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Gradient governed growth : the effect of viscosity ratio on stochastic simulations of the Saffman-Taylor instability

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Résumé. — Nous proposons un modèle stochastique pour simuler l'écoulement de deux fluides dans un milieu poreux. Nous supposons que la loi de Darcy est valable pour décrire les écoulements de chacun des deux fluides, l'un étant injecté dans le milieu et l'autre déplacé par ce dernier. A chaque pas de temps, nous résolvons numériquement l'équation de Laplace, afin d'obtenir les valeurs du champ de pression. Ensuite, l'interface entre les fluides est déplacée d'un pas constant en un seul point, choisi avec une probabilité proportionnelle au gradient de pression. Nous examinons la gamme complète des contrastes de viscosité entre les deux fluides, couvrant ainsi les cas d'écoulements stables et instables. Nous mesurons dans chaque cas la vitesse de croissance de la longueur de l'interface. Lorsque la viscosité du fluide injecté est faible devant celle du fluide déplacé, le modèle prédit le développement d'une instabilité de l'interface sous forme de digitations, comparable aux résultats donnés par le modèle d'agrégation par diffusion limitée proposé par Witten et Sander. Si l'on augmente la viscosité du fluide injecté, l'épaisseur des digitations augmente et leur croissance en longueur est moins rapide.

Abstract. — We report numerical simulations related to two-fluid flow in porous media. We assume that Darcy's law holds for the flow of each of the two fluids, one of which we are injecting into the porous medium, the other being that displaced. We therefore initially solve Laplace's equation for the pressure. At each time step we then advance the interface between the two fluids by a discrete step at a single point chosen with *probability* proportional to the pressure gradient. We examine the full range of viscosity ratios, covering both stable and unstable cases, and measure the rate of growth of the length of the interface between the two fluids. When the viscosity of the injected fluid is low compared with that of the fluid which is being displaced, the predicted fingering instability is similar to results obtained *via* the Witten and Sander model of diffusion limited aggregation. As the viscosity of the injected fluid increases, the fingers become thicker and they grow more slowly in length.

1. Introduction.

It is standard practice, in the petroleum industry, to inject water into oil-bearing rocks in order to sweep out the oil. It is usually found that only part of the total oil is recovered : the water tends to finger through the oil rather than sweep it towards a production well. This fingering instability is well known [1], and occurs because the viscosity of the water is lower than that of the oil. It is clearly worth investigating for commercial reasons, but is also of scientific interest because multi-phase flow in porous media is still poorly understood. In this paper, we present theoretical work which takes into account not only the viscosity of the oil, but also that of the water ; the random properties of the porous medium are also, in some sense, included.

Darcy's law, which predicts that fluid flow within a porous medium is proportional to the pressure gradient, works well when only a single fluid is present.

It implies, when the fluid is incompressible, that the pressure is governed by Laplace's equation. The connections between potential theory and the theory of random walks [e.g. 2, 3] have therefore led several authors [4-6] to point out the relevance of diffusion-limited aggregation (DLA) [7] to multi-phase flow in either a Hele-Shaw cell or in a porous medium. In this analogy the growing aggregate corresponds to an injected fluid, which is displacing the fluid initially in the porous medium. However, the natural boundary conditions for DLA are those of an aggregate at constant potential. The aggregate therefore corresponds to an invading fluid of zero viscosity, with zero interfacial tension between the invading fluid and that originally in the porous medium. Kadanoff [6] has discussed modifications to DLA which introduce interfacial tension. Here our interest lies in the effect of the viscosity ratio. Rather than take DLA as our starting point for the solution of Laplace's equation,

it is easier for us to use classical numerical schemes to resolve the pressure field. Motion of the interface is then assumed to depend on the pressure gradients thus obtained. Such a scheme has been used by Niemeyer *et al.* [8], and has been called diffusion-limited growth (DLG) by Meakin [9].

Darcy's law predicts that the volume flux \mathbf{Q} of fluid with viscosity μ flowing in a porous medium is proportional to the pressure gradient ∇p :

$$\mathbf{Q} \propto \mu^{-1} \nabla p.$$

If the fluid is incompressible, the pressure satisfies Laplace's equation. Niemeyer *et al.* [8] solved the Laplace equation outside their growing agglomerate, which was at constant potential as it represented the discharge pattern of dielectric breakdown. Here we are interested in two fluids, with different viscosities, and must therefore solve the Laplace equation on both sides of the interface. The normal flux $\mathbf{Q} \cdot \mathbf{n}$ must be continuous across the interface, as must the pressure p , since we assume that there is no interfacial tension between the fluids. These assumptions are the standard assumptions of continuum mechanics. The fluid velocity is well defined everywhere, including the boundary. Our calculations depart from the classical scheme only when we introduce a degree of randomness into the motion of the interface. This is described in section 2 below.

2. The advance of the interface.

In a continuum approach, we would solve for p , and thus ∇p , everywhere. The entire interface between the two fluids would then be advanced at each time step by a distance proportional to ∇p . Calculations, using either finite elements or finite differences, would be designed to avoid as much as possible the effects of a discrete numerical scheme. However, in a porous medium the fluid flows through a series of discrete pores with random sizes. We therefore include an element of discreteness/randomness by assuming that each point of the interface is either advanced by fixed amount during one time step, or it is not. We assume that advance occurs at a single point, which is chosen with a *probability* proportional to the local pressure gradient. The expected displacement at each time step is therefore proportional to the classical continuum velocity.

Picking just one point at each time step keeps the growth process close to that of DLA, as it corresponds to the arrival of one incoming particle at a time. Simultaneous motion of the interface at several places could either result in the advance of the entire interface by a constant amount (assuming that each point along the boundary is advanced by at most one step at any time step), or could perhaps approach the classical limit if we add a multitude of infinitesimal particles at each time step. Rather than investigate these possibilities, we pick just one point of advance. This corresponds to injection of fluid into the porous

medium at a constant rate. If instead we were to consider a constant pressure drop, then the number of points at which the interface advances each time step would change with time. As fingers with low viscosity approach the low pressure outlet, the pressure gradient at their tips will increase.

We work in 2 dimensions. We consider a porous medium which initially contains fluid with viscosity μ_1 , and inject fluid with viscosity μ_2 from the left-hand side. The porous medium is discretized by a rectangular grid, and at any time step each cell contains either fluid 1 or 2. The interface between the fluids is thus halfway between the grid-points. \mathbf{n} , the normal to the interface, is therefore parallel to one of the co-ordinate axes and the volume flux between points i and $i + 1$, occupied by fluids 2 and 1, becomes

$$Q \propto \frac{p_i - p_{i+1}}{\mu_1 + \mu_2}.$$

Incompressibility requires that the sum of the 4 volume fluxes at any one grid point be zero. This gives the standard 5-point discrete version of Laplace's equation at interior points of each fluid, and a modified version at the interface between the two fluids.

Once the interface has moved to its new position, we must re-solve Laplace's equation for the pressure. Gauss-Seidel over-relaxation was adopted, since we already have a good estimate of the solution from the previous time-step. The number of iterations required depends on both the position at which a new point has been added to the boundary, and on the viscosity ratio $\kappa = \mu_1/\mu_2$. If we advance the boundary at a point where the pressure gradient is low, then the pressure field will be little changed, while if the viscosity ratio is unity, the replacement of fluid 1 by fluid 2 will make no difference whatsoever to the pressure.

Up to six sweeps were made of the 9×9 gridblocks surrounding the new point, followed by relaxation over the entire mesh until the residuals ε_{ij} satisfied

$$\sum_{i,j} \varepsilon_{ij}^2 < \varepsilon.$$

ε was usually 2.56×10^{-4} , and halving this value made no qualitative difference to the results. The number of iterations required varied typically between 1 and 15, and it took approximately one hour to add 2 000 points. The over-relaxation parameters were taken to be 1.4 for the small grid and 1.6 for the entire mesh; the speed could perhaps be increased if more use was made of restricted meshes and of coarse grid scales.

We work on a mesh with dimension 160×160 . The pressure p was chosen to be 1 on the left-hand boundary $x = 0$, the side at which fluid is injected. $p = 0$ at the right hand boundary, and the pressure gradient normal to the two remaining sides is zero. Our aim is to avoid the effect of the side walls as much as possible: however, we have not been able to use

the large meshes (typically 10^3 wide) used by Meakin [10] in his 2-dimensional study of DLA at a line. We start with a straight interface parallel to the y axis, normal to the direction of flow. This was positioned 5 units away from the left-hand edge in order to reduce boundary effects.

3. Results and discussion.

Figure 1 shows typical outlines of the invading fluid after injection of 3 000 points. The first 2 000 points are coloured black. Each additional block of 320 points is then coloured grey, white, black, grey... Figure 1a corresponds to a viscosity ratio $\kappa = 10^4$. It is similar to results of DLA, though the fingers are perhaps somewhat thicker and closed loops are more frequent. Because of these differences, the calculations were repeated with more accurate solutions of Laplace's equation (ε taken to be one half its usual value). The results had the same general form. Note that the model implemented here has a tendency to create solid blocks of injected fluid. A point surrounded on three sides by injected fluid can be invaded from any one of three directions, and is thus counted three times when choosing the point at which the interface advances.

As we reduce the viscosity ratio, so the fingers become thicker and fewer in number. Eventually, at a viscosity ratio of unity ($\kappa = 1.01$ in Fig. 1f) there are no longer any fingers. The pressure gradient is uniform, since the two fluids have the same viscosity, and in particular, it is constant everywhere along the interface. At each time step the front advances at a point chosen at random; for our 160×160 grid, the probability of any one point being chosen is $\beta = 1/160$. Thus, for a given value of y , the x co-ordinate of the front will have a binomial distribution, with mean $\bar{x} = \beta t$, variance $\overline{(x - \bar{x})^2} = \beta t(1 - \beta)$. (This approximation assumes independence of the various y values, neglecting the fact that the total number of particles is equal to the number of time steps t .) Note that this differs from the modified DLA model [10] in which particles are fired in straight lines, rather than allowed to perform a random walk. Such particles can stick when they slide past part of the agglomerate. Here, since the pressure gradient in the y direction is zero, no sideways growth is possible.

Finally, in figure 1g, the injected fluid is ten times more viscous than the fluid it displaces. The front is stable.

It is impossible, in DLA, for diffusing particles to reach the interior of regions completely surrounded by invading fluid. In our work such growth should not occur if the invading fluid is completely inviscid, and if Laplace's equation is solved sufficiently accurately. Even at lower viscosity ratios, volume conservation should prevent the attrition of an island of the original fluid surrounded by injected fluid. For computational convenience, we ignore this problem and allow attrition — a simplification which has

similarly been found useful in studies of invasion percolation [11]. Thus our « residual oil saturation » (the fraction of fluid 1 remaining in the rock) is ultimately zero, though at high viscosity ratios this would take a long time to achieve. We have made no attempt to include interfacial tension in this work. Experiments in porous media [12] do indeed show that as the capillary number increases (i.e. as the capillary forces become small compared with viscous forces) the residual oil saturation is reduced, though never to a value as low as zero. Because of this unrealistic feature, and because of the 2-dimensional nature of the calculations, we have not attempted to interpret the density profiles shown on figure 1 in terms of Buckley-Leverett saturation profiles [13]. A more careful study would also need to consider whether islands of fluid 1 should be swept along by the injected fluid. The motion of such « oil ganglia » has been reviewed by Payatakes [14].

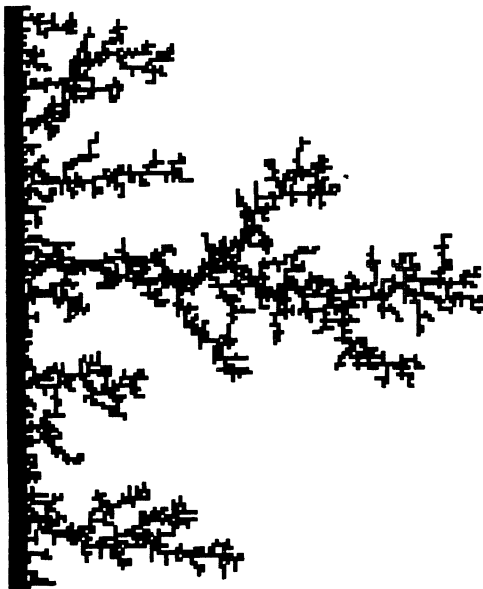
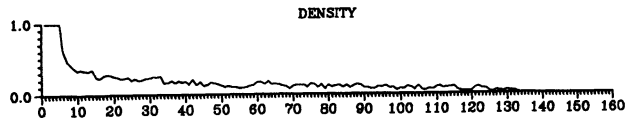
At each time step we record the x co-ordinate (the co-ordinate in the direction of flow) of the point which is added to the area of invading fluid. These co-ordinates are plotted, as a function of time, in figure 2, for a run with viscosity ratio $\kappa = 10^4$. Growth is concentrated at the tips of the fingers, and the scale of figure 2 is such that the plotted points appear to overlap when growth is rapid. The slope of the upper boundary of the plot represents the rate of growth of the fastest growing fingers. Other dense concentrations of points, with lower slopes, represent slower growing fingers and eventually disappear. The scattered, isolated points in the bottom right-hand corner of figure 2 indicate that the interface between the two fluids continues to advance, albeit slowly, at points far behind the finger tips. Although growth at any one such boundary point is unlikely, because of the low pressure gradients, these points are so numerous that a few of them do indeed grow.

Meakin [10] computed the root mean square thickness s_{rms} of his DLA aggregates. A log-log plot indicates that s_{rms} increases asymptotically as t^α . Values for the exponent α , measured over the last 500 time steps and averaged over 5 runs, are given in table I. The convergence of α is slow, and the result for a viscosity ratio $\kappa = 100$ suggests that our values for α have not in fact converged, presumably because of our relatively small grid size. Thus we cannot comment on the difference between our value $\alpha = 1.2$ for high viscosity ratios, and Meakin's value $\alpha = 1.3$. In general, we improve our estimate of α by looking at only the last 500 time steps. However, this represents only 3 complete rows of 160 gridblocks, and the result for the stable case ($\kappa = 0.1$) is poor.

The form of the injected fluid at high viscosity ratios is qualitatively similar to results obtained with DLA, suggesting that the area invaded may be described by a fractal dimension. At lower viscosity ratios, however, there is only a small zone of fingering, and it is this zone which we would like to characterize.

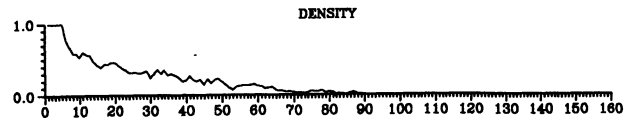
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 INITIAL VISCOSITY=10000.001
 INITIAL SEED=4035
 NUMBER OF STEPS=3000
 NUMBER OF POINTS PER STEP=1

NI=160
 NJ=160



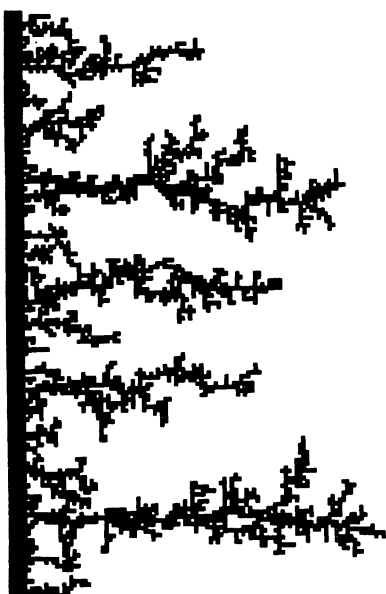
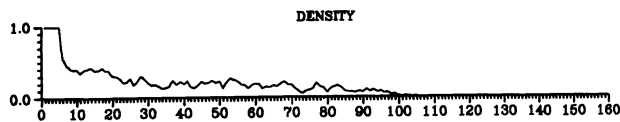
c) INJECTED VISCOSITY=1.000
 INITIAL VISCOSITY=100.000
 INITIAL SEED=4035
 NUMBER OF STEPS=3000
 NUMBER OF POINTS PER STEP=1

NI=160
 NJ=160



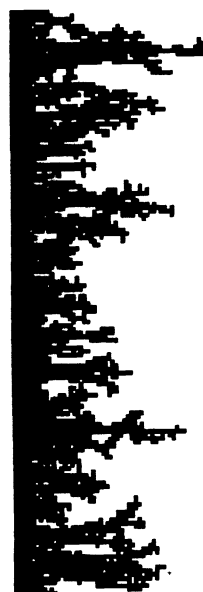
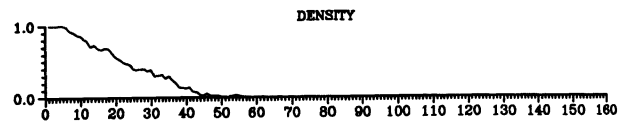
b) INJECTED VISCOSITY=1.000
 INITIAL VISCOSITY=1000.000
 INITIAL SEED=4035
 NUMBER OF STEPS=3000
 NUMBER OF POINTS PER STEP=1

NI=160
 NJ=160



d) INJECTED VISCOSITY=1.000
 INITIAL VISCOSITY=10.000
 INITIAL SEED=4037
 NUMBER OF STEPS=3000
 NUMBER OF POINTS PER STEP=1

NI=160
 NJ=160

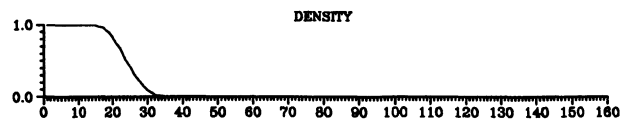
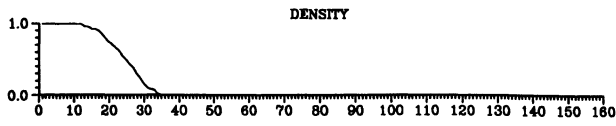


c) INJECTED VISCOSITY=1.000
 INITIAL VISCOSITY=2.000
 INITIAL SEED=4035
 NUMBER OF STEPS=3000
 NUMBER OF POINTS PER STEP=1

NI=160
 NJ=160

f) INJECTED VISCOSITY=1.000
 INITIAL VISCOSITY=1.010
 INITIAL SEED=4035
 NUMBER OF STEPS=3000
 NUMBER OF POINTS PER STEP=1

NI=160
 NJ=160



g) INJECTED VISCOSITY=1.000
 INITIAL VISCOSITY=0.100
 INITIAL SEED=4035
 NUMBER OF STEPS=3000
 NUMBER OF POINTS PER STEP=1

NI=160
 NJ=160

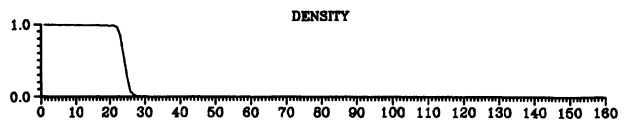


Fig. 1. — Typical examples of the shape of the growing zone of injected fluid after the addition of 3 000 points, for various values of the viscosity ratio $\kappa =$ (a) 10^4 ; (b) 10^3 ; (c) 10^2 ; (d) 10; (e) 2; (f) 1.01; (g) 0.1. The grid size is 160×160 . Above each figure is a plot of the fraction of sites occupied by the injected fluid, as a function of x .

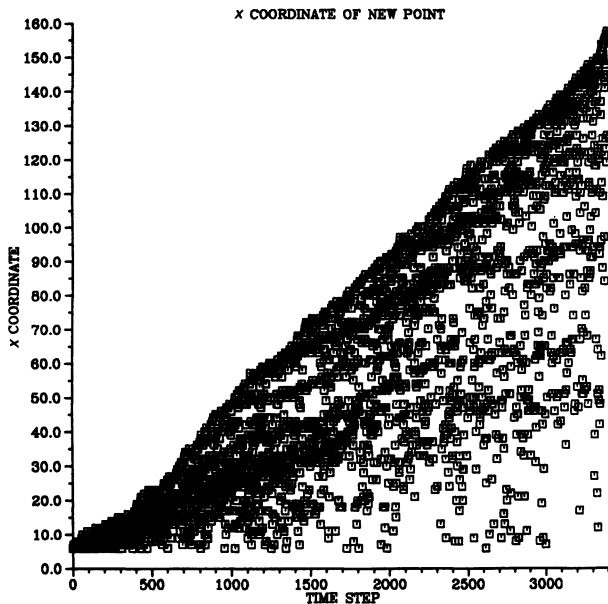


Fig. 2. — The x coordinate of each point as a function of the time at which it is added. $\kappa = 10^4$.

Table I. — The rms thickness exponent α for various viscosity ratios κ . α is determined from the last 500 time steps of each run, and the mean value over 5 simulations is quoted.

Viscosity ratio κ	rms thickness exponent α
—	—
0.10	1.10 ± 0.03
1.01	1.02 ± 0.05
2.0	1.00 ± 0.05
10.0	0.91 ± 0.04
10^2	1.06 ± 0.08
10^3	1.23 ± 0.08
10^4	1.19 ± 0.10

We therefore examine the principal interface between the fluids, ignoring the boundaries of any islands surrounded by the invading fluid.

If the boundary is a fractal with dimension D , then, measuring it with a yardstick of length r , we would expect to obtain a length which varies as r^{1-D} . For the high viscosity ratios, this is indeed the case, and, considering $\kappa = 10^3$ and 10^4 together, we obtain $D = 1.48 \pm 0.02$, where the error is the standard deviation based on 5 runs at each viscosity ratio, with D computed after the addition of 3000 points. At lower viscosity ratios this approach does not appear to be useful, since the apparent value of D decreases during the course of the simulation. Results are shown on figure 3 for the case $\kappa = 1.01$, together with, for comparison, $\kappa = 10^4$ and 10^3 . The long straight fingers of figure 1f become even longer as

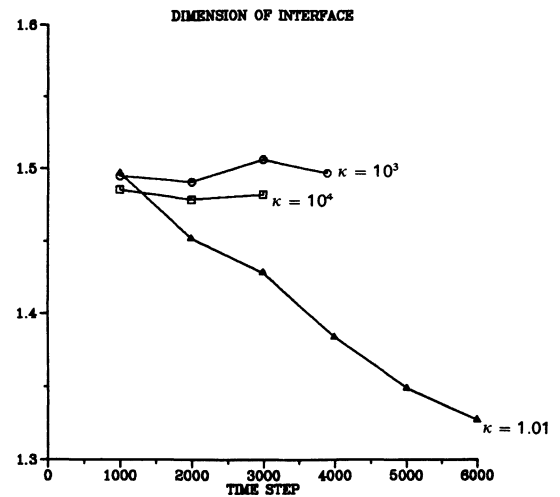


Fig. 3. — The Hausdorff dimension of the principal interface between the two fluids (ignoring closed loops), for three values of the viscosity ratio κ . The curves are obtained from averages of 5 different runs.

t increases, and the length of the interface varies less rapidly as we change the length of the yardstick. The corresponding development of the fingers in figures 1d and 1f is shown in figure 4. Measurements of the density correlation would similarly be time dependent. The mass of invading fluid increases linearly with time, but, when κ is unity, the size of the mixing zone increases only as \sqrt{t} , as discussed above. The decline in the value of D is less rapid at higher values of κ , but is still noticeable even when $\kappa = 100$. In the stable case, $\kappa = 0.1$, the length of the boundary quickly settles to approximately 160 once $r \geq 3$ (in units such that the distance between gridpoints is 1).

As an alternative, we may examine the curvilinear length of the interface (measured on the smallest scale, $r = 1$), again taking only the principal boundary and ignoring closed loops. This restriction occasionally causes fluctuations in the length, when fingers join and enclose a large island of fluid. Results are shown on figure 5. We see that, with the possible exception of the highest viscosity ratios, the length of the boundary increases more slowly than linearly with time (and a log-log plot does not give convincing straight lines). Growth is markedly slower at low viscosity ratios, and in the stable case ($\kappa = 0.1$) the length of the interface is approximately constant. The numerical computations of Tryggvason and Aref [15] predict linear growth, though their lengths include double-sided surfaces between fingers which have merged. The linear rate of penetration of our finger-tips, as depicted by the slope of figure 2, agrees with both [15] and with the long-time development of miscible fingers studied experimentally by Wooding [16].

The relevance of Laplace's equation to homogeneous flow in porous media is well known. When studying multi-fluid flow, however, an extension of the form

a) INJECTED VISCOSITY=1.000
INITIAL VISCOSITY=10.000
INITIAL SEED=4037
NUMBER OF STEPS=6000
NUMBER OF POINTS PER STEP=1

NI=160
NJ=160

b) INJECTED VISCOSITY=1.000
INITIAL VISCOSITY=1.010
INITIAL SEED=4036
NUMBER OF STEPS=6000
NUMBER OF POINTS PER STEP=1

NI=160
NJ=160

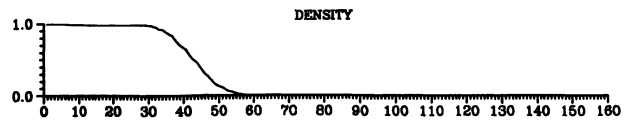
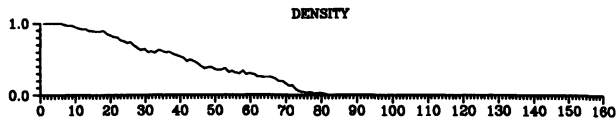


Fig. 4. — Figures 1d and 1f continued by the addition of a further 3 000 points, making 6 000 in all. (a) $\kappa = 10$ (cf. Fig. 1d); (b) $\kappa = 1.01$ (cf. Fig. 1f).

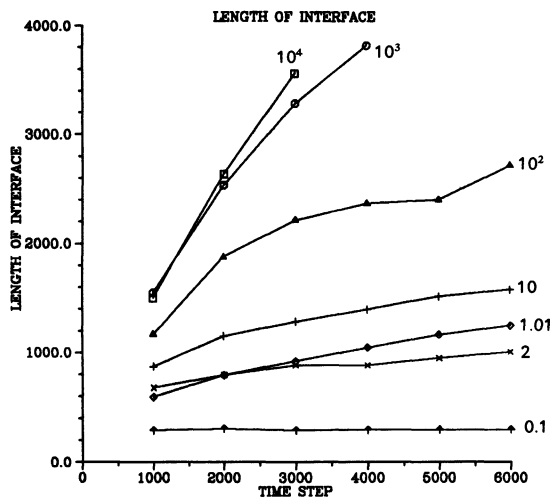


Fig. 5. — The length of the principal interface (neglecting closed loops), for various values of the viscosity ratio κ , as a function of time.

examined here is speculative and can be confirmed only by experiments. At present we can do no better than to draw the reader's attention to the qualitative similarities between figures 1, 4 and the experimental results reported in [17, 18, 13]. These results were obtained in packed bead beds, using both miscible and immiscible fluids. Other experiments in etched glass networks [19] similarly show the effect of viscosity ratio on the stability of the interface.

Acknowledgments.

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