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# Claw-free circular-perfect graphs

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## Abstract

The circular chromatic number of a graph is a well-studied refinement of the chromatic number. Circular-perfect graphs is a superclass of perfect graphs defined by means of this more general coloring concept. This paper studies claw-free circular-perfect graphs. A consequence of the strong perfect graph theorem is that minimal circular-imperfect graphs  $G$  have  $\min\{\alpha(G), \omega(G)\} = 2$ . In contrast to this result, it is shown in [9] that minimal circular-imperfect graphs  $G$  can have arbitrarily large independence number and arbitrarily large clique number. We prove that claw-free minimal circular-imperfect graphs  $G$  have  $\min\{\alpha(G), \omega(G)\} \leq 3$ .

*Keywords:* circular-perfect graph, claw-free graph

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Let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ , then a  $k$ -coloring of  $G$  is a mapping  $f : V \rightarrow \{1, \dots, k\}$  with  $f(u) \neq f(v)$  if  $uv \in E$ , i.e., adjacent vertices receive different colors. The minimum  $k$  for which  $G$  admits a  $k$ -coloring is called the *chromatic number*  $\chi(G)$ . The *clique number*  $\omega(G)$  (resp. *independence number*  $\alpha(G)$ ) of  $G$  is the order of a largest clique (resp. independent set) of  $G$ , i.e., the maximum number of pairwise adjacent (resp. non-adjacent) vertices of  $G$ .

The circular chromatic number and circular clique number of graphs are refinements of the chromatic number and the clique number. Suppose  $G = (V, E)$  is a graph with at least one edge, and  $k \geq 2d$  are positive integers. A  $(k, d)$ -circular coloring of  $G$  is a mapping  $f : V \rightarrow \{0, \dots, k-1\}$  with  $d \leq |f(u) - f(v)| \leq k-d$  if  $uv \in E$ . The *circular chromatic number*  $\chi_c(G)$  is the minimum  $\frac{k}{d}$  taken over all  $(k, d)$ -circular colorings of  $G$ . Since every  $(k, 1)$ -circular coloring is a usual  $k$ -coloring of  $G$ , we have  $\chi_c(G) \leq \chi(G)$ . On the other hand, it is known [13] and easy to see that for any graph  $G$ ,  $\chi_c(G) > \chi(G) - 1$ , and hence  $\chi(G) = \lceil \chi_c(G) \rceil$ . So  $\chi_c(G)$  is a refinement of  $\chi(G)$ .

Let  $K_{k/d}$  with  $k \geq 2d$  denote the graph with the  $k$  vertices  $0, \dots, k-1$  and edges  $ij$  such that  $d \leq |i-j| \leq k-d$ . The graphs  $K_{k/d}$  are called *circular cliques*. Circular cliques include all cliques  $K_t = K_{t/1}$ , all odd antiholes  $\overline{C}_{2t+1} = K_{(2t+1)/2}$ , and all odd holes  $C_{2t+1} = K_{(2t+1)/t}$ . The *circular clique number* is defined as  $\omega_c(G) = \max\{\frac{k}{d} : K_{k/d} \subseteq G, \gcd(k, d) = 1\}$ . It follows from the definition that  $\omega(G) \leq \omega_c(G)$ . It is also known [17] that for any graph  $G$ ,  $\omega_c(G) < \omega(G) + 1$ , and hence  $\omega(G) = \lfloor \omega_c(G) \rfloor$ . Therefore  $\omega_c(G)$  is a refinement of  $\omega(G)$ .

Obviously  $\omega(G)$  is a lower bound for  $\chi(G)$ . A graph  $G$  is *perfect* if each induced subgraph  $G' \subseteq G$  has  $\omega(G') = \chi(G')$ . Similarly,  $\omega_c(G)$  is a lower bound for  $\chi_c(G)$ . A graph  $G$  is called *circular-perfect* [17] if each induced subgraph  $G' \subseteq G$  has  $\chi_c(G') = \omega_c(G')$ .

Perfect graphs have been studied extensively since the concept and two conjectures (the weak and the strong perfect graph conjectures) were proposed by Berge [1] in 1961. The weak perfect graph conjecture was settled by Lovász [8] in 1972. Recently, the strong perfect graph conjecture has been settled by Chudnovsky, Robertson, Seymour and Thomas in [2], which gives a characterization of perfect graphs by means of forbidden induced subgraphs: a graph  $G$  is perfect if and only if  $G$  contains neither chordless odd cycles  $C_{2k+1}$  with  $k \geq 2$ , nor their complements  $\overline{C}_{2k+1}$ .

It follows from the definitions that for any graph  $G$ ,  $\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq$

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$\chi(G)$ . Therefore every perfect graph is circular-perfect. However, odd cycles and their complements are circular-perfect graphs but not perfect graphs. So the class of circular-perfect graphs is a proper superclass of the class of perfect graphs.

Is there a simple characterization of circular-perfect graphs by means of forbidden induced subgraphs? It is shown in [14] that the line graph  $L(G)$  of a cubic graph  $G$  is circular-perfect if and only if  $G$  is 3-edge colourable. Thus such a characterization of circular-perfect graphs implies a characterization of critically non-3-edge colourable cubic graphs, which is known to be a difficult problem. So it is unlikely that there is a simple forbidden induced subgraph characterization of circular-perfect graphs. Some sufficient conditions for a graph to be circular-perfect were obtained in [16,17]. Classes of (minimal) circular-imperfect graphs were constructed in [9,11,12,15]. Minimal circular-imperfect line graphs were studied in [14]. In this paper, we study claw-free circular-perfect graphs.

A graph  $G$  is *claw-free* if  $K_{1,3}$  is not an induced subgraph of  $G$ . Claw-free graphs is a superclass of line graphs and has been studied extensively in the literature. Recently, Chudnovsky and Seymour [4,3] presented a structural characterization of claw-free graphs. A graph  $G$  for which the neighbourhood of each vertex can be covered by two cliques is called a *quasi-line graph*. We use their characterization, restricted to quasi-line graphs, to prove a structural property of minimal circular-imperfect graphs. One consequence of the strong perfect graph theorem is that minimal imperfect graphs  $G$  have  $\min\{\omega(G), \alpha(G)\} = 2$ . It was asked in [11] whether  $\min\{\omega(G), \alpha(G)\}$  is bounded for minimal circular-imperfect graphs  $G$ . This question was answered in the negative in [9], where it is proved that for any positive integer  $k$ , there is a minimal circular-imperfect graph  $G$  with  $\min\{\omega(G), \alpha(G)\} \geq k$ .

We show that if restricted to claw-free graphs, the question above has a positive answer: if  $G$  is a claw-free minimal circular-imperfect graph, then  $\min\{\omega(G), \alpha(G)\} \leq 3$ .

Before the strong perfect graph conjecture becomes a theorem, the conjecture was confirmed for claw-free graphs in [7][10]. Our result above implies an alternative proof of this result, of course without making use of the strong perfect graph theorem [2].

## Sketch of the proof

Suppose  $G$  is a claw-free graph with independence number at least 3. It was proved by Fouquet [6] that for any vertex  $x$  of  $G$ , the neighborhood  $N_G(x)$  of  $x$  either contains an induced  $C_5$ , or can be covered with two cliques. If  $G$  is circular-perfect, then  $N_G(x)$  does not contain an induced  $C_5$ , for otherwise  $G$  contains the odd wheel

$W_5$ , which is circular-imperfect. Thus we have the following observation: *if  $G$  is a claw-free circular-perfect graph with independence number at least 3, then  $G$  is a quasi-line graph.*

It turns out that claw-free circular-perfect graphs with an induced odd antihole of size at least 7 have a basic structure:

**Theorem 1.** *If  $G$  is a connected claw-free circular-perfect graph with an induced odd antihole  $H$  of size at least 7 then  $G \setminus H$  is a clique. Furthermore  $\alpha(G) = 2$ .*

Since every minimal circular-imperfect graph is 2-connected, we have the following corollary:

**Corollary 1.** *If  $G$  is a claw-free minimal circular-imperfect graph and contains an induced odd antihole  $H$  of size at least 7, then  $\alpha(G) \leq 3$ .*

It remains to study claw-free circular perfect graphs  $G$  that do not contain an odd antihole of order at least 7. Due to Theorem 1,  $G$  has independence number at least 3, and is therefore, as mentioned above, quasi-line. We establish that if  $G$  has clique number at least 4 then  $G$  has an independent set  $I$  that intersects each maximum clique.

We prove a stronger statement:

**Theorem 2.** *If  $G$  is a quasi-line graph,  $\omega(G) = k \geq 4$  and for every vertex  $x$ ,  $G - x$  has a  $k$ -colouring, then either  $G$  is the complement of a circular clique or  $G$  has a stable set which intersects every maximum clique of  $G$ .*

As a consequence, a claw-free graph  $G$  with  $\omega(G) = k \geq 4$  and  $\alpha(G) \geq 4$  can not be minimal circular-imperfect. Because otherwise,  $G$  is not the complement of a circular-clique [5], is quasi-line since it does not contain the odd wheel  $W_5$  (which is already minimal circular-imperfect). Hence there is an independent set  $I$  intersecting each maximum clique. Since  $G - I$  is circular-perfect, we have  $\omega_c(G - I) = \chi_c(G - I)$ . Due to Corollary 1,  $G$ , and thus  $G - I$ , does not contain  $K_{(2p+1)/2}$  for  $p \geq 3$ . It follows that  $\omega_c(G - I) = \omega(G - I) = k - 1$  and hence  $\chi(G - I) = \chi_c(G - I) = k - 1$ . But then  $\chi(G) = \omega(G) = k$ , and hence  $G$  is circular-perfect.

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