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Kalman filter for discrete-time stochastic linear systems subject to intermittent unknown inputs

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Abstract— State estimation of stochastic discrete-time linear systems subject to persistent unknown inputs has been widely studied but only few works have been dedicated to the case where unknown inputs may be simultaneously or sequentially active or inactive. In this paper, a Kalman filter approach is proposed for state estimation of systems with unknown intermittent inputs. The design is based on the minimisation of the trace of the state estimation error covariance matrix under the constraint that the state estimation error is decoupled from the unknown inputs corrupting the system at the current time. The necessary and sufficient stability conditions are established considering the upper bound of the prediction error covariance matrix.

Index Terms— Kalman filter, intermittent unknown inputs, linear system, covariance matrices.

I. INTRODUCTION

KALMAN filtering plays an essential role in systems theory and has found a wide range of applications [19] and [29]. The problem of optimal state filtering in the presence of persistent unknown inputs, representing unknown disturbances or unmodeled dynamics, has received considerable attention in the last three decades. The most common approach consists in representing the unknown inputs as a deterministic or stochastic bias model, augment the state of the system with the bias, and apply the Kalman filter to the augmented state model of the system and finally reduce the computational time requirement [10], [1], [20], [17], [18] and [23]. When no prior information about the unknown input is available, an optimal recursive state filter is presented in [24] so that the state estimation error is decoupled from unknown inputs. Another approach which consists in transforming a standard system with unknown inputs into a singular system without unknown inputs was introduced in [6]. Other optimal filters closer to the standard Kalman filter have been derived by minimising the estimation error covariance matrix with respect to a reduced state feedback gain representing the

degrees of freedom in the design [5], [7] and [15]. The closely related problem of joint input and state estimation for linear discrete-time systems, which are of great importance for Fault Tolerant Control (FTC) when each component of the unknown inputs vector represents actuator or component faults [2], has been recently studied in [11], [12] and [8]. Note that the unknown inputs decoupling constraint used in [24] can be viewed as a limit case of a more general assignment taken into account in the design of stochastic detection filters [26], [22] and [21] for Fault Detection and Isolation (FDI).

Recent technological advances are revolutionizing our ability to build massively distributed Networked Control Systems (NCS). They present challenging problems arising from the fact that sensors, actuators and controllers exchange information via a digital communication network. The most common network effect can be categorized into additional network induced delays, packets losses, jitters,... In the framework of this paper, we are mainly concerned by packets losses, leading to intermittent data transmission. The state of the art on design control systems that take into account the effects of packet loss and packet delay in networked control systems have been surveyed in [16]. In particular, Kalman filtering with random lost of observations represented by Markovian or Bernoulli processes has been studied in [9] and [30], extended later to include both packet loss and random delay [27], [28] and more recently in [31] when the arrival of observations is driven by a semi-Markov chain.

This paper proposes a state filtering strategy for discrete-time linear stochastic systems in the case where the arrival of unknown inputs is described by a known binary sequence. Instead of using the parameterized approach proposed in [7] which requires to pre-compute off-line the structure of the state feedback gain for each combinatorial situation of the binary sequence, the intermittent unknown inputs decoupling constraint will be parameterized from two constant size matrices, called the free part and the constrained part of the filter gain. The constrained gain, structurally dependent on the binary sequence, will be linked to the intermittent unknown inputs estimator. From a two-stage optimisation strategy very similar to those described in [10], the free gain and the constrained gain will be both used to minimize the trace of the state estimation error covariance matrix and the trace of the unknown inputs estimation error covariance matrix. The obtained filter, called the Intermittent unknown Input Kalman Filter (IIKF), will take the form of a standard Kalman filter

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updated online from the intermittent unknown inputs estimation. The stability of the IIKF will be studied on the upper bound of the prediction error covariance matrix via the stability and convergence conditions established for the Unknown Input Kalman Filter (UIKF) designed under persistent unknown inputs.

The paper is organised as follows: Section 2 explains the state filtering problem in the presence of intermittent unknown inputs. Section 3 solves the state filtering problem and studies the filter's stability before conclusions in Section 4.

II. PROBLEM STATEMENT

Let us consider the following linear discrete-time stochastic systems

$$x_{k+1} = Ax_k + Bu_k + Fd_k^\theta + w_k \quad (1.a)$$

$$y_k = Cx_k + v_k \quad (1.b)$$

where $x_k \in \mathfrak{R}^n$, $u_k \in \mathfrak{R}^d$, $y_k \in \mathfrak{R}^m$ and $d_k^\theta \in \mathfrak{R}^q$ are the state, control, measurement and unknown input vectors with $q \leq m$. Matrices A , B , C and F are of appropriate dimensions with $\text{rank}(CF) = \text{rank}(F) = q$. The process and sensor noises $w_k \in \mathfrak{R}^n$ and $v_k \in \mathfrak{R}^m$ are zero mean uncorrelated Gaussian random sequences with

$$E \left\{ \begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_j \\ v_j \end{bmatrix}^T \right\} = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \delta_{k,j} \quad \text{with } W \geq 0 \quad (1.c)$$

The initial state x_0 , assumed to be uncorrelated with w_k and v_k , is a Gaussian random variable with $E\{x_0\} = \bar{x}_0$ and $P_0 = E\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\} \geq 0$.

The term Fd_k^θ in (1.a) is used to describe additive unknown inputs as for example interconnecting external inputs in the context of large scale NCS. It is further assumed that d_k^θ , due to possible unreliable data transmissions, is an intermittent unknown inputs vector

$$d_k^\theta = \begin{bmatrix} \rho_k^1 d_k^1 & \dots & \rho_k^i d_k^i & \dots & \rho_k^q d_k^q \end{bmatrix}^T \quad (1.d)$$

depending on the known binary variables

$$\theta_k = \{\rho_k^1, \dots, \rho_k^i, \dots, \rho_k^q\} \quad (1.e)$$

where $\rho_k^i = 1$ when the i th component of d_k^θ is active and $\rho_k^i = 0$ otherwise.

As illustrated by Fig. 1, u_k represents the control signals sent by the local controller to the plant and d_k^θ the control signals sent by remote controller with $\rho_k^i = 1$ when d_k^i is received by the plant or $\rho_k^i = 0$ when d_k^i is lost through the unreliable network. The unreliable network is a multichannel network where each component of d_k^θ may be lost independently from the others. The acknowledgement signals θ_k indicating the status of reception/delivery (TCP for example) and the distribution matrix F of d_k^θ are both assumed known to the local controller.

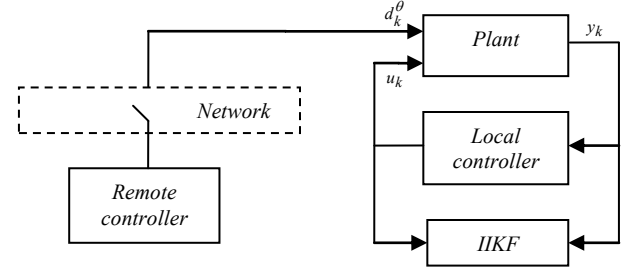


Fig. 1. Intermittent unknown inputs in NCS.

The arrival sequence $\{\theta_j\}_0^{k-1}$ of unknown inputs is considered as deterministic in the design of the following state filter

$$\hat{x}_{k/k}^\theta = \hat{x}_{k/k-1}^\theta + K_k^\theta (y_k - C\hat{x}_{k/k-1}^\theta) \quad (2.a)$$

$$P_{k/k}^\theta = (I - K_k^\theta C) P_{k/k-1}^\theta (I - K_k^\theta C)^T + K_k^\theta K_k^{\theta T} \quad (2.b)$$

$$\hat{x}_{k+1/k}^\theta = A\hat{x}_{k/k}^\theta + Bu_k \quad (2.c)$$

$$P_{k+1/k}^\theta = AP_{k/k}^\theta A^T + W \quad (2.d)$$

where $\hat{x}_{k/k-1}^\theta$ is the state prediction of covariance matrix $P_{k/k-1}^\theta = E\{(x_k - \hat{x}_{k/k-1}^\theta)(x_k - \hat{x}_{k/k-1}^\theta)^T\}$ based on measurements available until time $k-1$ and $\{\theta_j\}_0^{k-1}$, where $\hat{x}_{k/k}^\theta$ is the state estimate of covariance matrix $P_{k/k}^\theta = E\{(x_k - \hat{x}_{k/k}^\theta)(x_k - \hat{x}_{k/k}^\theta)^T\}$ based on measurements available until time k and $\{\theta_j\}_0^{k-1}$. From (1) and (2), the state prediction error $e_{k+1/k}^\theta = x_{k+1} - \hat{x}_{k+1/k}^\theta$ and the state estimation error $e_{k/k}^\theta = x_k - \hat{x}_{k/k}^\theta$ propagate as

$$e_{k+1/k}^\theta = Ae_{k/k}^\theta + Fd_k^\theta + w_k \quad (3.a)$$

$$e_{k/k}^\theta = (I - K_k^\theta C)e_{k/k-1}^\theta - K_k^\theta v_k \quad (3.b)$$

Under $E\{e_{k/k-1}^\theta\} = Fd_{k-1}^\theta$ with $F = [f^1 \dots f^i \dots f^q]$, we have $E\{e_{k/k}^\theta\} = (I - K_k^\theta C)E\{e_{k/k-1}^\theta\} = 0$ (and thus $E\{e_{k+1/k}^\theta\} = Fd_k^\theta$) if and only if the state feedback gain $K_k^\theta \in \mathfrak{R}^{n,m}$ satisfies the unknown inputs decoupling constraint $(I - K_k^\theta C)Fd_{k-1}^\theta = 0$ in [24] relaxed under (1.d) as

$$K_k^\theta CF_{k-1}^\theta = F_{k-1}^\theta \quad \text{with } F_{k-1}^\theta = \begin{bmatrix} \rho_{k-1}^1 f^1 & \dots & \rho_{k-1}^i f^i & \dots & \rho_{k-1}^q f^q \end{bmatrix} \quad (4)$$

Under $\text{rank}(CF) = \text{rank}(F) = q$ the existence condition $\text{rank}(CF_{k-1}^\theta) = \text{rank}(F_{k-1}^\theta) = s_k$ with $s_k = \sum_{i=1}^q \rho_{k-1}^i$ for a solution to (4) is satisfied $\forall \theta_{k-1}$. This paper proposes to parameterize the solution to equation (4) from constant size matrices as follows

$$K_k^\theta = K_k^0 + \mu_k^\theta G_k^\theta \quad (5)$$

with $K_k^0 \in \mathfrak{R}^{n,m}$, $\mu_k^\theta = (I - K_k^0 C)F_{k-1}^\theta$ and $G_k^\theta \in \mathfrak{R}^{q,m}$. Substituting (5) in (4), we obtain

$$\mu_k^\theta G_k^\theta CF_{k-1}^\theta = \mu_k^\theta \quad (6)$$

With $I_{k-1}^\theta = \text{diag}[\rho_{k-1}^1, \dots, \rho_{k-1}^i, \dots, \rho_{k-1}^q]$ so that $\mu_k^\theta I_{k-1}^\theta = \mu_k^\theta$ since μ_k^θ has a special structure with zero columns when d_{k-1}^θ has

zero components, we deduce from (6) that the state feedback gain (5) satisfies (4) if and only if G_k^θ satisfies

$$G_k^\theta C F_{k-1}^\theta = I_{k-1}^\theta \quad (7)$$

Suggested by the structure of the state feedback gain (5), let

$$\hat{d}_{k-1/k}^\theta = G_k^\theta (y_k - C \hat{x}_{k-1/k}^\theta) \quad (8.a)$$

$$Q_{k-1/k}^\theta = E \left\{ \begin{bmatrix} e_{k-1/k}^\theta \\ e_{k-1/k}^{\theta T} \end{bmatrix} \right\} \quad (8.b)$$

with $e_{k-1/k}^\theta = \hat{d}_{k-1/k}^\theta - d_{k-1}^\theta$ where $E \left\{ \hat{d}_{k-1/k}^\theta \right\} = G_k^\theta C E \left\{ e_{k-1/k}^\theta \right\} = d_{k-1}^\theta$ under (4) and (7). The state estimator (2) and the intermittent unknown inputs estimator (8) will be designed by minimising the trace of $P_{k/k}^\theta$ and $Q_{k-1/k}^\theta$ with respect to K_k^θ and G_k^θ under (4) and (7). The necessary and sufficient conditions so that $\lim_{k \rightarrow \infty} P_{k+1/k}^\theta < \infty$ for any sequence $\{\theta_j\}_0^\infty$ will be established.

III. KALMAN FILTER FOR STOCHASTIC LINEAR SYSTEM SUBJECT TO INTERMITTENT UNKNOWN INPUTS

The proposed IKF is described by theorem 3.1.

Theorem 3.1: The Unbiased Minimum Variance (UMV) state estimate $\hat{x}_{k/k}^\theta$ of covariance $P_{k/k}^\theta$ is generated by the following modified Kalman filter

$$\hat{x}_{k/k}^\theta = (I - K_k^\theta C) (\hat{x}_{k-1/k}^\theta + F_{k-1}^\theta \hat{d}_{k-1/k}^\theta) + K_k^\theta y_k \quad (9.a)$$

$$P_{k/k}^\theta = (I - K_k^\theta C) (P_{k-1/k}^\theta + F_{k-1}^\theta Q_{k-1/k}^\theta F_{k-1}^{\theta T}) (I - K_k^\theta C)^T + K_k^\theta K_k^{\theta T} \quad (9.b)$$

$$\hat{x}_{k+1/k}^\theta = A \hat{x}_{k/k}^\theta + B u_k \quad P_{k+1/k}^\theta = A P_{k/k}^\theta A^T + W \quad (10)$$

with $K_k^\theta = P_{k/k-1}^\theta C (C P_{k/k-1}^\theta C^T + I)^{-1}$, updated online from the additive quantities $F_{k-1}^\theta \hat{d}_{k-1/k}^\theta$ and $F_{k-1}^\theta Q_{k-1/k}^\theta F_{k-1}^{\theta T}$ depending on the unknown inputs estimate $\hat{d}_{k-1/k}^\theta$ of covariance $Q_{k-1/k}^\theta$ given by

$$\begin{aligned} \hat{d}_{k-1/k}^\theta &= G_k^\theta (y_k - C \hat{x}_{k-1/k}^\theta) \\ Q_{k-1/k}^\theta &= [(C F_{k-1}^\theta)^T (C P_{k-1/k}^\theta C^T + I)^{-1} C F_{k-1}^\theta]^\dagger \end{aligned} \quad (11)$$

with $G_k^\theta = Q_{k-1/k}^\theta (C F_{k-1}^\theta)^T (C P_{k-1/k}^\theta C^T + I)^{-1}$. The i th component $\hat{d}_{k-1/k}^{\theta i}$ of $\hat{d}_{k-1/k}^\theta$ represents the estimate of $\rho_{k-1}^i d_{k-1}^i$ (with $\hat{d}_{k-1/k}^{\theta i} = 0$ when $\rho_{k-1}^i = 0$) and the i th component $Q_{k-1/k}^{\theta i}$ on the diagonal part of $Q_{k-1/k}^\theta$ represents the variance of $\hat{d}_{k-1/k}^{\theta i}$ (with $Q_{k-1/k}^{\theta i} = 0$ when $\rho_{k-1}^i = 0$). The IKF is initialized by $\hat{x}_{0/-1}^\theta = \bar{x}_0$, $P_{0/-1}^\theta = P_0 \geq 0$ and $\theta_{-1} = \{0, \dots, 0\}$.

Proof. Let $X_k^\theta = \begin{bmatrix} x_k^\theta \\ d_{k-1}^{\theta T} \end{bmatrix}$, $\hat{X}_{k/k}^\theta = \begin{bmatrix} \hat{x}_{k/k}^\theta \\ \hat{d}_{k-1/k}^{\theta T} \end{bmatrix}$, $\Omega_{k/k}^\theta = E \left\{ \begin{bmatrix} e_{k/k}^\theta \\ e_{k-1/k}^\theta \end{bmatrix} \begin{bmatrix} e_{k/k}^{\theta T} \\ e_{k-1/k}^{\theta T} \end{bmatrix} \right\}$ and $L_k^\theta = \begin{bmatrix} K_k^{\theta T} \\ G_k^{\theta T} \end{bmatrix}$. The state estimator (2) and the unknown inputs estimator (8) can then be jointly expressed

$$\hat{X}_{k/k}^\theta = \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{X}_{k-1/k}^\theta + L_k^\theta (y_k - C \hat{x}_{k-1/k}^\theta) \quad (12.a)$$

$$\Omega_{k/k}^\theta = \begin{bmatrix} I \\ 0 \end{bmatrix} - L_k^\theta C \begin{bmatrix} P_{k-1/k}^\theta \\ 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} - L_k^\theta C^T + L_k^\theta L_k^{\theta T} \quad (12.b)$$

$$\hat{x}_{k+1/k}^\theta = [A \ 0] \hat{X}_{k/k}^\theta + B u_k \quad P_{k+1/k}^\theta = [A \ 0] \Omega_{k/k}^\theta [A \ 0]^T + W \quad (12.c)$$

Under $\text{tr}(P_{k/k-1}^\theta)$ minimum, $\hat{X}_{k/k}^\theta$ is the UMV estimate of X_k^θ (and thus $\text{tr}(P_{k+1/k}^\theta)$ minimum) if and only if the augmented gain $L_k^\theta = \begin{bmatrix} K_k^{\theta T} \\ G_k^{\theta T} \end{bmatrix}$ is solution to the constrained optimization problem

$$\min_{L_k^\theta} \text{tr}(\Omega_{k/k}^\theta) \quad \text{s.t.} \quad L_k^\theta C F_{k-1}^\theta = \begin{bmatrix} F_{k-1}^\theta \\ I_{k-1}^\theta \end{bmatrix} \quad (13)$$

The solution to (13) is difficult to obtain since K_k^θ depends on G_k^θ from $K_k^\theta = K_k^0 + \mu_k^\theta G_k^\theta$. Let $\bar{X}_k^\theta = T_k X_k^\theta$, $\hat{\bar{X}}_{k/k}^\theta = T_k \hat{X}_{k/k}^\theta$, $\bar{\Omega}_{k/k}^\theta = T_k \Omega_{k/k}^\theta T_k^T$ and $\bar{L}_k^\theta = T_k L_k^\theta$ where T_k is an arbitrary non-singular transformation matrix of appropriate dimension. The filter (12) can then be equivalently rewritten

$$\hat{\bar{X}}_{k/k}^\theta = T_k \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{\bar{X}}_{k-1/k}^\theta + \bar{L}_k^\theta (y_k - C \hat{\bar{x}}_{k-1/k}^\theta) \quad (14.a)$$

$$\bar{\Omega}_{k/k}^\theta = \begin{bmatrix} I \\ 0 \end{bmatrix} - \bar{L}_k^\theta C \begin{bmatrix} P_{k-1/k}^\theta \\ 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} - \bar{L}_k^\theta C^T + \bar{L}_k^\theta \bar{L}_k^{\theta T} \quad (14.b)$$

$$\hat{\bar{x}}_{k+1/k}^\theta = [A \ 0] T_k^{-1} \hat{\bar{X}}_{k/k}^\theta + B u_k \quad (14.c)$$

$$P_{k+1/k}^\theta = [A \ 0] T_k^{-1} \bar{\Omega}_{k/k}^\theta T_k^T [A \ 0]^T + W \quad (14.d)$$

Under $\text{tr}(P_{k/k-1}^\theta)$ minimum, $\hat{\bar{X}}_{k/k}^\theta$ is the UMV estimate of \bar{X}_k^θ (and thus $\text{tr}(P_{k+1/k}^\theta)$ minimum) if and only if the transformed gain \bar{L}_k^θ is solution to

$$\min_{\bar{L}_k^\theta} \text{tr}(\bar{\Omega}_{k/k}^\theta) \quad \text{s.t.} \quad \bar{L}_k^\theta C F_{k-1}^\theta = T_k \begin{bmatrix} F_{k-1}^\theta \\ I_{k-1}^\theta \end{bmatrix} \quad (15)$$

With $T_k = \begin{bmatrix} I & -\mu_k^\theta \\ 0 & I \end{bmatrix}$ so that $\bar{L}_k^\theta = T_k \begin{bmatrix} K_k^{\theta T} \\ G_k^{\theta T} \end{bmatrix} = \begin{bmatrix} K_k^{\theta T} \\ G_k^{\theta T} \end{bmatrix}$ where

K_k^0 is now decoupled from G_k^θ , we can verify that the transformed algebraic constraints in (15) reduce to $G_k^\theta C F_{k-1}^\theta = I_{k-1}^\theta$. So, after straightforward manipulations, (15) becomes

$$\min_{\begin{bmatrix} K_k^0 \\ G_k^\theta \end{bmatrix}} \text{tr}(\bar{\Omega}_{k/k}^\theta) \quad \text{s.t.} \quad G_k^\theta C F_{k-1}^\theta = I_{k-1}^\theta \quad (16)$$

$$\text{with } \bar{\Omega}_{k/k}^\theta = \begin{bmatrix} P_{k/k}^0 & (P_{k/k-1}^0 C^T - K_k^0 H_k) G_k^{\theta T} \\ G_k^\theta (P_{k/k-1}^0 C^T - K_k^0 H_k)^T & Q_{k-1/k}^\theta \end{bmatrix} \quad (17)$$

where $P_{k/k}^0 = (I - K_k^0 C) P_{k/k-1}^0 (I - K_k^0 C)^T + K_k^0 K_k^{\theta T}$, $Q_{k-1/k}^\theta = G_k^\theta H_k G_k^{\theta T}$ and $H_k = C P_{k/k-1}^0 C^T + I$. With $\text{tr}(\bar{\Omega}_{k/k}^\theta) = \text{tr}(P_{k/k}^0) + \text{tr}(Q_{k-1/k}^\theta)$ obtained from (17), we deduce that the global solution to (16) corresponds to the local solutions of the following decoupled optimisation problems

$$\min_{K_k^0} \text{tr}(P_{k/k}^0) \quad (18.a)$$

and

$$\min_{G_k^\theta} \text{tr}(Q_{k-1/k}^\theta) \quad \text{s.t.} \quad G_k^\theta C F_{k-1}^\theta = I_{k-1}^\theta \quad \text{for } i=1, \dots, q \quad (18.b)$$

where $F_{k-1}^{\theta i}$ and $I_{k-1}^{\theta i}$ represent the i th column of F_{k-1}^θ and I_{k-1}^θ .

The unique solution to (18.a) coincides with the Kalman filter's gain $K_k^0 = P_{k/k-1}^0 C (C P_{k/k-1}^0 C^T + I)^{-1}$. The existence of the i th

constraint in (18.b) is conditioned by $\rho_{k-1}^i=1$ and the solution to (18.b) can be derived by minimising

$$\Phi_k^\theta = 0.5tr(G_k^\theta H_k G_k^{\theta T}) + \sum_{i=1}^q \lambda_k^{\theta i} (C_k^\theta C F_{k-1}^{\theta i} - I_{k-1}^{\theta i}) \quad (19)$$

where $\lambda_k^{\theta i} \in \mathfrak{R}^{q,1}$ is the Lagrange parameters vector satisfying $\lambda_k^{\theta i} = 0$ when $\rho_{k-1}^i = 0$ and $\lambda_k^{\theta i} \neq 0$ when $\rho_{k-1}^i = 1$. The optimality conditions of Φ_k^θ gives

$$\frac{\partial \Phi_k^\theta}{\partial C_k^\theta} = G_k^\theta H_k + \sum_{i=1}^q \lambda_k^{\theta i} (C F_{k-1}^{\theta i})^T = 0 \quad (20.a)$$

$$\frac{\partial \Phi_k^\theta}{\partial \lambda_k^{\theta i}} = G_k^\theta C F_{k-1}^{\theta i} - I_{k-1}^{\theta i} = 0 \text{ if } \rho_{k-1}^i = 1 \text{ for } i=1, \dots, q \quad (20.b)$$

The solution $G_k^\theta = \left\{ \sum_{i=1}^q \lambda_k^{\theta i} (C F_{k-1}^{\theta i})^T \right\} H_k^{-1}$ to (20.a) substituted in (20.b) gives

$$-\lambda_k^{\theta i} (C F_{k-1}^{\theta i})^T H_k^{-1} C F_{k-1}^{\theta i} = I_{k-1}^{\theta i} \quad (21)$$

where $\lambda_k^\theta = \begin{bmatrix} \lambda_k^{\theta 1} & \lambda_k^{\theta 2} & \lambda_k^{\theta q} \end{bmatrix} \in \mathfrak{R}^{q,q}$ represents the Lagrange parameters matrix. The solution to (21), expressed $\lambda_k^{\theta i} = -I_{k-1}^{\theta i} [(C F_{k-1}^{\theta i})^T H_k^{-1} C F_{k-1}^{\theta i}]^+ = [(C F_{k-1}^{\theta i})^T H_k^{-1} C F_{k-1}^{\theta i}]^+$ from $F_{k-1}^{\theta i} I_{k-1}^{\theta i} = F_{k-1}^{\theta i}$, substituted in $G_k^\theta = -\lambda_k^{\theta i} (C F_{k-1}^{\theta i})^T H_k^{-1}$ gives

$$G_k^\theta = [(C F_{k-1}^{\theta i})^T H_k^{-1} C F_{k-1}^{\theta i}]^+ (C F_{k-1}^{\theta i})^T H_k^{-1} \quad (22)$$

leading to $Q_{k-1/k}^\theta = G_k^\theta H_k G_k^{\theta T} = [(C F_{k-1}^{\theta i})^T H_k^{-1} C F_{k-1}^{\theta i}]^+$ via $X^+ X X^+ = X^+$ (X^+ is the unique Moore-Penrose generalized inverse of X).

After some manipulations, the IIKF of theorem 3.1 is recovered by substituting the inverse transformation $T_k^{-1} = \begin{bmatrix} I & \mu_k^\theta \\ 0 & I \end{bmatrix}$ in the optimized filter (14). Its optimality is ensured by the invariance property to linear transforms of optimal linear filters. The IIKF must be initialized with $\theta_{-1} = \{0, \dots, 0\}$ since \bar{x}_0 represents the unbiased prediction of the initial state x_0 .

End of the proof

From (9.b) and (10), we derive the IIKF's Riccati Difference Equation (RDE)

$$P_{k+1/k}^\theta = (A - K_k^p C) P_{k/k-1}^\theta + F_{k-1}^{\theta} Q_{k-1/k}^\theta F_{k-1}^{\theta T} (A - K_k^p C)^T + K_k^p K_k^p T + W \quad (23)$$

with $K_k^p = A K_k^0 = A P_{k/k-1}^\theta C^T (C P_{k/k-1}^\theta C^T + I)^{-1}$.

Theorem 3.2: Under the necessary and sufficient conditions

$$\text{rank} \begin{bmatrix} -E + A & F \\ C & 0 \end{bmatrix} = n + q, \forall |z| \geq 1 \quad (24.a)$$

$$\text{rank} \begin{bmatrix} -I e^{j\omega} + A & W^{1/2} \end{bmatrix} = n, \forall \omega \in [0, 2\pi] \quad (24.b)$$

we have $\lim_{k \rightarrow \infty} P_{k+1/k}^\theta < \infty$ for any arrival sequence $\{\theta_j\}_0^\infty$ of unknown inputs.

Proof. Under persistent unknown inputs i.e. $\theta_k = \{1, \dots, 1\}$ $\forall k \geq 0$, we have $d_k^\theta = [d_k^1 \dots d_k^i \dots d_k^q]^T \forall k \geq 0$ or $F_k^\theta = F \forall k \geq 0$ and (23) gives

$$\bar{P}_{k+1/k}^\theta = (A - \bar{K}_k^p C) \bar{P}_{k/k-1}^\theta + F \bar{Q}_{k-1/k}^\theta F^T (A - \bar{K}_k^p C)^T + \bar{K}_k^p \bar{K}_k^p T + W \quad (25)$$

with $\bar{K}_k^p = A \bar{P}_{k/k-1}^\theta C^T \bar{H}_k^{-1}$, $\bar{Q}_{k-1/k}^\theta = [(C F)^T \bar{H}_k^{-1} C F]^T$, $\bar{H}_k = C \bar{P}_{k/k-1}^\theta C^T + I$ and $\bar{P}_{0/-1}^\theta = P_0 \geq 0$. The RDE (25) can be rewritten

$$\bar{P}_{k+1/k}^\theta = (A - \bar{K}_k C) \bar{P}_{k/k-1}^\theta (A - \bar{K}_k C)^T + \bar{K}_k \bar{K}_k^T + W \quad (26)$$

with $\bar{K}_k = A(\bar{K}_k^0 + (I - \bar{K}_k^0 C) F \bar{G}_k)$, $\bar{K}_k^0 = \bar{P}_{k/k-1}^\theta C^T (C \bar{P}_{k/k-1}^\theta C^T + I)^{-1}$ and $\bar{G}_k = [(C F)^T \bar{H}_k^{-1} C F]^T \bar{H}_k^{-1}$.

The appendix shows that the RDE (26) coincides with the time-invariant UIKF's RDE in [7] described by

$$P_{k+1/k} = (A - K_k C) P_{k/k-1} (A - K_k C)^T + K_k K_k^T + W \quad (27)$$

with $K_k = A(\bar{K}_k \Sigma + F \Pi)$, $\bar{K}_k = N P_{k/k-1} \bar{C}^T (\bar{C} P_{k/k-1} \bar{C}^T + \Sigma \Sigma^T)^{-1} \in \mathfrak{R}^{n, m-q}$, $\Pi = (C F)^+$, $N = I - F \bar{C}$, $\bar{C} = \Pi C$, $\bar{C} = \Sigma C$, $\Sigma = \beta(I - C F \Pi)$ and $\beta \in \mathfrak{R}^{m-q, m}$ so that $\text{rank}(\Sigma) = m - q$ with $q = \text{rank}(C F)$. This result also shows that the IIKF's RDE (23) is equivalent to a time-varying structure version of (27) with $K_k = A(\bar{K}_k^\theta \Sigma_k^\theta + F_{k-1}^\theta \Pi_k^\theta)$, $\bar{K}_k^\theta = N_k^\theta P_{k/k-1} \bar{C}_k^{\theta T} (\bar{C}_k^\theta P_{k/k-1} \bar{C}_k^{\theta T} + \Sigma_k^\theta \Sigma_k^{\theta T})^{-1}$, $\Pi_k^\theta = (C F_{k-1}^\theta)^+$, $N_k^\theta = I - F_{k-1}^\theta \bar{C}_k^\theta$, $\bar{C}_k^\theta = \Pi_k^\theta C$, $\bar{C}_k^\theta = \Sigma_k^\theta C$, $\Sigma_k^\theta = \beta_k^\theta (I - C F_{k-1}^\theta \Pi_k^\theta)$ and $\beta_k^\theta \in \mathfrak{R}^{m-s_k, m}$ so that $\text{rank}(\Sigma_k^\theta) = m - s_k$ where the variable size of the gain $\bar{K}_k^\theta \in \mathfrak{R}^{n, m-s_k}$

depends on $s_k = \sum_{i=1}^q \rho_{k-1}^i$ (contrary to the time-varying RDE, but fixed structure in [7], defined with $s_k = q, \forall k$). The degrees of freedom $\bar{K}_k^\theta \in \mathfrak{R}^{n, m-s_k}$ minimizing $\text{tr}(P_{k+1/k}^\theta)$ are always greater or equal to the degrees of freedom $\bar{K}_k \in \mathfrak{R}^{n, m-q}$ minimizing $\text{tr}(P_{k+1/k})$ since $s_k \leq q$. Without further mathematical development, this reasoning leads to $P_{k+1/k}^\theta \leq P_{k+1/k}$ for any sequence $\{\theta_j\}_0^{k-1}$. Under $P_{0/-1} = P_0 \geq 0$, the necessary and sufficient existence conditions (24) for an unique stabilising solution $P \geq 0$ to the following Algebraic Riccati Difference Equation (ARDE)

$$P = (A - K C) P (A - K C)^T + K K^T + W \quad (28)$$

associated to the time-invariant RDE (27) (all the modes of $A - K C$ inside the unique circle) have been established in [7]. Under (24), we conclude that $\lim_{k \rightarrow \infty} P_{k+1/k}^\theta \leq P < \infty$ for any sequence

$$\{\theta_j\}_0^\infty.$$

End of the proof

When the arrival sequence of each component of the unknown inputs vector follows a Bernoulli process with $\lambda = \text{Pr}\{\rho_k^i = 1\} \in [0, 1]$, the rank relations (24) ensure that there exists no critical value for the unknown inputs occurrence rate λ above which the estimation error covariance becomes unbounded.

IV. CONCLUSION

In this paper a Kalman Filter for discrete-time stochastic linear systems subject to intermittent unknown inputs has been presented. The decoupling constraint linked to the binary sequence describing the arrival of unknown inputs has been parameterized from constant size matrices in order to allow a two-stage optimisation strategy. The obtained filter has been presented as a standard Kalman filter updated online from the

intermittent unknown inputs estimation. When the arrival sequences of unknown inputs follow binary state Markov chains or Bernoulli processes, the study of less conservative stochastic stability conditions is currently under consideration by the authors.

Appendix: The RDE (26) coincides with the standard UIKF's RDE (27) i.e. $\bar{K}_k = K_k$ when $\bar{P}_{k/k-1} = P_{k/k-1}$: Define the non singular output transformation matrix

$$\bar{Z}_k = \begin{bmatrix} I & 0 \\ -\bar{J}_k & I \end{bmatrix} \Sigma \in \mathfrak{R}^{m,m} \quad (\text{A.1})$$

with $\bar{J}_k = \bar{C}\bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1}$ and $\bar{H}_k = \bar{C}\bar{P}_{k/k-1}\bar{C}^T + \Sigma\Sigma^T$ (\bar{J}_k is called the adaptive noise canceller gain in [21]) so that

$$\bar{Y}_k = \bar{Z}_k(\bar{C}\bar{P}_{k/k-1}\bar{C}^T + I)\bar{Z}_k^T = \begin{bmatrix} \bar{H}_k & 0 \\ 0 & \bar{H}_k \end{bmatrix} \quad (\text{A.2})$$

with $\bar{H}_k = \bar{C}\bar{S}_{k/k}\bar{C}^T + HH^T$ and $\bar{S}_{k/k} = \bar{P}_{k/k-1} - \bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1}\bar{C}\bar{P}_{k/k-1}$. The constrained gain $\bar{G}_k = [(CF)^T\bar{H}_k^{-1}CF]^{-1}(CF)^T\bar{H}_k^{-1}$ can be equivalently expressed

$$\begin{aligned} \bar{G}_k &= ((\bar{Z}_kCF)^T\bar{Y}_k^{-1}\bar{Z}_kCF)^{-1}(\bar{Z}_kCF)^T\bar{Y}_k^{-1}\bar{Z}_k \\ &= \left[\begin{bmatrix} \bar{H}_k & 0 \\ 0 & \bar{H}_k \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \left[\begin{bmatrix} \bar{H}_k & 0 \\ 0 & \bar{H}_k \end{bmatrix} \right] \bar{Z}_k = \Pi - \bar{J}_k\Sigma \end{aligned} \quad (\text{A.3})$$

By the same way, the free gain $\bar{K}_k^0 = \bar{P}_{k/k-1}\bar{C}^T(\bar{C}\bar{P}_{k/k-1}\bar{C}^T + I)^{-1}$ can be equivalently expressed

$$\begin{aligned} \bar{K}_k^0 &= \bar{P}_{k/k-1}\bar{C}^T\bar{Z}_k^{-1}\bar{Z}_k \\ &= \bar{P}_{k/k-1} \left[\begin{bmatrix} \bar{C}^T\bar{H}_k^{-1} & (\bar{C}^T - \bar{C}^T\bar{J}_k^T)\bar{H}_k^{-1} \end{bmatrix} \begin{bmatrix} \Sigma \\ \Pi - \bar{J}_k\Sigma \end{bmatrix} \right] \end{aligned} \quad (\text{A.4})$$

The gain $\bar{K}_k = A(\bar{K}_k^0 + (I - \bar{K}_k^0C)F\bar{G}_k)$ is rewritten $\bar{K}_k = A\bar{K}_k^0(I - CF(H - \bar{J}_k\Sigma)) + AF(H - \bar{J}_k\Sigma)$ and from (A.3) and (A.4), we derive the following expression

$$\begin{aligned} \bar{K}_k &= A\bar{P}_{k/k-1} \left[\begin{bmatrix} \bar{C}^T\bar{H}_k^{-1} & (\bar{C}^T - \bar{C}^T\bar{J}_k^T)\bar{H}_k^{-1} \end{bmatrix} \begin{bmatrix} \Sigma \\ H - \bar{J}_k\Sigma \end{bmatrix} \right] (I - CF(\Pi - \bar{J}_k\Sigma)) \\ &\quad + AF(\Pi - \bar{J}_k\Sigma) \end{aligned} \quad (\text{A.5})$$

From $\Sigma CF = 0$ and $\Pi CF = I$, we have $\Sigma(I - CF(\Pi - \bar{J}_k\Sigma)) = \Sigma$ and $(H - \bar{J}_k\Sigma)(I - CF(\Pi - \bar{J}_k\Sigma)) = (H - \bar{J}_k\Sigma) - (H - \bar{J}_k\Sigma)CF(\Pi - \bar{J}_k\Sigma)$

$$= (H - \bar{J}_k\Sigma) - (\Pi - \bar{J}_k\Sigma) = 0 \quad (\text{A.6})$$

leading to

$$\begin{aligned} \bar{K}_k &= A\bar{P}_{k/k-1} \left[\begin{bmatrix} \bar{C}^T\bar{H}_k^{-1} & (\bar{C}^T - \bar{C}^T\bar{J}_k^T)\bar{H}_k^{-1} \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \right] + AF(\Pi - \bar{J}_k\Sigma) \\ &= A\bar{P}_{k/k-1} \left[\begin{bmatrix} \bar{C}^T & \bar{C}^T - \bar{C}^T\bar{J}_k^T \end{bmatrix} \begin{bmatrix} \bar{H}_k^{-1}\Sigma \\ 0 \end{bmatrix} \right] + AF(\Pi - \bar{J}_k\Sigma) \\ &= A\bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1}\Sigma + AF(\Pi - \bar{J}_k\Sigma) \\ &= A(\bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1} - F\bar{J}_k)\Sigma + AFT\Pi \\ &= A(\bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1} - F\bar{C}\bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1})\Sigma + AFT\Pi \\ &= A(I - F\bar{C})\bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1}\Sigma + AFT\Pi = A\bar{N}\bar{P}_{k/k-1}\bar{C}^T\bar{H}_k^{-1}\Sigma + AFT\Pi \\ &= A(\bar{K}_k\Sigma + FT\Pi) = K_k \end{aligned} \quad (\text{A.7})$$

We conclude that $\bar{K}_k = K_k$ when $\bar{P}_{k/k-1} = P_{k/k-1}$ and thus

$$\bar{P}_{k/k-1} = P_{k/k-1} \quad \forall k \geq 0 \quad \text{when} \quad \bar{P}_{0/-1} = P_{0/-1}.$$

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