{n}))^2}{\sigma^2(\frac{k}{n}) \left| \frac{p}{n} \right|^{2H(\frac{k}{n})}} - 1 \right| \leq C_{\sigma, H}(p) \frac{\log n}{n^{\eta \wedge 1}}. \quad (5.1)$$

But a Taylor expansion implies for $k \in \mathcal{V}_{n, \alpha}(t_0)$,

$$\left| \frac{p}{n} \right|^{2H(\frac{k}{n})} = \left| \frac{p}{n} \right|^{2H(t_0)} \exp\left\{2\left(H(\frac{k}{n}) - H(t_0)\right) \log(p/n)\right\} = \left| \frac{p}{n} \right|^{2H(t_0)} \left(1 + O\left(\frac{\log n}{n^{\alpha(\eta \wedge 1)}}\right)\right)$$

with $O(\cdot)$ depending only on p and $H(\cdot)$. Moreover, $\sigma^2(\frac{k}{n}) = \sigma^2(t_0) + O\left(\frac{1}{n^{\alpha(\eta \wedge 1)}}\right)$ with $O(\cdot)$ depending only on $\sigma(\cdot)$. Thus, the inequality (5.1) can be replaced by the inequality (2.1) and Property 1 is proved. \square

Lemma 2. *For any $0 < t \leq t' \leq 1$, $(k, k') \in \mathcal{V}_{n, \alpha}(t) \times \mathcal{V}_{n, \alpha}(t')$, as $|k' - k|, n \rightarrow \infty$,*

$$\begin{aligned} \mathbb{E}[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n})] &= (V_n^a H(\frac{k}{n}), V_n^a \sigma(\frac{k}{n}), V_n^a \frac{k}{n}) \times R^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) \times (V_n^a H(\frac{k'}{n}), V_n^a \sigma(\frac{k'}{n}), V_n^a \frac{k'}{n})^\top \\ &+ \mathbf{1}_{\{m \geq 2\}} \frac{(\sum_{i=1}^q i^m a_i)^2}{2(m!)^2} \sigma^2(t) \left(\prod_{\ell=0}^{2m-1} (H(t) + H(t') - \ell) \right) \frac{|k' - k|^{H(t) + H(t') - 2m}}{n^{H(t) + H(t')}} + o\left(\frac{1}{n^{2(\eta \wedge m)}} + \frac{|k' - k|^{H(t) + H(t') - 2m}}{n^{H(t) + H(t')}}\right). \end{aligned} \quad (5.2)$$

Remark 6 (and proof of Property 2). Assumptions $H(\cdot) \in \mathcal{C}^\eta$, $\sigma(\cdot) \in \mathcal{C}^\eta$ imply

$$V_n^a H\left(\frac{k}{n}\right) = O(n^{-\eta \wedge m}), \quad V_n^a \sigma\left(\frac{k}{n}\right) = O(n^{-\eta \wedge m}). \quad (5.3)$$

If $t = t'$, from Lemma 1 and Cauchy-Schwarz Inequality, it is clear that there exists $C(q) \geq 0$ (depending only on q) such that for all $k, k' \in \{1, \dots, n - q\}$,

$$|\mathbb{E}[V_n^a Y\left(\frac{k}{n}\right) V_n^a Y\left(\frac{k'}{n}\right)]| \leq \frac{C(q)}{n^{H(\frac{k}{n}) + H(\frac{k'}{n})}}, \quad (5.4)$$

since $\text{Var}(V_n^a B_{H(\frac{k}{n})}(\frac{k}{n})) = -(1/2)n^{-2H(\frac{k}{n})} \sum_{0 \leq i, j \leq q} a_i a_j |j - i|^{2H(\frac{k}{n})} \leq (1/2)(q \sum_{i=0}^q |a_i|)^2 n^{-2H(\frac{k}{n})}$.

From (5.2), (5.3) and (5.4) it follows that, if $t = t'$, there exists $C_{\sigma, H}(t) > 0$ such that for all $k, k' \in \mathcal{V}_{n, \alpha}(t)$

$$|\mathbb{E}[V_n^a Y\left(\frac{k}{n}\right) V_n^a Y\left(\frac{k'}{n}\right)]| \leq C_{\sigma, H}(t) \left(\frac{1}{n^{2(\eta \wedge m)}} + \frac{(1 + |k' - k|)^{2H(t) - 2m}}{n^{2H(t)}} \right). \quad (5.5)$$

The proof of Property 2 is achieved.

Proof of Lemma 2. Denote

- $\delta_i := H\left(\frac{k}{n} + \frac{i}{n}\right) - H\left(\frac{k}{n}\right), \quad \delta'_j := H\left(\frac{k'}{n} + \frac{j}{n}\right) - H\left(\frac{k'}{n}\right), \quad \|\delta\| := \max_{1 \leq i \leq q} |\delta_i|, \quad \|\delta'\| := \max_{1 \leq j \leq q} |\delta'_j|,$
- $\beta_i := \sigma\left(\frac{k}{n} + \frac{i}{n}\right) - \sigma\left(\frac{k}{n}\right), \quad \beta'_j := \sigma\left(\frac{k'}{n} + \frac{j}{n}\right) - \sigma\left(\frac{k'}{n}\right), \quad \|\beta\| := \max_{1 \leq i \leq q} |\beta_i|, \quad \|\beta'\| := \max_{1 \leq j \leq q} |\beta'_j|.$

By the definitions in (2.3)-(2.5),

$$\begin{aligned} \mathbb{E}[V_n^a Y\left(\frac{k}{n}\right) V_n^a Y\left(\frac{k'}{n}\right)] &= \sum_{i, j=0}^q a_i a_j Q\left(\sigma\left(\frac{k+i}{n}\right), \sigma\left(\frac{k'+j}{n}\right), H\left(\frac{k+i}{n}\right), H\left(\frac{k'+j}{n}\right), \frac{k+i}{n}, \frac{k'+j}{n}\right) \\ &= \sum_{i, j=0}^q a_i a_j Q\left(\sigma\left(\frac{k}{n}\right) + \beta_i, \sigma\left(\frac{k'}{n}\right) + \beta'_j, H\left(\frac{k}{n}\right) + \delta_i, H\left(\frac{k'}{n}\right) + \delta'_j, \frac{k}{n} + \frac{i}{n}, \frac{k'}{n} + \frac{j}{n}\right), \end{aligned}$$

Similarly to (2.6), for any collection $\mathbf{p} = (p_1, \dots, p_6) \in \mathbb{N}^6$ of integers and $s \neq s'$, introduce partial derivatives of order $|\mathbf{p}| := p_1 + \dots + p_6$ of Q in (2.4):

$$R_{\sigma^{p_1}, \sigma'^{p_2}, \dots, s'^{p_6}}^{(|\mathbf{p}|)}(s, s') := \frac{\partial^{|\mathbf{p}|} Q}{\partial \sigma^{p_1} \partial \sigma'^{p_2} \dots \partial s'^{p_6}}(\sigma(s), \sigma(s'), H(s), H(s'), s, s'); \quad (5.6)$$

in particular, $R_H^{(1)}(s, s') = \frac{\partial Q}{\partial H}(\sigma(s), \sigma(s'), H(s), H(s'), s, s'), \dots, R_{s'}^{(1)}(s, s') = \frac{\partial Q}{\partial s'}(\sigma(s), \sigma(s'), H(s), H(s'), s, s')$. Without loss of generality, assume $k' - k > 2q$ in the sequel. Using the Taylor expansion of order $2m$ of Q in $(\sigma, \sigma', H, H', s, s')$, one obtains:

$$\mathbb{E}[V_n^a Y\left(\frac{k}{n}\right) V_n^a Y\left(\frac{k'}{n}\right)] = \sum_{0 \leq |\mathbf{p}| \leq 2m} J_n(\mathbf{p}) + \sum_{|\mathbf{p}|=2m+1} \tilde{J}_n(\mathbf{p}), \quad (5.7)$$

where

$$\begin{aligned} J_n(\mathbf{p}) &:= \frac{1}{p_1! \dots p_6!} R_{\sigma^{p_1}, \dots, s'^{p_6}}^{(|\mathbf{p}|)}\left(\frac{k}{n}, \frac{k'}{n}\right) \sum_{i, j=0}^q a_i a_j \beta_i^{p_1} (\beta'_j)^{p_2} \dots \left(\frac{j}{n}\right)^{p_6}, \\ \tilde{J}_n(\mathbf{p}) &\leq C \sup_{I(\frac{k}{n}, \frac{k'}{n})} |R_{\sigma^{p_1}, \dots, s'^{p_6}}^{(|\mathbf{p}|)}(s, s')| \sum_{i, j=0}^q |\beta_i^{p_1} (\beta'_j)^{p_2} \dots \left(\frac{j}{n}\right)^{p_6}| \end{aligned}$$

and where the set $I(\frac{k}{n}, \frac{k'}{n}) \subset (0, \infty)^2 \times (0, \infty)^2 \times (0, 1)^2$ is defined by

$$I\left(\frac{k}{n}, \frac{k'}{n}\right) := \left\{ (\sigma, \sigma', H, H', s, s') : |\sigma - \sigma\left(\frac{k}{n}\right)| \leq \|\beta\|, \dots, |H' - H\left(\frac{k'}{n}\right)| \leq \|\delta'\|, |s - \frac{k}{n}| \leq \frac{q}{n}, |s' - \frac{k'}{n}| \leq \frac{q}{n}, s < s' \right\}.$$

Note terms in (5.7) corresponding to $|\mathbf{p}| = 0$ and $|\mathbf{p}| = 1$ vanish because of the moment condition (1.6). In particular,

$$\sum_{|\mathbf{p}|=1} J_n(\mathbf{p}) = \sum_{i,j=0}^q a_i a_j \left(\beta_i R_\sigma^{(1)}\left(\frac{k}{n}, \frac{k'}{n}\right) + \cdots + \frac{j}{n} R_{s'}^{(1)}\left(\frac{k}{n}, \frac{k'}{n}\right) \right) = 0.$$

In a similar way,

$$\begin{aligned} \sum_{|\mathbf{p}|=2} J_n(\mathbf{p}) &= \frac{1}{2} \left(2 \left(\sum_{i=0}^q a_i \beta_i \right) \left(\sum_{j=0}^q a_j \beta'_j \right) R_{\sigma, \sigma'}^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) + \cdots + 2 \left(\sum_{i=0}^q a_i \frac{i}{n} \right) \left(\sum_{j=0}^q a_j \frac{j}{n} \right) R_{s, s'}^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) \right) \\ &= \left(V_n^a H\left(\frac{k}{n}\right), V_n^a \sigma\left(\frac{k}{n}\right), V_n^a \frac{k}{n} \right) \times R^{(2)}\left(\frac{k}{n}, \frac{k'}{n}\right) \times \left(V_n^a H\left(\frac{k'}{n}\right), V_n^a \sigma\left(\frac{k'}{n}\right), V_n^a \frac{k'}{n} \right)^\top, \end{aligned} \quad (5.8)$$

since $\sum_{i=0}^q a_i \beta_i = V_n^a H\left(\frac{k}{n}\right), \dots, \sum_{j=0}^q a_j \beta'_j = V_n^a H\left(\frac{k'}{n}\right)$. Note that the partial derivatives in (5.6) which do not involve arguments s and s' are bounded while each differentiation in s or s' contributes a factor $|\frac{k'-k}{n}|^{-1}$; in other words,

$$\sup_{I\left(\frac{k}{n}, \frac{k'}{n}\right)} \left| R_{\sigma^{p_1}, \dots, \sigma^{p_6}}^{(|\mathbf{p}|)}(s, s') \right| \leq C \left(1 + \left| \frac{k'-k}{n} \right|^{H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) - p_5 - p_6} \right) \leq C \left(1 + \left| \frac{k'-k}{n} \right|^{H(t) + H(t') - p_5 - p_6} \right) \quad (5.9)$$

since $(|H(t) - H(k/n)| + |H(t') - H(k'/n)|) \log(|k' - k|/n) = o(1)$. From (5.3) and (5.9) we obtain

$$\left| \sum_{|\mathbf{p}|=2} J_n(\mathbf{p}) \right| \leq C \left(\frac{1}{n^{2(\eta \wedge m)}} + \frac{|k' - k|^{H(t) + H(t') - 2}}{n^{H(t) + H(t')}} \mathbf{1}_{\{m=1\}} \right) \quad (5.10)$$

since $V_n^a \frac{k}{n} = 0$ for $m > 1$. We also have

$$\begin{aligned} J_n(0, 0, 0, 0, m, m) &= \frac{1}{2(m!)^2} \left(\sum_{i=1}^q i^m a_i \right)^2 \sigma\left(\frac{k}{n}\right) \sigma\left(\frac{k'}{n}\right) L\left(H\left(\frac{k}{n}\right), H\left(\frac{k'}{n}\right)\right) \\ &\quad \times \prod_{\ell=0}^{2m-1} \left(H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) - \ell \right) \left(\frac{|k - k'|}{n} \right)^{H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) - 2m} \\ &= \frac{1}{2} \left(\sum_{i=1}^q i^m a_i \right)^2 \sigma^2(t) \prod_{\ell=0}^{2m-1} \left(H(t) + H(t') - \ell \right) \left(\frac{|k' - k|^{H(t) + H(t') - 2m}}{n^{H(t) + H(t')}} \right) (1 + o(1)). \end{aligned} \quad (5.11)$$

It remains to prove that all the other terms in the Taylor expansion (5.7) are negligible with respect to (5.11) or (5.8) as estimated in (5.10). The above task is trivial in the case $t \neq t'$ so that we focus on the case $t' = t$ in the rest of the proof. Define

$$\Lambda_n := \frac{1}{n^{2(\eta \wedge m)}} + \frac{|k' - k|^{2H(t) - 2m}}{n^{2H(t)}}.$$

We shall prove below that

$$J_n(\mathbf{p}) = o(\Lambda_n), \quad 2 < |\mathbf{p}| \leq 2m, \quad (5.12)$$

$$\tilde{J}_n(\mathbf{p}) = o(\Lambda_n), \quad |\mathbf{p}| = 2m + 1. \quad (5.13)$$

Then the statement of the lemma for $t = t'$ follows from (5.7), (5.8) and (5.12), (5.13).

Let us prove (5.12). Define

$$U_n(\mathbf{p}) := \sum_{i=0}^q a_i \beta_i^{p_1} \delta_i^{p_3} \left(\frac{i}{n}\right)^{p_5}, \quad U'_n(\mathbf{p}) := \sum_{j=0}^q a_j (\beta'_j)^{p_2} (\delta'_j)^{p_3} \left(\frac{j}{n}\right)^{p_6}. \quad (5.14)$$

We claim that for any $\eta > 0$ and $m \geq 1$

$$|U_n(\mathbf{p})| \leq C \left(\frac{1}{n^{m \vee p_5}} + \frac{1}{n^{\eta + p_5}} \right) \quad (5.15)$$

and a similar bound holds for $U'_n(\mathbf{p})$. Let us check that (5.15) and (5.9) imply (5.12). Indeed, from the above relations and the definition of $J_n(\mathbf{p})$ we obtain

$$\begin{aligned} |J_n(\mathbf{p})| &\leq |R_{\sigma^{p_1}, \dots, t^{p_6}}(\frac{k}{n}, \frac{k'}{n})| |U_n(\mathbf{p})| |U'_n(\mathbf{p})| \\ &\leq C \left| \frac{k' - k}{n} \right|^{2H(t) - p_5 - p_6} \left(\frac{1}{n^{m \vee p_5}} + \frac{1}{n^{\eta + p_5}} \right) \left(\frac{1}{n^{m \vee p_6}} + \frac{1}{n^{\eta + p_6}} \right) \leq I_{11} + I_{12} + I_{21} + I_{22}, \end{aligned}$$

where

$$\begin{aligned} I_{11} &:= \left| \frac{k' - k}{n} \right|^{2H(t) - p_5 - p_6} \frac{1}{n^{2m}}, & I_{12} &:= \left| \frac{k' - k}{n} \right|^{2H(t) - p_5 - p_6} \frac{1}{n^{(m \vee p_6) + \eta + p_5}}, \\ I_{21} &:= \left| \frac{k' - k}{n} \right|^{2H(t) - p_5 - p_6} \frac{1}{n^{(m \vee p_5) + \eta + p_6}}, & I_{22} &:= \left| \frac{k' - k}{n} \right|^{2H(t) - p_5 - p_6} \frac{1}{n^{2\eta + p_5 + p_6}}. \end{aligned}$$

It suffices to prove (5.12) for I_{ij} , $i, j = 1, 2$.

Let $H := H(t)$, $p := p_5 + p_6$. For I_{11} , (5.12) follows from $\left| \frac{k' - k}{n} \right|^{2H - p} \frac{1}{n^{2m}} = o\left(\frac{|k' - k|^{2H - 2m}}{n^{2H}}\right)$, which is immediate from $p < 2m$ and $|k' - k| = o(n)$.

Consider I_{22} . Assume first that $\eta < m$. Observe that in this case,

$$\Lambda_n \leq 2 \begin{cases} n^{-2\eta}, & |k' - k| \geq n^{(\eta - H)/(m - H)}, \\ |k' - k|^{2H - 2m} n^{-2H}, & |k' - k| \leq n^{(\eta - H)/(m - H)}. \end{cases} \quad (5.16)$$

Therefore in the case $|k' - k| \geq n^{(\eta - H)/(m - H)}$, the bound (5.12) for I_{22} translates to $|k' - k|^{2H - p} = o(n^{2H})$ which is obvious by $p \geq 0$ and $|k' - k| = o(n)$. Next, let $|k' - k| \leq n^{(\eta - H)/(m - H)}$, $p \geq 1$, then the bound (5.12) for I_{22} becomes $|k' - k| = o(n^{2\eta/(2m - p)})$ which holds by $(\eta - H)/(m - H) < 2\eta/(2m - p)$ since $H < \eta < m$. Next, let $\eta \geq m$. Then the first term in the definition of Λ_n is negligible with respect to the second term and the corresponding relation reduces to

$$\frac{|k' - k|^{2H - p_5 - p_6}}{n^{2\eta + 2H}} = o\left(\frac{|k' - k|^{2H - 2m}}{n^{2H}}\right), \quad (5.17)$$

which is obvious by $|k' - k| = o(n)$ and $\eta \geq m$. This proves (5.12) for I_{22} .

Next, consider I_{12} . Let $\eta < m$ and $|k' - k| \geq n^{(\eta - H)/(m - H)}$. Firstly, let $p_6 \leq m$. Then using (5.16) the bound (5.12) for I_{12} translates to $n^{(\eta - m + p_6 - 2H)/(p_5 + p_6 - 2H)} = o(|k' - k|)$ or inequality $(\eta - m + p_6 - 2H)/(p_5 + p_6 - 2H) < (\eta - H)/(m - H)$, which is equivalent to

$$0 < p_5(\eta - H) + (m - \eta)(m + H - p_6). \quad (5.18)$$

The last inequality is immediate by $m > \eta > H$ and $p_6 \leq m$. Secondly, let $p_6 > m$. Then using (5.16) the bound (5.12) for I_{12} follows from $n^{(\eta - 2H)/(p_5 + p_6 - 2H)} = o(|k' - k|)$ or inequality $(\eta - 2H)/(p_5 + p_6 - 2H) < (\eta - H)/(m - H)$. Using $p_5 + p_6 \geq m + 1$, the previous inequality reduces to $0 < (\eta - H) + (m - \eta)H$ which is obvious by $m > \eta > H$.

Next, consider (5.12) for I_{12} and $\eta < m$, $|k' - k| \leq n^{(\eta - H)/(m - H)}$. Then using (5.16), relation (5.12) again reduces to the same inequalities (5.18) (for $p_6 \leq m$) and $0 < (\eta - H) + (m - \eta)H$ (for $p_6 > m$), as above.

It remains to show (5.12) for I_{12} and $\eta \geq m$, in which case it reduces to

$$\frac{|k' - k|^{2H - p_5 - p_6}}{n^{(m \vee p_6) - p_6 + \eta + 2H}} = o\left(\frac{|k' - k|^{2H - 2m}}{n^{2H}}\right)$$

c.f. (5.17). The above relation verifies easily for $\eta \geq m$ and $|k' - k| = o(n)$. This proves (5.12) for J_{12} . Since consideration of I_{21} is completely analogous, the proof of (5.12) is now complete.

Let us prove the claim (5.15). Write $\delta_i = \delta_{0i} + \Gamma_i$, $\beta_i = \beta_{0i} + \Sigma_i$, where

$$\Gamma_i := \sum_{r=1}^{[\eta]} \frac{H_r}{r!} \left(\frac{i}{n}\right)^r, \quad \Sigma_i := \sum_{r=1}^{[\eta]} \frac{\sigma_r}{r!} \left(\frac{i}{n}\right)^r, \quad H_r := H^{(r)}\left(\frac{k}{n}\right), \quad \sigma_r := \sigma^{(r)}\left(\frac{k}{n}\right).$$

By Assumption C(η),

$$\max(|\delta_{0i}|, |\beta_{0i}|) \leq C n^{-\eta}, \quad \max(|H_r|, |\sigma_r|) \leq C. \quad (5.19)$$

From binomial expansion,

$$U_n(\mathbf{p}) = \sum \binom{p_1}{\tau_{01}, \dots, \tau_{[\eta]1}} \binom{p_3}{\tau_{03}, \dots, \tau_{[\eta]3}} \left(\prod_{r=1}^{[\eta]} \sigma_r^{\tau_{r1}} H_r^{\tau_{r3}} \right) \sum_{i=0}^q a_i \beta_{0i}^{\tau_{01}} \delta_{0i}^{\tau_{03}} \left(\frac{i}{n} \right)^{p_5 + \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3})},$$

where the first sum is taken over all decompositions

$$p_1 = \tau_{01} + \dots + \tau_{[\eta]1}, \quad p_3 = \tau_{03} + \dots + \tau_{[\eta]3}$$

into nonnegative integers $\tau_{ij} \in \mathbb{N}$, $i = 0, 1, \dots, [\eta]$, $j = 1, 3$. Let $\tau := \tau_{01} + \tau_{03}$. According to (1.6), for any $s = 0, 1, \dots$

$$\left| \sum_{i=0}^q a_i \left(\frac{i}{n} \right)^s \right| \leq C n^{-s} \mathbf{1}(s \geq m).$$

Therefore from (5.19) it follows that

$$\left| \sum_{i=0}^q a_i \beta_{0i}^{\tau_{01}} \delta_{0i}^{\tau_{03}} \left(\frac{i}{n} \right)^{p_5 + \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3})} \right| \leq C \begin{cases} n^{-\eta\tau - p_5 - \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3})}, & \tau \geq 1, \\ n^{-p_5 - \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3})}, & \tau = 0, p_5 + \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3}) \geq m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $\tau = 0$, we have $\sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3}) \geq \sum_{r=0}^{[\eta]} (\tau_{r1} + \tau_{r3}) = p_1 + p_3$. Hence,

$$n^{-p_5 - \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3})} \mathbf{1}(\tau = 0, p_5 + \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3}) \geq m) \leq (n^{-m}) \wedge (n^{-p_1 - p_3 - p_5}) \leq n^{-(m \vee p_5)}.$$

Therefore

$$\left| \sum_{i=0}^q a_i \beta_{0i}^{\tau_{01}} \delta_{0i}^{\tau_{03}} \left(\frac{i}{n} \right)^{p_5 + \sum_{r=1}^{[\eta]} r(\tau_{r1} + \tau_{r3})} \right| \leq C(n^{-\eta - p_5} + n^{-(m \vee p_5)}),$$

proving (5.15) and (5.12).

It remains to prove (5.13). Let $\tilde{U}_n(\mathbf{p}) := \sum_{i=0}^q |a_i \beta_i^{p_1} \delta_i^{p_3} \left(\frac{i}{n} \right)^{p_5}|$. Since $\|\beta\| + \|\delta\| \leq C n^{-(\eta \wedge 1)}$, see (5.19), it immediately follows that

$$\tilde{U}_n(\mathbf{p}) \leq C n^{-(\eta \wedge 1)(p_1 + p_3) - p_5} \quad (5.20)$$

and a similar bound holds for $\tilde{U}'_n(\mathbf{p}) := \sum_{j=0}^q |a_j (\beta'_j)^{p_2} (\delta'_j)^{p_4} \left(\frac{j}{n} \right)^{p_6}|$. Relations (5.20) and (5.9) imply

$$\tilde{J}_n(\mathbf{p}) \leq C \left(1 + \left| \frac{k' - k}{n} \right|^{2H-p} \right) n^{-(\eta \wedge 1)(2m+1-p) - p} \quad (5.21)$$

where $p = p_5 + p_6 \in \{0, 1, \dots, 2m+1\}$ and $|\mathbf{p}| = 2m+1$. Relation (5.13) now can be verified directly, by inspecting the cases $\eta < 1, \eta \geq 1, p = 0, 1, 2m, 2m+1$ and $1 < p < 2m$ separately. This ends the proof of Lemma 2. \square

Proof of Theorem 1. The proof is based on a moment bound (Lemma 1) established in [5]. First, for $t_0 \in (0, 1)$, consider

$$S_n^{(i)} := \frac{1}{v_{n,\alpha}(t_0)} \sum_{k \in \mathcal{V}_{n,\alpha}(t_0)} \frac{|V_n^{a^{(i)}} Z(k/n)|^2}{n^{2H(t_0)}} \quad \text{and} \quad S_n^* := \frac{1}{v_{n,\alpha}(t_0)} \sum_{k \in \mathcal{V}_{n,\alpha}(t_0)} \frac{|V_n^{a^*} Z(k/n) + V_n^{a^*} Z((k+1)/n)|}{|V_n^{a^*} Z(k/n)| + |V_n^{a^*} Z((k+1)/n)|}.$$

The study of the asymptotic properties of $(S_n^{(i)})_{1 \leq i \leq p}$ and S_n^* will induce the asymptotic properties of $\widehat{H}_{n,\alpha}^{(QV)}(t_0)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t_0)$. Now for $k \in \{0, 1, \dots, n-q-1\}$, $i = \{1, \dots, p\}$ define

$$(\sigma_n^{(i)}(k))^2 := \text{Var} [V_n^{a^{(i)}} Z(k/n)] \quad \text{and} \quad \rho_n^{(i,j)}(k) := \frac{\text{Cov} [V_n^{a^{(i)}} Z(k/n), V_n^{a^{(j)}} Z(k/n)]}{\sigma_n^{(i)}(k) \sigma_n^{(j)}(k)}$$

and

$$\sigma_n^{*2}(k) := \text{Var} [V_n^{a^*} Z(k/n)] \quad \text{and} \quad \rho_n^*(k) := \frac{\text{Cov} [V_n^{a^*} Z(k/n), V_n^{a^*} Z((k+1)/n)]}{\sigma_n^*(k) \sigma_n^*(k+1)}.$$

Now, with $m_n := v_{n,\alpha}(t_0) \sim 2n^{1-\alpha}$ ($n \rightarrow \infty$) and $j_n = [nt_0 - n^{1-\alpha}]$,

- $S_n^{(i)} = \frac{1}{m_n} \sum_{j=0}^{m_n-1} f_{j,m_n}^{(i)}(\mathbf{Y}_{m_n}^{(i)}(j))$ where $\begin{cases} \mathbf{Y}_{m_n}^{(i)}(j) := \frac{V_n^{a^{(i)}} Z((j_n + j)/n)}{\sigma_n^{(i)}(j_n + j)} \\ f_{j,m_n}^{(i)}(y) := n^{-2H(t_0)} (\sigma_n^{(i)}(j_n + j))^2 y^2 \end{cases}$
- $S_n^* = \frac{1}{m_n} \sum_{j=0}^{m_n-1} f_{j,m_n}^*(\mathbf{Y}_{m_n}^*(j))$ where $\mathbf{Y}_{m_n}^*(j) := (Y_{m_n}^{1*}(j), Y_{m_n}^{2*}(j)) \in \mathbb{R}^2$ and $\begin{cases} Y_{m_n}^{1*}(j) := \frac{V_n^{a^*} Z((j_n + j)/n)}{\sigma_n^*(j_n + j)} \\ Y_{m_n}^{2*}(j) := -\frac{V_n^{a^*} Z((j_n + j)/n)}{\sigma_n^*(j_n + j)} \frac{\rho_n^*(j_n + j)}{\sqrt{1 - \rho_n^{*2}(j_n + j)}} + \frac{V_n^{a^*} Z((j_n + j + 1)/n)}{\sigma_n^*(j_n + j + 1)} \frac{1}{\sqrt{1 - \rho_n^{*2}(j_n + j)}} \\ f_{j,m_n}^*(y^{(1)}, y^{(2)}) := \frac{\left| y^{(1)} + \frac{\sigma_n^*(j_n + j + 1)}{\sigma_n^*(j_n + j)} \left(\rho_n^*(j_n + j) y^{(1)} + \sqrt{1 - \rho_n^{*2}(j_n + j)} y^{(2)} \right) \right|}{\left| y^{(1)} + \frac{\sigma_n^*(j_n + j + 1)}{\sigma_n^*(j_n + j)} \left| \rho_n^*(j_n + j) y^{(1)} + \sqrt{1 - \rho_n^{*2}(j_n + j)} y^{(2)} \right| \right|} \end{cases} .$

Therefore, in both the case $\mathbf{Y}_{m_n}^{(i)}(j)$ and $\mathbf{Y}_{m_n}^*(j)$ are standardized Gaussian vectors and $f_{j,m_n}^{(i)}$ and f_{j,m_n}^* are functions with Hermite rank at least 2 for any j, m_n . Now, we use the common notation $S_n, \mathbf{Y}_{m_n}(j)$ and f_{j,m_n} without specifying the exponent $^{(i)}$ or * . Using the same method as in the proof of Theorem 1 in [5], under Assumptions **(A1)** and **(A2)**, one obtains that there exist $C > 0$ and $\kappa > 0$ such that $\mathbb{E}(S_n - \mathbb{E}S_n)^4 \leq Cn^{-\kappa}$. Indeed, let $\ell = \lceil m_n^{\theta/\gamma} \rceil$. Write

$$S_n - \mathbb{E}S_n = \frac{1}{m_n} \sum_{j=0}^{\ell-1} T_n^{(\ell)}(j), \quad \text{where } T_n^{(\ell)}(j) := \sum_{k=0}^{\lfloor (m_n - j)/\ell \rfloor} \eta_{m_n}(k\ell + j)$$

and $\eta_{m_n}(i) := f_{i,m_n}(\mathbf{Y}_{m_n}(i)) - \mathbb{E}[f_{i,m_n}(\mathbf{Y}_{m_n}(i))]$.

Then

$$\mathbb{E}(S_n - \mathbb{E}S_n)^4 \leq \left(\frac{\ell}{m_n}\right)^4 \max_{0 \leq j < \ell} \mathbb{E}(T_n^{(\ell)}(j))^4 \quad \text{and} \quad \mathbb{E}(T_n^{(\ell)}(j))^4 \leq C(\Sigma_4 + \Sigma_3 + \Sigma_{2,1} + \Sigma_{2,2}), \quad (5.22)$$

where

$$\begin{aligned} \Sigma_4 &:= \sum' |\mathbb{E}[\eta_{m_n}(k_1\ell + j) \cdots \eta_{m_n}(k_4\ell + j)]|, \\ \Sigma_3 &:= \sum' |\mathbb{E}[\eta_{m_n}(k_1\ell + j) \eta_{m_n}(k_2\ell + j) \eta_{m_n}^2(k_3\ell + j)]|, \\ \Sigma_{2,1} &:= \sum' |\mathbb{E}[\eta_{m_n}^3(k_1\ell + j) \eta_{m_n}(k_2\ell + j)]|, \\ \Sigma_{2,2} &:= \sum \mathbb{E}[\eta_{m_n}^2(k_1\ell + j) \eta_{m_n}^2(k_2\ell + j)], \end{aligned}$$

where \sum' (respectively, \sum) stands for the sum over all *different* integers $0 \leq k_i < \lfloor m_n/\ell \rfloor, k_i \neq k_j (i \neq j)$ (respectively, over all $0 \leq k_1, k_2 < \lfloor m_n/\ell \rfloor$). But, since $\mathbb{E}|\eta_{m_n}(k)|^p \leq C_p$ for all k, m_n and $p > 0$,

$$\Sigma_{2,i} \leq C(m_n/\ell)^2 \quad (i = 1, 2). \quad (5.23)$$

Let us estimate Σ_3, Σ_4 . Let $Q_{m_n, \ell}$ be defined such that:

$$Q_{m_n}^{(\ell)}(j) := \max_{1 \leq i \leq \lfloor m_n/\ell \rfloor} \sum_{k=1, k \neq i}^{\lfloor m_n/\ell \rfloor} \bar{\rho}_{m_n}^2(k\ell + j, i\ell + j), \quad (5.24)$$

where $\bar{\rho}_{m_n}(k, j) := |\mathbb{E}[Y_{m_n}(k)Y_{m_n}(j)]|$ in the case of $S_n^{(i)}$ and $\bar{\rho}_{m_n}(k, j) := \max_{u,v=1,2} |\mathbb{E}[Y_{m_n}^{(u)}(k)Y_{m_n}^{(v)}(j)]|$ in the case of S_n^* . But from Assumption **(A.2)** and since $H(\cdot)$ and $c(\cdot)$ are continuous functions such that $0 < \underline{H} \leq H(t) \leq \bar{H} < 1$ and $c(t) > 0$ for all $t \in [0, 1]$, then $\bar{\rho}_{m_n}(k, j) \leq \min(1, C m_n^\theta |j - k|^{-\gamma})$ (the case of S_n^* is detailed in [5]). Therefore

$$\begin{aligned} \max_{0 \leq j < \ell} Q_{m_n}^{(\ell)}(j) &\leq C m_n^{2\theta} \ell^{-2\gamma} \max_{0 \leq j < \ell} \max_{1 \leq i \leq \lfloor m_n/\ell \rfloor} \sum_{k=1, k \neq i}^{\lfloor m_n/\ell \rfloor} |k - i|^{-2\gamma} \\ &\leq 2C m_n^{2\theta} \ell^{-2\gamma} \sum_{k'=1}^{\infty} |k'|^{-2\gamma} \leq C \end{aligned} \quad (5.25)$$

because $\ell = \lceil m_n^{\theta/\gamma} \rceil$ and $\gamma > 1/2$. Thus, since the Hermite rank of f_{j,m_n} is at least $m = 2$ for any j, m_n , Σ_4 and Σ_3 can be bounded using Lemma 1 of [5] with $N = \lceil m_n/\ell \rceil$. For Σ_4 , with $p = \alpha = 4$, $m = 2$, $Q_N = \max_{0 \leq j < \ell} Q_{m_n}^{(\ell)}(j)$, then

$$\Sigma_4 \leq C (m_n/\ell)^{4-\frac{4}{2}} \left(\max_{0 \leq j < \ell} Q_{m_n}^{(\ell)}(j) \right)^{\frac{4}{2}} \leq C (m_n/\ell)^2. \quad (5.26)$$

Similarly, for Σ_3 , with $p = 3$, $\alpha = 2$ and $m = 2$,

$$\Sigma_3 \leq C (m_n/\ell)^{3-\frac{2}{2}} \left(\max_{0 \leq j < \ell} Q_{m_n}^{(\ell)}(j) \right)^{\frac{2}{2}} \leq C (m_n/\ell)^2. \quad (5.27)$$

Combining (5.23), (5.27) and (5.26) with (5.22), we obtain

$$\mathbb{E}(T_n^{(\ell)}(j))^4 \leq C (m_n/\ell)^2 \implies \mathbb{E}(S_n - \mathbb{E}S_n)^4 \leq C \left(\frac{\ell}{m_n} \right)^2.$$

Now, since $\ell = \lceil m_n^{\theta/\gamma} \rceil$ and $m_n \sim 2n^{1-\alpha}$, then $\mathbb{E}(S_n - \mathbb{E}S_n)^4 \leq C n^{2(1-\alpha)(\theta/\gamma-1)}$. When $\theta < \gamma$, it is clear that

$$S_n - \mathbb{E}S_n \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

With $\theta/\gamma < 1 - \frac{1}{2(1-\alpha)}$ (Assumption **(A.2)**), which is equivalent to $\alpha < \frac{\gamma-2\theta}{2(\gamma-\theta)}$, there exists $\kappa = 2(1-\alpha)(1-\theta/\gamma) > 1$ such that $\mathbb{E}(S_n - \mathbb{E}S_n)^4 \leq C n^{-\kappa}$ and Borel-Cantelli Lemma implies

$$S_n - \mathbb{E}S_n \xrightarrow[n \rightarrow \infty]{a.s.} 0. \quad (5.28)$$

Finally, it remains to evaluate $\mathbb{E}S_n$. But,

$$\mathbb{E}S_n = \frac{1}{m_n} \sum_{j=0}^{m_n-1} \mathbb{E}[f_{j,m_n}(\mathbf{Y}_{m_n}(j))] = \int_0^1 \mathbb{E}[f_{\lceil m_n t \rceil, m_n}(\mathbf{Y}_{m_n}(\lceil m_n t \rceil))] dt. \quad (5.29)$$

Moreover, for all $t \in [0, 1]$, from Assumption **(A.1)**, it can be established that $f_{\lceil m_n t \rceil, m_n}^{(i)} \xrightarrow[n \rightarrow \infty]{} \phi_{t_0}^{(i)}$ and $f_{\lceil m_n t \rceil, m_n}^* \xrightarrow[n \rightarrow \infty]{} \phi_{t_0}^*$ where,

$$\begin{aligned} \phi_{t_0}^{(i)}(y) &:= \lim_{n \rightarrow \infty} f_{\lceil m_n t \rceil, m_n}^{(i)}(y) = \left(\lim_{n \rightarrow \infty} n^{-2H(t_0)} (\sigma_n^{(i)}(\lceil nt_0 - n^{1-\alpha} \rceil + \lceil m_n t \rceil))^2 \right) y^2 \\ &= c(t_0) \left(-\frac{1}{2} \sum_{k,k'=0}^{iq} a_k^{(i)} a_{k'}^{(i)} |k' - k|^{2H(t_0)} \right) y^2 = K(t_0) i^{2H(t_0)} y^2 \end{aligned} \quad (5.30)$$

$$\begin{aligned} \phi_{t_0}^*(y^{(1)}, y^{(2)}) &:= \lim_{n \rightarrow \infty} f_{\lceil m_n t \rceil, m_n}^*(y^{(1)}, y^{(2)}) \\ &= \frac{|y^{(1)} + (\rho_2(H(t_0))y^{(1)} + \sqrt{1 - \rho_2^2(H(t_0))}y^{(2)})|}{|y^{(1)}| + |\rho_2(H(t_0))y^{(1)} + \sqrt{1 - \rho_2^2(H(t_0))}y^{(2)}|} \end{aligned} \quad (5.31)$$

(this last limit was established in the proof of Theorem 1 in [5]) with

$$K(t_0) := -\frac{c(t_0)}{2} \sum_{k,k'=0}^q a_k^{(1)} a_{k'}^{(1)} |k' - k|^{2H(t_0)} \quad \text{and} \quad \rho_2(H) := \frac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}}.$$

Therefore from Lebesgue Theorem (in both the cases, $f_{\lceil m_n t \rceil, m_n}$ is bounded by a $\mathbb{L}^1([0, 1])$ -function),

$$\begin{aligned} \int_0^1 \mathbb{E}[f_{\lceil m_n t \rceil, m_n}^{(i)}(\mathbf{Y}_{m_n}(\lceil m_n t \rceil))] dt &\xrightarrow[n \rightarrow \infty]{} K(t_0) i^{2H(t_0)} \int_0^1 1 dt = K(t_0) i^{2H(t_0)}; \\ \int_0^1 \mathbb{E}[f_{\lceil m_n t \rceil, m_n}^*(\mathbf{Y}_{m_n}(\lceil m_n t \rceil))] dt &\xrightarrow[n \rightarrow \infty]{} \int_0^1 \Lambda_2(H(t_0)) dt = \Lambda_2(H(t_0)), \end{aligned}$$

with $\Lambda_2(H) = \mathbb{E}[\phi_{t_0}^*(Z_1, Z_2)]$ where (Z_1, Z_2) is a standardized Gaussian vector, *i.e.* with the expression of $\rho_2(H)$ provided above (see [5] for details),

$$\Lambda_2(H) := \frac{1}{\pi} \arccos(-\rho_2(H)) + \frac{1}{\pi} \sqrt{\frac{1 + \rho_2(H)}{1 - \rho_2(H)}} \log \left(\frac{2}{1 + \rho_2(H)} \right). \quad (5.32)$$

With (5.29) and (5.28), we deduce that for all $t_0 \in (0, 1)$ (here we only indicates the end of the proof for strong consistency but the results are the same when $\xrightarrow[n \rightarrow \infty]{\mathcal{P}}$ replaces $\xrightarrow[n \rightarrow \infty]{a.s.}$),

$$S_n^{(i)} \xrightarrow[n \rightarrow \infty]{a.s.} K(t_0) i^{2H(t_0)} \quad \text{and} \quad S_n^* \xrightarrow[n \rightarrow \infty]{a.s.} \Lambda_2(H(t_0)).$$

Now it is clear that since $x \rightarrow \log(x)$ and $x \rightarrow (\Lambda_2)^{-1}(x)$ are continuous functions,

$$(\log S_n^{(i)})_{1 \leq i \leq p} \xrightarrow[n \rightarrow \infty]{a.s.} (2H(t_0) \log i + \log K(t_0))_{1 \leq i \leq p} \quad \text{and} \quad (\Lambda_2)^{-1}(S_n^*) \xrightarrow[n \rightarrow \infty]{a.s.} H(t_0),$$

and the strong consistency of $\widehat{H}_{n,\alpha}^{(QV)}(t_0)$ and $\widehat{H}_{n,\alpha}^{(IR)}(t_0)$ follows (the expression of $\widehat{H}_{n,\alpha}^{(QV)}(t_0)$ is obtained by the linear regression of $(\log(n^{2H(t_0)} S_n^{(i)}))_{1 \leq i \leq p}$ onto $(\log i)_{1 \leq i \leq p}$ and therefore the terms $\log K(t_0)$ and $\log n^{2H(t_0)}$ disappear). \square

Proof of Theorem 2. The proof is based on a CLT for multidimensional Gaussian triangular arrays (Theorem 1 of [5]) and follows the same steps than the proof of Corollary 4.3. established in [6]. We are going to use the notation of the proof of Theorem 1, except that in the case of $(S_n^{(i)})_{1 \leq i \leq p}$ a multidimensional CLT has to be proved. For ease of writing we will only consider the bidimensional case: let $(i_1, i_2) \in \{1, \dots, p\}^2$, $(u_{i_1}, u_{i_2}) \in \mathbb{R}^2$ and

$$S_n^{(i_1, i_2)} := u_{i_1} S_n^{(i_1)} + u_{i_2} S_n^{(i_2)} = \frac{1}{m_n} \sum_{j=0}^{m_n-1} f_{j, m_n}^{(i_1, i_2)}(\mathbf{Y}_{m_n}^{(i_1, i_2)}(j))$$

where $\mathbf{Y}_{m_n}^{(i_1, i_2)}(j) := (Y_{m_n}^{1, (i_1, i_2)}(j), Y_{m_n}^{2, (i_1, i_2)}(j)) \in \mathbb{R}^2$ and

$$\left\{ \begin{array}{l} Y_{m_n}^{1, (i_1, i_2)}(j) := \frac{V_n^{a(i_1)} Z((j_n + j)/n)}{\sigma_n^{(i_1)}(j_n + j)} \\ Y_{m_n}^{2, (i_1, i_2)}(j) := \frac{1}{\sqrt{1 - (\rho_n^{(i_1, i_2)}(j_n + j))^2}} \left(-\rho_n^{(i_1, i_2)}(j_n + j) \frac{V_n^{a(i_1)} Z((j_n + j)/n)}{\sigma_n^{(i_1)}(j_n + j)} + \frac{V_n^{a(i_2)} Z((j_n + j)/n)}{\sigma_n^{(i_2)}(j_n + j)} \right) \\ f_{j, m_n}^*(y^{(1)}, y^{(2)}) := n^{-2H(t_0)} \left(u_{i_1} (\sigma_n^{(i_1)}(j_n + j))^2 (y^{(1)})^2 \right. \\ \left. + u_{i_2} (\sigma_n^{(i_2)}(j_n + j))^2 \left(\rho_n^{(i_1, i_2)}(j_n + j) y^{(1)} + \sqrt{1 - (\rho_n^{(i_1, i_2)}(j_n + j))^2} y^{(2)} \right)^2 \right) \end{array} \right.$$

Now we use the common notation S_n , $\mathbf{Y}_{m_n}(j)$ and f_{j, m_n} . First, as in the previous proof, the asymptotic behavior of $f_{[m_n t], m_n}$ when $n \rightarrow \infty$ has to be studied. This requires to study the following asymptotic behaviors of $\sigma_n^{(i)}(j_n + [m_n t])$, $\rho_n^{(i_1, i_2)}(j_n + [m_n t])$, $\sigma_n^*(j_n + [m_n t])$ and $\rho_n^*(j_n + [m_n t])$. Using Assumption **(A.1')**, since $H(\cdot)$ and $c(\cdot)$ are $\mathcal{C}^\eta([0, 1])$ functions and $n^{(1-\alpha)/2} n^{-\eta} \log n \xrightarrow[n \rightarrow \infty]{} 0$ since α is supposed to be such that $\alpha > (1 + 2(\eta \wedge 1))^{-1}$, technical computations imply that:

$$\begin{aligned} n^{(1-\alpha)/2} \sup_{\tau \in [t_0 - n^{-\alpha}, t_0 + n^{-\alpha}]} \left| n^{2H(\tau)} (\sigma_n^{(i)})^2([n\tau]) - K(\tau) i^{2H(\tau)} \right| &\xrightarrow[n \rightarrow \infty]{} 0 \\ n^{(1-\alpha)/2} \sup_{\tau \in [t_0 - n^{-\alpha}, t_0 + n^{-\alpha}]} \left| \rho_n^{(i_1, i_2)}([n\tau]) - \rho^{(i_1, i_2)}(\tau) \right| &\xrightarrow[n \rightarrow \infty]{} 0 \\ n^{(1-\alpha)/2} \sup_{\tau \in [t_0 - n^{-\alpha}, t_0 + n^{-\alpha}]} \left| \frac{\sigma_n^*([n\tau] + 1)}{\sigma_n^*([n\tau])} - 1 \right| &\xrightarrow[n \rightarrow \infty]{} 0 \\ n^{(1-\alpha)/2} \sup_{\tau \in [t_0 - n^{-\alpha}, t_0 + n^{-\alpha}]} \left| \rho_n^*([n\tau]) - \rho_2(H(\tau)) \right| &\xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

with $\rho^{(i_1, i_2)}(\tau) := -\frac{1}{2} \sum_{k_1, k_2=0}^q a_{k_1}^{(1)} a_{k_2}^{(1)} |k_1 i_1 - k_2 i_2|^{2H(\tau)}$. Moreover we have $n^{(1-\alpha)/2} n^{-\alpha \eta} \log n \xrightarrow[n \rightarrow \infty]{} 0$ since $\alpha > (1 + 2(\eta \wedge 1))^{-1}$ and therefore

$$\begin{aligned} n^{(1-\alpha)/2} \sup_{\tau \in [t_0 - n^{-\alpha}, t_0 + n^{-\alpha}]} \left| K(\tau) - K(t_0) \right| &\xrightarrow[n \rightarrow \infty]{} 0 \\ n^{(1-\alpha)/2} \sup_{\tau \in [t_0 - n^{-\alpha}, t_0 + n^{-\alpha}]} \left| \rho^{(i_1, i_2)}(\tau) - \rho^{(i_1, i_2)}(t_0) \right| &\xrightarrow[n \rightarrow \infty]{} 0 \\ n^{(1-\alpha)/2} \sup_{\tau \in [t_0 - n^{-\alpha}, t_0 + n^{-\alpha}]} \left| \rho_2(H(\tau)) - \rho_2(H(t_0)) \right| &\xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

Finally, for all $y \in \mathbb{R}^2$,

$$n^{(1-\alpha)/2} \sup_{t \in [0,1]} |f_{[m_n t], m_n}(y) - \phi_{t_0}(y)| \xrightarrow[n \rightarrow \infty]{} 0 \quad (5.33)$$

with $\phi_{t_0}^*$ specified in (5.31) and

$$\phi_{t_0}^{(i_1, i_2)}(y^{(1)}, y^{(2)}) := K(t_0) \left(u_{i_1} i_1^{2H(t_0)} (y^{(1)})^2 + u_{i_2} i_2^{2H(t_0)} \left(\rho^{(i_1, i_2)}(t_0) y^{(1)} + \sqrt{1 - (\rho^{(i_1, i_2)}(t_0))^2} y^{(2)} \right)^2 \right).$$

Now the CLT for multidimensional triangular arrays of [5] can be applied to the sample means of $\tilde{f}_{k, m_n}(\mathbf{Y}_{m_n}(j)) := f_{k, m_n}(\mathbf{Y}_{m_n}(j)) - \mathbb{E}[f_{k, m_n}(Y)]$ with Y a \mathbb{R}^2 standardized Gaussian vector. Since $\theta = 0$, then condition (6.3) is satisfied and Lemme 6.1 part (iii) can be applied, with the Hermite rank of \tilde{f}_{k, m_n} and ψ_{t_0} at least 2. It is clear from its definition in both the cases that for all $J \in \mathbb{N}$, and with Assumption **(A.1')**, that $(\mathbf{Y}_{m_n}([m_n t] + j))_{-J \leq j \leq J} \xrightarrow[n \rightarrow \infty]{f.d.d.} (\mathbf{W}_t(j))_{-J \leq j \leq J}$ where $(\mathbf{W}_t(j))_{j \in \mathbb{Z}}$ is a stationary Gaussian process, and therefore condition (6.8) is satisfied. Thus,

$$n^{(1-\alpha)/2} (S_n - ES_n) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \gamma^2(t_0)),$$

with $\gamma^2(t_0) := \int_0^1 dt \left(\sum_{j \in \mathbb{Z}} \mathbb{E}[\phi_{t_0}(\mathbf{W}_t(0)) \phi_{t_0}(\mathbf{W}_t(j))] \right)$. Finally, using (5.33), it is clear that

$$n^{(1-\alpha)/2} (ES_n - \int_0^1 \mathbb{E}[\phi_{t_0}(Y)] dt) \xrightarrow[n \rightarrow \infty]{} 0,$$

and therefore

$$n^{(1-\alpha)/2} (S_n - \int_0^1 \mathbb{E}[\phi_{t_0}(Y)] dt) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \gamma^2(t_0)).$$

Then with the expression of $\gamma^2(t_0)$ in both the cases and the Delta-Method, it follows that:

$$\begin{aligned} n^{(1-\alpha)/2} (\log S_n^{(i)} - (2H(t_0) \log i + \log K(t_0)))_{1 \leq i \leq p} &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Gamma(H(t_0))) \\ n^{(1-\alpha)/2} ((\Lambda_2)^{-1}(S_n^*) - H(t_0)) &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma_2(H(t_0))) \end{aligned}$$

with $H \rightarrow \Sigma_2(H)$ provided and drawn in [6] and $\Gamma(H(t_0))$ such that

$$\Gamma(H) := 2 \left(\sum_{j \in \mathbb{Z}} \left(\frac{\sum_{k_1, k_2=0}^q a_{k_1}^{(1)} a_{k_2}^{(1)} |i_1 k_1 - i_2 k_2 + j|^{2H}}{\sum_{k_1, k_2=0}^q a_{k_1}^{(1)} a_{k_2}^{(1)} |i_1 k_1 - i_2 k_2|^{2H}} \right)^2 \right)_{1 \leq i_1, i_2 \leq p}. \quad (5.34)$$

In the case of $\widehat{H}_{n, \alpha}^{(QV)}(t_0)$, using again the linear regression of $(\log S_n^{(i)})$ onto $(\log i)$ which is the same as the linear regression of $(\log(n^{2H(t_0)} S_n^{(i)}))$ onto $(\log i)$ concerning the estimation of the slope, Theorem 2 is proved. \square

Proof of Corollary 1. From Property 1, one obtains that under Assumption **C**(η), for all $t_0 \in (0, 1)$, and all $p \in \mathbb{N}$,

$$\max_{k \in \mathcal{V}_{n, \alpha}(t_0)} \left| \frac{\mathbb{E}(Y(\frac{k+p}{n}) - Y(\frac{k}{n}))^2}{(\frac{p}{n})^{2H(t_0)}} - \sigma^2(t_0) \right| \leq C_{H, \sigma}(p) \frac{\log n}{n^{\alpha(\eta \wedge 1)}}$$

Therefore, Assumption **(A.1)** is satisfied for all $\alpha \in (0, 1)$ while Assumption **(A.1')** is satisfied when

$$\alpha > \frac{1}{1 + 2(\eta \wedge 1)}. \quad (5.35)$$

In the other hand, Assumption **(A.2)** can be deduced from Property 2. Indeed, Property 2 induces that there exists $C > 0$ not depending on k, k' and n such that for all $k, k' \in \mathcal{V}_{n, \alpha}(t_0)$,

$$\left| \text{Cor} \left[V_n^a Y(\frac{k}{n}) V_n^a Y(\frac{k'}{n}) \right] \right| \leq C \left(\frac{1}{(|k' - k| + 1)^{m-H(t_0)}} + \frac{1}{n^{(\eta \wedge m) - H(t_0)}} \right)^2. \quad (5.36)$$

(i) Assumption **(A.1)** is satisfied for all $\alpha \in (0, 1)$.

(ii) Let us check conditions of Theorem 1 (ii) for $Z = Y$. As previously, Assumption **(A.1)** is satisfied for all $\alpha \in (0, 1)$. To verify **(A.2)**, for α satisfying (3.6), choose

$$\max\left(\frac{1}{2}, 2 \min\left(\frac{(\eta \wedge m) - H(t_0)}{1 - \alpha}, m - H(t_0)\right)\right) < \gamma < 4 \min((\eta \wedge m) - H(t_0), (m - H(t_0))(1 - \alpha)) \quad (5.37)$$

(note that from (3.6), $4((\eta \wedge 1) - H(t_0)) > \frac{1}{2}$, $4(m - H(t_0))(1 - \alpha) > \frac{1}{2}$ and $4((\eta \wedge m) - H(t_0)) > \frac{2((\eta \wedge m) - H(t_0))}{1 - \alpha}$, $4(m - H(t_0))(1 - \alpha) > 2m - 2H(t_0)$ since $\alpha < 1/2$ so that the interval in (5.37) is nonempty) and then define

$$\theta := \gamma - 2 \min\left(\frac{(\eta \wedge m) - H(t_0)}{1 - \alpha}, m - H(t_0)\right). \quad (5.38)$$

This choice (5.37)-(5.38) satisfies $0 \leq \theta < \gamma/2$, $\gamma > 1/2$ required by **(A.2)**. Moreover, $\alpha < \frac{\gamma - 2\theta}{2(\gamma - \theta)}$ as it is required in Theorem 1, part (ii). It remains to check that the inequality (5.36) induced by Property 2 satisfies (3.3), *i.e.* that

$$\frac{1}{k^{2m - 2H(t_0)}} \leq C \frac{n^{(1-\alpha)\theta}}{k^\gamma} \quad \text{and} \quad \frac{1}{n^{2((\eta \wedge m) - H(t_0))}} \leq C \frac{n^{(1-\alpha)\theta}}{k^\gamma} \quad (5.39)$$

hold for all $1 \leq k \leq n^{1-\alpha}$. These inequalities hold if they hold for $k = n^{1-\alpha}$; then this is obvious from the definition of $\gamma - \theta$ in (5.38).

(iii) Let us check conditions of Theorem 2 for $Z = Y$. Condition $\alpha > 1 - 2(\eta \wedge 1)$ is included in the condition $\alpha > (1 + 2(\eta \wedge 1))^{-1}$ of the corollary. Assumption **(A.2)** with $\theta = 0$ and $\gamma > 1/2$ follows from Property 2 since $2m - 2H(t_0) > 1/2$ (valid for $m = 1$ since $H(t_0) < 3/4$) and

$$\frac{1}{n^{2((\eta \wedge m) - H(t_0))}} = o\left(\frac{1}{n^{(1-\alpha)\gamma}}\right)$$

for $\gamma > 1/2$ sufficiently close to $1/2$ in view of the condition $\alpha > 1 - 4((\eta \wedge m) - H(t_0))$ of the corollary. Finally, **(A.2)** is satisfied by Property 2 since $(\eta \wedge 1) > \frac{1-\alpha}{2}$ and $2((\eta \wedge m) - H(t_0)) > \frac{1-\alpha}{2}$ follows from the conditions of the corollary. \square

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