COMMUTATIVE LOCAL RINGS OF BOUNDED MODULE TYPE

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Abstract. Let R be a local ring of bounded module type. It is shown that R is an almost maximal valuation ring if there exists a non-maximal prime ideal J such that R/J is an almost maximal valuation domain. We deduce from this that R is almost maximal if one of the following conditions is satisfied: R is a $\mathbb{Q}$-algebra of Krull dimension ≤ 1 or the maximal ideal of R is the union of all non-maximal prime ideals.

In this paper, R is an associative and commutative ring with identity. We will say that R has bounded module type if, for some positive integer n, every finitely generated R-module is a direct sum of submodules generated by at most n elements. The problem of investigating commutative rings of bounded module type has been studied by R.B. Warfield [9], R. Wiegand [10], B. Midgarden and S. Wiegand [6], P. Zanardo [11], P. Vámos [8] and the author [1]. In [9], R.B. Warfield proved that every local ring of bounded module type is a valuation ring. By theorems due to D.T. Gill [4] and J.P. Lafon [5], a valuation ring is almost maximal if and only if every finitely generated module is a direct sum of cyclics. So, the following question can be proposed :

Is a local ring R of bounded module type if and only if R is an almost maximal valuation ring ?

Positive answers are given by P. Zanardo in [11] for the class of totally branched and discrete valuation domains, by P. Vámos in [8] for $\mathbb{Q}$-algebra valuation domains and in [1] by the author for valuation rings with a finitely generated maximal ideal.

In this paper, we prove that if R is a valuation ring of bounded module type, then R/I is complete in its ideal topology for every nonzero and nonarchimedean ideal I, and R_J is almost maximal for every nonmaximal prime ideal J. To obtain these results, as in [1], we adapt to the nondomain case Zanardo's methods used in [11]. Moreover, we extend results obtained by P. Vámos in [8] : every valuation ring R of bounded module type is almost henselian, and if R is a $\mathbb{Q}$-algebra with Krull dimension greater than one, then R is almost maximal. Finally we obtain also a positive answer for valuation rings such that the maximal ideal is the union of all nonmaximal prime ideals.

For definitions and general facts about valuation rings and their modules we refer to the book by Fuchs and Salce [2]. The symbol $A \subset B$ denotes that A is a subset of B, possibly $A = B$. We recall some definitions and results which will be used in the sequel. An R-module M is called **fp-injective** if and only if $\mathrm{Ext}^1(F, M) = 0$ for every finitely presented R-module F. A **self fp-injective ring** R is fp-injective as R-module. For an R-module M and an ideal I of R, we set $M[I] = \{x \in M \mid I \subset (0 : x)\}$. If M is fp-injective then $rM = M[(0 : r)]$ for every $r \in R$. It is obvious that $rM \subset M[(0 : r)]$. Now let $x \in M[(0 : r)]$. It follows that we can define an homomorphism $f : Rr \to M$ by $f(tr) = tx$. Since M is fp-injective, f can be extended to R hence we get $x = f(r) = rf(1)$. Conversely if $rM = M[(0 : r)]$, then every homomorphism $f : Rr \to M$ can be extended to R, hence $\mathrm{Ext}^1(R/rR, M) = 0$. We recall that R is a **valuation ring** if and only if its ideals are totally ordered by inclusion.

Proposition 1. *Let R be a valuation ring and P its maximal ideal. Then :*

(1) *R is self fp-injective if and only if each nonunit of R is a zero-divisor.*
In this case, if I and J are ideals of R the following assertions hold :
(2) *If $(0 : P) \neq 0$ then $I = \big(0 : (0 : I)\big)$ and if $I \subsetneq J$ we have $(0 : J) \subsetneq (0 : I)$.*
(3) *If $(0 : P) = 0$ then $I \subsetneq \big(0 : (0 : I)\big)$ (respectively $I \subsetneq J$ and $(0 : J) = (0 : I)$) if and only if there exists $r \in R$ such that $I = Pr$ and $\big(0 : (0 : I)\big) = Rr$ (respectively $J = Rr$).*
(4) *If $I \subsetneq J$ then $\big(0 : (0 : I)\big) \subset J$.*

Proof. From [3, Lemma 3] it follows that every nonunit of R is a zero-divisor if and only if $Rr = \big(0 : (0 : r)\big)$ for every $r \in R$. Since R is a valuation ring then every finitely presented module is a direct sum of cyclic modules ([9, Theorem 1] or [5, Proposition II.2]). Hence R is self fp-injective if and only if $\mathrm{Ext}^1(R/r, R) = 0$ for every $r \in R$. From above we deduce that this last condition is equivalent to $\big(0 : (0 : r)\big) = Rr$ for every $r \in R$. The assertions 2. 3. and 4. follow from of [4, Proposition 1.3]. □

From now on, R will denote a valuation ring, P its maximal ideal and E the R-injective hull of R.

Lemma 2. *Let I be a nonzero proper ideal of R and $p \in P$. Then :*

(1) *$pI = I$ if and only if $(I : p) = I$.*
(2) *$\forall r \in P$, $pI \neq I$ and $rI \neq 0$ implies $prI \neq rI$.*
(3) *If $pI = I$ then :*
   i) *$\forall n \in \mathbb{N}$ $p^nI = I$ and $(I : p^n) = I$*
   ii) *$\forall a \in I$, $p(Ra : I) = (Ra : I)$*
   iii) *$\forall a \in I$, if $p^m a \neq 0$ then $(Ra : I) = (Rap^m : I)$.*

Proof.
1) Suppose $pI = I$. Let $r \in (I : p)$. If $rp = 0$, from $pI \neq 0$ it follows that $(0 : p) \subset I$ and $r \in I$. If $rp \neq 0$ then $rp = tp$ for $t \in I$ whence we get $Rt = Rr$ and $r \in I$.
Suppose $(I : p) = I$. Then $p \notin I$ whence for each $r \in I$ $\exists t \in P$ such that $r = pt$. We have $t \in (I : p) = I$.
2) If $prI = rI$ then $\forall a \in I$, $\exists b \in I$ such that $rpb = ra$. If $ra = 0$, we have $a \in (0 : r) \subset pI$ since $rI \neq 0$. If $ra \neq 0$, we obtain that $Ra = Rpb \subset pI$.
3) For iii) see the proof of [1, Lemma 2.3]. ii) Let $r \in \big((Ra : I) : p\big)$. Then $prI \subset Ra$ and $rI \subset Ra$. Hence $\big((Ra : I) : p\big) = (Ra : I)$ and by using 1) we have $p(Ra : I) = (Ra : I)$. □

If I is an ideal of R, by using Lemma 2, we follow [2] by calling I **archimedean ideal** if $(I : p) \neq I$, for every $p \in P$. Then I is archimedean if and only if R/I is a self fp-injective ring (Proposition 1).

R is called **maximal** if and only if every totally ordered family $\mathcal{F}$ of cosets $\{r_\lambda + I_\lambda \mid \lambda \in \Lambda\}$ has a nonempty intersection, and R is called almost maximal if the above condition holds whenever $I = \bigcap_{\lambda\in\Lambda} I_\lambda \neq 0$. When $I = Pr$ for some $r \in R$, then there exists $\lambda \in \Lambda$ such that $I = I_\lambda$ (else $r \in \bigcap_{\lambda\in\Lambda} I_\lambda$). In this case we deduce that $\mathcal{F}$ has a nonempty intersection.

The **ideal topology** of R is the linear topology where the family $\mathcal{J}$ of all nonzero ideals of R is a base of neighborhoods about 0. We said that R is complete in its ideal topology if and only if the canonical homomorphism $\phi : R \to \varprojlim_{I\in\mathcal{J}} R/I$ is an isomorphism. If $\mathcal{F} = \{I_\lambda \mid \lambda \in \Delta\}$ is a family of nonzero ideals such that $\bigcap_{\lambda\in\Lambda} I_\lambda = 0$, then it is easy to verify that $\mathcal{F}$ is also a base of neighborhoods about 0 in the ideal topology. Consequently R is complete in its ideal topology if and only if every totally ordered family of cosets $\{r_\lambda + I_\lambda \mid \lambda \in \Lambda\}$, with $\bigcap_{\lambda\in\Lambda} I_\lambda = 0$, has a nonempty intersection. Thus R is maximal (respectively almost maximal) if and only if R/I is complete in its ideal topology for each proper ideal (respectively nonzero proper ideal) I, such that $I \neq Pr$ for any $r \in R$.

Let now e be in E not in R. As in [7] the breadth ideal $B(e)$ of e is defined as follows : $B(e) = \{r \in R \mid e \notin R + rE\}$. The referee suggested me the following proposition that is similar to [7, Proposition 1.4].

Proposition 3. *Suppose R is self fp-injective and let I be a proper ideal of R, such that $I \neq Pr$ for every $r \in R$. Then R/I is not complete in its ideal topology if and only if there exists $e \in E \setminus R$ such that $I = B(e)$. Moreover we have the following properties :*

(1) $I = B(e) = (0 : (R : e))$,
(2) $(R : e) = P(0 : I)$ *and* $(R : e)$ *is not finitely generated,*
(3) *and e can be chosen such that* $(0 : e) = 0$.

Proof. Since R/I is not complete in its ideal topology there exists a totally ordered family of cosets $\{r_\lambda + I_\lambda \mid \lambda \in \Lambda\}$ with empty intersection such that $I = \bigcap_{\lambda\in\Lambda} I_\lambda$. Let $J = \bigcup_{\lambda\in\Lambda} (0 : I_\lambda)$. We define $f : J \to R$ by $f(c) = r_\lambda c$ for each c in $(0 : I_\lambda)$. If f can be extended to an endomorphism of R, then, by the same proof (the beginning of the proof) as in [4, Theorem 2.3] we obtain that $\bigcap_{\lambda\in\Lambda} r_\lambda + I_\lambda \neq \phi$. Thus f cannot be extended to R, but there exists $e \in E \setminus R$ such that $f(c) = ce$ for each c in J. It is obvious that $J \subset (R : e)$. Let $r \in (R : e)$. We consider the homomorphism $g : Rr \to R$ defined by $g(r) = re$. Since R is self fp-injective, g can be extended to R, hence there exists $u \in R$ such that $g(r) = re = ru$. If $J \subset Rr$, then g is an extension of f and we obtain a contradiction. Thus $Rr \subsetneq J$, $J = (R : e)$ and $(R : e)$ is not finitely generated. Since $I \subsetneq I_\lambda$ for each $\lambda \in \Lambda$, then $J \subset (0 : I)$. If $(0 : I)$ is not finitely generated then $J \subset (0 : I) = P(0 : I)$. If $(0 : I)$ is finitely generated, then $J \subsetneq (0 : I)$ hence $J \subset P(0 : I)$. Now the following equality and inclusions hold : $I = (0 : (0 : I)) \subset (0 : J) \subset \bigcap_{\lambda\in\Lambda}(0 : (0 : I_\lambda))$. Let

$r \in R \setminus I$. Then there exists $\lambda \in \Lambda$ such that $r \notin I_\lambda$. If $I_\mu \subsetneq I_\lambda$ for some $\mu \in \Lambda$, then $(0 : (0 : I_\mu)) \subset I_\lambda$. We get that $r \notin \bigcap_{\alpha \in \Lambda} (0 : (0 : I_\alpha))$ and the equalities $I = (0 : J) = (0 : (R : e))$. We deduce that $(0 : (0 : J)) = (0 : I)$. When $J \neq Pr$ for any $r \in R$, we have $J = (0 : I) = P(0 : I)$ since J is not finitely generated. If $J = Pr$ for some $r \in R$, we get also $J = P(0 : I)$.

Conversely let $e \in E \setminus R$ and $f : (R : e) \to R$ be the homomorphism defined by $f(r) = re$. If f can be extended to $g : R \to R$, then we obtain that $(R : e) \subset (0 : e - g(1))$. But $(R : e - g(1)) = (R : e)$ and since E is an essential extension of R, there exists $r \in (R : e)$ such that $r(e - g(1)) \neq 0$. Thus we get a contradiction. Hence f cannot be extended to R and $(R : e)$ is not finitely generated. Let $\{c_\lambda \mid \lambda \in \Lambda\}$ be a set of generators of $(R : e)$. Since R is self fp-injective the restriction of f to Rc_λ can be extended to R. Then there exists $r_\lambda \in R$ such that $f(c_\lambda) = r_\lambda c_\lambda$. Let $I_\lambda = (0 : c_\lambda)$. If $c_\lambda \in Rc_\mu$, then $(r_\lambda - r_\mu)c_\lambda = 0$ hence $r_\mu \in r_\lambda + I_\lambda$. Assume that $\bigcap_{\lambda \in \Lambda} (r_\lambda + I_\lambda) \neq \phi$. By the same proof (the end of the proof) as in [4, Theorem 2.3] we deduce that f can be extended to R, hence we get a contradiction. Let $I = \bigcap_{\lambda \in \Lambda} I_\lambda$. Thus R/I is not complete in its ideal topology. The inclusion $\big(0 : (R : e)\big) \subset I$ is obvious. Let $r \notin \big(0 : (R : e)\big)$. Then there exists $\lambda \in \Lambda$ such that $rc_\lambda \neq 0$. Consequently $r \notin (0 : c_\lambda)$ and we get $\big(0 : (R : e)\big) = I$.

To complete the proof, we must prove that $B(e) = \big(0 : (R : e)\big)$. Let $r \in R \setminus B(e)$. Then there exist $u \in R$ and $x \in E$ such that $e = u + rx$. For each $t \in (0 : r)$ we have $te = tu$. If $(R : e) \subset (0 : r)$ then the homomorphism $f : (R : e) \to R$ defined by $f(t) = te$ can be extended to R. It is not possible. Thus $(0 : r) \subsetneq (R : e)$. Since $(R : e)$ is not finitely generated we get $\big(0 : (R : e)\big) \subsetneq Rr$. Conversely let $r \in R \setminus \big(0 : (R : e)\big)$. If $(0 : P) \neq 0$ then $(0 : r) \subsetneq (R : e)$. If $(0 : P) = 0$ denote $I = \big(0 : (R : e)\big)$. Since $I \neq Pr$, we deduce that $(0 : r) \subsetneq (0 : I)$. If $(0 : I)$ is not finitely generated then $(0 : r) \subsetneq P(0 : I) = (R : e)$. If $(0 : I) = Rt$ for some $t \in R$, then $(R : e) = Pt$. We have not $(0 : r) = Pt$, else $rt \in (0 : P)$ and $rt \neq 0$. Hence $(0 : r) \subsetneq (R : e)$. Let $c \in (R : e) \setminus (0 : r)$. Then the restriction to Rc of the homomorphism f defined above can be extended to R since R is self fp-injective. Thus there exists $u \in R$ such that $tu = te$ for each $t \in (0 : r)$. Since E is injective, then $E[(0 : r)] = rE$ hence there exists $x \in E$ such that $e - u = rx$. We obtain that $r \notin B(e)$. As in [1, proposition 1.3 ii)] we have $(0 : e) = 0$ or $\big(0 : (1 - e)\big) = 0$. Obviously $(R : e) = (R : 1 - e)$ and $B(1 - e) = B(e)$. □

Recall that an R-module M has **Goldie dimension** n if the injective hull $E(M)$ is a direct sum of n indecomposable injective modules. We denote $g(M)$ the Goldie dimension of M and $\mu(M)$ the minimal number of generators of M.

Proposition 4. *Let R be a valuation ring and I a nonarchimedean and nonzero ideal of R. If R/I is not complete in its ideal topology then, for every $n \in \mathbb{N}^*$, there exists an indecomposable R-module M with $\mu(M) = n + 1$ and $g(M) = n$.*

The following two lemmas are needed for the proof of this proposition.

Lemma 5. *Suppose R is self fp-injective. Let I be a nonzero ideal of R, $x \in E\setminus R$ and $a \in R \setminus I$. If $ax \in IE$ then $x \in (I : a)E$.*

Proof. Suppose $ax = 0$. Let $b \in I$, $b \neq 0$. Then there exists $d \in (I : a)$ such that $b = ad$. Since $b \neq 0$ then $(0 : d) \subset Ra \subset (0 : x)$. Thus there exists $y \in E$ such that $x = dy$.

If $ax \neq 0$, there exist c$\in I$ and $y \in E$ such that $ax = cy$. Since $a \notin I$, there exists $d \in (I : a)$ such that $c = ad$. Since $ax \neq 0$, we have $(0 : d) \subset Ra \subset (0 : x - dy)$. Thus there exists $z \in E$ such that $x - dy = dz$.

Hence $x \in dE \subset (I : a)E$. □

Lemma 6. *Assume R is self fp-injective and $(0 : P) \neq 0$. Let $e \in E \setminus R$, $(0 : e) = 0$, $I = B(e)$, $a \in I$, $a \neq 0$ and $J = (Ra : I)$. We suppose that I is a nonarchimedean ideal. Let $p \in P$ such that $pI = I$. Then :*

i) *$\forall b \in J$, $\exists e_b \in R \setminus P$ with $(e - e_b) \in (Ra : b)E$.*

ii) *$IE = \bigcap_{b\in J} (Ra : b)E$.*

iii) *Let $c, d \in R$ with $c + de \in IE$. Then $c \in Rp^m$ and $d \in Rp^m$ for every $m \in \mathbb{N}^*$.*

Proof. For i) and ii) see proof of [1, Lemma 2.4].

iii) $pI = I \Rightarrow p \notin I$. If $c \notin I$ or $d \notin I$ we show as in [1, Lemma 2.4. iii)] that $Rc = Rd$. There exists $v \in R\setminus P$ such that $d = vc$. Now suppose $\exists k \in \mathbb{N}^*$ such that $c \notin Rp^k$. So we have $p^k = uc$ for some $u \in P$, and $u(c+de) = uc(1+ve) = p^k(1+ve) \in IE$. By Lemma 5 $(1+ve) \in (I : p^k)E = IE$. We deduce that $e \in R + IE$. By Proposition 3 we obtain a contradiction. □

Proof of proposition 4. Let $r \in I$, $r \neq 0$. We replace R by R/rP and assume that R is self fp-injective and $(0 : P) \neq 0$. By Proposition 3, there exists $e \in E \setminus R$ such that $(0 : e) = 0$ and $(R : e) = P(0 : I)$. Since I is nonarchimedean, there exists $p \in P$ such that $pI = I$. From Lemma 2, we deduce that $p(0 : I) = (0 : I) = P(0 : I) = (R : e)$. Let us fix $n \in \mathbb{N}^*$. Since $p^{2(n-1)}I = I$, there exists $a \in I$ such that $p^{2(n-1)}a \neq 0$. For every k, $1 \leq k \leq n$, we define $A_k = Rap^{2(k-1)}$. Let us now define n elements of E not in R in the following way : $e_1 = e$ and for every k, $2 \leq k \leq n$, $e_k = 1 + p^{k-1}e$. Then we have $(R : e_k) = (R : p^{k-1}e) = \big((R : e) : p^{k-1}\big)$. By Lemma 2(1 and 3ii) $(R : e_k) = (R : e)$ and $B(e_k) = I$ for every k, $1 \leq k \leq n$. Let $J = (Ra : I)$. Then by lemma 2, $J = (A_k : I), \forall k, 1 \leq k \leq n$.

By using Lemma 6, for every integer k, $1 \leq k \leq n$, there exists a family $\{e_k^b \mid b \in J\}$ of units of R, such that $(e_k - e_k^b) \in (A_k : b)E$.

Now we define an R-module M with $\{x_0, x_1, \dots, x_n\}$ as spanning set and with the following relations :

– $(0 : x_k) = A_k$ for every k, $1 \leq k \leq n$

– $(0 : x_0) = A_n$

– $\forall b \in J$, $bx_0 = b\left(\sum_{k=1}^{n} e_k^b x_k\right)$.

We prove that M is indecomposable, with $\mu(M) = n+1$ and $g(M) = n$, as in [1, Proposition 2.6], where [1, Lemma 2.4] is replaced by Lemma 6. □

As P. Vámos in [8], we call a local ring R **almost henselian** if every proper factor ring is a henselian ring.

Theorem 7. *Let R be a local ring of bounded module type. Then we have the following :*

(1) *R is a valuation ring.*
(2) a) *For every nonzero and nonarchimedean ideal I of R, R/I is complete in its ideal topology.*
  b) *If 0 is not prime and nonarchimedean then R is also complete.*
(3) *R is an almost henselian ring.*
(4) *For every nonmaximal prime ideal J, R_J is almost maximal.*

Proof. 1. See [9, Theorem 2].

2.a) is a consequence of proposition 4.

2.b) Let $(I_\alpha)_{\alpha\in\Lambda}$ be a family of ideals such that $\bigcap_{\alpha\in\Lambda} I_\alpha = 0$. If $t \notin I_\alpha \quad \forall\alpha\in\Lambda$, then we have $(0:t) = \bigcap_{\alpha\in\Lambda}(I_\alpha : t)$ and we can easily prove that 0 nonarchimedean implies $(0:t)$ nonarchimedean. By the same proof as in [3, Proposition 1] () we obtain that R is complete if R is not a domain.

3. Let J be the nilradical of R. From [8, Theorem 8] we deduce that R/J is almost henselian. If $J \neq 0$, then R/J is complete. Hence R/J is henselian and since J is a nilideal, we deduce that R is henselian.

4. Let I be a nonzero ideal of $R_J, I \subset JR_J$, $\phi : R \to R_J$ the canonical homomorphism and $I^c = \phi^{-1}(I)$. If $s \in P\setminus J$ and $r \in I^c$, then there exists $t \in P$ such that $r = st$ and $\frac{t}{1} = \frac{r}{s} \in I$. We deduce that $sI^c = I^c$ hence I^c is a nonarchimedean ideal of R. Since R/I^c is complete in its ideal topology, by the same proof as in [3, Lemma 2] we prove that R_J/I is also complete. Hence R_J is almost maximal. □

Proposition 8. *Let R be a valuation ring. Suppose there exists a nonmaximal prime ideal J such that R/J is almost maximal. Then, for every archimedean ideal I, R/I is complete in its ideal topology.*

Proof. Suppose there exists an archimedean ideal I, $I \neq Pr$ for each $r \in R$, such that R/I is not complete in its ideal topology. If $I \neq 0$ we can replace R by R/I and assume that $I = 0$, R self fp-injective and $(0:P) = 0$. By proposition 3 there exists $e \in E\setminus R$ such that $B(e) = 0$. Let J be a nonmaximal prime ideal and $t \in P\setminus J$. Then $0 \neq (0:t) \subset J$. Let $s \in (0:t)$, $s \neq 0$. Since $B(e) = 0$ there exist $u \in R$ and $x \in E$ such that $e = u + sx$. We deduce easily that $(R:x) = s(R:e) = sP$ and $B(x) = (0:sP) = (0:s)$. By proposition 3 $R/B(x)$ is not complete in its ideal topology and $J \subsetneq Rt \subset B(x)$. We obtain that R/J is not almost maximal. □

From this proposition and Theorem 7 we deduce the following corollary :

Corollary 9. *Let R be a local ring of bounded module type. Suppose there exists a nonmaximal prime ideal J such that R/J is almost maximal.*
Then R is an almost maximal valuation ring.

Now we can give a positive answer to our question for some classes of valuation rings.

Theorem 10. *Let R be a local ring of bounded module type. Suppose that one of the three following conditions is verified :*

(1) *P is finitely generated.*
(2) *R is a $\mathbb{Q}$-algebra with Krull dimension ≥ 1.*
(3) *P is the union of all nonmaximal prime ideals of R.*

Then R is an almost maximal valuation ring.

Proof. 1) It is the main result of [1]. We can deduce easily this result from Theorem 7 and Corollary 9 because, if R is not artinian, then $J = \bigcap_{n\in\mathbb{N}} P^n$ is a prime ideal and R/J is a discrete rank one valuation domain. Consequently R/J is almost maximal.

2) Let J be a nonmaximal prime ideal of R. By [8, Theorem 8] R/J is almost maximal.

3) Let J be a nonmaximal and nonzero prime ideal. We replace R by R/J and assume that R_Q is maximal for every nonmaximal prime ideal Q. Let$K$ be the field of fractions of R, and $X = \mathrm{Spec}R \setminus \{P\}$. If $x \in K \setminus R$ then $x = \frac{1}{s}$ where $s \in P$. Since $P = \bigcup_{Q\in X} Q$, $\exists Q \in X$ such that $s \in Q$. We deduce that $x \notin R_Q$ and $R = \bigcap_{Q\in X} R_Q$. By [12, proposition 4] R is linearly compact in the inverse limit topology. Since R is a Hansdorff space in this linear topology then every nonzero ideal is open and also closed. Hence R is linearly compact in the discrete topology. □

Remark 11. Let $J = \bigcup_{Q\in X} Q$. If $J \neq P$, then R/J is an archimedean valuation domain. From corollary 9, if R is of bounded module type, then R is almost maximal if and only if R/J is almost maximal. So, to give a definitive answer to our question, we must solve this problem for archimedean valuation rings.

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