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NONEXISTENCE OF POSITIVE SUPERSOLUTIONS OF ELLIPTIC EQUATIONS VIA THE MAXIMUM PRINCIPLE

SCOTT N. ARMSTRONG AND BOYAN SIRAKOV

ABSTRACT. We introduce a new method for proving the nonexistence of positive supersolutions of elliptic inequalities in the whole space \mathbb{R}^n as well as exterior domains. The simplicity and robustness of our maximum principle-based argument provides for its applicability to many elliptic inequalities and systems, including quasilinear operators such as the p -Laplacian, and non-divergence form fully nonlinear operators such as Bellman-Isaacs operators. Our method gives new and optimal results in terms of the nonlinear functions appearing in the inequalities.

1. INTRODUCTION

A well-studied problem in the theory of the elliptic partial differential equations is that of determining for which nonnegative, nonlinear functions $f = f(s, x)$ there exists a positive solution or supersolution $u > 0$ of the equation

$$(1.1) \quad -Q[u] = f(u, x),$$

in some subset of \mathbb{R}^n ; here Q denotes a second-order elliptic differential operator. A model case is the semilinear inequality

$$(1.2) \quad -\Delta u \geq f(u),$$

where f is a positive continuous function defined on $(0, \infty)$. There is a vast literature on the problem of obtaining sufficient conditions on f to ensure the nonexistence of positive supersolutions of such equations, both in \mathbb{R}^n and in subsets of \mathbb{R}^n , which encompasses many different choices of operators Q and nonlinear functions f .

In this paper we introduce a new method for proving the nonexistence of supersolutions of such equations in unbounded domains. It has the advantage of being both simple and robust, allowing us to prove new and essentially optimal results for wide classes of equations and systems of equations of type (1.1). In particular, we extend many of the previous Liouville results by substantially relaxing the hypotheses on f required for nonexistence. Namely, we impose only “local” conditions on the behavior of $f(s, x)$, near $s = 0$ or $s = \infty$, and for large $|x|$. Furthermore, our approach unites many previously known but seemingly disparate results by demonstrating that they follow from essentially the same argument.

Our method depends only on properties related to the maximum principle which are shared by many elliptic operators for which the solvability of (1.1) has been studied. Consequently, our technique applies to inequalities in both divergence

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and nondivergence form, interpreted in the appropriate (classical, weak Sobolev, or viscosity) sense.

To give a flavor of our results, let us consider the differential inequality (1.2) in dimensions $n \geq 3$, on the exterior domain $\mathbb{R}^n \setminus B$, where $B \subset \mathbb{R}^n$ is any ball. Under only the hypotheses that $f : (0, \infty) \rightarrow (0, \infty)$ is positive and continuous, as well as

$$(1.3) \quad 0 < \liminf_{s \searrow 0} s^{-n/(n-2)} f(s) \leq \infty,$$

we show that there does not exist a positive (classical, viscosity or weak Sobolev) solution of (1.2). Even this result is new.

The sufficiency of condition (1.3) for nonexistence validates the intuition that, since we expect positive supersolutions of (1.2) to vanish at infinity in dimensions $n \geq 3$ for a positive $f > 0$, it is only the behavior of $f(s)$ near $s = 0$ that should determine whether or not supersolutions exist. In the next section we prove this result and give the corresponding result in dimension $n = 2$, in which case it is the behavior of $f(s)$ at infinity which determines solvability.

The following is a rough list of the properties we assume the operator Q possesses, and on which our method relies:

- (H1) Q satisfies a weak comparison principle;
- (H2) the equations $-Q[\Phi] = 0$ and $-Q[\tilde{\Phi}] = 0$ have solutions in $\mathbb{R}^n \setminus \{0\}$ which are asymptotically homogeneous and positive (resp. negative) at infinity. In most cases, Φ and $\tilde{\Phi}$ are the *fundamental solutions* of Q , and in many familiar cases $\Phi = -\tilde{\Phi}$;
- (H3) nonnegative solutions of $-Q[u] \geq h(x) \geq 0$ have a lower bound (on compact subsets of the underlying domain) in terms of the measure of a set on which h is greater than a positive constant;
- (H4) nonnegative solutions of $-Q[u] \geq 0$ satisfy a weak Harnack inequality, or at least a “very weak” Harnack inequality; and
- (H5) the operator Q possesses some homogeneity.

Specific details on these hypotheses and on some operators which satisfy them are given in Section 3. For the Laplacian, (H1) and (H2) are very standard, (H3) follows from the positivity of Green’s function, (H4) is a consequence of the mean value property, and (H5) is obvious. These properties are verified for instance by quasilinear operators of p -Laplacian type with solutions interpreted in the weak Sobolev sense, and by fully nonlinear Isaacs operators with solutions interpreted in the viscosity sense.

We now make the following deliberately vague assertion:

Suppose Q has the properties (H1)-(H5) above, and the behavior of $f(s, x)$ near $s = 0$ (and in some cases $s = \infty$) compares appropriately with that of the functions Φ and $\tilde{\Phi}$ for large $|x|$. Then there does not exist a positive solution of the inequality

$$(1.4) \quad -Q[u] \geq f(u, x) \quad \text{in any exterior domain of } \mathbb{R}^n.$$

Obviously a nonexistence result in exterior domains implies the nonexistence of solutions in \mathbb{R}^n as well as the absence of singular solutions in \mathbb{R}^n with arbitrary singularities in a bounded set.

Rigorous statements with precise assumptions on f will be given in Section 4. As we will see, the above statement is optimal in the sense that if a model nonlinearity f does not satisfy its hypotheses, then (1.1) has positive supersolutions. The technique we introduce here extends very easily to *systems* of inequalities in unbounded domains.

Let us now give a brief account of the previous results on the subject. Due to the large number of works in the linear and quasilinear settings, we make no attempt to create an exhaustive bibliography here. Much more complete accounts can be found in the book of Veron [37] and the surveys articles of Mitidieri and Pohozaev [25] and Kondratiev, Liskevich, and Sobol [20]. In 1980, Gidas gave a simple proof of the fact that the equation $-\Delta u = u^\sigma$ has no solutions in \mathbb{R}^n , provided $\sigma \leq n/(n-2)$. Condition (1.3) appeared first in Ni and Serrin [27], where the nonexistence of *decaying* radial solutions to some quasilinear inequalities like $-\Delta_p u \geq u^\sigma$ in \mathbb{R}^n for $\sigma \leq n(p-1)/(n-p)$ was proved. In two important papers, Bidaut-Veron [4] and Bidaut-Veron and Pohozaev [3] extended these results by dropping the restrictions on the behavior of a supersolution u . In particular, they showed that the inequality $-\Delta_p u \geq |x|^{-\gamma} u^\sigma$ has no positive solution in any exterior domain provided that $\sigma \leq (n-\gamma)(p-1)/(n-p)$. For more nonexistence results for positive solutions of quasilinear inequalities with pure power right-hand sides, we refer to Birindelli and Mitidieri [7], Serrin and Zou [31], Liskevich, Skrypnik, and Skrypnik [24]. Liouville-type results for semilinear inequalities in nondivergence form can be found in the paper of Kondratiev, Liskevich, and Sobol [21].

Fully nonlinear inequalities of the form $F(D^2u) \geq u^\sigma$, where F is an Isaacs operator, were first studied by Cutri and Leoni [13], and later by Felmer and Quaas [17], in the case of a rotationally invariant F and a solution in the whole space. These results were recently extended in [1], by a different method, to more general operators, to exterior domains, and with less restrictive hypotheses on f . In particular, it was shown in [1] that the inequality $F(D^2u) \geq u^\sigma$ has no positive solutions in any exterior domain in \mathbb{R}^n , provided that $\sigma \leq (\alpha^* + 2)/\alpha^*$ (or $\alpha^* \leq 0$), where $\alpha^* = \alpha^*(F)$ characterizes the homogeneity of the upward-pointing fundamental solution of the operator F (as found in [2]).

As far as systems of inequalities are concerned, Liouville results were obtained by Mitidieri [26] for the case of a whole space, Bidaut-Veron [5] for quasilinear systems in exterior domains, and Quaas and Sirakov [29] for fully nonlinear systems in the whole space. For elliptic systems, the literature is more sparse with most results concerning only systems with pure power right-hand sides such as the Lane-Emden system $-\Delta u = v^\sigma$, $-\Delta v = u^\rho$.

Despite the great variety of approaches and methods, most of the previous results cited above required a global hypothesis on the function f , usually that f be a power function or a linear combination of power functions. A notable exception is the very recent work of D'Ambrosio and Mitidieri [15], who obtained various nonexistence results for divergence-form quasilinear inequalities in the whole space with only a local hypothesis on the function $f(s)$ near $s = 0$ as in (1.3). Their method is based on sophisticated integral inequalities and depends on the assumption that the inequality holds in the whole space.

There is a very large literature concerning Liouville results for solutions (not supersolutions) of equations of the form $-Q[u] = f(u)$ in \mathbb{R}^n , which started with

the well-known work by Gidas and Spruck [18]. These results are in general quite delicate and depend on the precise behavior of f on the whole interval $(0, \infty)$, as well as on the equality being verified in the whole space. We also note that there has been a large amount of work on proving Liouville type results for positive solutions of “coercive” elliptic inequalities $Q[u] \geq f(u, x)$ (as opposed to $-Q[u] \geq f(u, x)$). We do not consider these problems here.

This paper is organized as follows. In Section 2 we present the main ideas by proving a Liouville result in the simple particular case of (1.2). We collect some preliminary observations in Section 3, including a precise list of the properties (H1)-(H5) above as well as some estimates for the minima of positive supersolutions of $-Q[u] \geq 0$ over annuli. Our main results for scalar equations are presented in Section 4. We conclude in Section 5 with applications to systems of inequalities.

2. A SIMPLE SEMILINEAR INEQUALITY

In this section, we illustrate our main ideas by considering the model semilinear inequality

$$(2.1) \quad -\Delta u \geq f(u)$$

in exterior domains in dimension $n \geq 3$, and under the assumption that the nonlinearity $f = f(s)$ is positive and continuous on $(0, \infty)$. We will show that the additional hypothesis

$$(2.2) \quad \liminf_{s \searrow 0} s^{-n/(n-2)} f(s) > 0$$

implies that the inequality (2.1) has no positive solution in any exterior domain. Notice that we impose no requirements on the behavior of $f(s)$ away from $s = 0$, apart from continuity and positivity.

It is easily checked that for $q > n/(n-2)$, the function

$$u(x) := c_q (1 + |x|^2)^{-1/(q-1)}, \quad c_q := \left(\frac{(2n-4)(q-1) - 4}{(q-1)^2} \right)^{1/(q-1)},$$

is a smooth supersolution of $-\Delta u = u^q$ in \mathbb{R}^n . Moreover, the function $\tilde{u}(x) := c_q |x|^{-2/(q-1)}$ is a *solution* of the equation in $\mathbb{R}^n \setminus \{0\}$. Thus the following theorem is seen to be optimal in a certain sense.

Theorem 2.1. *Assume that $n \geq 3$ and the nonlinearity $f : (0, \infty) \rightarrow (0, \infty)$ is continuous and satisfies (2.2). Then the differential inequality (2.1) has no positive solution in any exterior domain of \mathbb{R}^n .*

We have left the statement of Theorem 2.1 intentionally vague as to the notion of supersolution, since the result holds regardless of whether we consider supersolutions in the classical, weak, or viscosity sense.

Several classical facts regarding the Laplacian operator are required for the proof of Theorem 2.1, and we recall them now. It is convenient for us to state them for annuli.

A basic ingredient in the proof of Theorem 2.1 is the following “quantitative” strong maximum principle.

Lemma 2.2. *Assume $h \in L^\infty(B_3 \setminus B_{1/2})$ is nonnegative, and $u \geq 0$ satisfies*

$$-\Delta u \geq h \quad \text{in } B_3 \setminus B_{1/2}.$$

Then for each $A \subset B_2 \setminus B_1$,

$$\inf_{B_2 \setminus B_1} u \geq c|A| \inf_A h.$$

This lemma is a simple consequence of the fact that Green's function for the Laplacian with respect to any domain is strictly positive away from the boundary of the domain, which yields

$$\inf_{B_2 \setminus B_1} u \geq c \int_{B_2 \setminus B_1} h(x) dx,$$

for some $c > 0$ depending only on the dimension n . See for example [8, Lemma 3.2] for a more general statement.

In order to show that only the behaviour of f at zero matters for the nonexistence result to hold, we will employ the following consequence of the mean value theorem.

Lemma 2.3. *For every $0 < \nu < 1$, there exists a constant $\bar{C} = \bar{C}(n, \nu) > 1$ such that for any positive superharmonic function u in $B_3 \setminus \bar{B}_{1/2}$ and any $x_0 \in B_2 \setminus B_1$, we have*

$$|\{u \leq \bar{C}u(x_0)\} \cap (B_2 \setminus B_1)| \geq \nu |B_2 \setminus B_1|.$$

Lemma 2.3 is clearly weaker than the weak Harnack inequality. A proof of the latter can be found in [19, Theorem 8.18] for weak solutions, and in [11, Theorem 4.8] for viscosity solutions.

Applying the comparison principle to a positive superharmonic function and the fundamental solution $\Phi(x) = |x|^{2-n}$ of Laplace's equation yields the following simple lemma, which is well-known. For the reader's convenience, we recall an elementary proof.

Lemma 2.4. *Suppose that $u > 0$ is superharmonic in an exterior domain Ω of \mathbb{R}^n , with $n \geq 3$. Then there are constants $C, c > 0$, depending only on u , such that*

$$(2.3) \quad cr^{2-n} \leq \inf_{B_{2r} \setminus B_r} u \leq C \quad \text{for every sufficiently large } r > 0.$$

Proof. Fix $r_0 > 0$ such that $\mathbb{R}^n \setminus B_{r_0} \subseteq \Omega$. Select $c > 0$ so small that $u \geq c\Phi$ on ∂B_{r_0} . Then for each $\varepsilon > 0$, there exists $\bar{R} = \bar{R}(\varepsilon) > r_0$ such that $u + \varepsilon \geq \varepsilon \geq \Phi$ in $\mathbb{R}^n \setminus B_{\bar{R}}$. Applying the maximum principle to

$$-\Delta(u + \varepsilon) \geq 0 = -\Delta\Phi$$

in $B_R \setminus B_{r_0}$, for each $R > \bar{R}(\varepsilon)$, we conclude that $u + \varepsilon \geq \Phi$ in $\mathbb{R}^n \setminus B_{r_0}$. Letting $\varepsilon \rightarrow 0$ we obtain $u \geq \Phi$ in $\mathbb{R}^n \setminus B_{r_0}$, which gives the first inequality in (2.3).

For the second inequality in (2.3), observe that for every $r > r_0$

$$u(x) \geq \Psi_r(x) := \left(\inf_{B_{2r} \setminus B_r} u \right) (1 - r_0^{n-2}|x|^{2-n}) \quad \text{for every } x \in \partial(B_r \setminus B_{r_0}),$$

as well as $-\Delta u \geq 0 = -\Delta\Psi_r$ in $\mathbb{R}^n \setminus B_{r_0}$. By the maximum principle we deduce that $u \geq \Psi_r(x)$ in $B_r \setminus B_{r_0}$. In particular, for every $r > 4r_0$, we have

$$\inf_{B_{4r_0} \setminus B_{2r_0}} u \geq \left(\inf_{B_{2r} \setminus B_r} u \right) (1 - 2^{2-n}),$$

which yields the second inequality in (2.3). \square

Let us now combine the three lemmas above into a proof of Theorem 2.1.

Proof of Theorem 2.1. Let us suppose that $u > 0$ is a supersolution of (2.1) in $\mathbb{R}^n \setminus B_{r_0}$, for some $r_0 > 0$. For each $r > 2r_0$, denote $u_r(x) := u(rx)$ and observe that u_r is a supersolution of

$$-\Delta u_r \geq r^2 f(u_r) \quad \text{in } \mathbb{R}^n \setminus B_{r_0/r}.$$

For each $r > R$, define the quantity

$$m(r) := \inf_{B_2 \setminus B_1} u_r = \inf_{B_{2r} \setminus B_r} u.$$

Set $A_r := \{x \in B_2 \setminus B_1 : m(r) \leq u_r(x) \leq \bar{C}m(r)\}$, where $\bar{C} > 1$ is as in Lemma 2.3. Then Lemma 2.3 implies that

$$|A_r| \geq (1/2)|B_2 \setminus B_1|$$

and thus applying Lemma 2.2 with $h(x) := r^2 f(u_r(x))$ produces the estimate

$$m(r) \geq \frac{1}{2} cr^2 |B_2 \setminus B_1| \min_{s \in [m(r), \bar{C}m(r)]} f(s) \quad \text{for every } r \geq 2r_0,$$

where $c > 0$ is as in Lemma 2.2. Applying Lemma 2.4 we obtain

$$(2.4) \quad cr^2 \min_{[m(r), \bar{C}m(r)]} f \leq m(r) \leq C.$$

Since $f > 0$ on $(0, \infty)$, it follows immediately from (2.4) that

$$\lim_{r \rightarrow \infty} m(r) = 0.$$

Hence if r is sufficiently large, (2.2) and (2.4) imply

$$cr^2(m(r))^{n/(n-2)} \leq m(r).$$

We may rewrite this inequality as

$$(2.5) \quad m(r) \leq Cr^{2-n} \quad \text{for every sufficiently large } r \geq 2r_0.$$

Recall that by Lemma 2.4 we also have, for some $c > 0$,

$$(2.6) \quad m(r) \geq cr^{2-n} \quad \text{for every } r > 2r_0.$$

Let us define the quantity

$$\rho(r) := \inf_{\partial B_r} \frac{u}{\Phi} > 0, \quad \text{for } r > 2r_0, \quad \Phi = |x|^{2-n}.$$

Observe that for every $r > r_0$ and $\varepsilon > 0$, we may choose $R > r$ large enough that

$$u(x) + \varepsilon \geq \rho(r)\Phi(x) \quad \text{on } \partial(B_R \setminus B_r).$$

By the maximum principle, $u(x) + \varepsilon \geq \rho(r)\Phi(x)$ in $B_R \setminus B_r$. Sending $R \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we discover that

$$(2.7) \quad u(x) \geq \rho(r)\Phi(x) \quad \text{in } \mathbb{R}^n \setminus B_r,$$

that is, $\rho(r) = \inf_{\mathbb{R}^n \setminus B_r} u/\Phi$. Therefore the map $r \mapsto \rho(r)$ is nondecreasing. For every $r > 2r_0$, define the function

$$v_r(x) := u(rx) - \rho(r/2)\Phi(rx).$$

Observe that by (2.7) we have $v_r \geq 0$ in $\mathbb{R}^n \setminus B_{1/2}$, and

$$-\Delta v_r \geq r^2 f(u_r) \quad \text{in } \mathbb{R}^n \setminus B_{1/2}.$$

Using again Lemmas 2.3 and 2.2 together with (2.2), (2.4) and (2.6), we deduce that

$$\inf_{B_2 \setminus B_1} v_r \geq ar^2(m(r))^{n/(n-2)} \geq ar^{2-n} \quad \text{for every sufficiently large } r \geq 2r_0,$$

where $a > 0$ does not depend on r . In particular,

$$u(rx) \geq (\rho(r/2) + a) \Phi(rx) \quad \text{on } B_2 \setminus B_1.$$

That is, $\rho(r) \geq \rho(r/2) + a$ for all sufficiently large r . Therefore we obtain that $\lim_{r \rightarrow \infty} \rho(r) = \infty$, which contradicts our inequality (2.5). \square

Remark 2.5. Note that if instead of (2.2) we assumed the stronger hypothesis $\liminf_{s \searrow 0} s^{-p} f(s) > 0$ for some $p < n/(n-2)$, then (2.5) is replaced by $m(r) \leq r^{-\beta}$ for some $\beta > n-2$, which immediately contradicts (2.6).

Remark 2.6. If in addition to (2.2) we made the relatively mild assumption

$$(2.8) \quad \liminf_{s \rightarrow \infty} f(s) > 0$$

then we do not need Lemma 2.3, that is, we do not need to use a weak Harnack inequality. Indeed, it is obvious from the proof above that (2.8) prevents $m(r)$ from going to infinity as $r \rightarrow \infty$.

The proof of the following analogue of Theorem 2.1 for two dimensions is postponed until Section 4, where we obtain it as a consequence of Corollary 4.2.

Theorem 2.7. *Let f be a positive, continuous function on $(0, \infty)$ which satisfies*

$$(2.9) \quad \lim_{s \rightarrow \infty} e^{as} f(s) = \infty \quad \text{for every } a > 0.$$

Then the inequality (2.1) has no positive solution in any exterior domain of \mathbb{R}^2 .

Observe that (2.9) is a condition on $f(s)$ near $s = \infty$, as opposed to near zero. This difference from condition (2.2) is due to the behavior of the fundamental solution of Laplace's equation near infinity in dimension $n = 2$ versus higher dimensions. See Section 4, in particular Corollary 4.2, for a much more detailed study of this phenomenon.

This above result, which to our knowledge is new, is also sharp. Indeed, notice that for any $a > 0$, the function

$$u(x) := \frac{2}{a} (\log |x| + \log (\log |x|))$$

is a smooth positive solution of the equation

$$-\Delta u = e^{-au} \quad \text{in } \mathbb{R}^2 \setminus B_3.$$

As is well-known, there is no positive solution of $-\Delta u \geq 0$ in $\mathbb{R}^2 \setminus \{0\}$, except for constant functions (see also Theorem 4.3 for a more general statement).

3. PRELIMINARIES

3.1. Several properties of supersolutions. In this section we confirm that the hypotheses of our main result are satisfied by the p -Laplacian operator and fully nonlinear Isaacs operators.

Recall the p -Laplacian is defined by

$$\Delta_p u := \operatorname{div}(|Du|^{p-2} Du), \quad 1 < p < \infty.$$

For the sake of simplicity, we do not consider more general quasilinear operators, although our techniques apply for instance to operators of the more general form $Q[u] = \operatorname{div}(A(x, Du))$, with A satisfying hypotheses (1.1)-(1.4) in [14].

A *uniformly elliptic Isaacs operator* F is a function $F : \mathcal{S}_n \rightarrow \mathbb{R}$ satisfying the uniform ellipticity condition

$$(3.1) \quad \mathcal{M}_{\lambda, \Lambda}^-(M - N) \leq F(M) - F(N) \leq \mathcal{M}_{\lambda, \Lambda}^+(M - N) \quad \text{for all } M, N \in \mathcal{S}_n,$$

and which is positively homogeneous of order one:

$$(3.2) \quad F(tM) = tF(M) \quad \text{for all } t \geq 0, M \in \mathcal{S}_n.$$

Here \mathcal{S}_n is the set of n -by- n symmetric matrices, and $\mathcal{M}_{\lambda, \Lambda}^+$ and $\mathcal{M}_{\lambda, \Lambda}^-$ are the Pucci extremal operators defined for instance in [11]. The inequalities (3.1) imply that F is uniformly elliptic. Equivalent to (3.1) and (3.2) is the requirement that F be a lim-sup of linear, uniformly elliptic operators

$$F(D^2u) = \inf_{\alpha} \sup_{\beta} \operatorname{trace}(A_{\alpha\beta} D^2u)$$

over a collection of matrices $\{A^{\alpha\beta}\}$ such that $\lambda I_n \leq A^{\alpha\beta} \leq \Lambda I_n$ for all α and β . Consult [11] for more on fully nonlinear, uniformly elliptic equations.

Our notion of solution is chosen to suit the particular operator under consideration. The p -Laplacian is of divergence form, and thus we use the weak integral formulation. More precisely, recall that a *weak supersolution* of the quasilinear equation

$$(3.3) \quad -\operatorname{div}A(x, Du) = f(u, x)$$

in a domain $\Omega \subseteq \mathbb{R}^n$ is a function $u \in W_{\operatorname{loc}}^{1,p}(\Omega)$ with the property that for all nonnegative smooth $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_{\Omega} A(x, Du) \cdot D\varphi \, dx \geq \int_{\Omega} f(u, x)\varphi \, dx.$$

When $Q[u] = F(D^2u)$ for an Isaacs operator F , the appropriate notion of solution is that of viscosity solution. Recall that a *viscosity solution* of the inequality

$$-F(D^2u) \geq (\leq) f(u, x) \quad \text{in } \Omega$$

is a function $u \in C(\Omega)$ such that for each $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ for which the map $x \mapsto u(x) - \varphi(x)$ has a local minimum (maximum) at x_0 , we have

$$-F(D^2\varphi(x_0)) \geq (\leq) f(u(x_0), x_0).$$

Henceforth, when we write a differential inequality such as $-Q[u] \geq f(u, x)$, we intend that it be interpreted in the appropriate sense.

We now present a list of properties which these operators share and upon which our method is based. We will confirm below that the following hold in the case that Q is the p -Laplacian operator or an Isaacs operator:

- (H1) Q satisfies the *weak comparison principle*; that is, if $-Q[u] \leq 0 \leq -Q[v]$ in a bounded domain Ω , and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω ;
- (H2) Q has *fundamental solutions*: there exist functions $\Phi, \tilde{\Phi}$ which satisfy $-Q[\Phi] = 0 = -Q[\tilde{\Phi}]$ in $\mathbb{R}^n \setminus \{0\}$, and are approximately homogeneous in the sense of (3.6) below;

- (H3) Q satisfies the *quantitative strong comparison principle*: if $-Q[v] \geq \chi_\omega \geq 0 = -Q[w]$ in a bounded Ω and some compact subset $\omega \subset \Omega$ of positive measure, then $v > w + c_0$ in any $K \subset\subset \Omega$, where $c_0 > 0$ depends only on Q, K, Ω , and a lower bound for $|\omega|$;
- (H4) Q satisfies the *very weak Harnack inequality*: if $-Q[v] \geq 0$ in a bounded Ω and $K \subset\subset \Omega$, then for each $0 < \tau < 1$ there exists $\bar{C} = \bar{C}(\tau, Q, K, \Omega) > 1$ such that for any point $x_0 \in K$, we have $|\{u \leq \bar{C}u(x_0)\} \cap K| \geq \tau |K|$;
- (H5) Q has no zero order term and possesses some homogeneity: precisely, we have $Q[u + c] = Q[u]$ for each constant $c \in \mathbb{R}$, and for some $p > 1$, $Q[tu] = t^{p-1}Q[u]$ for every $t \geq 0$; and if u satisfies $-Q[u] \geq f(u, x)$ in Ω and we set $u_r(x) := u(rx)$, then $-Q[u_r] \geq r^p f(rx, u_r)$ in $\Omega_r := \Omega/r$.

The last hypothesis can be weakened, as will be obvious from the proofs below. Namely, we can assume that if u satisfies $-Q[u] \geq f(u, x)$ in Ω , then $-Q_r[u_r] \geq r^p f(u_r, rx)$ on Ω_r , for some operator Q_r which satisfies the same hypotheses as Q , with constants independent of r , and such that for some $\beta > 0$ the operator $Q^t[u] := t^{-\beta}Q[tu]$ satisfies the same hypotheses as Q with constants independent of $t > 0$.

Let us now recall that both the p -Laplacian and Isaacs operators satisfy conditions (H1)-(H5). We begin by recalling the weak comparison principle. For the p -Laplacian, we refer for example to [28, Corollary 3.4.2], while for Isaacs operators, this is a particular case of results in [11, 12].

Proposition 3.1. *Let Q denote the p -Laplacian or an Isaacs operator. Suppose that Ω is a bounded domain, and u and v satisfy the inequalities*

$$-Q[u] \leq 0 \leq -Q[v] \quad \text{in } \Omega,$$

and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω .

Another important property for our purposes is the availability of solutions of $-Q[u] \leq 0$ with given behavior at infinity. Sharp nonexistence results are obtained by using the fundamental solutions of $-Q[u] = 0$. For $\alpha \in \mathbb{R}$, we denote

$$(3.4) \quad \xi_\alpha(x) := \begin{cases} |x|^{-\alpha} & \text{if } \alpha > 0, \\ -\log|x| & \text{if } \alpha = 0, \\ -|x|^{-\alpha} & \text{if } \alpha < 0, \end{cases}$$

Proposition 3.2. *Let Q denote the p -Laplacian or an Isaacs operator. Then there exist numbers $\alpha^*, \tilde{\alpha}^* \in (-1, \infty)$ and functions $\Phi, \tilde{\Phi}$ such that*

$$(3.5) \quad -Q[\Phi] = 0 = -Q[\tilde{\Phi}] \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

and

$$(3.6) \quad \begin{aligned} c\xi_{\alpha^*} &\leq \Phi \leq C\xi_{\alpha^*}, & \text{if } \alpha^* \neq 0 \\ c\xi_{\tilde{\alpha}^*} &\leq -\tilde{\Phi} \leq C\xi_{\tilde{\alpha}^*}, & \text{if } \tilde{\alpha}^* \neq 0 \\ -C + \xi_0 &\leq \Phi \text{ (resp. } -\tilde{\Phi}) \leq C + \xi_0, & \text{if } \alpha^* = 0 \text{ (resp. } \tilde{\alpha}^* = 0), \end{aligned}$$

for some positive constants $c, C > 0$.

It is well-known (and can be easily checked) that the p -Laplacian satisfies the statement above with $\alpha^* = \tilde{\alpha}^* = (n-p)/(p-1)$ and $\Phi = -\tilde{\Phi} = \xi_{(n-p)/(p-1)}$. For the reader interested in extending the results in this paper to more general quasilinear

operators, we note that results on the existence and behavior of singular solutions of quasilinear equalities can be found in the classical work of Serrin [30].

For Isaacs operators the question of existence, uniqueness, and properties of fundamental solutions was studied in detail in the recent work [2]. In particular, the result above is a consequence of Theorem 1.2 in that paper. We remark that for nonlinear Isaacs operators we have $\alpha^* \neq \tilde{\alpha}^*$, except in very particular cases. This is due to the fact that Isaacs operators are not odd, in general. For the Pucci extremal operators, for example, we have

$$\alpha^* \left(\mathcal{M}_{\lambda, \Lambda}^- \right) = \tilde{\alpha}^* \left(\mathcal{M}_{\lambda, \Lambda}^+ \right) = \frac{\Lambda}{\lambda} (n-1) - 1,$$

and

$$\alpha^* \left(\mathcal{M}_{\lambda, \Lambda}^+ \right) = \tilde{\alpha}^* \left(\mathcal{M}_{\lambda, \Lambda}^- \right) = \frac{\lambda}{\Lambda} (n-1) - 1.$$

Central for our method is the following quantitative (uniform) strong maximum principle. This result, while well-known (and fundamental to the regularity theory of linear elliptic equations of Krylov and Safonov, for example) is surprisingly underutilized in the theory of elliptic equations.

Theorem 3.3. *Let Q denote the p -Laplacian or an Isaacs operator. Assume that K and A are compact subsets of a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, with $|A| > 0$. Suppose that v is nonnegative in Ω and satisfies*

$$-Q[v] \geq \chi_A \quad \text{in } \Omega,$$

where χ_A denotes the characteristic function of A .

(i) *Then there exists a constant $c_0 = c_0(Q, |A|, \Omega, K) > 0$ such that*

$$v \geq c_0 \quad \text{on } K.$$

(ii) *Suppose in addition that $v \geq \Phi \geq 0$ on $\partial\Omega$, where Φ is as in the previous theorem, and $0 \notin \Omega$. Then there exists a constant $c_0 = c_0(Q, |A|, \Omega, K) > 0$ such that*

$$v \geq \Phi + c_0 \quad \text{on } K.$$

Proof. For an Isaacs operator, we have

$$-\mathcal{M}_{\lambda, \Lambda}^-(D^2v) \geq \chi_A \quad \text{and} \quad -\mathcal{M}_{\lambda, \Lambda}^-(D^2(v - \Phi)) \geq \chi_A \quad \text{in } \Omega,$$

and thus both (i) and (ii) are consequences of Theorem 3.1, the usual strong maximum principle [11, Theorem 4.9], and [29, Theorem A.1]. The result can also be obtained as a consequence of the results of Krylov and Safonov for linear equations [22, Chapter 4]; such an argument can be found in the Appendix of [29].

Let us give the proof for the p -Laplacian. Suppose that (i) or (ii) is false so that there exists a sequence of compact subsets $A_j \subseteq \Omega$ with $\inf_j |A_j| > 0$, and a sequence of positive functions v_j such that $-\Delta_p v_j \geq \chi_{A_j}$ in Ω and

$$(3.7) \quad \text{either } v_j(x_j) \rightarrow 0 \quad \text{or } v_j(x_j) - \Phi(x_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

for some sequence of points $x_j \in K$. Let \tilde{v}_j solve the Dirichlet problem

$$-\Delta_p \tilde{v}_j = \chi_{A_j} \quad \text{in } \Omega, \quad \tilde{v}_j = 0 \quad (\text{or } \tilde{v}_j = \Phi) \quad \text{on } \partial\Omega.$$

Then by Theorem 3.1, $v_j \geq \tilde{v}_j$ in Ω , and so we can replace v_j by \tilde{v}_j . For all $\varphi \in C_0^\infty(\Omega)$ we have

$$(3.8) \quad \int_{\Omega} |Dv_j|^{p-2} Dv_j \cdot D\varphi \, dx = \int_{A_j} \varphi \, dx.$$

According to the $C^{1,\alpha}$ estimates for the p -Laplace equation (see [34, 16, 23]), we deduce that v_j is bounded in $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$. Therefore we may extract a subsequence of v_j which converges to a function v_0 in $C^1(\overline{\Omega})$. We may pass to limits in (3.8) to obtain $-\Delta_p v_0 \geq 0$ in Ω , as well as $v_0 \geq 0$ on $\partial\Omega$ (or $v_0 \geq \Phi$ on $\partial\Omega$). By the strong maximum principle (see [33], or Theorem 3.4 below) we conclude that either $v_0 \equiv 0$ or $v_0 > 0$ in Ω . In the second case, by the strong comparison principle (see Theorem 1.4 in [14]), either $v_0 \equiv \Phi$ or $v_0 > \Phi$ in Ω . To apply the strong comparison principle here, we must note that the gradient of $\Phi = \xi_{(n-p)/(p-1)}$ never vanishes in Ω .

By passing to limits in (3.7), we obtain $v_0 \equiv 0$ in Ω , or in the second case $v_0 \equiv \Phi$ in Ω . In either case, v_0 is p -harmonic, so that a passage to the limit in (3.8) gives

$$\lim_{j \rightarrow \infty} \int_{A_j} \varphi \, dx \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

for each $\varphi \in C_0^\infty(\Omega)$. By taking $\varphi \geq 1$ except on a very small subset of Ω , this is easily seen to be a contradiction, according to $\inf_j |A_j| > 0$. □

The final ingredient of our proofs of the Liouville results is the weak Harnack inequality. For weak solutions of degenerate quasilinear equations it is due to Serrin [32] and Trudinger [35]. In the nondivergence framework it was proved by Krylov and Safonov for strong solutions (see [22]), see also [36], while for viscosity solutions of Isaacs equations it was obtained by Caffarelli [10]; see also Theorem 4.8 in [11].

Theorem 3.4. *Let $u \geq 0$ and $-Q[u] \geq 0$ in a bounded domain Ω , where Q is the p -Laplacian or an Isaacs operator. Then there exists $\gamma > 0$ depending only on Q and n , such that for each compact $K \subset \Omega$ we have*

$$\left(\int_K u^\gamma \, dx \right)^{1/\gamma} \leq C \inf_K u.$$

for some positive constant C , which depends only on n, Q, K, Ω .

Remark 3.5. In some cases the use of this theorem can be avoided, at the expense of strengthening the hypotheses on f , see for instance the remark after the proof of Theorem 2.1.

Remark 3.6. We actually use only the following weaker result: *for each $\gamma < 1$ there exists a constant $\bar{C} = \bar{C}(n, Q, \gamma) > 1$ such that for any nonnegative weak supersolution u of $-Q[u] \geq 0$ in the annulus $B_3 \setminus B_{1/2}$, and any $x_0 \in B_2 \setminus B_1$, we have*

$$|\{u \leq \bar{C}u(x_0)\} \cap (B_2 \setminus B_1)| \geq \gamma |B_2 \setminus B_1|.$$

This is a consequence of the “very weak” Harnack inequality, which states that for every $\gamma > 0$, there exists a constant $\bar{c} = \bar{c}(n, Q, \gamma, \Omega, K) \in (0, 1)$ such that for any nonnegative weak supersolution u of $-Q[u] \geq 0$ in Ω ,

$$|\{u \geq 1\} \cap K| \geq \gamma |K| \quad \text{implies} \quad u \geq \bar{c} \text{ in } K.$$

This fact, though a consequence of the weak Harnack inequality, is interesting in its own right. For instance, it admits a proof which is considerably simpler than the proof of the weak Harnack inequality while being sufficient to imply the Hölder estimates for solutions of $-Q[u] = 0$.

The reader is advised that in the rest of the paper only properties (H1)-(H5) will be used. In other words, the Liouville theorems stated in Section 4 are proved for any Q such that the inequalities $-Q[u] \geq (\leq) f(x, u)$ can be interpreted in such a way that properties (H1)-(H5), or a subset of them, are satisfied.

3.2. Properties of minima of supersolutions on annuli. Our method for proving Liouville theorems is based on the study of minima of supersolutions in annuli. In this section we obtain some preliminary estimates by comparing supersolutions of $-Q[u] \geq 0$ with the fundamental solutions of Q from property (H2).

Note that Φ and $\tilde{\Phi}$ can be assumed to never vanish in $\mathbb{R}^n \setminus B_{r_0}$, since if needed we simply add or subtract a constant from these functions. With this in mind, let us define the quantities

$$(3.9) \quad m(r) := \inf_{B_{2r} \setminus B_r} u, \quad \rho(r) := \inf_{B_{2r} \setminus B_r} \frac{u}{\Phi}, \quad \tilde{\rho}(r) := \inf_{B_{2r} \setminus B_r} \frac{u}{\tilde{\Phi}}.$$

Lemma 3.7. *Assume Q satisfies (H1) and (H2). Suppose that $r_0 > 0$ and $u \geq 0$ satisfy*

$$-Q[u] \geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{r_0}.$$

Then for some $r_1 > r_0$,

$$(3.10) \quad \begin{cases} r \mapsto \rho(r) \text{ is nondecreasing on } (r_0, \infty), & \text{if } \alpha^* > 0, \\ r \mapsto m(r) \text{ is nondecreasing on } (r_0, \infty), & \text{if } \alpha^* \leq 0, \\ r \mapsto m(r) \text{ is bounded on } (r_1, \infty), & \text{if } \tilde{\alpha}^* > 0, \\ r \mapsto \tilde{\rho}(r) \text{ is bounded on } (r_1, \infty), & \text{if } \tilde{\alpha}^* \leq 0. \end{cases}$$

Proof. First consider the case that $\alpha^* > 0$. Then $\Phi > 0$ and $\Phi(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Observe that for every $r > r_0$ and $\varepsilon > 0$, we may choose $R > r$ large enough that

$$u(x) + \varepsilon \geq \rho(r)\Phi(x) \quad \text{on } \partial(B_R \setminus B_r).$$

By the maximum principle,

$$u(x) + \varepsilon \geq \rho(r)\Phi(x) \quad \text{in } B_R \setminus B_r.$$

Sending $R \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we discover that

$$u(x) \geq \rho(r)\Phi(x) \quad \text{in } \mathbb{R}^n \setminus B_r, \quad \text{hence} \quad \rho(r) := \inf_{\mathbb{R}^n \setminus B_r} \frac{u}{\Phi}.$$

The desired monotonicity of $r \mapsto \rho(r)$ follows.

Next, suppose that $\alpha^* \leq 0$. Recall that in this case $\Phi(x) < 0$ for $|x| \geq r_0$ and $\Phi(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. Thus for every $\delta > 0$, we can find $R > 0$ so large that

$$u \geq m(r) + \delta\Phi \quad \text{on } \partial(B_R \setminus B_r).$$

Using the maximum principle and sending $R \rightarrow \infty$, we deduce that

$$u \geq m(r) + \delta\Phi \quad \text{in } \mathbb{R}^n \setminus B_r.$$

Now let $\delta \rightarrow 0$ to obtain $m(r) := \inf_{\mathbb{R}^n \setminus B_r} u$, and hence the monotonicity of $r \mapsto m(r)$ on the interval (r_0, ∞) .

Suppose that $\tilde{\alpha}^* > 0$. Then $\tilde{\Phi} < 0$, and we may normalize $\tilde{\Phi}$ so that

$$\max_{\partial B_{r_0}} \tilde{\Phi} = -1.$$

For any $r > r_0$, we clearly have

$$u(x) \geq \left(\inf_{B_{2r} \setminus B_r} u \right) \left(1 + \tilde{\Phi}(x) \right) \quad \text{for each } x \in \partial(B_r \setminus B_{r_0}).$$

By the maximum principle,

$$u \geq \left(\inf_{B_{2r} \setminus B_r} u \right) \left(1 + \tilde{\Phi} \right) \quad \text{in } B_r \setminus B_{r_0},$$

for each $r > r_0$. In particular, if k is fixed sufficiently large so that

$$\max_{\partial B_{kr_0}} u \geq \left(\inf_{B_{2r} \setminus B_r} u \right) \left(1 - ck^{-\tilde{\alpha}^*} \right) \geq \frac{1}{2} \left(\inf_{B_{2r} \setminus B_r} u \right),$$

we obtain the third statement in (3.10), for $r \geq kr_0$.

Finally, we consider the case that $\tilde{\alpha}^* \leq 0$. Observe that

$$u(x) \geq \tilde{\rho}(r) \left(\tilde{\Phi}(x) - \max_{\partial B_{r_0}} \tilde{\Phi} \right) \quad \text{for each } x \in \partial(B_r \setminus B_{r_0})$$

for any $r > r_0$. By the maximum principle,

$$u(x) \geq \tilde{\rho}(r) \left(\tilde{\Phi}(x) - \max_{\partial B_{r_0}} \tilde{\Phi} \right) \quad \text{in } B_r \setminus B_{r_0}.$$

Hence

$$\max_{\partial B_{kr_0}} u \geq \tilde{\rho}(r) \left(\min_{\partial B_{kr_0}} \tilde{\Phi} - \max_{\partial B_{r_0}} \tilde{\Phi} \right).$$

We fix k sufficiently large so that the quantity in the last parentheses be larger than one (recall we are in a case when $\tilde{\Phi} \rightarrow \infty$ as $|x| \rightarrow \infty$), and the second part of (3.10) follows. The lemma is proved. \square

The following bounds on $m(r)$ are an immediate consequence of (3.10).

Lemma 3.8. *Assume Q satisfies (H1) and (H2). Suppose that $r_0 > 0$ and $u \geq 0$ satisfy*

$$-Q[u] \geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{r_0}.$$

Then for some $c, C > 0$ depending only on Q, n, u , but not on r ,

$$(3.11) \quad \begin{cases} m(r) \geq cr^{-\alpha^*} & \text{if } \alpha^* > 0, \\ m(r) \geq c & \text{if } \alpha^* \leq 0, \end{cases} \quad \text{and} \quad \begin{cases} m(r) \leq C & \text{if } \tilde{\alpha}^* > 0, \\ m(r) \leq C \log r & \text{if } \tilde{\alpha}^* = 0, \\ m(r) \leq Cr^{-\tilde{\alpha}^*} & \text{if } \tilde{\alpha}^* < 0. \end{cases}$$

In some situations the map $r \mapsto m(r)$ is a nonincreasing function, in contrast with some conclusions of Lemma 3.7.

Lemma 3.9. *Assume Q satisfies (H1) and (H2). Suppose $u \geq 0$ satisfies either*

$$-Q[u] \geq 0 \quad \text{in } B_R \quad \text{or} \quad \begin{cases} -Q[u] \geq 0 & \text{in } B_R \setminus \{0\}, \\ \tilde{\alpha}^* \geq 0. \end{cases}$$

Then $r \mapsto m(r)$ is nonincreasing on $(0, R)$.

Proof. The first statement is obvious, since the maximum principle implies that $m(r) = \inf_{B_r} u$. Let us prove the second statement. By subtracting a constant from $\tilde{\Phi}$ in the case that $\tilde{\alpha}^* = 0$, we may assume that $\tilde{\Phi} < 0$ in $B_R \setminus \{0\}$. Since $\tilde{\Phi}(x) \rightarrow -\infty$ as $x \rightarrow 0$, for every $0 < r < R$ and $\delta > 0$, there exists $0 < \varepsilon < r$ small enough that

$$u \geq m(r) + \delta \tilde{\Phi} \quad \text{on } \partial(B_r \setminus B_\varepsilon).$$

By the maximum principle,

$$u \geq m(r) + \delta \tilde{\Phi} \quad \text{in } B_r \setminus B_\varepsilon.$$

Sending $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, we deduce that

$$u \geq m(r) \quad \text{in } B_r \setminus \{0\}, \quad \text{hence} \quad m(r) = \inf_{B_r \setminus \{0\}} u.$$

The monotonicity of the map $r \mapsto m(r)$ on the interval $(0, R)$ follows. \square

4. THE LIOUVILLE THEOREMS

This section contains our main result on the nonexistence of solutions of elliptic inequalities. As we pointed out in the previous section, all results will be announced under some or all of the hypotheses (H1)-(H5) on the elliptic operator Q . These properties hold for solutions of quasilinear inequalities of p -Laplacian type, as well as for viscosity supersolutions of fully nonlinear equations of Isaacs type. Therefore all the following results will be valid for such supersolutions and inequalities.

We pursue with this choice of exposition to emphasize the independence of the method on the particular type of operators and weak solutions that we consider. We believe that modifications of our arguments will yield analogous results for inequalities involving other types of operators, for instance mean-curvature type operators, nondivergence form extensions of the p -Laplacian studied by Birindelli and Demengel [6], nonlinear integral operators (c.f. [9]), and so on.

4.1. Statement of the main result. We begin by providing a brief overview of the main ideas in the proof of our main result, Theorem 4.1, which will also motivate the complicated hypotheses (f1)-(f4), below. Assume that we have a positive solution $u > 0$ of the inequality

$$(4.1) \quad -Q[u] \geq f(u, x) \quad \text{in } \mathbb{R}^n \setminus B_{r_0}.$$

Setting $u_r(x) = u(rx)$ for $r \geq 2r_0$ and using (H5), we see that u_r is a solution of

$$-Q[u_r] \geq r^p f(u_r, rx) \geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{1/2},$$

where $p > 1$ is as in (H5). Then property (H4) implies that the set

$$A_r := \{x \in B_2 \setminus B_1 : m(r) \leq u_r(x) \leq \bar{C}m(r)\}, \quad r > 2r_0,$$

is such that $|A_r| \geq (1/2)|B_2 \setminus B_1|$, provided $\bar{C} > 1$ is large enough. Then by (H3),

$$(4.2) \quad m(r) \geq cr^p \inf_{m(r) \leq s \leq \bar{C}m(r), x \in A_r} f(s, rx).$$

The idea is to discover hypotheses on f which imply that (4.2) is incompatible with the bounds on $m(r)$ obtained from Lemma 3.8. First, if the simple nondegeneracy condition (f2) below is in force, then we immediately obtain from (4.2) that either $m(r) \rightarrow 0$ or else $m(r) \rightarrow \infty$ as $r \rightarrow \infty$. We then impose conditions on f to rule out both of these alternatives; these are, respectively, (f3) and (f4) below. In light of

(4.2), we see that the former need concern only the behavior of $f(s, x)$ near $s = 0$ and $|x| = \infty$, and the latter the behavior of $f(s, x)$ near $s = \infty$ and $|x| = \infty$.

Our precise hypotheses on the function $f = f(s, x)$ are as follows:

- (f1) $f : (0, \infty) \times (\mathbb{R}^n \setminus B_{r_0}) \rightarrow [0, \infty)$ is continuous;
- (f2) $|x|^p f(s, x) \rightarrow \infty$ as $|x| \rightarrow \infty$ locally uniformly in $s \in (0, \infty)$;
- (f3) either $\alpha^* \leq 0$, or else $\alpha^* > 0$ and there exists a constant $\mu > 0$ such that if we define

$$\Psi_k(x) := |x|^p \inf_{k\Phi(x) \leq s \leq \mu} s^{1-p} f(s, x) \quad \text{and} \quad h(k) := \liminf_{|x| \rightarrow \infty} \Psi_k(x),$$

then $0 < h(k) \leq \infty$ for each $k > 0$, and

$$\lim_{k \rightarrow \infty} h(k) = \infty.$$

- (f4) either $\tilde{\alpha}^* > 0$, or else $\tilde{\alpha}^* \leq 0$ and there exists a constant $\mu > 0$ such that if we define

$$\tilde{\Psi}_k(x) := |x|^p \inf_{\mu \leq s \leq k\tilde{\Phi}(x)} s^{1-p} f(s, x) \quad \text{and} \quad \tilde{h}(k) := \liminf_{|x| \rightarrow \infty} \tilde{\Psi}_k(x),$$

then $0 < \tilde{h}(k) \leq \infty$ for each $k > 0$, and

$$\lim_{k \rightarrow 0} \tilde{h}(k) = \infty.$$

Observe that (f3) is void if $\alpha^* \leq 0$, while (f4) is void in the case that $\tilde{\alpha}^* > 0$. We recall that for the p -Laplacian operator, we have

$$\alpha^* = \tilde{\alpha}^* = \frac{n-p}{p-1},$$

while for an Isaacs operator with ellipticity Λ/λ , in general $\alpha^* \neq \tilde{\alpha}^*$, and each of α^* and $\tilde{\alpha}^*$ can be any number in the interval

$$\left[\frac{\lambda}{\Lambda}(n-1) - 1, \frac{\Lambda}{\lambda}(n-1) - 1 \right].$$

Our main result is:

Theorem 4.1. *Assume that $n \geq 2$, and Q and f satisfy (H1)-(H5) as well as (f1)-(f4), above. Then there does not exist a positive supersolution $u > 0$ of (4.1), for any $r_0 > 1$.*

We prove Theorem 4.1 in the following subsection, and conclude the present one by stating a consequence for nonlinearities f of the simple form

$$f(s, x) = |x|^{-\gamma} g(s).$$

For such f , we observe at once that conditions (f1) and (f2) are together equivalent to the statement

$$(4.3) \quad g : (0, \infty) \rightarrow (0, \infty) \quad \text{is continuous, and} \quad \gamma < p.$$

We claim that, together with (4.3), a sufficient condition for (f3) is

$$(4.4) \quad \text{if } \alpha^* > 0, \quad \text{then } \liminf_{s \searrow 0} s^{-\sigma^*} g(s) > 0, \quad \text{for } \sigma^* := (p-1) + \frac{p-\gamma}{\alpha^*}.$$

Observe first that (4.4) implies that

$$\eta = \eta(\mu) := \inf_{0 < s < \mu} s^{-\sigma^*} g(s) > 0, \quad \text{for every } \mu > 0.$$

Thus with $\Psi_k(x)$ as in (f3), we have for each $\mu > 0$ and all sufficiently large $|x|$,

$$\Psi_k(x) = |x|^p \inf_{k\Phi(x) \leq s \leq \mu} s^{1-p} f(s, x) \geq \eta \inf_{k\Phi(x) \leq s \leq \mu} |x|^{p-\gamma} s^{1-p+\sigma^*}.$$

Since $1 - p + \sigma^* = (p - \gamma)/\alpha^* > 0$, the last infimum above is attained at $s = k\Phi(x)$, and hence we obtain by the asymptotic homogeneity of Φ ,

$$\Psi_k(x) \geq c\eta k^{1-p+\sigma^*} |x|^{p-\gamma-\alpha^*(1-p+\sigma^*)} = \eta k^{1-p+\sigma^*} \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

where we have also used $1 - p + \sigma^* - (p - \gamma)/\alpha^* = 0$. This confirms that (f3) holds. A similar analysis on the validity of (f4) when $f(s, x) = |x|^{-\gamma} g(s)$ yields the following corollary, which contains Theorems 2.1 and 2.7 as very particular cases.

Corollary 4.2. *Assume that $n \geq 2$ and Q satisfies (H1)-(H5). Then the differential inequality*

$$-Q[u] \geq |x|^{-\gamma} g(u), \quad \gamma < p,$$

has no positive solution in any exterior domain, provided the function $g : (0, \infty) \rightarrow (0, \infty)$ is continuous and satisfies

$$(4.5) \quad \text{if } \alpha^* > 0, \quad \text{then} \quad \liminf_{s \searrow 0} s^{-\sigma^*} g(s) > 0, \quad \text{for } \sigma^* := (p-1) + \frac{p-\gamma}{\alpha^*},$$

$$(4.6) \quad \text{if } \tilde{\alpha}^* = 0, \quad \text{then} \quad \liminf_{s \rightarrow \infty} e^{as} g(s) > 0, \quad \text{for every } a > 0,$$

$$(4.7) \quad \text{if } \tilde{\alpha}^* < 0, \quad \text{then} \quad \liminf_{s \rightarrow \infty} s^{-\tilde{\sigma}^*} g(s) > 0, \quad \text{for } \tilde{\sigma}^* := (p-1) + \frac{p-\gamma}{\tilde{\alpha}^*}.$$

Observe that

$$-\infty < \tilde{\sigma}^* < p-1 < \sigma^* < \infty.$$

Applied to the model nonlinearity $f(s, x) = |x|^{-\gamma} s^\sigma$, the conditions (4.5), (4.6) and (4.7) are sharp. Indeed, it was shown in [1] that the inequality

$$(4.8) \quad -Q[u] \geq |x|^{-\gamma} u^\sigma$$

has a positive solution in $\mathbb{R}^n \setminus \{0\}$ if $\sigma > \sigma^*$, and even in whole space \mathbb{R}^n in the case $\gamma \leq 0$. The argument in [1] can be easily modified to show that (4.6) and (4.7) are similarly sharp. For example, in the case $\tilde{\alpha}^* = 0$, then the function $u(x) = \tilde{\Phi}(x) + \log \tilde{\Phi}(x)$ is a supersolution of the inequality

$$-Q[u] \geq e^{-au}$$

in some exterior domain, for some $a > 0$. We multiply u by a positive constant, we can have any $a > 0$ we wish.

Notice also that we have $\tilde{\sigma}^* > 0$ when $0 < p + \tilde{\alpha}^*(p-1) \leq \gamma < p$. For such values of γ and $\tilde{\alpha}^* < 0$, we see that there exist elliptic operators such that sublinear inequalities with nonlinearities that behave at infinity like u^σ , $\sigma \in (0, \tilde{\sigma}^*)$, may have positive solutions.

Finally, as mentioned above both (f3) and (f4) are void in the case $\alpha^* \leq 0$ and $\tilde{\alpha}^* > 0$, and we have the nonexistence of supersolutions in exterior domains under the modest hypotheses (f1) and (f2). In fact, in this case a simpler argument totally eliminates any hypotheses apart from the nonnegativity of f , provided the inequality holds in $\mathbb{R}^n \setminus \{0\}$. We conclude this subsection with another Liouville theorem, which is an immediate consequence of Lemmas 3.7 and 3.9.

Theorem 4.3. *Assume Q satisfies (H1) and (H2). Suppose $u \geq 0$ satisfies either*

$$\begin{cases} -Q[u] \geq 0 & \text{in } \mathbb{R}^n, \\ \alpha^* \leq 0. \end{cases} \quad \text{or} \quad \begin{cases} -Q[u] \geq 0 & \text{in } \mathbb{R}^n \setminus \{0\}, \\ \alpha^* \leq 0, \tilde{\alpha}^* \geq 0. \end{cases}$$

Then u is constant.

Proof. By adding a constant to u , we may suppose that $\inf u = 0$. According to Lemmas 3.7 and 3.9, the map $r \mapsto m(r)$ is constant on $(0, \infty)$, and hence $m(r) \equiv 0$. The strong maximum principle then implies that $u \equiv 0$. \square

The fact that there do not exist superharmonic functions in $\mathbb{R}^2 \setminus \{0\}$ which are bounded below is a very particular case of this theorem. The first part of Theorem 4.3 was obtained for the Pucci minimal operator in [13].

The sharpness of Theorem 4.3 illustrates the difference between nonexistence results in the whole space and in more general unbounded domains. For instance, the inequality $-\Delta_p u \geq 0$ has no positive solutions in \mathbb{R}^n for every $p \geq n$, while the same inequality has no positive solutions in the punctured space $\mathbb{R}^n \setminus \{0\}$ only in the case $p = n$.

4.2. Proof of the Main Result. We now give the

Proof of Theorem 4.1. To obtain a contradiction, let us suppose that $u > 0$ is a solution of the differential inequality

$$-Q[u] \geq f(u, x) \quad \text{in } \mathbb{R}^n \setminus B_{r_0},$$

for some $r_0 > 1$. For each $r > 2r_0$, denote $u_r(x) := u(rx)$, and observe that (H5) says u_r is a supersolution of

$$-Q[u_r] \geq r^p f(u_r, rx) \quad \text{in } \mathbb{R}^n \setminus B_{1/2}.$$

As before, set $m(r) := \inf_{B_{2r} \setminus B_r} u = \inf_{B_2 \setminus B_1} u_r$ for $r > 2r_0$. Let \bar{C} be as in (H4) with $\tau = 1/2$, $K = \bar{B}_2 \setminus B_1$, and $\Omega = B_3 \setminus B_{1/2}$. According to (H4), for each $r > 2r_0$ the set $A_r := (B_2 \setminus B_1) \cap \{m(r) \leq u_r \leq \bar{C}m(r)\}$ has measure at least $\frac{1}{2}|B_2 \setminus B_1|$. Then (H3) implies that for some $c > 0$,

$$(4.9) \quad m(r)^{p-1} \geq cr^p \inf \{f(s, x) : r \leq |x| \leq 2r, m(r) \leq s \leq \bar{C}m(r)\}$$

for every $r > 2R$. Owing to hypothesis (f2), we immediately deduce that

$$(4.10) \quad \text{either } m(r) \rightarrow 0 \quad \text{or} \quad m(r) \rightarrow +\infty \quad \text{as } r \rightarrow \infty.$$

Indeed, if we had a subsequence $r_j \rightarrow \infty$ such that $m(r_j) \rightarrow a \in (0, \infty)$, then by sending $r = r_j \rightarrow \infty$ in (4.9) we obtain a contradiction to (f2). We will complete the proof of by showing that the alternatives in (4.10) are contradicted by (f3) and (f4), respectively.

Case 1: $m(r) \rightarrow 0$ as $r \rightarrow \infty$. If $\alpha^* \leq 0$, then we may immediately appeal to Lemma 3.8 to obtain a contradiction. So we need only consider the case that $\alpha^* > 0$, for which Lemma 3.8 provides the lower bound

$$(4.11) \quad m(r) \geq cr^{-\alpha^*} \quad \text{for all } r > 2r_0.$$

We next establish the upper bound

$$(4.12) \quad m(r) \leq Cr^{-\alpha^*} \quad \text{for all sufficiently large } r > 2r_0.$$

Let $k > 0$ and $r > 2r_0$ be very large, and suppose that $m(r) \geq kr^{-\alpha^*}$. Then assuming that $r > 0$ is large enough that $\bar{C}m(r) \leq \mu$, and using (4.9), we have

$$\begin{aligned} \inf_{r \leq |x| \leq 2r} \Psi_{Ck}(x) &\leq \inf \left\{ s^{1-p} |x|^p f(s, x) : r \leq |x| \leq 2r, k|x|^{-\alpha^*} \leq s \leq \mu \right\} \\ &\leq \inf \left\{ s^{1-p} |x|^p f(s, x) : r \leq |x| \leq 2r, m(r) \leq s \leq \bar{C}m(r) \right\} \\ &\leq Cm(r)^{1-p} \inf \left\{ r^p f(s, x) : r \leq |x| \leq 2r, m(r) \leq s \leq \bar{C}m(r) \right\} \\ &\leq C. \end{aligned}$$

Owing to (f3), this is clearly impossible if $k > 0$ and $r > 2r_0$ are large enough. Thus we obtain the upper bound (4.12), and we have the two-sided estimate

$$(4.13) \quad cr^{-\alpha^*} \leq m(r) \leq Cr^{-\alpha^*}$$

for large $r > 2r_0$.

According to Lemma 3.7, the map $r \mapsto \rho(r)$ is nondecreasing. Thus for every $r > 2r_0$, the functions

$$v_r(x) := r^{\alpha^*} u(rx), \quad w_r(x) := \rho(r)\Phi(x) = \rho(r)r^{\alpha^*}\Phi(rx)$$

satisfy $v_r \geq w_r$ in $\mathbb{R}^n \setminus B_1$, and we have

$$-Q[v_r] \geq r^{p+\alpha^*(p-1)} f(u_r, rx) \geq 0 = -Q[w_r] \quad \text{in } \mathbb{R}^n \setminus B_{1/2}.$$

Note that $c \leq v_r \leq C$ on A_r , by (4.13). Using (4.9), (4.13), and (f3), for large enough $r > 2r_0$ we have

$$\begin{aligned} \inf_{A_r} \left(r^{p+\alpha^*(p-1)} f(u_r, rx) \right) &\geq r^p \inf_{A_r} \left\{ \frac{f(r^{-\alpha^*} v_r, rx)}{(r^{-\alpha^*} v_r)^{p-1}} \right\} \\ &\geq c \inf \left\{ s^{1-p} |y|^p f(s, y) : r \leq |y| \leq 2r, m(r) \leq s \leq \bar{C}m(r) \right\} \\ &\geq c \inf \left\{ s^{-1} |y|^p f(s, y) : r \leq |y|, cr^{-\alpha^*} \leq s \leq \mu, \right\} \\ &= c \inf_{x \geq r} \Psi_c(x) \\ &\geq c. \end{aligned}$$

Hence for such r ,

$$-Q[c^{1/(1-p)} v_r] \geq \chi_{A_r} \geq 0 = -Q[c^{1/(1-p)} w_r] \quad \text{in } B_5 \setminus B_1.$$

According to (H3), for sufficiently large $r > 2r_0$ we have

$$v_r \geq w_r + c_0 = (\rho(r) + c_1)\Phi \quad \text{in } B_4 \setminus B_2.$$

for some $c_1 > 0$ which does not depend on r . Unwinding the definitions, we discover that $u \geq (\rho(r) + c_1)\Phi$ in $B_{4r} \setminus B_{2r}$. In particular, $\rho(2r) \geq \rho(r) + c_1$, and we deduce that $\rho(r) \rightarrow \infty$ as $r \rightarrow \infty$. This contradicts the second inequality in (4.13), since obviously $\rho(r) \leq Cm(r) \max_{r \leq |x| \leq 2r} \Phi \leq Cm(r)r^{\alpha^*}$. The proof in the case $\lim_{r \rightarrow 0} m(r) = 0$ is complete.

Case 2: $m(r) \rightarrow \infty$ as $r \rightarrow \infty$. If $\tilde{\alpha}^* > 0$, we obtain an immediate contradiction by applying Lemma 3.8, so we may suppose that $\tilde{\alpha}^* \leq 0$ and the second alternative in (f4) is in force. We may assume that $\tilde{\Phi}$ is normalized so that $\max_{\partial B_{r_0}} \tilde{\Phi} = 1$, as well as $\tilde{\Phi} > 0$ on $\mathbb{R}^n \setminus B_{r_0}$ in the case that $\tilde{\alpha}^* = 0$.

Lemma 3.8 gives the upper bound

$$(4.14) \quad m(r) \leq C \max_{|x|=r} \tilde{\Phi}(x) \leq C \min_{|x|=r} \tilde{\Phi}(x), \quad r > 2r_0,$$

using the asymptotic homogeneity of $\tilde{\Phi}$. We next establish the lower bound

$$c \max_{|x|=r} \tilde{\Phi}(x) \leq m(r),$$

Let $k > 0$, and assume that $r > 2r_0$ is large enough that $m(r) \geq \mu$, and suppose that $m(r) \leq k \min_{|x|=r} \tilde{\Phi}(x)$. Then this, (4.9), and our assumption that $\lim_{r \rightarrow \infty} m(r) = \infty$ imply that for sufficiently large $r > 2r_0$ we have

$$\begin{aligned} \inf_{r \leq |x| \leq 2r} \tilde{\Psi}_{\bar{C}k}(x) &\leq \inf \left\{ s^{1-p} |x|^p f(s, x) : r \leq |x| \leq 2r, \mu \leq s \leq \bar{C}k \tilde{\Phi}(x) \right\} \\ &\leq \inf \left\{ s^{1-p} r^p f(s, x) : r \leq |x| \leq 2r, m(r) \leq s \leq \bar{C}m(r) \right\} \\ &\leq C m(r)^{1-p} \inf \left\{ r^p f(s, x) : r \leq |x| \leq 2r, m(r) \leq s \leq \bar{C}m(r) \right\} \\ &\leq C. \end{aligned}$$

This contradicts (f4) if $k > 0$ is chosen sufficiently small, and we have the lower bound

$$c \max_{|x|=r} \tilde{\Phi}(x) \leq c \min_{|x|=r} \tilde{\Phi}(x) \leq m(r).$$

Recalling (4.14), we have the two-sided estimate

$$(4.15) \quad c \max_{|x|=r} \tilde{\Phi}(x) \leq m(r) \leq C \min_{|x|=r} \tilde{\Phi}(x) \quad \text{for sufficiently large } r > r_0.$$

Define the quantity

$$\omega(r) := \inf_{\partial B_r} \frac{u}{\tilde{\Phi} - 1}, \quad r > r_0.$$

By the maximum principle, we have that

$$u \geq \omega(r) (\tilde{\Phi} - 1) \quad \text{in } B_r \setminus B_{r_0}.$$

Thus for $r > 2r_0$, the functions

$$v_r(x) := u(rx), \quad w_r := \omega(2r) (\tilde{\Phi}(rx) - 1) \geq 0 \quad \text{in } B_2 \setminus B_{1/2},$$

are such that $v_r \geq w_r$ and satisfy the differential inequalities

$$-Q[v_r] \geq r^p f(u_r, rx) \geq 0 = -Q[w_r] \quad \text{in } \mathbb{R}^n \setminus B_{1/2}.$$

Using (4.9), (4.15), and (f4), we see that for sufficiently large $r > 2r_0$,

$$\begin{aligned} \inf_{A_r} (r^p f(u_r, rx)) &\geq \frac{1}{2^p} \inf \left\{ |y|^p f(s, y) : r \leq |y| \leq 2r, m(r) \leq s \leq \bar{C}m(r) \right\} \\ &\geq c m(r)^{p-1} \inf \left\{ s^{1-p} |y|^p f(s, y) : \mu \leq s \leq C \tilde{\Phi}(y), r \leq |y| \right\} \\ &\geq c m(r)^{p-1}. \end{aligned}$$

In particular, for such $r > 2r_0$,

$$-Q[v_r] \geq c m(r)^{p-1} \chi_{A_r} \geq 0 = -Q[w_r] \quad \text{in } B_2 \setminus B_{1/2r_0}.$$

Applying (H3) and (H5), we find that

$$v_r \geq c_1 m(r) + w_r \quad \text{on } \partial B_1.$$

for some $c_1 > 0$ which does not depend on r . Using the definition of v_r , we discover that for sufficiently large $r > 2r_0$,

$$u(x) - \omega(2r) (\tilde{\Phi} - 1) \geq c_1 m(r) \geq c_2 \tilde{\Phi} \quad \text{on } \partial B_r,$$

and therefore

$$u(x) - (\omega(2r) + c_2) (\tilde{\Phi} - 1) > 0 \quad \text{on } \partial B_r.$$

for some $c_2 > 0$ which does not depend on r . It follows that $\omega(r) \geq \omega(2r) + c_2$ for all sufficiently large $r > 2r_0$, and hence $\omega(r) \rightarrow -\infty$ as $r \rightarrow \infty$. This is an obvious contradiction, since $\omega > 0$. Our proof is complete. \square

5. APPLICATIONS TO SYSTEMS OF INEQUALITIES

5.1. Systems of elliptic inequalities. Our approach permits generalizations to systems of the form

$$(5.1) \quad -Q_i[u_i] \geq f_i(u_1, \dots, u_N, x), \quad i = 1, \dots, N,$$

where f is a positive, continuous function on $(\mathbb{R}^n \setminus B_{r_0}) \times (0, \infty)^N$. The method essentially reduces the question of existence of positive solutions of (5.1) to that of systems of certain algebraic inequalities.

Assuming that Q_i satisfies (H1)-(H5) with constants $p_i, \alpha_i^*, \tilde{\alpha}_i^*$, and so on, we may use the Harnack inequality and the quantitative strong maximum principle as before to obtain

$$(5.2) \quad m_i(r) \geq cr^{p_i} \inf_{(s_1, \dots, s_N, x) \in A_r} f_i(s_1, \dots, s_N, x) \quad \text{for every } 1 \leq i \leq N,$$

where we have set $m_i(r) := \inf_{r \leq |x| \leq 2r} u_i(x)$ as well as

$$A_r := \{(s_1, \dots, s_N, x) : r \leq |x| \leq 2r, m_i(r) \leq s_i \leq \tilde{C}m_i(r) \text{ for } i = 1, \dots, N\}.$$

The game is then to impose hypotheses on the functions f_i which ensure that the inequalities (5.2) are incompatible with those of Lemma 3.8 for large r . In the ‘‘critical’’ cases, we typically obtain a two-sided bound on some $m_j(r)$ for large r , rescale the function u_j , and then proceed as in the proof of Theorem 4.1 to obtain a contradiction.

For instance, if we consider the system

$$(5.3) \quad \begin{cases} -Q_1[u] \geq |x|^{-\gamma} u^a v^b, \\ -Q_2[v] \geq |x|^{-\delta} u^c v^d, \end{cases}$$

in some exterior domain of \mathbb{R}^n , then we obtain

$$(5.4) \quad \begin{cases} m_1^{a-1}(r) m_2^b(r) \leq Cr^{\gamma-p_1}, \\ m_1^c(r) m_2^{d-1}(r) \leq Cr^{\delta-p_2}, \end{cases}$$

for sufficiently large r . We may then combine (5.4) with the inequalities given by Lemma 3.8 in order to determine the set of parameters for which these inequalities are incompatible. Notice that we may take parameters $a, b, c, d \in \mathbb{R}$, and in particular we can consider systems with various singularities. It is also possible to consider operators Q_1 and Q_2 which are of a different nature; e.g., Q_1 may be the p -Laplacian while Q_2 is an Isaacs operator.

Naturally, any attempt at stating a very general result for a system of the form (5.1) is immediately met with a combinatorial explosion of cases to consider (e.g., the various signs of α_i^* and $\tilde{\alpha}_i^*$, corresponding requirements on the functions f_i as some s_j are going to zero while others are at infinity, etc). While it will be apparent that our techniques are sufficiently flexible to yield nonexistence results

for such general systems, with an eye toward the clarity of our presentation we study only some special cases here.

Let us calculate the set of parameters $(\sigma_1, \sigma_2) \in \mathbb{R}^2$ for which the extended Lane-Emden system

$$(5.5) \quad \begin{cases} -Q_1[u] \geq v^{\sigma_1}, \\ -Q_2[v] \geq u^{\sigma_2}, \end{cases}$$

has no positive solutions in any exterior domain of \mathbb{R}^n . Considering first the case that $\sigma_1, \sigma_2 \geq 0$, we obtain from (5.4) the bounds

$$(5.6) \quad \begin{cases} r^{p_1 + \sigma_1 p_2} m_1^{\sigma_1 \sigma_2 - 1} \leq C, \\ r^{p_2 + \sigma_2 p_1} m_2^{\sigma_1 \sigma_2 - 1} \leq C. \end{cases}$$

If $\sigma_1 \sigma_2 \leq 1$, then sending $r \rightarrow \infty$ in (5.6) with the second set of inequalities in Lemma 3.8 immediately yields a contradiction, since we recall that $\tilde{\alpha}^* > -1$ and $p_i > 1$. If $\sigma_1 \sigma_2 > 1$ then Lemma 3.8 yields

$$cr^{-\max\{\alpha_1^*, 0\}} \leq m_1(r) \leq Cr^{-\frac{p_1 + \sigma_1 p_2}{\sigma_1 \sigma_2 - 1}}, \quad cr^{-\max\{\alpha_2^*, 0\}} \leq m_2(r) \leq Cr^{-\frac{p_2 + \sigma_2 p_1}{\sigma_1 \sigma_2 - 1}}$$

This is of course a contradiction, if

$$\alpha_1^* < \frac{p_1 + \sigma_1 p_2}{\sigma_1 \sigma_2 - 1} \quad \text{or} \quad \alpha_2^* < \frac{p_2 + \sigma_2 p_1}{\sigma_1 \sigma_2 - 1}.$$

If neither of these strict inequalities holds, but equality holds say in the first, then we may obtain

$$\alpha_1^* + p_1 - \sigma_1 \alpha_2^* \geq 0, \quad \alpha_2^* + p_2 - \sigma_2 \alpha_1^* \geq 0.$$

The rescaled functions

$$u_r = r^{\alpha_1^*} u(rx), \quad v_r = r^{\alpha_2^*} v(rx),$$

satisfy the system

$$(5.7) \quad \begin{cases} -Q_1[u_r] \geq r^{\alpha_1^* + p_1 - \sigma_1 \alpha_2^*} v_r^{\sigma_1}, \\ -Q_2[v_r] \geq r^{\alpha_2^* + p_2 - \sigma_2 \alpha_1^*} u_r^{\sigma_2}, \end{cases}$$

in an exterior domain which does not depend on $r \geq 1$. Moreover, the very weak Harnack inequality implies that for sufficiently large r we have $0 < c \leq u_r \leq C$ on a subset of $B_2 \setminus B_1$ with measure bounded below by a positive constant which does not depend on r . By applying the quantitative strong maximum principle to the second, then to the first equation in this system, and using the equality $\alpha_1^* + p_1 - \sigma_1 \alpha_2^* + \sigma_1(\alpha_2^* + p_2 - \sigma_2 \alpha_1^*) = 0$, we obtain

$$-Q_1[u_r] \geq c > 0$$

on a subset of $B_2 \setminus B_1$ which has measure bounded below by a positive constant independently of r . We now proceed as in the proof of Theorem 4.1 to deduce from (5.15) that $r^{\alpha_1^*} m_1(r) \rightarrow \infty$ as $r \rightarrow \infty$, a contradiction. This completes the argument in the case that $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$, and we have found that we have nonexistence of positive solutions provided that

$$\sigma_1 \sigma_2 \leq 1, \quad \text{or} \quad \min \left\{ \alpha_1^* - \frac{p_1 + \sigma_1 p_2}{\sigma_1 \sigma_2 - 1}, \alpha_2^* - \frac{p_2 + \sigma_2 p_1}{\sigma_1 \sigma_2 - 1} \right\} \leq 0.$$

Consider now the case that $\sigma_1 < 0$ and $\sigma_2 < 0$. Lemma 3.8 yields an immediate contradiction if either $\tilde{\alpha}_1^* \geq 0$ or $\tilde{\alpha}_2^* \geq 0$. Thus we assume that $\tilde{\alpha}_1^* < 0$ and $\tilde{\alpha}_2^* < 0$, and obtain

$$\begin{cases} cr^{-\tilde{\alpha}_1^*} \geq m_1(r) \geq cr^{p_1} m_2^{\sigma_1} \geq r^{p_1 - \tilde{\alpha}_2^* \sigma_1} \\ cr^{-\tilde{\alpha}_2^*} \geq m_2(r) \geq cr^{p_2} m_1^{\sigma_2} \geq r^{p_2 - \tilde{\alpha}_1^* \sigma_2} \end{cases}$$

and we get a contradiction provided

$$p_1 + \tilde{\alpha}_2^*(-\sigma_1) + \tilde{\alpha}_1^* > 0 \quad \text{or} \quad p_2 + \tilde{\alpha}_1^*(-\sigma_2) + \tilde{\alpha}_2^* > 0.$$

If one of these inequalities is an equality, we rescale u or v as above, replacing α_i by $\tilde{\alpha}_i$, and Φ_i by $\tilde{\Phi}_i$, to reach a contradiction in the same manner.

Finally, in the case that $\sigma_1 < 0$, $\sigma_2 \geq 0$, we get

$$\begin{cases} cr^{-\tilde{\alpha}_1^*} \geq m_1(r) \geq cr^{p_1} m_2^{\sigma_1} \geq r^{p_1 - \tilde{\alpha}_2^* \sigma_1} \\ cr^{-\tilde{\alpha}_2^*} \geq m_2(r) \geq r^{\frac{p_2 + \sigma_2 p_1}{1 - \sigma_1 \sigma_2}} \end{cases}$$

from which the arguments above give a contradiction provided that

$$p_1 + \tilde{\alpha}_2^*(-\sigma_1) + \tilde{\alpha}_1^* \geq 0 \quad \text{or} \quad \frac{p_2 + \sigma_2 p_1}{1 - \sigma_1 \sigma_2} \geq -\tilde{\alpha}_2^*.$$

5.2. Autonomous systems of three or more inequalities. Let us now consider a system of the form

$$(5.8) \quad \begin{cases} -Q_1[u_1] \geq f_k(u_k), \\ -Q_2[u_2] \geq f_1(u_1), \\ \vdots \\ -Q_k[u_k] \geq f_{k-1}(u_{k-1}). \end{cases}$$

For simplicity we assume that $p_i = p > 1$ for all i , as well as

$$(5.9) \quad \tilde{\alpha}_i^* > 0 \quad \text{for every } i = 1, \dots, k.$$

The later hypothesis renders it unnecessary to form an analogue of condition (f4). Let us state the hypotheses on the functions f_i which will ensure nonexistence of positive solutions of (5.8). We assume that

$$(5.10) \quad \text{the nonlinearity } f_i : (0, \infty) \rightarrow (0, \infty) \quad \text{is continuous for each } i = 1, \dots, k,$$

as well as

$$(5.11) \quad 0 < \liminf_{s \searrow 0} s^{-\sigma_i} f_i(s) \leq \infty \quad \text{for each } i = 1, \dots, k.$$

for some exponents $\sigma_1, \dots, \sigma_k > 0$. We will see later that we may assume without loss of generality that the geometric mean of $\sigma_1, \dots, \sigma_k$ is at least $p - 1$. Therefore let us denote

$$(5.12) \quad D := \prod_{i=1}^k \sigma_i - (p-1)^k$$

and assume for the moment that $D > 0$. For each $1 \leq i \leq k$, define the constant

$$(5.13) \quad \beta_i := \frac{p}{D} \sum_{j=0}^{k-1} \left((p-1)^j \prod_{l=0}^{k-2-j} \sigma_{k+i-1-l} \right),$$

where for notational convenience for $i > k$ we set $\sigma_i := \sigma_{(i \bmod k)}$, $u_i := u_{(i \bmod k)}$, and so on, and we define an empty product to have the value of 1. For example, β_1 is given by the expression

$$\beta_1 = \frac{p}{\sigma_1 \sigma_2 \cdots \sigma_k - (p-1)^k} \left((p-1)^{k-1} + (p-1)^{k-2} \sigma_k \right. \\ \left. + (p-1)^{k-3} \sigma_k \sigma_{k-1} + \cdots + (p-1) \sigma_k \cdots \sigma_3 + \sigma_k \cdots \sigma_2 \right)$$

We will argue that the system (5.8) has no solution $u_1, u_2, \dots, u_k > 0$ in any exterior domain of \mathbb{R}^n provided that

$$\min_{1 \leq i \leq k} (\alpha_i - \beta_i) \leq 0.$$

It clearly suffices to show that

$$\alpha_1 \leq \beta_1 \implies \text{nonexistence of a positive solution of (5.8).}$$

Arguing by contradiction, we assume that (5.8) has a solution $u_1, u_2, \dots, u_k > 0$ in some exterior domain $\mathbb{R}^n \setminus B_{r_0}$ but that $\alpha_1 \leq \beta_1$. Denote

$$u_{i,r}(x) := u_i(rx) \quad \text{and} \quad m_i(r) := \inf_{B_{2r} \setminus B_r} u_i = \inf_{B_2 \setminus B_1} u_{i,r}, \quad r > 2r_0, \quad i = 1, \dots, k.$$

For every $r > 2r_0$ and $1 \leq i \leq k$,

$$-Q_{i+1}[u_{i+1,r}] \geq r^p f_i(u_{i,r}) \quad \text{in } \mathbb{R}^n \setminus B_{1/2}.$$

Arguing as in the proof of Theorem 4.1, for $r > 2r_0$ and $1 \leq i \leq k$ we obtain

$$(5.14) \quad m_{i+1}(r)^{p-1} \geq cr^p \inf \{f_i(s) : m_i(r) \leq s \leq \bar{C}m_i(r)\}, \quad \text{for all } i = 1, \dots, k,$$

where $c > 0$ and $\bar{C} > 1$ can be taken independent of i as well as r . By our hypothesis (5.9) and Lemma 3.8 we deduce that $m_i(r) \leq C$ for all $r > 2r_0$ and all i . Thus (5.14) implies that

$$\inf \{f_i(s) : m_i(r) \leq s \leq \bar{C}m_i(r)\} \leq Cr^{-p} m_{i+1}(r)^{p-1} \rightarrow 0$$

as $r \rightarrow \infty$. Since f_i is positive and continuous on $(0, \infty)$, and $m_i(r) \leq C$, we deduce that $m_i(r) \rightarrow 0$ for all i . Therefore (5.11) and (5.14) imply that for all sufficiently large $r > 2r_0$ and each i ,

$$m_{i+1}(r) \geq cr^{\frac{p}{p-1}} m_i(r)^{\frac{\sigma_i}{p-1}}$$

By induction we have for sufficiently large $r > 2r_0$,

$$\begin{aligned} m_1(r) &\geq cr^{\frac{p}{p-1}} m_k(r)^{\frac{\sigma_k}{p-1}} \\ &\geq cr^{\frac{p}{p-1}} \left(cr^{\frac{p}{p-1}} m_{k-1}(r)^{\frac{\sigma_{k-1}}{p-1}} \right)^{\frac{\sigma_k}{p-1}} \\ &= cr^{\frac{p}{p-1}} \left(1 + \frac{\sigma_k}{p-1} \right) m_{k-1}(r)^{\frac{\sigma_k \sigma_{k-1}}{(p-1)^2}} \\ &\geq cr^{\frac{p}{p-1}} \left(1 + \frac{\sigma_k}{p-1} \right) \left(cr^{\frac{p}{p-1}} m_{k-2}(r)^{\frac{\sigma_{k-2}}{p-1}} \right)^{\frac{\sigma_k \sigma_{k-1}}{(p-1)^2}} \\ &= cr^{\frac{p}{p-1}} \left(1 + \frac{\sigma_k}{p-1} + \frac{\sigma_k \sigma_{k-1}}{(p-1)^2} \right) m_{k-2}(r)^{\frac{\sigma_k \sigma_{k-1} \sigma_{k-2}}{(p-1)^3}} \\ &\vdots \\ &\geq cr^A m_1(r)^B \end{aligned}$$

where we have written

$$A := \frac{p}{p-1} \left(1 + \frac{\sigma_k}{p-1} + \frac{\sigma_k \sigma_{k-1}}{(p-1)^2} + \dots + \frac{\sigma_k \sigma_{k-1} \dots \sigma_2}{(p-1)^{k-1}} \right)$$

and

$$B := \frac{\sigma_k \sigma_{k-1} \dots \sigma_1}{(p-1)^k}.$$

Observing that $A/(B-1) = \beta_1$ and rearranging the inequality above, we discover

$$m_1(r) \leq Cr^{-\beta_1} \quad \text{for sufficiently large } r > 2r_0.$$

Since we have the lower bound $m_1(r) \geq cr^{-\alpha_1}$, we deduce an immediate contradiction in the case that $\beta_1 > \alpha_1$. Thus we may assume that $\beta_1 = \alpha_1$. Hence $r^A m_1(r)^B \geq cm_1(r)$. Thus the string of inequalities above may be reversed, that is, we have

$$m_{i+1}(r) \leq Cr^{\frac{p}{p-1}} m_i(r)^{\frac{\sigma_i}{p-1}}$$

for sufficiently large r and all i . Using this for $i = k$, we discover that on the set $A_r := \{x \in B_2 \setminus B_1 : m_k(r) \leq u_k(rx) \leq \bar{C}m_k(r)\}$ we have

$$-Q_1[u_{1,r}] \geq r^p f_k(u_{k,r}) \geq cr^p m_k(r)^{\sigma_k} \geq cm_1(r)^{p-1} \geq cr^{-\alpha_1(p-1)}.$$

Defining $v_{1,r} := r^{\alpha_1} u_{1,r}$, we obtain

$$(5.15) \quad -Q_1[v_{1,r}] \geq c \quad \text{in } A_r.$$

We now proceed as in the proof of Theorem 4.1 to deduce from the inequality (5.15) that $r^{\alpha_1} m_1(r) \rightarrow \infty$ as $r \rightarrow \infty$, a contradiction.

Finally, notice that if $D \leq 0$, then we may simply replace σ_1 by a larger number so that $D > 0$, but D is small enough that $\beta_1 > \alpha_1$. Notice that the hypothesis (5.11) weakens as σ_1 increases. We have proved the following theorem:

Theorem 5.1. *Fix $p > 1$ and an integer $k \geq 2$, and suppose that for each $i = 1, \dots, k$ the elliptic operator $Q_i[\cdot]$ satisfies the hypotheses of Section 4 with constants α_i^* and $\tilde{\alpha}_i^*$, and such that (5.9) holds. Let $\sigma_1, \dots, \sigma_k > 0$ and f_i satisfy (5.10), (5.11), D be given by (5.12), and β_i given by (5.13). Suppose that either $D \leq 0$, or*

$$D > 0 \quad \text{and} \quad \min_{1 \leq i \leq k} (\alpha_i - \beta_i) \leq 0.$$

Then, denoting $f_0 := f_k$ and $u_0 := u_k$, the system of inequalities

$$-Q_i[u_i] \geq f_{i-1}(u_{i-1}), \quad 1 \leq i \leq k$$

has no positive solution $u_1, \dots, u_k > 0$ in any exterior domain of \mathbb{R}^n .

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