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High gain observers with updated gain and homogeneous correction terms

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Abstract

Exploiting dynamic scaling and homogeneity in the bi-limit, we develop a new class of high gain observers which incorporate a gain update law and nonlinear output error injection terms. A broader class of systems can be addressed and the observer gain is better fitted to the incremental rate of the nonlinearities. The expected improved performance is illustrated.

Key words: High-gain observers, Homogeneity in the bi-limit, Dynamic scaling.

1 Introduction

We extend the standard high-gain observer (see (Gauthier and Kupka, 2001) and references therein) in two directions : homogeneity and gain adaptation. Our motivation comes from considering the system :

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f_2(x_1, x_2, u), \quad y = x_1, \quad (1)$$

with

$$f_2(x_1, x_2, u) = g(x_1)x_2 + x_2^{1+p} + u,$$

where $p \geq 0$ is a real number, g is a locally Lipschitz function and u is a known input.

When $p = 0$, we have :

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq |g(x_1) + 1| |x_2 - \hat{x}_2|. \quad (2)$$

The term $|g(x_1) + 1|$ is the output dependent incremental rate of the non-linearity. Systems with nonlinearities satisfying inequalities like (2) have already been studied in (Praly, 2003) (see also (Krishnamurthy et al., 2003)) and we know that a high gain observer can be used provided the gain is updated.

When p is in the interval $(0, 1)$, inequality (2) becomes :

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq \quad (3)$$

$$(|g(x_1)| + (1+p)|\hat{x}_2|^p) |x_2 - \hat{x}_2| + |x_2 - \hat{x}_2|^{1+p}.$$

The term, $|x_2 - \hat{x}_2|^{1+p}$ is a rational power of the norm

of the error $|x_2 - \hat{x}_2|$. To deal with this term we use the homogeneous in the bi-limit observer introduced in (Andrieu et al., 2008-SICON).

In the following we address the problem of state observation for systems whose dynamics admit a global explicit observability canonical form (Gauthier and Kupka, 2001, Equation (20)) in which the nonlinearities have increments bounded as in (3). However, we restrict our attention to estimating the state only of those solutions which are bounded in positive time.

Our new observer uses a less conservative estimate of the nonlinearities increments. From this we expect the possibility of achieving better performance. This is confirmed via simulations of an academic model of a bioreactor.

2 Main theoretical result

We consider systems whose dynamics are :

$$\begin{cases} \dot{x}_1 = f_1(u, y) + a_1(y)x_2 + \delta_1(t), \\ \vdots \\ \dot{x}_i = f_i(u, y, x_2, \dots, x_i) + a_i(y)x_{i+1} + \delta_i(t), \\ \vdots \\ \dot{x}_n = f_n(u, y, x_2, \dots, x_n) + \delta_n(t), \\ y = x_1 + \delta_y(t), \end{cases} \quad (4)$$

where y is the measured output in \mathbb{R} and the functions a_i and f_i are locally Lipschitz. u is a vector in \mathbb{R}^m representing the known inputs and a finite number of their derivatives. The vector $\delta = (\delta_1, \dots, \delta_n)$ represents the unknown inputs and δ_y is a measurement noise.

To simplify notations, let :

$$w^r = \text{sign}(w) |w|^r.$$

so that, for instance, to recover the usual quadratic function we must write $|x^2|$ or $|x|^2$. We let also :

$$\begin{aligned} \mathcal{S} \cdot x &= (x_2, \dots, x_n, 0)^T, \\ f(u, y, x) &= (f_1(u, y, x), \dots, f_n(u, y, x)), \\ A(y) &= \text{diag}(a_1(y), \dots, a_n(y)), \end{aligned}$$

where a_n is to be selected so that (5) below holds.

Theorem 1 *Suppose there exists a continuous function \mathbf{a} satisfying, with ρ , $\underline{\alpha}$ and $\overline{\alpha}$ constant and for j in $\{1, \dots, n\}$,*

$$0 < \rho \leq \mathbf{a}(y), \quad 0 < \underline{\alpha} \leq \frac{a_j(y)}{a(y)} \leq \overline{\alpha} \quad \forall y \in \mathbb{R}, \quad (5)$$

a real number d_∞ in $[0, \frac{1}{n-1})$, a positive real number c_∞ , a continuous function Γ and real numbers v_j in $[0, \frac{1}{j-1})$, for $j = 2, \dots, n$, such that, for all i in $\{2, \dots, n\}$ and all (\hat{x}, x, y, u) in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$, we have :

$$\begin{aligned} &|f_i(u, y, \hat{x}_2, \dots, \hat{x}_i) - f_i(u, y, x_2, \dots, x_i)| \\ &\leq \Gamma(u, y) \left(1 + \sum_{j=2}^n |\hat{x}_j|^{v_j} \right) \sum_{j=2}^i |\hat{x}_j - x_j| \\ &\quad + c_\infty \sum_{j=2}^i |\hat{x}_j - x_j|^{\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)}}. \end{aligned} \quad (6)$$

Then, for all sufficiently small strictly positive real numbers b , there exists a function K such that, for all sufficiently small strictly positive real number φ_1 and sufficiently large real numbers φ_2 and φ_3 , we can find functions β_W and β_L of class \mathcal{KL} and functions γ_W and γ_L of class \mathcal{K} such that the observer

$$\dot{\hat{x}} = A(y)\mathcal{S}\hat{x} + f(u, y, \hat{x}) + L \mathfrak{L} A(y) K \left(\frac{\hat{x}_1 - y}{L^b} \right), \quad (7)$$

$$\dot{L} = L [\varphi_1(\varphi_2 - L) + \varphi_3 \Omega(u, y, \hat{x})], \quad (8)$$

with :

$$\Omega(u, y, \hat{x}) = \Gamma(u, y) \left(1 + \sum_{j=2}^n |\hat{x}_j|^{v_j} \right), \quad (9)$$

$$\mathfrak{L} = \text{diag}(L^b, \dots, L^{n+b-1}), \quad (10)$$

initialized with $L(0) \geq \varphi_2$, has the following property : For each solution $t \mapsto x(t)$ of (4) right maximally defined on $[0, T)$, the observer solution is defined on the same interval and the error estimate $\mathbf{e} = \hat{x} - x$ satisfies :

$$|\mathfrak{L}(t)^{-1} \mathbf{e}(t)| \leq \beta_W (\mathfrak{L}(0)^{-1} \mathbf{e}(0), t) \quad (11)$$

$$+ \sup_{s \in [0, t]} \gamma_W \left(\left| \left(\frac{\delta(s)}{\varphi_2} \right) \right| \right) \quad \forall t \in [0, T)$$

where L satisfies :

$$\begin{aligned} L(t) &\leq 4\varphi_2 + \beta_L \left(\left(\frac{\mathbf{e}(0)}{L(0)} \right), t \right) \\ &\quad + \sup_{s \in [0, t]} \gamma_L \left(\left| \left(\frac{\delta(s)}{\varphi_2} \right) \right| \right) \\ &\quad + \sup_{s \in [0, t]} \gamma_L \left(\left| \left(\begin{array}{c} \mathbf{a}(y(s))\delta_y(s) \\ \Gamma(u(s), y(s)) \\ x(s) \end{array} \right) \right| \right). \end{aligned} \quad (12)$$

2.1 Discussion on the assumptions

The form (4) is a particular case of the implicit form obtained in (Gauthier and Kupka, 2001, Equation (20)). The functions a_i and f_i in (4), are not uniquely defined. We can get other functions by changing coordinates and, in this way, possibly satisfy conditions (6).

To understand the meaning of (6), we observe that, for any C^1 function f , there exist always two functions \mathcal{U} and Δ such that we have :

$$|f(a, b+c) - f(a, b)| \leq \mathcal{U}(a, b) |c| + \Delta(c).$$

Hence, in essence, (6) imposes two restrictions :

- the function Ω , defined in (9), is a bound on the local incremental rate \mathcal{U} .
- a fractional power limitation, $\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)}$ with d_∞ in $[0, \frac{1}{n-1})$, on the growth of Δ which bounds function increments for large argument increments.

For system (1), inequality (3) is in the form (6) with $d_\infty = p$, $\Gamma(u, y) = (|g(y)| + 1 + p)$ and $v_2 = p$. Hence, Theorem 1 applies when p is in the interval $[0, 1)$. Actually, when $p > 1$ and $u = 0$, there does not exist any observer guaranteeing convergence of the estimation error within the domain of existence of the solutions (see (Astolfi and Praly, 2006, Proposition 1)).

2.2 Discussion on the result

With (11) and (12) but with the presence of $\sup_s |x(s)|$, Theorem 1 says that the observer (7),(8) gives, at least for bounded solutions, an estimation error converging to a ball centered at the origin and with radius depending on the asymptotic L^∞ -norm of the disturbances δ and δ_y and therefore converging to the origin if these disturbances are vanishing.

Although we restrict our attention to bounded solutions, we are not back to the global Lipschitz case since the ‘‘Lipschitz constant’’ is solution dependent and therefore unavailable for observer design. It has to be learned on line and this is what L is doing in (8). The update law for L is very similar to the one introduced in (Praly, 2003) (see also (Krishnamurthy et al., 2003)). The difference is in the fact that (8) depends also on \hat{x} and u and not

only on y and we need the restrictions on v_j to deal with this dependence on \hat{x} .

If Ω were differentiable along the solutions, the update law (8) would give :

$$\overline{L - \left(\varphi_2 + \frac{\varphi_3}{\varphi_1} \Omega \right)} = \frac{\varphi_3}{\varphi_1} \dot{\Omega} - \varphi_1 L \left[L - \left(\varphi_2 + \frac{\varphi_3}{\varphi_1} \Omega \right) \right]. \quad (13)$$

This says that L would track $\varphi_2 + \frac{\varphi_3}{\varphi_1} \Omega$ up to an error proportional to the magnitude of $\dot{\Omega}$. We expect improved performance from this tracking property (see Section 3).

2.3 Comparison with published results

High gain observers have a long history. The prototype result is (Gauthier and Kupka, 2001, Theorem 6.2.2). It deals with systems admitting an observability canonical representation more general than (4) by being implicit in x_{i+1} . But there the right hand side of inequality (6) is supposed to be $\Gamma \sum_{j=2}^i |\hat{x}_j - x_j|$ with Γ constant.

The case where Γ may depend on u and y can be handled with updating the gain as in (8). This extends what can be found in (Praly, 2003) when the a_i are constant and in (Krishnamurthy et al., 2003) when the a_i are y -dependent.

The idea of having homogeneous (in the classical weighted sense) correction terms has been introduced in (Qian, 2005) for a pure chain of integrator, i.e. when the a_i 's are constant and the f_i are zero.

Another observer is proposed in (Lei et al., 2005), for systems with bounded solutions and admitting the same form (4) with the a_i 's constant and $f_1 = \dots = f_{n-1} = 0$ but with no restriction on f_n . However this is obtained by having a gain which grows monotonically with time along the solutions.

3 Discussion and illustration

To illustrate the interest for applications of our observer and the tracking property noticed in (13), we consider the same "academic" bioreactor as the one studied in (Gauthier et al., 1992). Its dynamics are described, in normalized variables and time, by the Contois model :

$$\dot{\eta}_1 = \frac{\eta_1 \eta_2}{\hbar \eta_1 + \eta_2} - u \eta_1, \quad \dot{\eta}_2 = -\frac{\eta_1 \eta_2}{\hbar \eta_1 + \eta_2} + u(1 - \eta_2) \quad (14)$$

where $y = \eta_1$ is measured. The parameter \hbar is a positive real number and the control input u is in the interval $\mathcal{M}_u = [u_{\min}, u_{\max}] \subset (0, 1)$. In (Gauthier et al., 1992), it is observed that the following set is forward invariant :

$$\mathcal{M}_\eta = \{(\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1 \geq \epsilon_1, \eta_2 \geq \epsilon_2, \eta_1 + \eta_2 \leq 1\},$$

where, $\epsilon_1 = \frac{(1-u_{\max})\epsilon_2}{\hbar u_{\max}}$, and $u_{\min} \geq \frac{\epsilon_2}{\hbar(1-\epsilon_2)+\epsilon_2}$. This guarantees that the bioreactor state remains in a known compact set.

Following (Gauthier et al., 1992), we change the coordinates as :

$$(\eta_1, \eta_2) \mapsto (x_1, x_2) = F(\eta_1, \eta_2) = \left(\eta_1, \frac{\eta_1 \eta_2}{\hbar \eta_1 + \eta_2} \right).$$

with x evolving in $\mathcal{M}_x = F(\mathcal{M}_\eta)$. In these new coordinates the system is in the explicit observability canonical form :

$$\dot{x}_1 = x_2 - u x_1, \quad \dot{x}_2 = f_2(x_1, x_2, u), \quad y = \eta_1,$$

with,

$$f_2(x_1, x_2, u) = m_0 + m_1 x_2 + m_2 x_2^2 + m_3 x_2^3 \quad (15)$$

where :

$$m_0 = \frac{u}{\hbar}, \quad m_1 = -u - \frac{1}{\hbar} - \frac{2u}{\hbar x_1},$$

$$m_2 = \frac{2}{\hbar x_1} + \frac{u}{\hbar x_1^2}, \quad m_3 = \frac{\hbar-1}{\hbar x_1^2}.$$

Note that, for all (x_1, x_2, u) in $\mathcal{M}_x \times \mathcal{M}_u$, we have :

$$\underline{x}_2(x_1) = x_1 \frac{\epsilon_2}{\hbar x_1 + \epsilon_2} \leq x_2 \leq x_1 \frac{1-x_1}{1-x_1+\hbar x_1} = \bar{x}_2(x_1).$$

Hence, without loss of generality, to evaluate f_2 in (15), we can replace (x_1, x_2) by (x_{1s}, x_{2s}) defined as

$$x_{1s} = \max\{\epsilon_1, \min\{1 - \epsilon_2, x_1\}\},$$

$$x_{2s} = \max\{\underline{x}_2(x_{1s}), \min\{\bar{x}_2(x_{1s}), x_2\}\}$$

and therefore assume that f_2 is globally Lipschitz.

For a nominal high gain observer, as in (Gauthier et al., 1992), the nonlinearity increment is bounded as :

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq df_{2\max} |x_2 - \hat{x}_2|.$$

where, from the Mean Value Theorem,

$$df_{2\max} = \max_{(u, x_1, x_2) \in \mathcal{M}_u \times \mathcal{M}_x} |m_1 + 2m_2 x_2 + 3m_3 x_2^2|.$$

For an updated high gain observer, the bound is :

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq \Omega_1(u, x_1, \hat{x}_2) |x_2 - \hat{x}_2|,$$

with

$$\Omega_1(u, x_1, \hat{x}_2) =$$

$$\max_{x_2 \in [\underline{x}_2(x_{1s}), \bar{x}_2(x_{1s})]} |m_1 + [m_2 + m_3 \hat{x}_2](\hat{x}_2 + x_2) + m_3 x_2^2|.$$

$$\hat{x}_{2s} = \max\{\underline{x}_2(x_{1s}), \min\{\bar{x}_2(x_{1s}), \hat{x}_2\}\}.$$

It follows that Theorem 1 applies with $d_\infty = c_\infty = 0$.

Finally for our observer with both updated gain and rational power error term, the bound is :

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq \Omega_2(u, x_1, \hat{x}_2) |x_2 - \hat{x}_2| + c_\infty |x_2 - \hat{x}_2|^{1+p},$$

with p in $(0, 1)$ and where

$$\Omega_2(u, x_1, \hat{x}_2) = \max_{x_2 \in [\underline{x}_2(x_{1s}), \bar{x}_2(x_{1s})]} |m_1 + \hat{x}_2^p ([m_2 + m_3 \hat{x}_2][\hat{x}_2^{1-p} + x_2^{1-p}] + m_3 x_2^{2-p})|$$

$$c_\infty = \max_{(u, x_1, x_2, \hat{x}_2) \in \mathcal{M}_u \times \mathcal{M}_x \times [\underline{x}_2(\epsilon_1), \bar{x}_2(1-\epsilon_2)]} |(m_2 + m_3 \hat{x}_2) x_2^{1-p} + m_3 x_2^{2-p}|$$

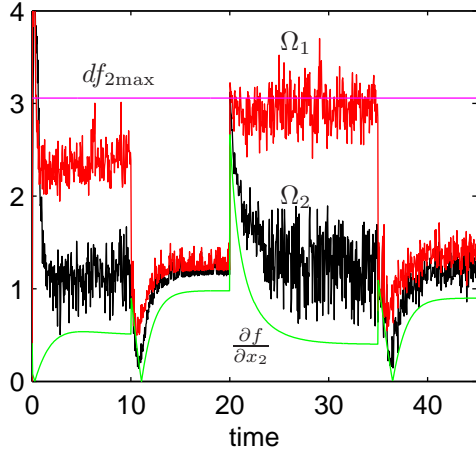


Fig. 1. Approximations of the local incremental rates

In this case, Theorem 1 gives the following observer :

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 - u y - L^{1+b} q_1 \left(\ell_1 \frac{[\hat{x}_1 - y]}{L^b} \right) , \\ \dot{\hat{x}}_2 = f_2(y, \hat{x}_{2s}, u) \\ \quad - L^{2+b} q_2 \left(\ell_2 q_1 \left(\ell_1 \frac{[\hat{x}_1 - y]}{L^b} \right) \right) , \\ \dot{L} = L [\varphi_1 (\varphi_2 - L) + \varphi_3 \Omega_2(u, y, \hat{x}_{2s})] , \end{cases}$$

where $q_1(s) = s + s^{\frac{1}{1-p}}$, $q_2(s) = s + s^{1+p}$ and b , φ_i and ℓ_i are parameters to be chosen.

Since we have, for all (x_1, x_2, u) in $\mathcal{M}_x \times \mathcal{M}_u$,

$$\frac{\partial f_2}{\partial x_2}(x_1, x_2, u) \leq \Omega_2(u, x_1, x_2) \leq \Omega_1(u, x_1, x_2) \leq df_{2 \max}$$

we expect the updated high gain observer to give better performance than the one without adaptation, and the new one proposed in this paper to give even better behavior in particular in presence of measurement noise.

3.1 Simulations

We illustrate the behavior of the observers with simulations. But this is no more than an illustration and we do not claim that our observer is the best one for this particular application¹.

The control input is selected as :

$$u(t) = \begin{cases} 0.410 & \text{if } t < 10 , \\ 0.02 & \text{if } 10 \leq t < 20 , \\ 0.6 & \text{if } 20 \leq t < 35 , \\ 0.1 & \text{if } 35 \leq t . \end{cases}$$

From this we have chosen $u_{\min} = 0.01$ and $u_{\max} = 0.7$ and ϵ_1 and ϵ_2 accordingly. Also, we have introduced two disturbances :

– the measurement disturbance is a Gaussian white

¹ A simple copy (without correction term) gives an observer which is not sensitive to measurement noise but on the other hand we cannot assign its speed of convergence.

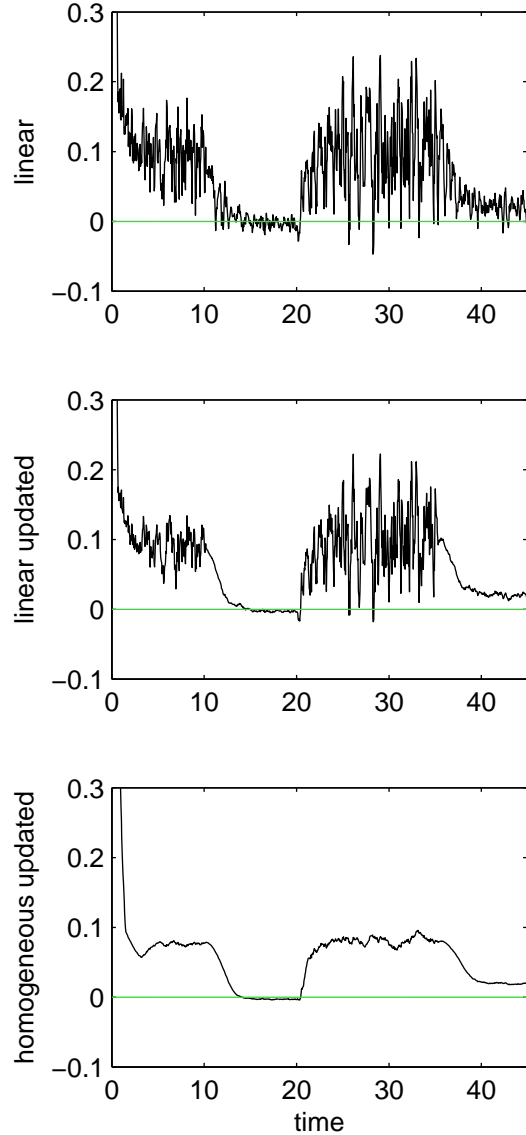


Fig. 2. Estimation error $\eta_2 - \hat{\eta}_2$ given by each observer.

noise with standard deviation equal to 10% of the η_1 domain $[\epsilon_1, 1 - \epsilon_2]$, i.e. = 0.05.

– a 20% error in \hbar . The value used for the system (14) is 1, whereas the one in the observers is 0.8

For the observers we have used the following values :

$$\begin{aligned} p &= 0.9 , \quad b = 0.410 , \\ \varphi_1 &= 0.03 , \quad \varphi_2 = 1 , \quad \varphi_3 = 3 , \quad \ell_1 = 0.01 , \quad \ell_2 = 0.01 . \end{aligned}$$

Figure 1 shows the values of the estimates of the local incremental rate of f_2 (i.e. $\frac{\partial f_2}{\partial x_2}$), $df_{2 \max}$ for the high-gain observer, Ω_1 for an updated high-gain observer, and Ω_2 for a homogeneous updated high-gain observer. In spite of the measurement noise, the predicted order $df_{2 \max} \geq \Omega_1 \geq \Omega_2 \geq \frac{\partial f_2}{\partial x_2}$ is observed in the mean.

Figure 2 displays the plot of the estimation error $\eta_2 - \hat{\eta}_2$, given by the observers with constant gain deduced from $df_{2\max}$ (top), with adapted gain deduced from Ω_1 (middle), and with adapted gain deduced from Ω_2 and homogeneity (bottom). In the three cases there is a bias, due to the error in \bar{h} , which increases with the estimates of the local incremental rate. We see also a strong correlation between the standard deviation of the error $\hat{\eta}_2 - \eta_2$ and the magnitude of these estimates respectively used, i.e. $df_{2\max}$, Ω_1 and Ω_2 . As expected the best result is given by the new observer based on Ω_2 .

4 Proof of Theorem 1

Theorem 1 is proved in Subsection 4.3. It needs some prerequisite summarized now and which can be found in (Andrieu et al., 2008-SICON).

4.1 Homogeneous approximation

Given a vector $r = (r_1, \dots, r_n)$ in $(\mathbb{R}_+/\{0\})^n$, we define the dilation of a vector x in \mathbb{R}^n as

$$\lambda^r \diamond x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)^T.$$

Definition 1

- A continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is said homogeneous in the 0-limit (respectively ∞ -limit) with associated triple (r_0, d_0, ϕ_0) (resp. $(r_\infty, d_\infty, \phi_\infty)$), where r_0 (resp. r_∞) in $(\mathbb{R}_+/\{0\})^n$ is the weight, d_0 (resp. d_∞) in \mathbb{R}_+ the degree and $\phi_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ (resp. $\phi_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$) the approximating function, if ϕ_0 (resp. ϕ_∞) is continuous and not identically zero and, for each compact set C in \mathbb{R}^n and each $\varepsilon > 0$, there exists λ^* such that we have :

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_0} \diamond x)}{\lambda^{d_0}} - \phi_0(x) \right| \leq \varepsilon \quad \forall \lambda \in (0, \lambda^*].$$

(respectively

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_\infty} \diamond x)}{\lambda^{d_\infty}} - \phi_\infty(x) \right| \leq \varepsilon \quad \forall \lambda \in [\lambda^*, +\infty) .)$$

- A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the 0-limit (resp. ∞ -limit) with associated triple (r_0, d_0, f_0) (resp. $(r_\infty, d_\infty, f_\infty)$), where $f_0 = \sum_{i=1}^n f_{0,i} \frac{\partial}{\partial x_i}$ (resp. $f_\infty = \sum_{i=1}^n f_{\infty,i} \frac{\partial}{\partial x_i}$), if, for each i in $\{1, \dots, n\}$, the function f_i is homogeneous in the 0-limit (resp. ∞ -limit) with associated triple $(r_0, d_0 + r_{0,i}, f_{0,i})$ ².

Definition 2 A continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ (or a vector field f) is said homogeneous in the bi-limit if it is homogeneous in the 0-limit and in the ∞ -limit.

² In the case of a vector field the degree d_0 can be negative as long as $d_0 + r_{0,i} \geq 0$ (resp. $(r_\infty, d_\infty + r_{\infty,i}, f_{\infty,i})$), for all $1 \leq i \leq n$.

4.2 Homogeneous in the bi-limit observer

Consider the following chain of integrators on \mathbb{R}^n :

$$\dot{\mathfrak{X}} = \mathfrak{A}(t) \mathfrak{S} \mathfrak{X}, \quad (16)$$

where $\mathfrak{A}(t) = \text{diag}(\mathfrak{a}_1(t), \dots, \mathfrak{a}_n(t))$, is a known time varying matrix with the \mathfrak{a}_i satisfying, with $\underline{\mathfrak{a}}$ and $\bar{\mathfrak{a}}$ constant,

$$0 < \underline{\mathfrak{a}} \leq \mathfrak{a}_i(t) \leq \bar{\mathfrak{a}} \quad \forall t. \quad (17)$$

With $d_0 = 0$ and d_∞ arbitrary in $\left[0, \frac{1}{n-1}\right)$, the system (16) is homogeneous in the bi-limit with the weights $r_0 = (r_{0,1}, \dots, r_{0,n})$ and $r_\infty = (r_{\infty,1}, \dots, r_{\infty,n})$ as :

$$r_{0,i} = 1, \quad r_{\infty,i} = 1 - d_\infty (n - i). \quad (18)$$

In (Andrieu et al., 2008-SICON), a new observer was proposed for system (16) for the particular case where $\mathfrak{a}_i(t) = 1$. Its design is done recursively together with the one of an appropriate error Lyapunov function W which is homogeneous in the bi-limit.

To combine this tool with gain updating we need an extra property on W which is a counterpart of (Praly, 2003, equation (16)) or (Krishnamurthy et al., 2003, Lemma A1). We have :

Theorem 2 Let d_∞ be in $[0, \frac{1}{n-1})$, d_W in $[2 + d_\infty, \infty)$ and $\mathfrak{B} = \text{diag}(\mathfrak{b}_1, \dots, \mathfrak{b}_n)$ with $\mathfrak{b}_j > 0$. If (17) holds, there exist a vector field $K : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is homogeneous in the bi-limit with associated weights r_0 and r_∞ , and a positive definite, proper and C^1 function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, homogeneous in the bi-limit with associated triples (r_0, d_W, W_0) and $(r_\infty, d_W, W_\infty)$, such that

- (1) The functions W_0 and W_∞ are positive definite and proper and, for each j in $\{1, \dots, n\}$, the function $\frac{\partial W}{\partial e_j}$ is homogeneous in the bi-limit with approximating functions $\frac{\partial W_0}{\partial e_j}$ and $\frac{\partial W_\infty}{\partial e_j}$.
- (2) There exist two positive real numbers c_1 and c_2 such that we have, for all (t, E) in $\mathbb{R} \times \mathbb{R}^n$,

$$\frac{\partial W}{\partial E}(E) \mathfrak{A}(t) (\mathfrak{S} E + K(e_1)) \quad (19)$$

$$\leq -c_1 \left(W(E) + W(E)^{\frac{d_W + d_\infty}{d_W}} \right),$$

$$\frac{\partial W}{\partial E}(E) \mathfrak{B} E \geq c_2 W(E). \quad (20)$$

For proving this result, the only difference compared with what is done in (Andrieu et al., 2008-SICON) is to multiply W_i by a sufficiently small positive real number σ_i before using it in the definition of W_{i-1} . The proof is omitted due to space limitation. It can be found in (Andrieu et al., 2008-HAL).

4.3 Proof of Theorem 1

Let \mathfrak{A} and \mathfrak{B} in Theorem 2 be (see (Praly, 2003)) :

$$\mathfrak{A}(t) = \frac{A(y(t))}{\mathfrak{a}(y(t))}, \quad \mathfrak{B} = \text{diag}(b, 1+b, \dots, n-1+b)^T,$$

where $y(t)$ is the evaluation of y along a solution and b is a positive real number satisfying, for all $1 \leq j \leq i \leq n$,

$$\frac{1 - d_\infty(n-i-1)}{1 - d_\infty(n-j)} < \frac{i+b}{j-1+b} < \frac{i}{j-1}, \quad (21)$$

$$\text{and} \quad 0 < b < \frac{1 - v_j(j-1)}{v_j}, \quad (22)$$

with d_∞ and v_j as given in the statement of Theorem 1.

From Theorem 2, we obtain a homogeneous in the bi-limit vector field K and a homogeneous in the bi-limit Lyapunov function W satisfying (19) and (20). This allows us to write the observer as in (7)-(8). Note that if $L(0) \geq \varphi_2$ then $L(t) \geq \varphi_2$ for all $t \geq 0$ in the domain of definition.

Properties of the estimation error. With \mathfrak{L} given in (10), let $E = (e_1, \dots, e_n)$ and τ be defined as :

$$E = \mathfrak{L}^{-1}(\hat{x} - x) = \mathfrak{L}^{-1} \boldsymbol{\epsilon} \quad , \quad d\tau = L dt. \quad (23)$$

Since we have :

$$\dot{\mathfrak{L}}^{-1} = -L^{-1} \dot{L} \mathfrak{B} \mathfrak{L}^{-1},$$

we get :

$$\frac{dE}{d\tau} = A(y) \left[SE + K \left(e_1 - \frac{\delta_y}{L^b} \right) \right] - L^{-1} \mathfrak{L}^{-1} \delta \quad (24)$$

$$+ \mathfrak{D}(L) - L^{-1} \frac{dL}{d\tau} \mathfrak{B} E,$$

with $\mathfrak{D}(L)$ defined as

$$\mathfrak{D}(L) = \left(\dots, \frac{f_i(u, y, \hat{x}) - f_i(u, y, x)}{L^{i+b}}, \dots \right).$$

Along the solutions of the system (24) we have :

$$\frac{dW(E)}{d\tau} = \frac{\partial W}{\partial E}(E) \mathfrak{a}(y) \mathfrak{A}(t) [SE + K(e_1)] \quad (25)$$

$$+ T_\delta + T_y + T_{GU} + T_{NL},$$

with the notations

$$T_\delta = -\frac{\partial W}{\partial E}(E) L^{-1} \mathfrak{L}^{-1} \delta,$$

$$T_y = \frac{\partial W}{\partial E}(E) \mathfrak{a}(y) \mathfrak{A}(t) \left[K \left(e_1 - \frac{\delta_y}{L^b} \right) - K(e_1) \right],$$

$$T_{GU} = -L^{-2} \dot{L} \frac{\partial W}{\partial E}(E) \mathfrak{B} E,$$

$$T_{NL} = \frac{\partial W}{\partial E}(E) \mathfrak{D}(L),$$

and, with (19), we have :

$$\frac{\partial W}{\partial E}(E) \mathfrak{a}(y) \mathfrak{A}(t) (SE + K(e_1)) \quad (26)$$

$$\leq -c_1 \mathfrak{a}(y) \left(W(E) + W(E) \frac{d_W + d_\infty}{d_W} \right).$$

Bounding T_δ . With weights 1 and $r_{\infty, i} + d_\infty$ for $\frac{\delta_i}{L^i}$, the function $(E, \frac{\delta_i}{L^i}) \mapsto \frac{\partial W}{\partial e_i}(E) \frac{\delta_i}{L^i}$ is homogeneous in the bi-limit with degrees d_W and $d_W + d_\infty$. Also W, W_0 and W_∞ are positive definite. Hence (see (Andrieu et

al., 2008-SICON, Corollary 2.15)) there exists a positive real number c_3 satisfying, for any E, i and $L \geq \varphi_2$,

$$\frac{\partial W}{\partial e_i}(E) \frac{\delta_i}{L^{i+b}} \leq \frac{c_3}{n\varphi_2^b} \left[W(E) + W(E) \frac{d_W + d_\infty}{d_W} \right]$$

$$+ \frac{c_3}{nL^b} \mathfrak{H}_i \left(\frac{\delta_i}{L^i} \right).$$

where $\mathfrak{H}_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the strictly increasing, homogeneous in bi-limit function defined as

$$\mathfrak{H}_i(s) = \left(1 + |s|^{\frac{d_W + d_\infty}{r_{\infty, i} + d_\infty}} \right) \frac{|s|^{d_W}}{1 + |s|^{d_W}}.$$

This gives :

$$T_\delta \leq \frac{c_3}{\varphi_2^b} \left[W(E) + W(E) \frac{d_W + d_\infty}{d_W} \right] + \frac{c_3}{nL^b} \sum_{i=1}^n \mathfrak{H}_i \left(\frac{\delta_i}{L^i} \right).$$

Bounding T_y . Let k_i be the i th component of K and μ and η be the functions :

$$\mu(\bar{\delta}) = |\bar{\delta}|^{d_W} + |\bar{\delta}|^{\frac{d_W + d_\infty}{r_{\infty, 1}}},$$

$$\eta(E, \bar{\delta}) = \frac{c_1}{4n\bar{\mathfrak{A}}} [W(E) + W(E) \frac{d_W + d_\infty}{d_W}]$$

$$- \frac{\partial W}{\partial e_i}(E) [k_i(e_1 + \bar{\delta}) - k_i(e_1)]$$

From the properties of K , with the weights 1 and $r_{\infty, 1}$ for $\bar{\delta}$, the functions $\bar{\delta} \mapsto \mu(\bar{\delta})$ and $(E, \bar{\delta}) \mapsto \eta(E, \bar{\delta})$ are homogeneous in the bi-limit with degree 1 and $d_W + d_\infty$ and approximating functions, respectively,

$$|\bar{\delta}|^{d_W}, \quad \frac{c_1}{4n\bar{\mathfrak{A}}} W_0(E) - \frac{\partial W_0}{\partial e_i}(E) [k_{i,0}(e_1 + \bar{\delta}) - k_{i,0}(e_1)]$$

and

$$|\bar{\delta}|^{\frac{d_W + d_\infty}{r_{\infty, 1}}}, \quad \frac{c_1}{4n\bar{\mathfrak{A}}} W_\infty(E) \frac{d_W + d_\infty}{d_W}$$

$$- \frac{\partial W_\infty}{\partial e_i}(E) [k_{i,\infty}(e_1 + \bar{\delta}) - k_{i,\infty}(e_1)].$$

Hence (see (Andrieu et al., 2008-SICON, Lemma 2.13)) there exists a positive real number c_4 satisfying

$$\frac{\partial W}{\partial e_i}(E) [k_i(e_1 + \bar{\delta}) - k_i(e_1)] \leq$$

$$\frac{c_1}{4n\bar{\mathfrak{A}}} \left[W(E) + W(E) \frac{d_W + d_\infty}{d_W} \right] + \frac{c_4}{n} \left[|\bar{\delta}|^{d_W} + |\bar{\delta}|^{\frac{d_W + d_\infty}{r_{\infty, 1}}} \right]$$

By letting $\bar{\delta} = -\frac{\delta_y}{L^b}$, this yields :

$$T_y \leq \frac{c_1}{4} \mathfrak{a}(y) \left[W(E) + W(E) \frac{d_W + d_\infty}{d_W} \right]$$

$$+ c_4 \bar{\mathfrak{A}} \mathfrak{a}(y) \left[\left| \frac{\delta_y}{L^b} \right|^{d_W} + \left| \frac{\delta_y}{L^b} \right|^{\frac{d_W + d_\infty}{r_{\infty, 1}}} \right].$$

Bounding T_{GU} . The function $E \mapsto \frac{\partial W}{\partial E}(E) \mathfrak{B} E$ is homogeneous in the bi-limit with associated weights r_0 and r_∞ and degrees $d_{W,0} = d_{W,\infty} = d_W$. Hence (see (Andrieu et al., 2008-SICON, Corollary 2.15)) there exists a positive real number c_5 satisfying :

$$\frac{\partial W}{\partial E}(E) \mathfrak{B} E \leq c_5 W(E) \quad \times \left(W(E) + W(E)^{\frac{d_W + d_\infty}{d_W}} \right)$$

With (20) and the definition of \dot{L} in (8), this yields :

$$T_{GU} \leq -c_2 \varphi_3 \frac{\Omega(u, y, \hat{x})}{L} W(E) + c_5 \varphi_1 W(E). \quad (27)$$

Bounding T_{NL} . With (6), (23) and (18) give :

$$|\mathfrak{D}_i(L)| \leq \Omega(u, y, \hat{x}) \sum_{j=2}^i L^{j-1-i} |e_j| \quad (28)$$

$$+ c_\infty L^{-i-b} \sum_{j=2}^i |L^{b+j-1} e_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}}.$$

With the inequalities (21) and (18), we know there exists a strictly positive real number ϵ_1 satisfying :

$$\varphi_2^{-\epsilon_1} \geq L^{-\epsilon_1} \geq L^{(b+j-1)\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}} - i - b} \quad \forall L \geq \varphi_2 \geq 1.$$

Consequently, for all $L \geq \varphi_2 \geq 1$, we have :

$$|\mathfrak{D}_i(L)| \leq \frac{\Omega(u, y, \hat{x})}{L} \sum_{j=2}^i |e_j| + c_\infty \varphi_2^{-\epsilon_1} \sum_{j=2}^i |e_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}}.$$

On another hand, since (18) and $0 = d_0 \leq d_\infty$ imply

$$\frac{r_{\infty,i} + d_\infty}{r_{\infty,i}} \geq 1 \quad i \in \{1 \dots n\}, \quad (29)$$

the functions $E \mapsto \left| \frac{\partial W}{\partial e_i}(E) \right| |e_j|$ and

$$E \mapsto \left| \frac{\partial W}{\partial e_i}(E) \right| \sum_{j=2}^i |e_j| + |e_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}}$$

are homogeneous in the bi-limit with weights r_∞ and r_0 and degrees d_W and $d_W + (j - i)d_\infty$ ($\leq d_W$) and d_W and $d_\infty + d_W$ respectively. Hence (see (Andrieu et al., 2008-SICON, Corollary 2.15)) there exists positive real numbers c_6 and c_7 satisfying, for all E in \mathbb{R}^n and i in $\{1, \dots, n\}$,

$$\left| \frac{\partial W}{\partial e_i}(E) \right| |e_j| \leq \frac{c_6}{n^2} W(E) \quad \forall j \in \{1, \dots, i\}$$

$$\left| \frac{\partial W}{\partial e_i}(E) \right| \left(\sum_{j=2}^i |e_j| + |e_j|^{\frac{r_{\infty,i} + d_\infty}{r_{\infty,j}}} \right) \leq \frac{c_7}{n} \left(W(E) + W(E)^{\frac{d_\infty + d_W}{d_W}} \right).$$

This gives

$$|T_{NL}| \leq c_6 \frac{\Omega(u, y, \hat{x})}{L} W(E) \quad (30)$$

$$+ c_\infty c_7 \varphi_2^{-\epsilon_1} \left(W(E) + W(E)^{\frac{d_\infty + d_W}{d_W}} \right).$$

Bound on $\frac{dW(E)}{d\tau}$. Using inequality (26), and the bounds on T_δ , T_y , T_{GU} and T_{NL} , we obtain :

$$\frac{dW(E)}{d\tau} \leq (c_6 - c_2 \varphi_3) \frac{\Omega(u, y, \hat{x})}{L} W(E)$$

$$+ \left(\frac{c_3}{\varphi_2^b} + \frac{c_\infty c_7}{\varphi_2^{\epsilon_1}} + c_5 \varphi_1 - \frac{3c_1}{4} \mathbf{a}(y) \right)$$

$$+ \frac{c_3}{nL^b} \sum_{i=1}^n \mathfrak{H}_i \left(\frac{\delta_i}{L^i} \right) + \frac{c_4 \bar{\mathfrak{A}}}{L^b} \mathbf{a}(y) \left[|\delta_y|^{d_W} + |\delta_y|^{\frac{d_W + d_\infty}{r_{\infty,1}}} \right].$$

Therefore, with (5) and $L > \varphi_2$, by choosing φ_2 and φ_3 sufficiently large and φ_1 sufficiently small, we get :

$$\frac{dW(E)}{dt} \leq -\kappa L W(E) + L U,$$

where $\kappa = \frac{c_1 \rho}{2}$ and

$$U(t) = \frac{c_3}{n \varphi_2^b} \sum_{i=1}^n \mathfrak{H}_i \left(\frac{\delta_i(t)}{L(t)^i} \right)$$

$$+ \frac{c_4 \bar{\mathfrak{A}} \rho}{\varphi_2^b} \left[\left| \frac{\mathbf{a}(y(t)) \delta_y(t)}{\rho} \right|^{d_W} + \left| \frac{\mathbf{a}(y(t)) \delta_y(t)}{\rho} \right|^{\frac{d_W + d_\infty}{r_{\infty,1}}} \right].$$

Integrating this inequality, it yields on the time of existence of the solutions :

$$W(E(t)) \leq \exp \left(-\kappa \int_s^t L(s) \right) W(E(s)) + \int_s^t \exp \left(-\kappa \int_r^t L(u) du \right) L(r) U(r) dr.$$

And since $L > \varphi_2 \geq 1$, this implies

$$W(E(t)) \leq \exp(-\kappa(t-s)) W(E(s)) + \frac{1}{\kappa} \sup_{r \in [s,t]} U(r).$$

The function W being proper, it yields an ISS property between the inputs $\frac{\delta_i}{L^i}$'s and $\mathbf{a}(y) \delta_y$ and the state E . Hence, inequality (11) follows on the time domain of existence of the solutions from the definition (23) of e_i .

Behavior of L . We can rewrite (9) in ³ :

$$\Omega(u, y, \hat{x}) \leq \Gamma(u, y) \left(1 + \sum_{j=2}^n |x_j|^{v_j} + L^{v_j(b+j-1)} |e_j|^{v_j} \right).$$

Then, with (22) and $L \geq 1$, we get $\epsilon_2 > 0$ satisfying :

$$\Omega(u, y, \hat{x}) \leq \Gamma(u, y) \left(1 + \sum_{j=2}^n |x_j|^{v_j} + L^{1-\epsilon_2} |e_j|^{v_j} \right).$$

Consequently (8) gives

$$\dot{L} \leq L \left(\varphi_1 \varphi_2 - \varphi_1 L + \varphi_3 \Gamma(u, y) + \varphi_3 \Gamma(u, y) \sum_{j=2}^n |x_j|^{v_j} + L^{1-\epsilon_2} |e_j|^{v_j} \right).$$

$$\leq -\frac{\varphi_1}{2} L^2$$

$$-L \left(\frac{\varphi_1}{4} L - \varphi_1 \varphi_2 - \varphi_3 \Gamma(u, y) - \varphi_3 \Gamma(u, y) \sum_{j=2}^n |x_j|^{v_j} - \left(\frac{4n\varphi_3}{\varphi_1} \right)^{\frac{1}{\epsilon_2}} \Gamma(u, y)^{\frac{2}{\epsilon_2}} - \left(\frac{4n\varphi_3}{\varphi_1} \right)^{\frac{1}{\epsilon_2}} \sum_{j=2}^n |e_j|^{\frac{2v_j}{\epsilon_2}} \right).$$

³ Since v_j is smaller than 1, we have, for all $(a, b) \in \mathbb{R}_{+}^2$, $\frac{a}{(a^{v_j} + b^{v_j})^{\frac{1}{v_j}}} \leq \frac{a^{v_j}}{a^{v_j} + b^{v_j}} \leq 1$ and $(a+b)^{v_j} \leq a^{v_j} + b^{v_j}$. (31)

This implies the existence of a class \mathcal{KL} function β_1 and a class \mathcal{K} function γ_1 such that, along any solution, we have on its domain of existence

$$\max\{L(t) - 4\varphi_2, 0\} \leq \beta_1(L(0), t) + \sup_{s \leq t} \gamma_1(|(\Gamma(u(s), y(s)), x(s), E(s))|) .$$

Hence having a cascade of two ISS systems, (12) and (11) holds on $[0, \mathcal{T}]$.

5 Conclusion

We have presented a modification of the classical high gain observer with the introduction of a gain updating and of a homogeneous in the bi-limit correction term. We have shown that this extends the domain of applicability and proved convergence for bounded solutions. We have also shown by means of an example, that the modification may improve performance by allowing a better fit of the incremental rate of the nonlinearities.

This work has been performed while the first author was a member of the CAP Group of Imperial College London.

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