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The epimorphisms of the category Haus are exactly the image-dense morphisms

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1 Introduction

In this document, we define the category **Haus** as the category whose objects are Hausdorff topological spaces and morphisms are the continuous maps. The category **Top** is the category whose objects are the topological spaces and the morphisms are the continuous maps. The definition of epimorphisms and the details about category theory can be found in [1].

2 Image-dense morphisms of Haus are epimorphisms

Proposition 1. *Let A and B be two Hausdorff topological spaces and let $A \xrightarrow{f} B$ be a continuous map. If the closure $\overline{f(A)}$ of $f(A)$ is such $B = \overline{f(A)}$, then f is an epimorphism.*

Proof. Let $B \xrightarrow{g} C$ and $B \xrightarrow{h} C$ two continuous maps between two Hausdorff topological spaces such $g \neq h$. Then there exists $b \in B$ such $g(b) \neq h(b)$. As C is Hausdorff, then there exists two disjoint open subsets \mathcal{O}_1 and \mathcal{O}_2 of C such $g(b) \in \mathcal{O}_1$ and $h(b) \in \mathcal{O}_2$. We deduce that $b \in g^{-1}(\mathcal{O}_1) \cap h^{-1}(\mathcal{O}_2)$. As $B = \overline{f(A)}$, there exists $a \in A$ such $f(a) \in g^{-1}(\mathcal{O}_1) \cap h^{-1}(\mathcal{O}_2)$. And as \mathcal{O}_1 and \mathcal{O}_2 are disjoint, $g \circ f(a) \neq h \circ f(a)$. Consequently, f is an epimorphism. \square

3 The category Haus is a reflective and full subcategory of Top

The fullness of **Haus** in **Top** is trivial. We have to prove the reflectivity.

Proposition 2. *Let C be a topological space and let \sim be the equivalence relation defined by:*

$$x \sim y \Leftrightarrow \text{for any continuous map } C \xrightarrow{g} D \text{ where } D \text{ Hausdorff, } g(x) = g(y),$$

then the projection:

$$C \xrightarrow{r} C / \sim,$$

where C/\sim is provided with the quotient-topology, is a reflection from the **Top**-object C to the **Haus**-object C/\sim .

Proof. Let $C \xrightarrow{f} D$ a continuous map to a Hausdorff space D . If $r(x) = r(y)$, then, by definition of the equivalence relation \sim , $f(x) = f(y)$. Consequently, any $C \xrightarrow{f} D$ is compatible with \sim and by consequence, there exists a unique function (ie. **Set**-morphism) \bar{f} such the following diagram:

$$\begin{array}{ccc} C & \xrightarrow{r} & C/\sim \\ \downarrow f & \swarrow \bar{f} & \\ D & & \end{array}$$

commutes. We have to show that \bar{f} is a continuous map and that C/\sim is a Hausdorff space.

\bar{f} is a continuous map:

We recall that \mathcal{O} is an open subset of C/\sim if and only if $r^{-1}(\mathcal{O})$ is an open subset of C (see [2], Theorem 4.1 of the chapitre I). Let U an open subset of D , then $r^{-1}(\bar{f}^{-1}(U)) = f^{-1}(U)$ is an open set of C . We deduce that $\bar{f}^{-1}(U)$ is an open set of C/\sim and by consequence, \bar{f} is continous.

C/\sim is a Hausdorff space:

Let $r(x) \neq r(y)$, then there exists $C \xrightarrow{f} D$ continuous map to D an Hausdorff space such $f(x) \neq f(y)$. D is Hausdorff, then there exists two disjoint open-sets \mathcal{O}_x and \mathcal{O}_y of D such $f(x) \in \mathcal{O}_x$ and $f(y) \in \mathcal{O}_y$. It implies that $r(x) \in \bar{f}^{-1}(\mathcal{O}_x)$ and $r(y) \in \bar{f}^{-1}(\mathcal{O}_y)$. The subsets $\bar{f}^{-1}(\mathcal{O}_x)$ and $\bar{f}^{-1}(\mathcal{O}_y)$ are clearly disjoint open subsets of C/\sim . \square

We will denote $H(C)$ the set C/\sim and it is called the ‘‘Hausdorff quotient’’ of C .

Corollary 1. *A topological space C is Hausdorff if and only if C and $H(C)$ are homeomorphic.*

Proof. As **Haus** is a full subcategory of **Top**, one can use the proposition I.4.20 of [1]. \square

4 The epimorphisms of Haus are image-dense morphisms

Proposition 3. *Let $A \xrightarrow{f} B$ an epimorphism of **Haus**, then $B = \overline{f(A)}$.*

Proof. Let $B \xrightarrow{q} B/\overline{f(A)}$ be the quotient morphism such $q(x) = q(y)$ if and only if $(x = y \text{ and } x, y \notin \overline{f(A)})$ or $(x, y \in \overline{f(A)})$ and $B/\overline{f(A)} \xrightarrow{r} H(B/\overline{f(A)})$ be the Hausdorff quotient of $B/\overline{f(A)}$.

We have $r \circ q \circ f$ is constant and as f is an epimorphism of **Haus**, then the **Haus**-morphism $r \circ q$ is constant. So, either q is constant or r is constant. If q is constant, then $B = \overline{f(A)}$.

Suppose that $B \neq \overline{f(A)}$ and r is constant. Let $x \in B \setminus \overline{f(A)}$. As $\overline{f(A)}$ is a closed subset of B , then there exists an open subset \mathcal{O} of B such $x \in \mathcal{O} \subset B \setminus \overline{f(A)}$. By definition of q , for x_1 and x_2 in \mathcal{O} , $q(x_1) = q(x_2)$ if and only if $x_1 = x_2$. It implies that $q^{-1}(q(\mathcal{O})) = \mathcal{O}$ and $q(\mathcal{O})$ is a Hausdorff open subset of $B/\overline{f(A)}$. We deduce that $r \circ q(\mathcal{O})$ has at least one point which is different from $r \circ q(\overline{f(A)})$. Consequently, r takes at least two different values; this fact is contradictory with r constant. \square

References

- [1] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories. The Joy of Cats*. 2004.
- [2] C. Godbillon. *Éléments de topologie algébrique*. Hermann. Editeurs des Sciences et des Arts, 1997.