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Bisets as categories, and tensor product of induced bimodules

Serge Bouc

Abstract : Bisets can be considered as categories. This note uses this point of view to give a simple proof of a Mackey-like formula expressing the tensor product of two induced bimodules.

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1. Introduction

Let R be a commutative ring, let G and H be finite groups, let X be a subgroup of $H \times G$, and M be an RX -module. If $m \in M$ and $(h, g) \in X$, set $h \cdot m \cdot g^{-1} = (h, g) \cdot m$: this is a slight extension of the usual correspondence between $R(H \times G)$ -modules and (RH, RG) -bimodules.

The object of this note is to give a simple proof of the following result :

1.1. Theorem : *Let R be a commutative ring, let G , H , and K be finite groups, let X be a subgroup of $H \times G$ and Y be a subgroup of $K \times H$. Let M be an RX -module, and N be an RY -module. Then there is an isomorphism of (RK, RG) -bimodules*

$$(\text{Ind}_Y^{K \times H} N) \otimes_{RH} (\text{Ind}_X^{H \times G} M) \cong \bigoplus_{t \in [p_2(Y) \setminus H/p_1(X)]} \text{Ind}_{Y *^{(t,1)} X}^{K \times G} (N \otimes_{k_2(Y) \cap {}^t k_1(X)} ({}^{(t,1)} M)) ,$$

where the notation is as follows (cf. [1]) :

$$p_1(X) = \{h \in H \mid \exists g \in G, (h, g) \in X\}, \quad k_1(X) = \{h \in H \mid (h, 1) \in X\}$$

$$p_2(Y) = \{h \in H \mid \exists k \in K, (k, h) \in Y\}, \quad k_2(Y) = \{h \in H \mid (1, h) \in Y\}$$

$$Y *^{(t,1)} X = \{(k, g) \in K \times G \mid \exists h \in H, (k, h) \in Y, (h^t, g) \in X\} .$$

The action of $(k, g) \in Y *^{(t,1)} X$ on $N \otimes_{k_2(Y) \cap {}^t k_1(X)} ({}^{(t,1)} M)$ is given by

$$k \cdot (n \otimes m) \cdot g^{-1} = (k \cdot n \cdot h^{-1}) \otimes (h^t \cdot m \cdot g^{-1}) ,$$

if $h \in H$ is chosen such that $(k, h) \in Y$ and $(h^t, g) \in X$.

2. Functors over bisets

Recall that when G and H are groups, an (H, G) -biset U is a set equipped with a left action of H and a right action of G which commute, i.e. such that $(hu)g = h(ug)$ for any $h \in H$, $u \in U$, and $g \in G$.

2.1. Notation : Let G and H be groups. When U is an (H, G) -biset, let $\langle U \rangle$ denote the following category :

- The objects of $\langle U \rangle$ are the elements of U .
- If $u, v \in U$, then

$$\text{Hom}_{\langle U \rangle}(u, v) = \{(h, g) \in H \times G \mid hu = vg\} .$$

- If $u, v, w \in U$, the composition of the morphisms $(h, g) : u \rightarrow v$ and $(h', g') : v \rightarrow w$ is the morphism $(h'h, g'g) : u \rightarrow w$.
- If $u \in U$, the identity morphism $\text{Id}_u : u \rightarrow u$ is the pair $(1, 1) \in G \times G$.

Note that the category $\langle U \rangle$ is a groupoid (any morphism is an isomorphism), and that for any $u \in U$, the group

$$A(u) = \text{Hom}_{\langle U \rangle}(u, u) = \{(h, g) \in H \times G \mid hu = ug\}$$

is a subgroup of $H \times G$.

A functor M from $\langle U \rangle$ to a category \mathcal{C} consists of a collection of objects $M(u)$ of \mathcal{C} , for $u \in U$, together with morphisms

$$M(h, g) : M(u) \rightarrow M(hug^{-1})$$

in the category \mathcal{C} , for $(h, g) \in H \times G$, fulfilling the usual functorial conditions. In particular, for each $u \in U$, there is a group homomorphism $A(u) \rightarrow \text{Aut}_{\mathcal{C}} M(u)$.

Functors from $\langle U \rangle$ to \mathcal{C} are the objects of a category $\text{Fun}(\langle U \rangle, \mathcal{C})$, in which the morphisms are natural transformation of functors.

2.2. Notation : When \mathcal{C} is a subcategory of the category **Sets** of sets, and M is a functor $\langle U \rangle \rightarrow \mathcal{C}$, the image of $m \in M(u)$ by the map $M(h, g) : M(u) \rightarrow M(hug^{-1})$, for $(h, g) \in H \times G$, will be denoted by hmg^{-1} .

In this case, a functor $M : \langle U \rangle \rightarrow \mathcal{C}$ is a collection of objects $M(u)$ of \mathcal{C} , for $u \in U$, together with morphisms $m \mapsto hmg^{-1} : M(u) \rightarrow M(hug^{-1})$ in \mathcal{C} ,

for $(h, g) \in H \times G$, such that $h'(hmg^{-1})g'^{-1} = (h'h)m(g'g)^{-1}$ and $1m1 = m$, for any $(h, g), (h', g')$ in $H \times G$, any $u \in U$, and any $m \in M(u)$.

2.3. Example : Suppose that $\mathcal{C} = \mathbf{Sets}$. Then the disjoint union $\bigsqcup M = \bigsqcup_{u \in U} M(u)$ becomes an (H, G) -biset, and the map $\bigsqcup M \rightarrow U$ sending elements of $M(u)$ to u , for $u \in U$, is a map of (H, G) -bisets. Conversely, if $\pi : S \rightarrow U$ is a map of (H, G) -bisets, then the assignment $u \mapsto \pi^{-1}(u)$ is a functor from $\langle U \rangle$ to \mathbf{Sets} .

In other words, a functor $\langle U \rangle \rightarrow \mathbf{Sets}$ is just an (H, G) -biset over U . More precisely, the category $\mathbf{Fun}(\langle U \rangle, \mathbf{Sets})$ of such functors is equivalent to the category of (H, G) -bisets over U .

2.4. Example : Let R be a commutative ring. In the remainder of this note, the category \mathcal{C} will be the category $R\text{-Mod}$ of (left) R -modules. If M is functor from $\langle U \rangle$ to $R\text{-Mod}$, then for each $u \in U$, the R -module $M(u)$ has a natural structure of $RA(u)$ -module.

Conversely, let $[H \backslash U / G]$ be a set of representatives of (H, G) -orbits on U . Equivalently $[H \backslash U / G]$ is a set of representatives of isomorphism classes in the category $\langle U \rangle$. Since $\langle U \rangle$ is a groupoid, it is equivalent to its full subcategory $[H \backslash U / G]$. In particular, this yields an equivalence of categories

$$(2.5) \quad \mathbf{Fun}(\langle U \rangle, R\text{-Mod}) \cong \prod_{u \in [H \backslash U / G]} RA(u)\text{-Mod} .$$

2.6. Remark : In the situation of Example 2.4, the direct sum

$$\Sigma(M) = \bigoplus_{u \in U} M(u)$$

has a natural structure of (RH, RG) -bimodule, i.e. using the usual group isomorphism $(h, g) \mapsto (h, g^{-1})$ from $H \times G^{op}$ to $H \times G$, of left $R(H \times G)$ -module.

Moreover, it is easy to see that there is an isomorphism of (RH, RG) -bimodules

$$\Sigma(M) \cong \bigoplus_{u \in [H \backslash U / G]} \text{Ind}_{A(u)}^{H \times G} M(u) .$$

3. Product of bisets, and product of functors

Let G, H and K be groups. If U is an (H, G) -biset and V is a (K, H) -biset, recall that the product (or *composition*) of V and U is the set

$$V \times_H U = (V \times U) / H ,$$

where the right action of H on $(V \times U)$ is defined by $(v, u) \cdot h = (vh, h^{-1}u)$, for $v \in V$, $u \in U$, and $h \in H$. The set $V \times_H U$ is a (K, G) -biset for the following action

$$\forall z \in K, \forall x \in G, \forall v \in V, \forall u \in U, \quad z \cdot (v, {}_H u) \cdot x = (zv, {}_H ux) \quad ,$$

where $(v, {}_H u)$ denotes the H -orbit of (v, u) .

3.1. Definition : *Let G , H , and K be finite groups. Let U be a finite (H, G) -biset, and V be a finite (K, H) -biset. If M is a functor $\langle U \rangle \rightarrow R\text{-Mod}$ and N is a functor $\langle V \rangle \rightarrow R\text{-Mod}$, the tensor product $N \otimes_H M$ is the functor $\langle V \times_H U \rangle \rightarrow R\text{-Mod}$ defined by*

$$(N \otimes_H M)(v, {}_H u) = \left(\bigoplus_{h \in H} N(vh) \otimes_R M(h^{-1}u) \right) / \mathcal{I}_{v,u} \quad ,$$

where $\mathcal{I}_{v,u}$ is the R -submodule generated by the elements of the form

$$[ny \otimes y^{-1}m]_{hy} - [n \otimes m]_h \quad ,$$

where $y \in H$, and where $[n \otimes m]_h$ denotes the element $n \otimes m$ of the component indexed by $h \in H$ in the direct sum, for $n \in N(vh)$, and $m \in M(h^{-1}u)$.

If $(k, g) \in K \times G$, then by definition

$$k [n \otimes m]_h g = [kn \otimes mg]_h \quad .$$

3.2. Remark : It follows from this definition that

$$(N \otimes_H M)((v, {}_H u)) \cong N(v) \otimes_{RH_{v,u}} M(u) \quad ,$$

where $H_{v,u}$ is the set of elements $h \in H$ such that $vh = v$ and $hu = u$.

3.3. Lemma : *There is an isomorphism of (RK, RG) -bimodules*

$$\Sigma(N) \otimes_{RH} \Sigma(M) \cong \Sigma(N \otimes_H M) \quad ,$$

sending (from right to left) the element $[n \otimes m]_h$ to $n \otimes_{RH} m$.

Proof : To be more precise, the map α from

$$\Sigma(N \otimes_H M) = \bigoplus_{(v, {}_H u) \in V \times_H U} \left(\bigoplus_{h \in H} N(vh) \otimes_R M(h^{-1}u) \right) / \mathcal{I}_{v,u}$$

sending the element $[n \otimes m]_h$ in the component indexed by $(v, {}_H u)$ to the element $n \otimes m$ of the tensor product

$$\Sigma(N) \otimes_{RH} \Sigma(M) = \left(\bigoplus_{v \in V} N(v) \right) \otimes_{RH} \left(\bigoplus_{u \in U} M(u) \right)$$

is well defined. To show that it is an isomorphism, define a map

$$\beta : \Sigma(N) \otimes_{RH} \Sigma(M) \rightarrow \Sigma(N \otimes_H M)$$

in the following way : choose a set S of representatives of the classes $(v, {}_H u)$. Now map the element $n \otimes_{RH} m \in N(v) \otimes M(u) \subseteq \Sigma(N) \otimes_{RH} \Sigma(M)$ to $[n \otimes m]_h$, where $h \in H$ is chosen such that $(vh^{-1}, hu) \in S$. Again, it is easy to see that this map is well defined, and that the maps α and β are mutual inverse isomorphisms of (RK, RG) -bimodules. \square

3.4. Corollary : *Let $G, H,$ and K be finite groups. Let X be a subgroup of $H \times G$ and Y be a subgroup of $K \times H$. Let M be an RX -module, and N be an RY -module. Then there is an isomorphism of (RK, RG) -bimodules*

$$(\text{Ind}_Y^{K \times H} N) \otimes_{RH} (\text{Ind}_X^{H \times G} M) \cong \bigoplus_{t \in [p_2(Y) \setminus H/p_1(X)]} \text{Ind}_{Y^{*(t,1)} X}^{K \times G} (N \otimes_{k_2(Y) \cap {}^t k_1(X)} ({}^{t,1} M)) .$$

Proof : Set $U = (H \times G)/X$. Then U is an (H, G) -biset by $h \cdot (t, s)X \cdot g = (ht, g^{-1}s)X$, and this biset is transitive. If u is the point X of U , then $A(u) = X$, and the equivalence of categories 2.5 reads

$$\text{Fun}(\langle U \rangle, R\text{-Mod}) \cong RX\text{-Mod} .$$

More precisely, for an RX -module M , this equivalence yields a functor $\tilde{M} : \langle U \rangle \rightarrow R\text{-Mod}$ in the following way : for any $(h, g) \in H \times G$, set

$$\tilde{M}((h, g)X) = M .$$

Next, fix a set S of representatives of elements of U , i.e. X -cosets in $H \times G$. For $(t, s) \in S$, and $(h, g) \in H \times G$, define a map

$$\tilde{M}(h, g) : \tilde{M}((t, s)X) = M \rightarrow \tilde{M}((ht, gs)X) = M$$

by $\tilde{M}(h, g)(m) = (y, x)m$, where (y, x) is the unique element of X such that $(ht, gs)(y, x)^{-1} \in S$.

Then it is easy to check that \tilde{M} is indeed a functor, and that there is an isomorphism of (RH, RG) -bimodules

$$\Sigma(\tilde{M}) \cong \text{Ind}_X^{H \times G} M .$$

Similarly, set $V = (K \times H)/Y$, and define a functor $\tilde{N}\langle V \rangle \rightarrow R\text{-Mod}$, using the RY -module N . Then the corollary is a straightforward consequence of the lemma, applied to the functors \tilde{M} and \tilde{N} , using Remark 2.6 and Remark 3.2. \square

References

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