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Gauge group of the standard model in $Cl_{1,5}$

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Abstract

ABSTRACT. Describing a wave with spin $1/2$, the Dirac equation is form invariant under a group which is not the Lorentz group. This $SL(2, \mathbb{C})$ group is a subgroup of $Cl_3^* = GL(2, \mathbb{C})$ which is the true group of form invariance of the Dirac equation. Firstly we use the Cl_3 algebra to read all features of the Dirac equation for a wave with spin $1/2$. We extend this to electromagnetic laws. Next we use both the Cl_3 algebra and the space-time algebra to get the gauge group of the electro-weak interactions, first in the leptonic case, electron+neutrino, next in the quark case. The complete wave for all objects of the first generation uses two supplementary dimensions of space and the Clifford algebra $Cl_{1,5}$. It is a function of the usual space-time with value into this enlarged algebra. The gauge group is then enlarged into a $U(1) \times SU(2) \times SU(3)$ Lie group in a way which gives automatically the insensitivity of electrons and neutrinos to strong interactions. This study gives new insights for many features of the standard model. It explains also how to get three generations and four kinds of neutrinos. We encounter not only two remarkable identities, we are able to explain several enigmas, like the existence of the Planck constant or why the great unification based on $SU(5)$ could not be successful. We consolidate both the standard model and the use of Clifford algebras as the true mathematical frame of quantum physics. We present in concluding remarks a simple solution to integrate together gravitation and quantum physics. Then the only great domain of physics which remains to study is the electromagnetism with magnetic monopoles.

keywords : invariance group, Dirac equation, electromagnetism, weak interactions, strong interactions, Clifford algebras, magnetic monopoles.

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1 Form invariance of the Dirac equation

The standard model uses fermions and bosons. All fermions are described with a Dirac equation. The Dirac wave ψ_e of the electron is made of two Pauli waves

$$\psi_e = \begin{pmatrix} \xi_e \\ \eta_e \end{pmatrix}; \quad \xi_e = \begin{pmatrix} \xi_{1e} \\ \xi_{2e} \end{pmatrix}; \quad \eta_e = \begin{pmatrix} \eta_{1e} \\ \eta_{2e} \end{pmatrix} \quad (1.1)$$

where $\xi_{1e}, \xi_{2e}, \eta_{1e}, \eta_{2e}$ are four functions of space and time with value in the complex field. The Dirac equation reads

$$0 = [\gamma^\mu (\partial_\mu + iqA_\mu) + im]\psi_e; \quad q = \frac{e}{\hbar c}; \quad m = \frac{m_0 c}{\hbar} \quad (1.2)$$

where γ^μ , $\mu = 0, 1, 2, 3$ are four complex matrices. Relativistic theory uses:

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad \gamma_j = -\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}; \quad I = \sigma_0 = \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.3)$$

$$\sigma_1 = -\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = -\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = -\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.4)$$

The electro-weak theory uses also

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \psi_R = \psi_e \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} \xi_e \\ 0 \end{pmatrix}; \quad \psi_L = \psi_e \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 \\ \eta_e \end{pmatrix} \quad (1.5)$$

then ξ_e and η_e are respectively right and left Weyl spinors.

About geometry the quantum theory says: if $\psi(x)$ are quantum states of an object and if we rotate the object described by these states with $x' = R(x)$ then states $\psi'(x')$ are transformed from $\psi(x)$ by a linear transformation Λ :

$$\psi'(x') = \Lambda[\psi(x)] \quad (1.6)$$

and the application

$$\mathbf{f} : R \mapsto \Lambda \quad (1.7)$$

is an homomorphism from the group $G_1 = \{R\}$ into the group $G_2 = \{\Lambda\}$. Physicists name this homomorphism "representation". What does this

become in the Dirac theory ? x and x' read

$$\begin{aligned} x &= x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \\ x' &= x'^\mu \sigma_\mu = \begin{pmatrix} x'^0 + x'^3 & x'^1 - ix'^2 \\ x'^1 + ix'^2 & x'^0 - x'^3 \end{pmatrix}. \end{aligned} \quad (1.8)$$

The Dirac theory uses complex matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

$$1 = ad - bc = \det(M) ; \quad M \in SL(2, \mathbb{C}) \quad (1.9)$$

Noting a^* the conjugate of a and with

$$M^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} ; \quad \overline{M} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} ; \quad \widehat{M} = \overline{M}^\dagger = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} \quad (1.10)$$

$$\det(M) = M\overline{M} = \overline{M}M = ad - bc = re^{i\theta} \quad (1.11)$$

the Dirac theory associates to M the R transformation satisfying

$$x' = R(x) = MxM^\dagger \quad (1.12)$$

because we get, since $r = 1$:

$$\begin{aligned} (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 &= \det(x') = \det(M) \det(x) \det(M^\dagger) \\ &= re^{i\theta} \det(x) re^{-i\theta} = r^2 \det(x) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \end{aligned} \quad (1.13)$$

Then R is a Lorentz rotation. Moreover with

$$x'^\mu = R_\nu^\mu x^\nu ; \quad N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} ; \quad \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix}. \quad (1.14)$$

we get [11] [6]:

$$R_0^0 = \frac{1}{2}(aa^* + bb^* + cc^* + dd^*) > 0 \quad (1.15)$$

$$\det(R_\nu^\mu) = r^4 \quad (1.16)$$

$$R_\nu^\mu \gamma^\nu = \widetilde{N} \gamma^\mu N, \quad \mu = 0, 1, 2, 3. \quad (1.17)$$

Then R is an element of the restricted Lorentz group \mathcal{L}_+^\uparrow and the application f satisfying

$$f : M \in SL(2, \mathbb{C}) \mapsto R \in \mathcal{L}_+^\uparrow \quad (1.18)$$

is an homomorphism. The form invariance of the Dirac equation comes from

$$\begin{aligned} 0 &= [\gamma^\nu (\partial_\nu + iqA_\nu) + im]\psi_e = [\gamma^\nu R_\nu^\mu (\partial'_\mu + iqA'_\mu) + im]\psi_e \\ &= [\widetilde{N} \gamma^\mu (\partial'_\mu + iqA'_\mu) N + im]\psi_e \end{aligned} \quad (1.19)$$

which implies, since $\det(M) = 1$, that

$$M\bar{M} = \bar{M}M = 1; \quad \bar{M} = M^{-1}; \quad \tilde{N} = N^{-1}. \quad (1.20)$$

This gives

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi_e = N^{-1}[\gamma^\mu(\partial'_\mu + iqA'_\mu) + im]N\psi_e. \quad (1.21)$$

The Dirac theory lets then

$$\psi'_e(x') = N\psi_e(x) \quad (1.22)$$

and since

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi_e \iff 0 = \gamma^\mu(\partial'_\mu + iqA'_\mu) + im]\psi'_e \quad (1.23)$$

the Dirac equation is said “form invariant” under the Lorentz rotation R . But there is a big cheating here, because we have changed the homomorphism \mathbf{f} defined in (1.7) into something completely different :

$$R \xleftarrow{f} M \xrightarrow{g} N \quad (1.24)$$

This is not a little problem, because $f(-M) = f(M)$; f is not invertible, and the composition $\mathbf{f} = g \circ f^{-1}$ with $f^{-1} : R \mapsto M$, $g : M \mapsto N$, that quantum theory calls “bi-valued representation” is just a trick, a joke, since f^{-1} is not an homomorphism. Ten years ago, revisiting this old trick, I noticed that relations (1.15), (1.16) and (1.17) are true even if $\det(M) \neq 1$. Then instead of (1.13) I got

$$(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = r^2[(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2] \quad (1.25)$$

And I understood that the fundamental group of form invariance of the Dirac wave is the central group in (1.24), the group of the M , usually named $GL(2, \mathbb{C})$. Since this group is also the multiplicative group of the invertible elements in Cl_3 , I note this group Cl_3^* . The central place of Cl_3 in (1.24) is linked to the fact that the true frame of the Dirac theory, and more generally of all electromagnetic laws, is Cl_3 .

2 Dirac equation and electromagnetism in Cl_3

We have previously explained [4][6][11] how this works. I have let

$$\phi_e = \sqrt{2}(\xi_e \quad -i\sigma_2\eta_e^*) = \sqrt{2} \begin{pmatrix} \xi_{1e} & -\eta_{2e}^* \\ \xi_{2e} & \eta_{1e}^* \end{pmatrix} \quad (2.1)$$

which implies

$$\det(\phi_e) = 2(\xi_{1e}\eta_{1e}^* + \xi_{2e}\eta_{2e}^*) = \Omega_1 + i\Omega_2 = \rho e^{i\beta} \quad (2.2)$$

The ϕ_e wave is then a function of space-time into $Cl_3 = M_2(\mathbb{C})$. We get

$$\hat{\phi}_e = \sqrt{2}(\eta_e \quad -i\sigma_2\xi_e^*) = \sqrt{2} \begin{pmatrix} \eta_{1e} & -\xi_{2e}^* \\ \eta_{2e} & \xi_{1e}^* \end{pmatrix}; \quad \bar{\phi}_e = \hat{\phi}_e^\dagger \quad (2.3)$$

$$\phi_e\bar{\phi}_e = \bar{\phi}_e\phi_e = \det(\phi_e) = \rho e^{i\beta} \quad (2.4)$$

where β is the Yvon-Takabayasi angle. (1.22) reads

$$\begin{pmatrix} \xi'_e \\ \eta'_e \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} \xi_e \\ \eta_e \end{pmatrix}; \quad \xi'_e = M\xi_e; \quad \eta'_e = \widehat{M}\eta_e. \quad (2.5)$$

It happens that we get, with any M and any η :

$$(-i\sigma_2)\eta'^* = (-i\sigma_2)\widehat{M}^*\eta^* = M(-i\sigma_2)\eta^* \quad (2.6)$$

The link that I made in (2.1) between ϕ_e and the Weyl spinors ξ_e, η_e is therefore invariant under Cl_3^* : if

$$\phi'_e(x') = \sqrt{2}(\xi'_e \quad (-i\sigma_2)\eta'^*) = \sqrt{2} \begin{pmatrix} \xi'_{1e} & -\eta'_{2e} \\ \xi'_{2e} & \eta'_{1e} \end{pmatrix} \quad (2.7)$$

we simply get

$$\phi'_e = M\phi_e; \quad \widehat{\phi}'_e = \widehat{M}\widehat{\phi}_e. \quad (2.8)$$

We have explained [11][6] that the Dirac equation reads in Cl_3 :

$$\nabla\widehat{\phi}_e\sigma_{21} + qA\widehat{\phi}_e + m\phi_e = 0; \quad \nabla = \sigma^\mu\partial_\mu; \quad A = \sigma^\mu A_\mu; \quad \sigma_{21} = \sigma_2\sigma_1. \quad (2.9)$$

Multiplying on the left by $\overline{\phi}_e$, I got three years ago the invariant form of the wave equation [6][8]:

$$\overline{\phi}_e(\nabla\widehat{\phi}_e)\sigma_{21} + \overline{\phi}_eqA\widehat{\phi}_e + m\rho e^{i\beta} = 0. \quad (2.10)$$

The first term is form invariant because, for any M satisfying (1.12), with

$$\nabla' = \sigma^\mu\partial'_\mu \quad (2.11)$$

I got the following relation [4] implying (1.17):

$$\nabla = \overline{M}\nabla'\widehat{M} \quad (2.12)$$

which gives with (2.8):

$$\overline{\phi}_e(\nabla\widehat{\phi}_e)\sigma_{21} = \overline{\phi}_e\overline{M}\nabla'\widehat{M}\widehat{\phi}_e\sigma_{21} = \overline{\phi}'_e(\nabla'\widehat{\phi}'_e)\sigma_{21}. \quad (2.13)$$

Next the form invariance of $\overline{\phi}_eqA\widehat{\phi}_e$ is necessary to satisfy both the form invariance and the electric gauge invariance of the Dirac equation. This means

$$\begin{aligned} \overline{\phi}_eqA\widehat{\phi}_e &= \overline{\phi}_e\overline{M}q'A'\widehat{M}\widehat{\phi}_e \\ qA &= \overline{M}q'A'\widehat{M}. \end{aligned} \quad (2.14)$$

Then qA which transforms like ∇ is named a ‘‘covariant vector’’ (in space-time), while vectors transforming like x , for instance $J = \phi_e\phi_e^\dagger$, are named ‘‘contravariant’’. This is now a physical distinction because, if $\det(M) \neq 1$ then $\overline{M} \neq M^{-1}$. Why the invariant form of the Dirac equation was not previously seen? I think that it is the presence of the $e^{i\beta}$ term in (2.10) which makes this equation not very well suited, but I knew for long how to improve this invariant equation: the non-linear homogeneous equation studied in my thesis [3] reads in Cl_3 :

$$\nabla\widehat{\phi}_e\sigma_{21} + qA\widehat{\phi}_e + me^{-i\beta}\phi_e = 0 \quad (2.15)$$

and is then equivalent to the form invariant equation

$$\overline{\phi}_e(\nabla\widehat{\phi}_e)\sigma_{21} + \overline{\phi}_eqA\widehat{\phi}_e + m\rho = 0. \quad (2.16)$$

This equation will be the starting point to get the gauge group of the standard model in the frame of Clifford algebra. Two of the eight numeric equations equivalent to the form invariant equation are remarkable and well known: the law of conservation of the current of probability $\partial_\mu J^\mu = 0$ and, still more important, the scalar part of (2.16) or (2.10) reads simply:

$$\mathcal{L} = 0 \quad (2.17)$$

where \mathcal{L} is the Lagrangian density of the Dirac equation. This Lagrangian density and the whole equation (2.16) are form invariant under Cl_3^* because (2.8) implies

$$\rho' e^{i\beta'} = \det(\phi'_e) = \det(M\phi_e) = \det(M) \det(\phi_e) = r\rho e^{i(\beta+\theta)} \quad (2.18)$$

$$\rho' = r\rho; \quad \beta' = \beta + \theta \pmod{2\pi}. \quad (2.19)$$

The form invariance of the wave equation is satisfied if and only if the mass term satisfies

$$m\rho = m'\rho' = m'r\rho \quad (2.20)$$

$$m = rm'. \quad (2.21)$$

What this means on the physical point of view ? When we go from classical mechanics to relativistic physics we replace the invariance under rotations by the invariance under the greater group of Lorentz rotations. Many things change. There are less invariant terms and it is the same here: I replaced the \mathcal{L}_+^\uparrow group (but really the $SL(2, \mathbb{C})$ group) by the $Cl_3^* = GL(2, \mathbb{C})$ group. Therefore m and ρ are no more invariant. Now it is only the $m\rho$ product which is invariant.

Now with

$$F = \vec{E} + i\vec{H}, \quad A = A^0 + \vec{A}; \quad B = B^0 + \vec{B} \quad (2.22)$$

$$j = j^0 + \vec{j}; \quad k = k^0 + \vec{k} \quad (2.23)$$

where \vec{E} is the electric field, \vec{H} is the magnetic field, A the space-time vector potential, B the magnetic potential, j the electric density of charge and current, k the magnetic density of charge and current. Laws of electromagnetism in the void with magnetic monopoles read (See [11] chapter 4)

$$F = \nabla\widehat{A} + i\widehat{B}; \quad \widehat{\nabla}F = \frac{4\pi}{c}\widehat{j} + i\widehat{k}. \quad (2.24)$$

I found more recently [8] how the electromagnetic field F (and all other boson fields) behaves under the transformation induced by a M in Cl_3^* :

$$F'(x') = MF(x)M^{-1} \quad (2.25)$$

$$A'(x') = MA(x)M^\dagger; \quad B'(x') = MB(x)M^\dagger \quad (2.26)$$

$$j(x) = \overline{M}j'(x')\widehat{M}; \quad k(x) = \overline{M}k'(x')\widehat{M} \quad (2.27)$$

The physical meaning of the contravariance of A and B is that potentials move with sources. I previously explained [11] [8] [6] how the covariance of qA , j and k is compatible with all laws of electromagnetism and relativistic mechanics. Using always complex quantities with an indefinite i , quantum physicists could not see the magnetic part of the potentials and currents that they used. The equation (2.25) explains by itself why only the $SL(2, \mathbb{C})$ part of Cl_3^* was previously seen. The P rotor which plays a central role in the Hestenes' work [15][16] and in the Boudet's work [1][2], actually an element of $SL(2, \mathbb{C})$, is defined such as

$$M = \sqrt{r}e^{i\theta/2}P. \quad (2.28)$$

This gives

$$\overline{M} = \sqrt{r}e^{i\theta/2}\overline{P} = \sqrt{r}e^{i\theta/2}P^{-1} \quad (2.29)$$

$$M^{-1} = \frac{1}{\sqrt{r}}e^{-i\theta/2}\overline{P} \quad (2.30)$$

$$F' = \sqrt{r}e^{i\theta/2}PF\frac{1}{\sqrt{r}}e^{-i\theta/2}\overline{P} = PF\overline{P} \quad (2.31)$$

and M transforms as if $r = 1$ and $\theta = 0$. The electromagnetic field (and more generally all gauge fields) transforms in such a way that we see only the relativistic Lorentz rotation induced by the P term. Due to the invariance of the velocity c of light: under the dilation R , product of a Lorentz rotation and of an homothety with ratio r induced by any M in Cl_3^* , I got [6][7]:

$$e' = r^2e; \quad h' = r^4h; \quad m'_0 = r^3m_0; \quad m' = r^{-1}m. \quad (2.32)$$

Electric charge, proper mass and Planck "constant" are changed in the dilation R induced by a M in Cl_3^* if $r \neq 1$. The dilation is the composition of the Lorentz rotation induced by P and an homothety with ratio r , in any order. We evidently get the results of restricted relativity if $r = 1$.

3 Electro-weak interactions

The electro-weak theory [17] needs three spinorial waves in the electron-neutrino case: the right ξ_e and the left η_e of the electron and the left spinor η_n of the electronic neutrino. The form invariance of the Dirac theory imposes to use a wave Ψ_l satisfying

$$\Psi_l = \begin{pmatrix} \phi_e & \phi_n \\ \hat{\phi}_n & \hat{\phi}_e \end{pmatrix}; \quad \phi_e = \sqrt{2} \begin{pmatrix} \xi_{1e} & -\eta_{2e}^* \\ \xi_{2e} & \eta_{1e}^* \end{pmatrix}; \quad \phi_n = \sqrt{2} \begin{pmatrix} 0 & -\eta_{2n}^* \\ 0 & \eta_{1n}^* \end{pmatrix} \quad (3.1)$$

$$\hat{\phi}_e = \sqrt{2} \begin{pmatrix} \eta_{1e} & -\xi_{2e}^* \\ \eta_{2e} & \xi_{1e}^* \end{pmatrix}; \quad \hat{\phi}_n = \sqrt{2} \begin{pmatrix} \eta_{1n} & 0 \\ \eta_{2n} & 0 \end{pmatrix}; \quad \eta_n = \begin{pmatrix} \eta_{1n} \\ \eta_{2n} \end{pmatrix}. \quad (3.2)$$

The wave is a function of space and time with value into the space-time algebra $Cl_{1,3}$. The standard model uses only a left wave for the neutrino, this may be seen in (3.1). I implicitly use the matrix representation (1.3)

which allows to see $Cl_{1,3}$ as a sub-algebra of $M_4(\mathbb{C})$. Under the dilation R with ratio r induced by M we have

$$\xi' = M\xi ; \quad \eta' = \widehat{M}\eta ; \quad \eta'_n = \widehat{M}\eta_n ; \quad \phi'_e = M\phi_e ; \quad \phi'_n = M\phi_n \quad (3.3)$$

$$\Psi'_l = \begin{pmatrix} \phi'_e & \phi'_n \\ \widehat{\phi}'_n & \widehat{\phi}'_e \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix} = N\Psi_l \quad (3.4)$$

The form (3.1) of the wave is compatible both with the form invariance of the Dirac theory and with the charge conjugation used in the standard model: the wave $\psi_{\bar{e}}$ of the positron satisfies

$$\psi_{\bar{e}} = i\gamma_2\psi_e^* \Leftrightarrow \widehat{\phi}_{\bar{e}} = \widehat{\phi}_e\sigma_1 \quad (3.5)$$

We can then think the Ψ_l wave as containing the electron wave ϕ_e , the neutrino wave ϕ_n and also the positron wave $\phi_{\bar{e}}$ and the antineutrino wave $\phi_{\bar{n}}$:

$$\Psi_l = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_{\bar{n}}\sigma_1 & \widehat{\phi}_{\bar{e}}\sigma_1 \end{pmatrix} ; \quad \phi_{\bar{e}} = \sqrt{2} \begin{pmatrix} \xi_{1\bar{e}} & -\eta_{2\bar{e}}^* \\ \xi_{2\bar{e}} & \eta_{1\bar{e}}^* \end{pmatrix} ; \quad \phi_{\bar{n}} = \sqrt{2} \begin{pmatrix} \xi_{1\bar{n}} & 0 \\ \xi_{2\bar{n}} & 0 \end{pmatrix} \quad (3.6)$$

And the antineutrino has consequently only a right wave. The multivector $\Psi_l(x)$ is usually an invertible element of the space-time algebra because (See [11] (6.250)), with:

$$a_1 = \det(\phi_e) = \phi_e\bar{\phi}_e = 2(\xi_{1e}\eta_{1e}^* + \xi_{2e}\eta_{2e}^*) \quad (3.7)$$

$$a_2 = 2(\xi_{1\bar{e}}\eta_{1n}^* + \xi_{2\bar{e}}\eta_{2n}^*) = 2(\eta_{2e}^*\eta_{1n}^* - \eta_{1e}^*\eta_{2n}^*) \quad (3.8)$$

$$a_3 = 2(\xi_{1e}\eta_{1n}^* + \xi_{2e}\eta_{2n}^*) \quad (3.9)$$

I got last September:

$$\det(\Psi_l) = a_1a_1^* + a_2a_2^*. \quad (3.10)$$

and the determinant of the $\Psi_l(x)$ matrix is usually not zero.

To get the gauge group of electro-weak interactions I simply used [8] two projectors P_{\pm} and four operators P_{μ} , $\mu = 0, 1, 2, 3$ satisfying:

$$P_{\pm}(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi\gamma_{21}) ; \quad \mathbf{i} = \gamma_0\gamma_1\gamma_2\gamma_3 ; \quad \gamma_{21} = \gamma_2\gamma_1 \quad (3.11)$$

$$P_0(\Psi) = \Psi\gamma_{21} + P_-(\Psi)\mathbf{i} \quad (3.12)$$

$$P_1(\Psi) = P_+(\Psi)\gamma_3\mathbf{i} \quad (3.13)$$

$$P_2(\Psi) = P_+(\Psi)\gamma_3 \quad (3.14)$$

$$P_3(\Psi) = P_+(\Psi)(-\mathbf{i}) \quad (3.15)$$

The covariant derivative of the Weinberg-Salam model:

$$D_{\mu} = \partial_{\mu} - ig_1\frac{Y}{2}B_{\mu} + ig_2T_jW_{\mu}^j \quad (3.16)$$

where Y is the weak hypercharge ($Y_L = -1$, $Y_R = -2$ for the electron), has a very simple translation in the $Cl_{1,3}$ frame:

$$\mathbf{D} = \boldsymbol{\partial} + \frac{g_1}{2}\mathbf{B}P_0 + \frac{g_2}{2}(\mathbf{W}^1P_1 + \mathbf{W}^2P_2 + \mathbf{W}^3P_3) ; \quad \mathbf{D} = \widetilde{N}\mathbf{D}'N \quad (3.17)$$

$$\mathbf{D} = \gamma^{\mu}D_{\mu} ; \quad \boldsymbol{\partial} = \gamma^{\mu}\partial_{\mu} ; \quad \mathbf{B} = \gamma^{\mu}B_{\mu} ; \quad \mathbf{W}^j = \gamma^{\mu}W_{\mu}^j \quad (3.18)$$

Because we get from these definitions:

$$D_\mu \xi_e = \partial_\mu \xi_e + i g_1 B_\mu \xi_e \quad (3.19)$$

$$D_\mu \eta_e = \partial_\mu \eta_e + i \frac{g_1}{2} B_\mu \eta_e - i \frac{g_2}{2} [(W_\mu^1 + i W_\mu^2) \eta_n - W_\mu^3 \eta_e] \quad (3.20)$$

$$D_\mu \eta_n = \partial_\mu \eta_n + i \frac{g_1}{2} B_\mu \eta_n - i \frac{g_2}{2} [(W_\mu^1 - i W_\mu^2) \eta_e + W_\mu^3 \eta_n] \quad (3.21)$$

which is equivalent to (3.16). The other features of the Weinberg-Salam model are then straightforward. The main remark to add is that the Weinberg-Salam θ_W angle satisfies

$$B + i W^3 = e^{i\theta_W} (A + i Z^0). \quad (3.22)$$

This means that the magnetic space-time vector potential B of (2.22) is linked to the Z^0 gauge boson. This is enough to prove that magnetic monopoles are necessarily present in the electro-weak theory. With the charge conjugation of the standard model (3.5) the covariant derivative (3.17) gives also with:

$$D = \sigma^\mu D_\mu ; \quad \nabla = \sigma^\mu \partial_\mu ; \quad B = \sigma^\mu B_\mu ; \quad W^j = \sigma^\mu W_\mu^j , \quad j = 1, 2, 3 \quad (3.23)$$

$$D_\mu \eta_{\bar{e}} = \partial_\mu \eta_{\bar{e}} - i g_1 B_\mu \eta_{\bar{e}} \quad (3.24)$$

$$D_\mu \xi_{\bar{e}} = \partial_\mu \xi_{\bar{e}} - i \frac{g_1}{2} B_\mu \xi_{\bar{e}} + i \frac{g_2}{2} [(W_\mu^1 - i W_\mu^2) \xi_{\bar{n}} - W_\mu^3 \xi_{\bar{e}}] \quad (3.25)$$

$$D_\mu \xi_{\bar{n}} = \partial_\mu \xi_{\bar{n}} - i \frac{g_1}{2} B_\mu \xi_{\bar{n}} + i \frac{g_2}{2} [(W_\mu^1 + i W_\mu^2) \xi_{\bar{e}} + W_\mu^3 \xi_{\bar{n}}] \quad (3.26)$$

We naturally get several features of the standard model: the charge conjugation changes the sign of the electric charge and changes left into right waves and vice-versa. The $U(1) \times SU(2)$ gauge group is obtained by exponentiation of the operators P_μ . If a^μ are four real parameters we let:

$$\exp(a^0 P_0) = \sum_{n=0}^{\infty} \frac{(a^0 P_0)^n}{n!} ; \quad \exp(a^j P_j) = \sum_{n=0}^{\infty} \frac{(a^1 P_1 + a^2 P_2 + a^3 P_3)^n}{n!}. \quad (3.27)$$

And we use the gauge transformation

$$\Psi'_l = [\exp(a^\mu P_\mu)](\Psi_l) \quad (3.28)$$

We have found only a few months ago the mass term compatible both with the form invariance and with this gauge invariance, the wave equation [10] reads

$$\mathbf{D}\Psi_l \gamma_{012} + m \rho_1 \chi_l = 0 ; \quad \gamma_{012} = \gamma_0 \gamma_1 \gamma_2 \quad (3.29)$$

where

$$\rho_1 = \sqrt{a_1 a_1^* + a_2 a_2^* + a_3 a_3^*} \quad (3.30)$$

$$\chi_l = \frac{1}{\rho_1^2} \begin{pmatrix} a_1^* \phi_e + a_2^* \phi_n \sigma_1 + a_3^* \phi_n & -a_2^* \phi_{eL} \sigma_1 + a_3^* \phi_{eR} \\ a_2 \widehat{\phi}_{eL} \sigma_1 + a_3 \widehat{\phi}_{eR} & a_1 \widehat{\phi}_e - a_2 \widehat{\phi}_n \sigma_1 + a_3 \widehat{\phi}_n \end{pmatrix} \quad (3.31)$$

$$\phi_{eR} = \phi_e \frac{1 + \sigma_3}{2} ; \quad \phi_{eL} = \phi_e \frac{1 - \sigma_3}{2} \quad (3.32)$$

This wave equation is equivalent to the invariant equation:

$$\tilde{\Psi}_l(\mathbf{D}\Psi_l)\gamma_{012} + m\rho_1\tilde{\Psi}_l\chi_l = 0 ; \quad \tilde{\Psi}_l = \begin{pmatrix} \bar{\phi}_e & \phi_n^\dagger \\ \bar{\phi}_n & \phi_e^\dagger \end{pmatrix}. \quad (3.33)$$

The form invariance under Cl_3^* of this equation results from (1.12), (2.8), (2.11), (2.12), (3.4) and (3.17). We get:

$$\tilde{\Psi}'_l(\mathbf{D}'\Psi'_l)\gamma_{012} = \tilde{\Psi}_l(\mathbf{D}\Psi_l)\gamma_{012} \quad (3.34)$$

and also [10]

$$m\rho_1 = m'r\rho_1 = m'\rho'_1 ; \quad \tilde{\Psi}'_l\chi'_l = \tilde{\Psi}_l\chi_l. \quad (3.35)$$

The wave equation is also gauge invariant under the gauge transformation (3.28) [10], becoming:

$$0 = \tilde{\Psi}'_l(\mathbf{D}'\Psi'_l)\gamma_{012} + m\rho\tilde{\Psi}'_l\chi' \quad (3.36)$$

$$\mathbf{D}' = \boldsymbol{\partial} + \frac{g_1}{2}\mathbf{B}'P_0 + \frac{g_2}{2}(\mathbf{W}'^1P_1 + \mathbf{W}'^2P_2 + \mathbf{W}'^3P_3) \quad (3.37)$$

$$B'_\mu = B_\mu - \frac{2}{g_1}\partial_\mu a^0 ; \quad \mathbf{B}' = \gamma^\mu B'_\mu \quad (3.38)$$

$$W'^j_\mu P_j = [\exp(a^k P_k)W^j_\mu P_j - \frac{2}{g_2}\partial_\mu[\exp(a^k P_k)]] \exp(-a^k P_k) \quad (3.39)$$

$$\mathbf{W}'^j = \gamma^\mu W'^j_\mu. \quad (3.40)$$

Since a wave equation exists, form invariant and gauge invariant, with a mass term, it is useless to build a complicated process of spontaneously broken symmetry to account for the mass term of the electron.

Two amongst the fourteen numeric equations equivalent to (3.33) are remarkable: the real part of (3.33) which is, like in the case of the alone electron:

$$\mathcal{L} = 0 ; \quad \mathcal{L} = \langle \tilde{\Psi}_l(\mathbf{D}\Psi_l)\gamma_{012} \rangle + m\rho_1 \quad (3.41)$$

We have then, for the pair electron-neutrino like for the alone electron, a double link between wave equation and Lagrangian formalism. It is well-known that the wave equation may be obtained by the Lagrange equations from a Lagrangian density. Now we see the reciprocal relation: the Lagrangian density comes as real part of the invariant wave equation and this explains why there is a principle of minimum.

The other remarkable numeric equation reads

$$\partial_\mu(D_0^\mu + D_n^\mu) = 0 ; \quad D_0 = \phi_e\phi_e^\dagger ; \quad D_n = \phi_n\phi_n^\dagger. \quad (3.42)$$

A conservative current exists, the total current $D_0 + D_n$. This current may be interpreted as the probability current, like in the case of the electron alone. But the interpretation as the density of probability of presence for the electron-particle is impossible since we have here both the electron and its neutrino.

The charge conjugation is simply the change of the differential term, both in the Lagrangian density and in the wave equation, and the exchange of right and left terms, the mass term remains unchanged (there are no negative energies). Charge conjugation changes a_1, a_2, a_3 , into:

$$a_{1c} = -a_1 ; \quad a_{2c} = -a_2 ; \quad a_{3c} = -a_3. \quad (3.43)$$

Then ρ_1 is unchanged and also χ_l which now reads

$$\chi_{\bar{l}} = \frac{1}{\rho_1^2} \begin{pmatrix} a_{1c}^* \phi_{\bar{e}} + a_{2c}^* \phi_{\bar{\nu}} \sigma_1 + a_{3c}^* \phi_{\bar{\nu}} & -a_{2c}^* \phi_{\bar{e}R} \sigma_1 + a_{3c}^* \phi_{\bar{e}L} \\ a_{2c} \widehat{\phi}_{\bar{e}R} \sigma_1 + a_{3c} \widehat{\phi}_{\bar{e}L} & a_{1c} \widehat{\phi}_{\bar{e}} - a_{2c} \widehat{\phi}_{\bar{\nu}} \sigma_1 + a_{3c} \widehat{\phi}_{\bar{\nu}} \end{pmatrix} \quad (3.44)$$

Instead of (3.1), (3.12) to (3.15), (3.17) we have:

$$\Psi_{\bar{l}} = \begin{pmatrix} \phi_{\bar{e}} & \phi_{\bar{\nu}} \\ \widehat{\phi}_{\bar{\nu}} & \widehat{\phi}_{\bar{e}} \end{pmatrix} \quad (3.45)$$

$$P_{c0}(\Psi_{\bar{l}}) = \Psi_{\bar{l}} \gamma_{21} - P_+(\Psi_{\bar{l}}) \mathbf{i} \quad (3.46)$$

$$P_{c1}(\Psi_{\bar{l}}) = P_-(\Psi_{\bar{l}}) \gamma_3 \mathbf{i} \quad (3.47)$$

$$P_{c2}(\Psi_{\bar{l}}) = P_-(\Psi_{\bar{l}}) \gamma_3 \quad (3.48)$$

$$P_{c3}(\Psi_{\bar{l}}) = P_-(\Psi_{\bar{l}}) \mathbf{i} \quad (3.49)$$

$$\mathbf{D}_c = -\boldsymbol{\partial} + \frac{g_1}{2} \mathbf{B} P_{c0} + \frac{g_2}{2} (\mathbf{W}^1 P_{c1} + \mathbf{W}^2 P_{c2} + \mathbf{W}^3 P_{c3}) \quad (3.50)$$

$$0 = \widetilde{\Psi}_{\bar{l}} (\mathbf{D}_c \Psi_{\bar{l}}) \gamma_{012} + m \rho_1 \widetilde{\Psi}_{\bar{l}} \chi_{\bar{l}} \quad (3.51)$$

Charge conjugation is then a pure quantum transformation, charges are conserved. Since only the differential term changes sign, the energy as coefficient of the time in the phase of the wave changes sign, but the density of energy T_0^0 remains positive (See [8] 5.3). This suppress the old problem of negative non-physical energy in quantum physics.

4 Electro-weak and strong interactions

The standard model adds to the leptons (electron and its neutrino) in the first “generation” two quarks u and d with three states each. Weak interactions acting only on left waves of quarks (and right waves of antiquarks) we have 8 left spinors instead of 2. To account for a multiplication by 4 is easy in Clifford algebras : it is enough to add 2 dimensions to the space. With our matrix representation it is enough to work with 8×8 matrices. So I read the wave of all fermions of the first generation as follows:

$$\Psi = \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix} ; \quad \Psi_r = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{ur} & \widehat{\phi}_{dr} \end{pmatrix} = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{\bar{u}r} \sigma_1 & \widehat{\phi}_{\bar{d}r} \sigma_1 \end{pmatrix} \quad (4.1)$$

$$\Psi_g = \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{ug} & \widehat{\phi}_{dg} \end{pmatrix} = \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{\bar{u}g} \sigma_1 & \widehat{\phi}_{\bar{d}g} \sigma_1 \end{pmatrix} \quad (4.2)$$

$$\Psi_b = \begin{pmatrix} \phi_{db} & \phi_{ub} \\ \widehat{\phi}_{ub} & \widehat{\phi}_{db} \end{pmatrix} = \begin{pmatrix} \phi_{db} & \phi_{ub} \\ \widehat{\phi}_{\bar{u}b} \sigma_1 & \widehat{\phi}_{\bar{d}b} \sigma_1 \end{pmatrix} \quad (4.3)$$

The Ψ wave is now a function of space and time with value into $Cl_{1,5} = Cl_{5,1}$ which is a sub-algebra (on the real field) of $Cl_{5,2} = M_8(\mathbb{C})$. The

covariant derivative (3.16) becomes

$$\underline{\mathbf{D}} = \underline{\partial} + \frac{g_1}{2} \underline{\mathbf{B}} P_0 + \frac{g_2}{2} (\underline{\mathbf{W}}^1 P_1 + \underline{\mathbf{W}}^2 P_2 + \underline{\mathbf{W}}^3 P_3) \quad (4.4)$$

$$\underline{\mathbf{D}} = \sum_{\mu=0}^3 L^\mu D_\mu ; \quad \underline{\partial} = \sum_{\mu=0}^3 L^\mu \partial_\mu ; \quad \underline{\mathbf{B}} = \sum_{\mu=0}^3 L^\mu B_\mu ; \quad \underline{\mathbf{W}}^j = \sum_{\mu=0}^3 L^\mu W_\mu^j \quad (4.5)$$

$$L_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} ; \quad \mu = 0, 1, 2, 3 \quad (4.6)$$

$$L_4 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix} ; \quad L_5 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \quad (4.7)$$

We use two projectors satisfying

$$\underline{P}_\pm(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi L_{21}) ; \quad \mathbf{i} = L_{0123} \quad (4.8)$$

Three operators act on quarks like on leptons:

$$\underline{P}_1(\Psi) = \underline{P}_+(\Psi) L_{35} \quad (4.9)$$

$$\underline{P}_2(\Psi) = \underline{P}_+(\Psi) L_{5012} \quad (4.10)$$

$$\underline{P}_3(\Psi) = \underline{P}_+(\Psi)(-\mathbf{i}). \quad (4.11)$$

The fourth operator acts differently on the lepton and on the quark sector:

$$\underline{P}_0(\Psi) = \begin{pmatrix} P_0(\Psi_l) & P'_0(\Psi_r) \\ P'_0(\Psi_g) & P_0(\Psi_b) \end{pmatrix} \quad (4.12)$$

$$P_0(\Psi_l) = \Psi_l \gamma_{21} + P_-(\Psi_l) \mathbf{i} \quad (4.13)$$

$$P'_0(\Psi_c) = -\frac{1}{3} \Psi_c \gamma_{21} + P_-(\Psi_c) \mathbf{i}, \quad c = r, g, b. \quad (4.14)$$

These definitions are absolutely all that you have to change to go from the lepton case into the quark case, to get the gauge group of electro-weak interactions. We proved in [11] 6.3 that this gives:

$$D_\mu \xi_d = \partial_\mu \xi_d - i \left(-\frac{1}{3}\right) g_1 B_\mu \xi_d \quad (4.15)$$

$$D_\mu \xi_u = \partial_\mu \xi_u - i \left(+\frac{2}{3}\right) g_1 B_\mu \xi_u \quad (4.16)$$

$$D_\mu \eta_{\bar{d}} = \partial_\mu \eta_{\bar{d}} - i \left(+\frac{1}{3}\right) g_1 B_\mu \eta_{\bar{d}} \quad (4.17)$$

$$D_\mu \eta_{\bar{u}} = \partial_\mu \eta_{\bar{u}} - i \left(-\frac{2}{3}\right) g_1 B_\mu \eta_{\bar{u}} \quad (4.18)$$

This means that changing the coefficient 1 of $\Psi \gamma_{21}$ into $-\frac{1}{3}$ is enough to get the correct charges of u and d quarks, the correct charges of antiquarks. Moreover we get a doublet of left waves for the quarks and a doublet of right waves for the antiquarks:

$$\psi_L = \begin{pmatrix} \eta_u \\ \eta_d \end{pmatrix} ; \quad \psi_R = \begin{pmatrix} \xi_{\bar{u}} \\ \xi_{\bar{d}} \end{pmatrix}, \quad (4.19)$$

which gives

$$D_\mu \psi_L = \partial_\mu \psi_L - i \frac{g_1}{6} B_\mu \psi_L - i \frac{g_2}{2} (W_\mu^1 \tau_1 + W_\mu^2 \tau_2 + W_\mu^3 \tau_3) \psi_L \quad (4.20)$$

$$D_\mu \psi_R = \partial_\mu \psi_R + i \frac{g_1}{6} B_\mu \psi_R - i \frac{g_2}{2} (W_\mu^1 \tau_1 - W_\mu^2 \tau_2 + W_\mu^3 \tau_3) \psi_R \quad (4.21)$$

$$\tau_1 = \gamma_0 ; \quad \tau_2 = \gamma_{123} ; \quad \tau_3 = \gamma_5. \quad (4.22)$$

Then all features of electro-weak interactions of leptons and quarks are simply obtained from the structure of the wave and from a few operators.

Now to get the generators of the $SU(3)$ gauge group of chromodynamics I consider two new projectors:

$$P^+ = \frac{1}{2}(I_8 + L_{012345}) = \begin{pmatrix} I_4 & 0 \\ 0 & 0 \end{pmatrix} ; \quad P^- = \frac{1}{2}(I_8 - L_{012345}) = \begin{pmatrix} 0 & 0 \\ 0 & I_4 \end{pmatrix} \quad (4.23)$$

and eight operators Γ_k , $k = 1, 2, \dots, 8$ so defined:

$$\Gamma_1(\Psi) = \frac{1}{2}(L_4 \Psi L_4 + L_{01235} \Psi L_{01235}) = \begin{pmatrix} 0 & g \\ r & 0 \end{pmatrix} \quad (4.24)$$

$$\Gamma_2(\Psi) = \frac{1}{2}(L_5 \Psi L_4 - L_{01234} \Psi L_{01235}) = \begin{pmatrix} 0 & -ig \\ ir & 0 \end{pmatrix} \quad (4.25)$$

$$\Gamma_3(\Psi) = P^+ \Psi P^- - P^- \Psi P^+ = \begin{pmatrix} 0 & r \\ -g & 0 \end{pmatrix} \quad (4.26)$$

$$\Gamma_4(\Psi) = L_{01253} \Psi P^- = \begin{pmatrix} 0 & b \\ 0 & r \end{pmatrix} ; \quad \Gamma_5(\Psi) = L_{01234} \Psi P^- = \begin{pmatrix} 0 & -ib \\ 0 & ir \end{pmatrix} \quad (4.27)$$

$$\Gamma_6(\Psi) = P^- \Psi L_{01253} = \begin{pmatrix} 0 & 0 \\ b & g \end{pmatrix} ; \quad \Gamma_7(\Psi) = -i P^- \Psi L_4 = \begin{pmatrix} 0 & 0 \\ -ib & ig \end{pmatrix} \quad (4.28)$$

$$\Gamma_8(\Psi) = \frac{1}{\sqrt{3}}(P^- \Psi L_{012345} + L_{012345} \Psi P^-) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & r \\ g & -2b \end{pmatrix}. \quad (4.29)$$

We explained in [8] how this is equivalent to the eight generators λ_k of $SU(3)$. Everywhere in (4.24) to (4.29) the eight matrices $\Gamma_k(\Psi)$ have a zero left up term, therefore all Γ_k project the wave on its quark sector. The physical translation is: leptons do not interact by strong interactions, they have only electromagnetic and weak interactions. What I add here to the standard model is: this comes from the structure itself of the quantum wave.

Now with

$$\begin{aligned} \underline{\mathbf{D}} &= \sum_{\mu=0}^3 L^\mu D_\mu ; \quad \underline{\boldsymbol{\partial}} = \sum_{\mu=0}^3 L^\mu \partial_\mu ; \quad \underline{\mathbf{B}} = \sum_{\mu=0}^3 L^\mu B_\mu \\ \underline{\mathbf{W}}^j &= \sum_{\mu=0}^3 L^\mu W_\mu^j ; \quad \underline{\mathbf{G}}^k = \sum_{\mu=0}^3 L^\mu G_\mu^k \end{aligned} \quad (4.30)$$

where the eight $\underline{\mathbf{G}}^k$ are named ‘‘gluons’’, the covariant derivative reads

$$\underline{\mathbf{D}} = \underline{\boldsymbol{\partial}} + \frac{g_1}{2} \underline{\mathbf{B}} P_0 + \frac{g_2}{2} \sum_{j=1}^3 \underline{\mathbf{W}}^j P_j + \frac{g_3}{2} \sum_{k=1}^8 \underline{\mathbf{G}}^k i \Gamma_k. \quad (4.31)$$

The gauge group is obtained by exponentiation. We use four numbers a^μ and eight numbers b^k . We let

$$S = S_0 + S_1 + S_2 \quad (4.32)$$

$$S_0 = a^0 \underline{P}_0 ; \quad S_1 = \sum_{j=1}^3 a^j \underline{P}_j ; \quad S_2 = \sum_{k=1}^8 b^k \underline{\mathbf{i}}\Gamma_k. \quad (4.33)$$

We get

$$\begin{aligned} \exp(S) &= \exp(S_0) \exp(S_1) \exp(S_2) = \exp(S_0) \exp(S_2) \exp(S_1) \\ &= \exp(S_2) \exp(S_1) \exp(S_0) = \dots \end{aligned} \quad (4.34)$$

in any order, because:

$$\underline{P}_0 \underline{P}_j = \underline{P}_j \underline{P}_0, \quad j = 1, 2, 3 \quad (4.35)$$

$$\underline{P}_\mu \underline{\mathbf{i}}\Gamma_k = \underline{\mathbf{i}}\Gamma_k \underline{P}_\mu, \quad \mu = 0, 1, 2, 3, \quad k = 1, 2 \dots 8 \quad (4.36)$$

Therefore the set

$$G = \{\exp(S)\} \quad (4.37)$$

is a $U(1) \times SU(2) \times SU(3)$ Lie group. The gauge transformation reads

$$\underline{\Psi}' = [\exp(S)](\underline{\Psi}) ; \quad \underline{\mathbf{D}} = L^\mu D_\mu ; \quad \underline{\mathbf{D}}' = L^\mu D'_\mu \quad (4.38)$$

$$\underline{\mathbf{D}}' = \underline{\boldsymbol{\theta}} + \frac{g_1}{2} \underline{\mathbf{B}}' \underline{P}_0 + \frac{g_2}{2} \sum_{j=1}^3 \underline{\mathbf{W}}'^j \underline{P}_j + \frac{g_3}{2} \sum_{k=1}^8 \underline{\mathbf{G}}'^k \underline{\mathbf{i}}\Gamma_k \quad (4.39)$$

$$B'_\mu = B_\mu - \frac{2}{g_1} \partial_\mu a^0 \quad (4.40)$$

$$W'^j \underline{P}_j = [\exp(S_1) W'_\mu^j \underline{P}_j - \frac{2}{g_2} \partial_\mu [\exp(S_1)]] \exp(-S_1) \quad (4.41)$$

$$G'^k \underline{\mathbf{i}}\Gamma_k = [\exp(S_2) G'^k \underline{\mathbf{i}}\Gamma_k - \frac{2}{g_3} \partial_\mu [\exp(S_2)]] \exp(-S_2) \quad (4.42)$$

The $SU(3)$ group of $\exp(S_2)$ operators, generated by projectors on the quark sector, acts only on this sector of the wave:

$$P^+ [\exp(S_2)](\underline{\Psi}) P^+ = P^+ \underline{\Psi} P^+ = \begin{pmatrix} \underline{\Psi}_l & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.43)$$

We then get the gauge group of the standard model, automatically, and not another group. It is possible to get operators exchanging $\underline{\Psi}_l$ and $\underline{\Psi}_c$, $c = r, g, b$ like Γ_1 exchanging $\underline{\Psi}_r$ and $\underline{\Psi}_g$ but the difference between P_0 and P'_0 forbids the commutativity. Then we cannot get a greater group than the preceding $U(1) \times SU(2) \times SU(3)$ gauge group.

We have still supposed nothing on $\phi_{ur}, \phi_{ug}, \phi_{ub}, \phi_{dr}, \phi_{dg}, \phi_{db}$. But the standard model uses only left waves for the particles in the case of electro-weak interactions (and right waves for the antiparticles). Why? I think possible to give a mathematical link to this physical situation. If u

and d quarks have only left wave, this means η waves, we have:

$$\begin{aligned}\Psi_l &= \sqrt{2} \begin{pmatrix} \xi_{1e} & -\eta_{2e}^* & 0 & -\eta_{2n}^* \\ \xi_{2e} & \eta_{1e}^* & 0 & \eta_{1n}^* \\ \eta_{1n} & 0 & \eta_{1e} & -\xi_{2e}^* \\ \eta_{2n} & 0 & \eta_{2e} & \xi_{1e}^* \end{pmatrix} \\ \Psi_c &= \sqrt{2} \begin{pmatrix} 0 & -\eta_{2dc}^* & 0 & -\eta_{2uc}^* \\ 0 & \eta_{1dc}^* & 0 & \eta_{1uc}^* \\ \eta_{1uc} & 0 & \eta_{1dc} & 0 \\ \eta_{2uc} & 0 & \eta_{2dc} & 0 \end{pmatrix}; \quad c = r, g, b.\end{aligned}\quad (4.44)$$

Now I define two matrices M_1 and M_2 :

$$\begin{aligned}M_1 &= \sqrt{2} \begin{pmatrix} \eta_{1e} & \eta_{1n} & \eta_{1dr} & \eta_{1ur} \\ \eta_{2e} & \eta_{2n} & \eta_{2dr} & \eta_{2ur} \\ \eta_{1dg} & \eta_{1ug} & \eta_{1db} & \eta_{1ub} \\ \eta_{2dg} & \eta_{2ug} & \eta_{2db} & \eta_{2ub} \end{pmatrix} \\ M_2 &= \sqrt{2} \begin{pmatrix} \eta_{1e} & -\xi_{2e}^* & \eta_{1dr} & \eta_{1ur} \\ \eta_{2e} & \xi_{1e}^* & \eta_{2dr} & \eta_{2ur} \\ \eta_{1dg} & 0 & \eta_{1db} & \eta_{1ub} \\ \eta_{2dg} & 0 & \eta_{2db} & \eta_{2ub} \end{pmatrix}\end{aligned}\quad (4.45)$$

and I got the remarkable identity [11]

$$\det(\Psi) = |\det(M_1)|^2 + |\det(M_2)|^2. \quad (4.46)$$

And usually $\det(\Psi) \neq 0$ and $\Psi(x)$ is invertible. We can see the wave Ψ , which implies by its structure itself the gauge group of the standard model, as having the maximum number (36) of degrees of freedom compatible with the existence of an inverse wave. And we need the existence of the inverse to allow the construction of the wave of systems of fermions (See [5] and [8] 4.4.1).

5 Three generations, four neutrinos

We know three generations of leptons and quarks and the standard model study separately the three generations. I saw many years ago the reason, which is simply that our physical space is three dimensional, and we get the wave equation of leptons three times. One of the three is (3.28) that reads:

$$\begin{aligned}0 &= \tilde{\Psi}_3(\mathbf{D}_3\Psi_3)\gamma_{012} + m_3\rho\tilde{\Psi}_3\chi_3 \\ \mathbf{D}_3 &= \mathbf{D}; \quad \Psi_3 = \Psi_l; \quad \chi_3 = \chi_l; \quad m_3 = m; \quad \rho = \rho_1\end{aligned}\quad (5.1)$$

$$0 = \tilde{\Psi}_1(\mathbf{D}_1\Psi_1)\gamma_{023} + m_1\rho\tilde{\Psi}_1\chi_1 \quad (5.2)$$

$$0 = \tilde{\Psi}_2(\mathbf{D}_2\Psi_2)\gamma_{031} + m_2\rho\tilde{\Psi}_2\chi_2 \quad (5.3)$$

To go from one generation to another one is simple: I permute indices 1,2,3 of σ_j everywhere in all preceding formulas with the circular permutation p or p^2 :

$$p : 1 \mapsto 2 \mapsto 3 \mapsto 1; \quad p^2 : 1 \mapsto 3 \mapsto 2 \mapsto 1. \quad (5.4)$$

I do not know if the muon is obtained by p or by p^2 (one chance on two!) If it is p , the wave of the pair muon-muonic neutrino follows (5.2) and this explains why a muon is like an electron, generally. But the covariant derivative is different, because in the place of (3.10) to (3.14) we must use

$$P_{\pm}^1(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi\gamma_{32}) \quad (5.5)$$

$$P_0^1(\Psi) = \Psi\gamma_{32} + P_-(\Psi)\mathbf{i} \quad (5.6)$$

$$P_1^1(\Psi) = P_+(\Psi)\gamma_1\mathbf{i} \quad (5.7)$$

$$P_2^1(\Psi) = P_+(\Psi)\gamma_1 \quad (5.8)$$

$$P_3^1(\Psi) = P_+(\Psi)(-\mathbf{i}) \quad (5.9)$$

To add two quarks with three colors each we need

$$P_0^1(\Psi_c) = -\frac{1}{3}\Psi_c\gamma_{32} + P_-(\Psi_c)\mathbf{i}; \quad c = r, g, b. \quad (5.10)$$

We must also change the link (3.5) between the wave of the particle and the wave of the antiparticle, link using a σ_1 for the first generation. The wave of the anti-muon must satisfy:

$$\widehat{\phi}_{\bar{\mu}} = \widehat{\phi}_{\mu}\sigma_2 \quad (5.11)$$

and we shall have a 3 index in the case of the third generation. We must also change the definition of left and right wave. For the second generation this becomes

$$\phi_{\mu L} = \phi_{\mu}\frac{1}{2}(1 - \sigma_1); \quad \phi_{\mu R} = \phi_{\mu}\frac{1}{2}(1 + \sigma_1) \quad (5.12)$$

and so on. We can then understand why the Lagrangian density, which comes from the scalar part of the invariant equation, must be calculated separately for the pair electron-electronic neutrino and for the pair muon-muonic neutrino or tau-tauic neutrino. Now since the Γ_k operators, generators of the $SU(3)$ group of chromodynamics, are unchanged by the circular permutation p used to pass from one generation to another, strong interactions are unperturbed by the change of generation. This allows physical quarks composing particles to mix the generations. For instance the physical quark \mathbf{d} present in protons and neutrons is thought as a mixing of the \mathbf{d} of the first generation and the quark \mathbf{s} that is the equivalent of \mathbf{d} in the second generation. Even if the wave of antiquarks is linked to the wave of the quarks, the mixing of waves of different generations, and the difference between what we call “left” and “right” in each generation, induce the wave of physical quarks to have both a left and a right wave.

If there are only three objects like σ_{21} , there is one other term with square -1 in Cl_3 , $i = \sigma_1\sigma_2\sigma_3$. this fourth term allows a fourth neutrino [9].

6 Concluding remarks

From the Schrödinger equation, which introduces quantum waves as functions of space and time with value into complex linear spaces, quantum

physicists have built an “axiomatic quantum mechanics” and the non-relativistic Schrödinger equation is one of these axioms. From the Dirac equation, which is not a consequence of the Schrödinger equation and is then out of this axiomatic theory, we can account for the existence of particles with spin 1/2. This existence implies a greater group of invariance for physical laws.

The mathematical frame necessitates only three Clifford algebras: Cl_3 , because the physical world is 3-dimensional, $Cl_{1,3}$ to get the wave of the pair electron-neutrino, $Cl_{1,5}$ to get the wave of the electron, its neutrino and two quarks of the first generation with three states of color each. All features of the standard model fit together, with very simple hypothesis. We can easily see, for instance, why the great unification with a $SU(5)$ gauge group was not successful. The standard model really works not with complex linear spaces, but with real Clifford algebra. It happens that the gauge group has a structure made of unitary groups, but there are no reason to privilege unitary groups. Adjoint matrices are actually reverse multivectors, but this is true only in Cl_3 . The invariant equations use reversion, then unitarity is accidental, not fundamental. Since the structure of the group comes from the structure of the wave, even if the constants g_1 , g_2 and g_3 are reunited at very high energy, this does not change the structure of the gauge group. The physical consequence is that a quark cannot transform into an electron, and the proton is indefinitely stable. The translation in the formalism of quantum field theory is: the baryonic number is conservative.

A greater group means greater constraints. One of the more visible is the difference between covariant and contravariant vectors in space-time. Another one, more important, very well-known, is the existence of the Planck “constant”, and I put the word into brackets, because it is not a constant, as we saw in (2.32). Evidently, if you begin a book by choosing $\hbar = 1$ you will never understand what is really this Planck factor. The form invariance of the wave under Lorentz dilations induced by all M matrices induces that it is the product $m\rho$ alone which is invariant: $m\rho = m'r\rho = m'\rho'$. What says to us the invariance of $m\rho$? It is the product of a reduced mass and a dilation ratio which is invariant. A reduced mass $m = \frac{m_0c}{\hbar}$ is proportional to the inverse of a space-time length, which is a frequency. This is exactly what says $E = h\nu$. The existence of the Planck’s constant is linked to the fact that m and ρ are not separately invariant, but only their product.

The invariance of the Lagrangian under all translations, as with the linear Dirac theory, induces the existence of a conservative impulse-energy tensor, the Tetrode’s tensor which is, in the case of the alone electron:

$$T_\nu^\mu = i\frac{\hbar}{2}c(\bar{\psi}\gamma^\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma^\mu\psi) - \delta_\nu^\mu\mathcal{L}. \quad (6.1)$$

Since the wave equation is homogeneous, the Lagrangian is null and we get:

$$T_\nu^\mu = i\frac{\hbar}{2}c(\bar{\psi}\gamma^\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma^\mu\psi). \quad (6.2)$$

For an electron in a stationary state with energy E we have:

$$\begin{aligned}\psi &= e^{-i\frac{Et}{\hbar}}\psi(x); \quad \bar{\psi} = e^{i\frac{Et}{\hbar}}\bar{\psi}(x) \\ \partial_0\psi &= -i\frac{E}{\hbar}\psi; \quad \partial_0\bar{\psi} = i\frac{E}{\hbar}\bar{\psi}.\end{aligned}\tag{6.3}$$

So we get:

$$T_0^0 = i\frac{\hbar}{2}c[\bar{\psi}\gamma^0(-i\frac{E}{\hbar})\psi - i\frac{E}{\hbar}\bar{\psi}\gamma^0\psi] = E\bar{\psi}\gamma^0\psi = EJ^0.\tag{6.4}$$

The condition normalizing the wave function

$$\iiint J^0 dv = 1\tag{6.5}$$

is then equivalent to

$$\iiint T_0^0 dv = E.\tag{6.6}$$

The left term of this equality is the total energy of the wave, whilst the right term is the energy of the electron, energy linked to its frequency. So it is not because we must get a probability density that the wave must be normalized. The wave is physically normalized because **the energy-frequency of the electron is equal to the total energy of its wave**. Since the energy E , linked to the frequency, is dependent on the gravitational field, we can consider $\frac{E}{c^2}$ as the gravitational mass of the electron. We also know that the tensor of Tetrode is linked to the Laplace law (see [6] B.2), then this tensor may be considered as giving the inertia of the electron. Equation (6.6) may actually be considered as **the equality between inertial and gravitational mass-energy**.

Since in all cases considered in quantum theory a density of probability exists, the preceding approach is certainly general : for any quantum system with a quantum wave, the energy-frequency (=gravitational mass-energy) is equal to the integral of the density of energy of the wave (=inertial mass-energy). The principle of equivalence between gravitational and inertial mass, basis of the General Relativity of A. Einstein, is then identical to the quantum principle of the square of the wave as a probability density. It is how the quantum world and the gravitation are united. This unification is only partial, because it is an integral on space, not on space-time, which gives physical quantities, a thing that Louis de Broglie noticed very early [12]. Another limitation to this unification: the Lagrangian density, real part of the wave equation, gives the whole wave equation with a condition at infinity easily satisfied by bound states but not established for propagating waves.

Since the three generations of fundamental fermions and the 12 gauge bosons have been yet studied, since our approach accounts for the form invariance and the gauge invariance of the standard model, we have no need of supersymmetry, or of great unification, of strings, branes or complicated Lie groups. We need neither new gauge boson, nor a fourth generation. Even the Higgs boson can take place in the construction of bosons from a couple fermion-antifermion (See [11] 4.4.2), construction initiated by L. de Broglie [13][14].

But there is a place, in the standard model, for complex space-time vectors potential. Complex space-time vectors are in fact sum of true vectors, and pseudo-vectors. These pseudo-vectors allows particles with a magnetic charge instead of the electric charge and are named magnetic monopoles. The study of these magnetic monopoles is only at the beginning (see [11] chapter 7). This is the only possible extension of the standard model.

This study also is only a beginning. It is possible to extend the wave equation for the pair electron-neutrino into a wave equation with mass term for all fermions of the first generation. This wave equation shall be published as soon as possible. Calculations are too long to be put into this presentation. We have not yet completely studied the second and the third generation, and the mixing of these generations in the physical leptons and quarks. Gauge bosons remain also largely to study. The experimental work on magnetic monopoles is also only beginning.

A little part of this text was presented in the thirty minutes that the organizers of ICCA10 in Tartu (Estonia) gave to me to explain all this. Then absolutely all my thanks go to Jacques Bertrand, the only physicist in the world who helped me to develop the present work.

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