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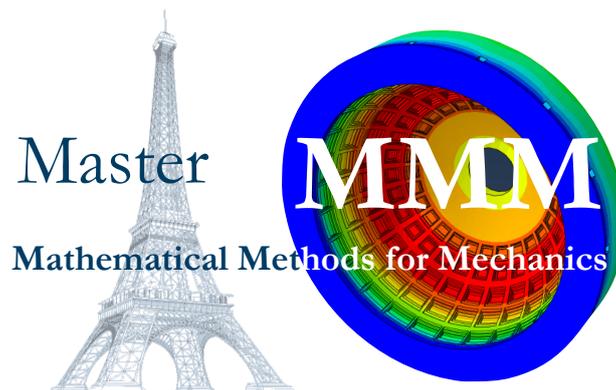
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Class notes of the course

# Modeling of Solids



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# Preface

This text is the support for the course of Modeling of Solids, of the Master of Mechanics of the University Paris-Saclay - Curriculum MMM: Mathematical Methods for Mechanics, held at Versailles.

The course is the continuation of the course Continuum Mechanics - Solids, and as such it is an introduction, for graduate students, to some typical topics of the theory of solid bodies.

The different arguments are dealt with in a simple, succinct way, the objective being to give to students the fundamentals of each argument. Only static problems are considered, being the dynamic of structures dealt with in other courses.

*Versailles, December 7, 2015*



# Contents

<b>Preface</b>	<b>iii</b>
<b>1 Some elements of differential geometry</b>	<b>1</b>
1.1 Curves of points, vectors and tensors . . . . .	1
1.2 Differentiation of a curve . . . . .	2
1.3 Integral of a curve of vectors . . . . .	3
1.4 The Frenet-Serret basis . . . . .	5
1.5 Curvature of a curve . . . . .	7
1.6 The Frenet-Serret formulae . . . . .	9
1.7 The torsion of a curve . . . . .	10
1.8 Osculating sphere and circle . . . . .	11
1.9 Exercices . . . . .	12
<b>2 Cables</b>	<b>15</b>
2.1 Introduction . . . . .	15
2.2 Mechanical definition of a cable . . . . .	15
2.3 The intrinsic balance equations . . . . .	18
2.4 Forces depending upon a potential . . . . .	18
2.5 Parallel and coplanar forces . . . . .	19
2.6 Concentrated forces . . . . .	19
2.7 Cables on surfaces . . . . .	20
2.8 Applications . . . . .	21
2.8.1 The catenary . . . . .	21
2.8.2 The suspension bridge . . . . .	27
2.8.3 The curve of uniform vertical load . . . . .	28
2.8.4 The cable coiled on a rough cylinder . . . . .	29
2.9 Elastic cables . . . . .	30
2.9.1 The catenary of an elastic cable . . . . .	31
2.10 Exercices . . . . .	33
<b>3 Arches</b>	<b>35</b>
3.1 Introduction . . . . .	35
3.2 Balance equations . . . . .	36
3.3 Compatibility equations . . . . .	38
3.4 Constitutive equations . . . . .	42
3.5 The problem of elastic equilibrium for the arches . . . . .	43
3.6 Transforming the equations of the arches . . . . .	43

3.6.1	Transforming the balance equations . . . . .	43
3.6.2	Transforming the elastic equilibrium equations . . . . .	45
3.6.3	The Euler-Bernoulli model for arches . . . . .	45
3.6.4	Arches of constant section . . . . .	46
3.6.5	Circular arches . . . . .	46
3.7	Examples . . . . .	47
3.7.1	Example 1 . . . . .	47
3.7.2	Example 2 . . . . .	48
3.7.3	Example 3 . . . . .	48
3.8	Exercices . . . . .	49
<b>4</b>	<b>Plates</b>	<b>53</b>
4.1	Problem definition: basic assumptions . . . . .	53
4.2	The Kirchhoff model for thin plates . . . . .	54
4.2.1	The displacement field . . . . .	54
4.2.2	Strain field . . . . .	57
4.2.3	Stress field . . . . .	57
4.2.4	Internal actions . . . . .	59
4.2.5	Uncoupling bending and extension . . . . .	60
4.2.6	Balance equations . . . . .	61
4.2.7	Elastic equilibrium equations . . . . .	63
4.2.8	Expressions for stresses . . . . .	64
4.2.9	Transverse shear stresses . . . . .	65
4.2.10	Boundary conditions . . . . .	66
4.2.10.1	Simply supported edges . . . . .	68
4.2.10.2	Clamped edges . . . . .	69
4.2.10.3	Free edges . . . . .	69
4.2.10.4	The Kelvin's reduction of the boundary conditions . . . . .	69
4.2.11	Some exact solutions . . . . .	71
4.2.11.1	Elliptic plate . . . . .	71
4.2.11.2	Circular plate . . . . .	72
4.2.11.3	Equilateral triangular plate . . . . .	72
4.2.11.4	Anticlastic bending . . . . .	72
4.2.11.5	Navier's method for rectangular plates . . . . .	75
4.2.11.6	Levy's method for rectangular plates . . . . .	76
4.3	The Reissner-Mindlin theory . . . . .	77
4.4	The Von Karman theory . . . . .	80
4.5	Exercices . . . . .	85
	<b>Suggested texts</b>	<b>87</b>

# Chapter 1

## Some elements of differential geometry

### 1.1 Curves of points, vectors and tensors

Be  $\mathcal{R} = \{o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  a reference frame of the euclidean space  $\mathcal{E}$ <sup>1</sup> and let us consider a point  $p = (p_1, p_2, p_3)$ . If the three coordinates are three continuous functions  $p_i(t)$  over the interval  $[t_1, t_2] \in \mathbb{R}$ , then the mapping  $p(t) : [t_1, t_2] \rightarrow \mathcal{E}$  is a *curve* in  $\mathcal{E}$ .

The independent variable  $t$  is the *parameter* and the equation

$$p(t) = (p_1(t), p_2(t), p_3(t)) \rightarrow \begin{cases} p_1 = p_1(t) \\ p_2 = p_2(t) \\ p_3 = p_3(t) \end{cases} \quad (1.1)$$

is the *parametric point equation of the curve*: to each value of  $t \in [t_1, t_2]$  it corresponds a point of the curve in  $\mathcal{E}$ , see Fig. 1.1. The curve is *smooth* whenever the functions  $p_i(t)$  are of class  $C^1$ .

The vector function  $\mathbf{r}(t) = p(t) - o$  is the *position vector* of point  $p$  in  $\mathcal{R}$ ; the equation

$$\mathbf{r}(t) = r_1(t)\mathbf{e}_1 + r_2(t)\mathbf{e}_2 + r_3(t)\mathbf{e}_3 \rightarrow \begin{cases} r_1 = r_1(t) \\ r_2 = r_2(t) \\ r_3 = r_3(t) \end{cases} \quad (1.2)$$

is the *parametric vector equation* of the curve: to each value of  $t \in [t_1, t_2]$  it corresponds a vector of  $\mathcal{V}$  that determines a point of the curve in  $\mathcal{E}$  through the operation  $p(t) = o + \mathbf{r}(t)$ .

---

<sup>1</sup> The Euclidean space  $\mathcal{E}$  is the ordinary three-dimensional space, whose elements are geometric points  $p$ . A reference frame  $\mathcal{R} = \{o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  in  $\mathcal{E}$  is composed by a point  $o$ , called the *origin* of the frame and by a basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  of the space vector  $\mathcal{V}$  associated to  $\mathcal{E}$ , called the *space of translations*: any vector  $\mathbf{v} \in \mathcal{V}$  is a function that operates on the points  $p \in \mathcal{E} : \forall \mathbf{v} \in \mathcal{V}, \mathbf{v}(p) = q$ . Concerning the basis associated to the reference frame  $\mathcal{R}$ , we assume it to be *orthonormal*:  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \forall i, j = 1, 2, 3$ . A *second-rank tensor* is any linear mapping  $\mathbf{L} : \mathcal{V} \rightarrow \mathcal{V}$ . The space of all the tensors forms a manifold, indicated by  $\text{Lin}(\mathcal{V})$ . Any tensor can be expressed, in  $\mathcal{R}$ , as a sum of dyads:  $\mathbf{L} = L_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ . A *dyad*  $\mathbf{a} \otimes \mathbf{b}$  is defined as  $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = \mathbf{b} \cdot \mathbf{v} \mathbf{a} \forall \mathbf{v} \in \mathcal{V}$ .

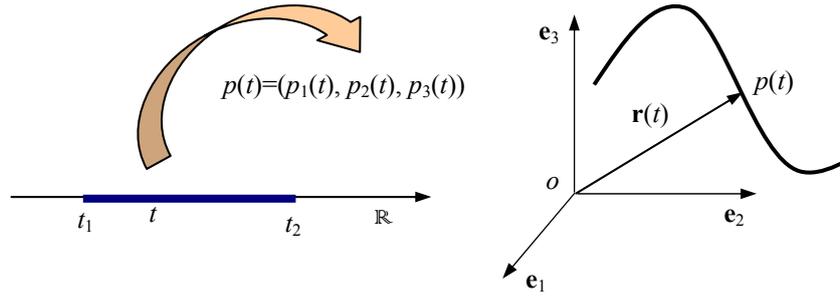


Figure 1.1: Mapping of a curve of points.

The expression  $p = p(t)$  is a *curve of points* while  $\mathbf{r} = \mathbf{r}(t)$  is a *curve of vectors*. In the same way, we can introduce a *curve of tensors*: if the components  $L_{ij}(t)$  are continuous functions of a parameter  $t$ , the mapping  $\mathbf{L}(t) : [t_1, t_2] \rightarrow \text{Lin}(\mathcal{V})$  defined as

$$\mathbf{L}(t) = L_{ij}(t)\mathbf{e}_i \otimes \mathbf{e}_j, \quad i, j = 1, 2, 3, \quad (1.3)$$

is a *curve of tensors*.

To be noticed that the choice of the parameter is *not unique*: the equation  $p = p[\tau(t)]$  still represents the same curve  $p = p(t)$ , through the *change of parameter*  $\tau = \tau(t)$ .

## 1.2 Differentiation of a curve

We define *derivative* of a curve of points  $p = p(t)$  in  $t = t_0$  the limit

$$\frac{dp(t)}{dt} = \lim_{\varepsilon \rightarrow 0} \frac{p(t_0 + \varepsilon) - p(t_0)}{\varepsilon}; \quad (1.4)$$

being defined as a difference of points  $\frac{dp(t)}{dt}$  is a vector, see fig. 1.2

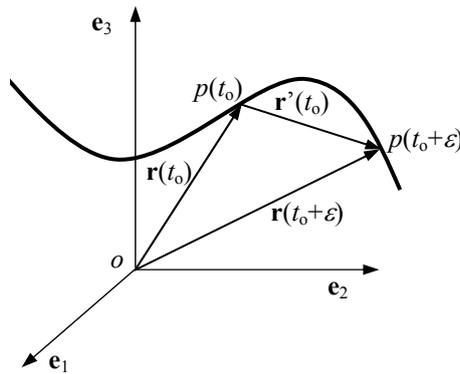


Figure 1.2: Derivative of a curve.

In a similar way we define the derivative of a curve of vectors,

$$\frac{d\mathbf{r}(t)}{dt} = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{r}(t_0 + \varepsilon) - \mathbf{r}(t_0)}{\varepsilon}; \quad (1.5)$$

and of a curve of tensors

$$\frac{d\mathbf{L}(t)}{dt} = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{L}(t_0 + \varepsilon) - \mathbf{L}(t_0)}{\varepsilon}, \quad (1.6)$$

being defined as differences of vectors or of tensors, the derivative of a curve of vectors is a vector and that of a curve of tensors is a tensor.

We indicate also  $\frac{dp}{dt}$  as  $p'$  and, if  $t$  is the time, as  $\dot{p}$ ; the same will be do also for the derivatives of vector and tensor curves.

The following rules for the differentiation of products or sums of curves can be easily shown using the above definitions:

$$\begin{aligned} (\mathbf{u} + \mathbf{v})' &= \mathbf{u}' + \mathbf{v}', \\ (\alpha\mathbf{v})' &= \alpha'\mathbf{v} + \alpha\mathbf{v}', \quad \alpha(t) : \mathbb{R} \rightarrow \mathbb{R}, \\ (\mathbf{u} \cdot \mathbf{v})' &= \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \\ (\mathbf{u} \times \mathbf{v})' &= \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}', \\ [\mathbf{v}(\alpha(t))]' &= \mathbf{v}'(\alpha(t)) \alpha'(t), \quad \alpha(t) : \mathbb{R} \rightarrow \mathbb{R}, \\ (\mathbf{u} \otimes \mathbf{v})' &= \mathbf{u}' \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{v}', \\ (\mathbf{L}\mathbf{u})' &= \mathbf{L}'\mathbf{u} + \mathbf{L}\mathbf{u}', \\ (\mathbf{L}\mathbf{M})' &= \mathbf{L}'\mathbf{M} + \mathbf{L}\mathbf{M}', \\ (\mathbf{L}^{-1})' &= -\mathbf{L}^{-1}\mathbf{L}'\mathbf{L}^{-1}, \\ (\det \mathbf{L})' &= \det \mathbf{L} \mathbf{L}' \cdot \mathbf{L}^{-\top}. \end{aligned} \quad (1.7)$$

An important case is that of a vector  $\mathbf{v}(t)$  whose norm  $v(t)$  is constant  $\forall t$ :

$$(v^2)' = (\mathbf{v} \cdot \mathbf{v})' = \mathbf{v}' \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}' = 2\mathbf{v}' \cdot \mathbf{v} = 0 : \quad (1.8)$$

the derivative of such a vector is orthogonal to it  $\forall t$ . The contrary is also true, as immediately apparent.

The *second derivative* of a curve is simply defined as the derivative of the derivative of a curve. In such a way, derivatives of any order can be defined and calculated.

Finally, using the above rules and assuming that the reference frame  $\mathcal{R}$  is independent from  $t$ , we get easily that

$$\begin{aligned} p'(t) &= p'_i(t) \mathbf{e}_i, \\ \mathbf{v}'(t) &= v'_i(t) \mathbf{e}_i, \\ \mathbf{L}'(t) &= L'_{ij}(t) \mathbf{e}_i \otimes \mathbf{e}_j. \end{aligned} \quad (1.9)$$

### 1.3 Integral of a curve of vectors

We define *integral of a curve of vectors* between  $a$  and  $b \in [t_1, t_2]$  the curve that is obtained integrating each component of the curve:

$$\int_a^b \mathbf{r}(t) dt = \int_a^b r_i(t) dt \mathbf{e}_i. \quad (1.10)$$

If the curve is regular, we can generalize the second fundamental theorem of the integral calculus

$$\mathbf{r}(t) = \mathbf{r}(a) + \int_a^t \mathbf{r}'(t^*) dt^*. \quad (1.11)$$

Because

$$\mathbf{r}(t) = p(t) - o, \quad \mathbf{r}'(t) = (p(t) - o)' = p'(t), \quad (1.12)$$

we get also

$$p(t) = p(a) + \int_a^t p'(t^*) dt^*. \quad (1.13)$$

The integral of a vector function is the generalization of the vector sum, see Fig. 1.3.

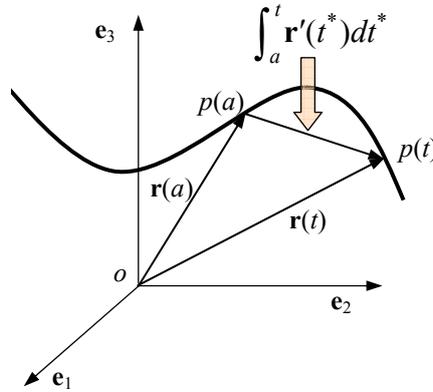


Figure 1.3: Integral of a vector curve.

A simple way to determine a point  $p(t)$  on a curve is to fix a point  $p_0$  on the curve and to measure the length  $s(t)$  of the arc of curve between  $p_0 = p(t = 0)$  and  $p(t)$ . This length  $s(t)$  is called a *curvilinear abscissa* and it can be shown that

$$s(t) = \int_{t_0}^t |(p(t^*) - o)'| dt^* = \int_{t_0}^t |\mathbf{r}'(t^*)| dt^*, \quad (1.14)$$

so that the total length  $\ell$  of a curve is

$$\ell = \int_a^b |\mathbf{r}'(t^*)| dt^*. \quad (1.15)$$

From eq. (1.14) we get

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left(\frac{dr_1}{dt}\right)^2 + \left(\frac{dr_2}{dt}\right)^2 + \left(\frac{dr_3}{dt}\right)^2} > 0 \quad (1.16)$$

so that  $s(t)$  is an increasing function of  $t$  and the length of an infinitesimal arc is

$$ds = \sqrt{dr_1^2 + dr_2^2 + dr_3^2} = \sqrt{dx^2 + dy^2 + dz^2}. \quad (1.17)$$

For a plane curve  $y = f(x)$ , we can always put  $t = x$ , which gives the parametric equation

$$p(t) = (t, f(t)), \quad (1.18)$$

or in vector form

$$\mathbf{r}(t) = t \mathbf{e}_1 + f(t) \mathbf{e}_2, \quad (1.19)$$

from which we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = |p'(t)| = \sqrt{1 + f'^2(t)}, \quad (1.20)$$

that gives the length of a plane curve between  $t = x_0$  and  $t = x$  as a function of the abscissa  $x$ :

$$s(x) = \int_{x_0}^x \sqrt{1 + f'^2(t)} dt. \quad (1.21)$$

## 1.4 The Frenet-Serret basis

We define the *tangent vector*  $\boldsymbol{\tau}(t)$  to a regular curve  $p = p(t)$  the vector

$$\boldsymbol{\tau}(t) = \frac{p'(t)}{|p'(t)|}. \quad (1.22)$$

By the definition of derivative, this unit vector is always oriented as the increasing values of  $t$ ; the straight line tangent to the curve in  $p_0 = p(t_0)$  has hence equation

$$q(\bar{t}) = p(t_0) + \bar{t} \boldsymbol{\tau}(t_0). \quad (1.23)$$

If the curvilinear abscissa  $s$  is chosen as parameter for the curve, through the change of parameter  $s = s(t)$  we get

$$\boldsymbol{\tau}(t) = \frac{p'(t)}{|p'(t)|} = \frac{p'[s(t)]}{|p'[s(t)]|} = \frac{1}{s'(t)} \frac{dp(s)}{ds} \frac{ds(t)}{dt} = \frac{dp(s)}{ds} \rightarrow \boldsymbol{\tau}(s) = p'(s). \quad (1.24)$$

So, if the parameter of the curve is  $s$ , the derivative of the curve is  $\boldsymbol{\tau}$ , i.e. it is automatically a unit vector. The above equation, in addition, shows that the change of parameter does not change the direction of the tangent, because just a scalar, the derivative of the parameter's change, multiplies the vector. Nevertheless, generally speaking, a change of parameter can change the orientation of the curve.

Because the norm of  $\boldsymbol{\tau}$  is constant, its derivative is a vector orthogonal to  $\boldsymbol{\tau}$ , see eq. (1.8). That is why we call *principal normal vector* to a curve the unit vector

$$\boldsymbol{\nu}(t) = \frac{\boldsymbol{\tau}'(t)}{|\boldsymbol{\tau}'(t)|}. \quad (1.25)$$

$\boldsymbol{\nu}$  is defined only on the points of the curve where  $\boldsymbol{\tau}' \neq \mathbf{o}$  which implies that  $\boldsymbol{\nu}$  is not defined on the points of a straight line. This simply means that there is not, among the

infinite unit normal vectors to a straight line, a normal with special properties, a *principal* one, uniquely linked to  $\boldsymbol{\tau}$ .

Unlike  $\boldsymbol{\tau}$ , whose orientation changes with the choice of the parameter,  $\boldsymbol{\nu}$  is an *intrinsic* local characteristic of the curve: *it is not affected by the choice of the parameter*. In fact, by its same definition,  $\boldsymbol{\nu}$  does not depend upon the reference frame; then, because the direction of  $\boldsymbol{\tau}$  is also independent upon the parameter's choice, the only factor that could affect  $\boldsymbol{\nu}$  is the orientation of the curve, that depends upon the parameter. But a change of the orientation affects, in (1.25), both  $\boldsymbol{\tau}$  and the sign of the increment  $dt$ , so that  $\boldsymbol{\tau}'(t) = d\boldsymbol{\tau}/dt$  does not change, neither  $\boldsymbol{\nu}$ , which is hence an intrinsic property of the curve.

The vector

$$\boldsymbol{\beta}(t) = \boldsymbol{\tau}(t) \times \boldsymbol{\nu}(t) \quad (1.26)$$

is called the *binormal vector*; by construction, it is orthogonal to  $\boldsymbol{\tau}$  and  $\boldsymbol{\nu}$  and it is a unit vector. In addition, it is evident that

$$\boldsymbol{\tau} \times \boldsymbol{\nu} \cdot \boldsymbol{\beta} = 1, \quad (1.27)$$

so the set  $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$  forms a positively oriented orthonormal basis that can be defined at any regular point of a curve with  $\boldsymbol{\tau}' \neq \mathbf{o}$ . Such a basis is called the *Frenet-Serret local basis*, local in the sense that it changes with the position along the curve. The plane  $\boldsymbol{\tau} - \boldsymbol{\nu}$  is the *osculating plane*, the plane  $\boldsymbol{\nu} - \boldsymbol{\beta}$  the *normal plane* and the plane  $\boldsymbol{\beta} - \boldsymbol{\tau}$  the *rectifying plane*, see Fig. 1.4. The osculating plane is particularly important: if we

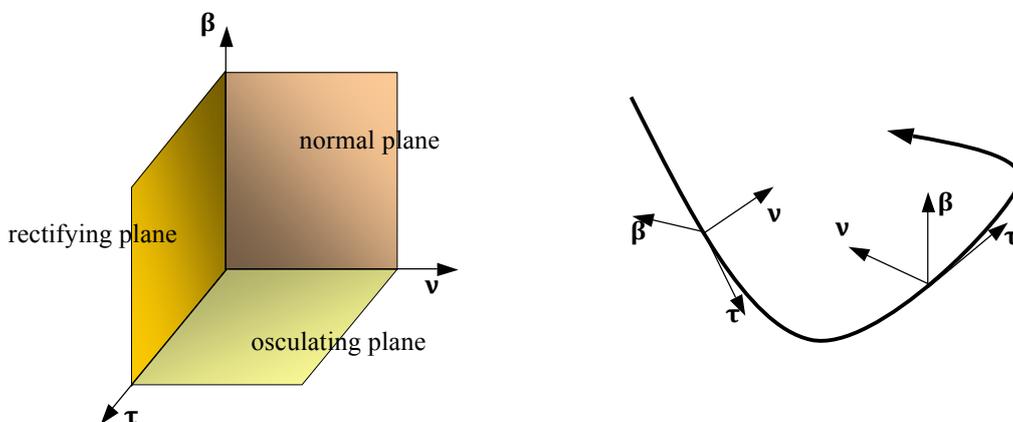


Figure 1.4: The Frenet-Serret basis.

consider a plane passing through three not aligned points of the curve, when these points become closer and closer, still remaining on the curve, the plane tends to the osculating plane: the osculating plane at a point of a curve is the plane that better approaches the curve near the point. A plane curve is entirely contained in the osculating plane, which is fixed.

The principal normal  $\boldsymbol{\nu}$  is always oriented towards the part of the space, with respect to the rectifying plane, where the curve is; in particular, for a plane curve,  $\boldsymbol{\nu}$  is always directed towards the concavity of the curve. To show it, it is sufficient to prove that the

vector  $p(t + \varepsilon) - p(t)$  forms with  $\boldsymbol{\nu}$  an angle  $\psi \leq \pi/2$ , i.e. that  $(p(t + \varepsilon) - p(t)) \cdot \boldsymbol{\nu} \geq 0$ . In fact,

$$\begin{aligned} p(t + \varepsilon) - p(t) &= \varepsilon p'(t) + \frac{1}{2}\varepsilon^2 p''(t) + o(\varepsilon^2) \rightarrow \\ (p(t + \varepsilon) - p(t)) \cdot \boldsymbol{\nu} &= \frac{1}{2}\varepsilon^2 p''(t) \cdot \boldsymbol{\nu} + o(\varepsilon^2), \end{aligned} \quad (1.28)$$

but

$$p''(t) \cdot \boldsymbol{\nu} = (\boldsymbol{\tau}'|p'| + \boldsymbol{\tau}|p''|) \cdot \boldsymbol{\nu} = (|\boldsymbol{\tau}'||p'| + \boldsymbol{\tau}|p''|) \cdot \boldsymbol{\nu} = |\boldsymbol{\tau}'||p'|, \quad (1.29)$$

so that, to within infinitesimal quantities of order  $o(\varepsilon^2)$ , we obtain

$$(p(t + \varepsilon) - p(t)) \cdot \boldsymbol{\nu} = \frac{1}{2}\varepsilon^2 |\boldsymbol{\tau}'||p'| \geq 0. \quad (1.30)$$

## 1.5 Curvature of a curve

It is important, in several situations, to evaluate how much a curve moves away from a straight line, in the neighborhood of a point. To do that, we calculate the angle formed by the tangents at two close points, determined by the curvilinear abscissae  $s$  and  $s + \varepsilon$ , and we measure the angle  $\chi(s, \varepsilon)$  that they form, see Fig. 1.5.

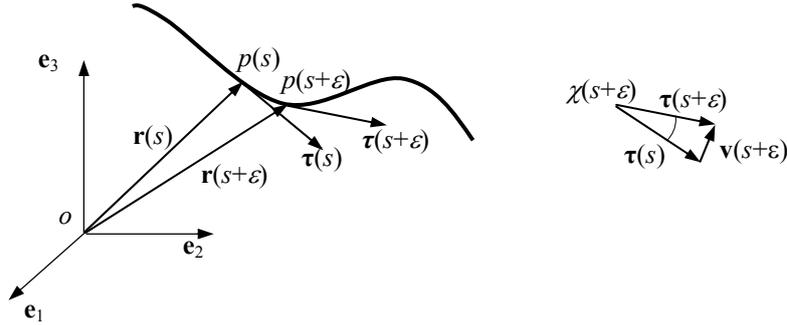


Figure 1.5: Curvature of a curve.

We then define *curvature* of the curve in  $p = p(s)$  the limit

$$c(s) = \lim_{\varepsilon \rightarrow 0} \left| \frac{\chi(s, \varepsilon)}{\varepsilon} \right|. \quad (1.31)$$

The curvature is hence a non-negative scalar that measures the rapidity of variation of the direction of the curve per unit length of the curve (that is why  $c(s)$  is defined as a function of the curvilinear abscissa); by its same definition, the curvature is an *intrinsic property* of the curve, i.e. independent from the parameter's choice. For a straight line, the curvature is identically null everywhere.

The curvature is linked to the second derivative of the curve:

$$\begin{aligned} c(s) &= \lim_{\varepsilon \rightarrow 0} \left| \frac{\chi(s, \varepsilon)}{\varepsilon} \right| = \lim_{\varepsilon \rightarrow 0} \left| \frac{\sin \chi(s, \varepsilon)}{\varepsilon} \right| = \lim_{\varepsilon \rightarrow 0} \left| \frac{2}{\varepsilon} \sin \frac{\chi(s, \varepsilon)}{2} \right| = \\ &= \lim_{\varepsilon \rightarrow 0} \left| \frac{\mathbf{v}(s, \varepsilon)}{\varepsilon} \right| = \lim_{\varepsilon \rightarrow 0} \left| \frac{\boldsymbol{\tau}(s + \varepsilon) - \boldsymbol{\tau}(s)}{\varepsilon} \right| = |\boldsymbol{\tau}'(s)| = |p''(s)|. \end{aligned} \quad (1.32)$$

Another formula for the calculation of  $c(s)$  can be obtained if we consider that

$$\frac{d\boldsymbol{\tau}[s(t)]}{dt} = \frac{d\boldsymbol{\tau}}{ds} \frac{ds}{dt} = \frac{d\boldsymbol{\tau}}{ds} |p'(t)| \rightarrow \frac{d\boldsymbol{\tau}}{ds} = \frac{1}{|p'(t)|} \frac{d\boldsymbol{\tau}}{dt}, \quad (1.33)$$

so that

$$c(s) = |\boldsymbol{\tau}'(s)| = \frac{1}{|p'(t)|} \left| \frac{d\boldsymbol{\tau}}{dt} \right| = \frac{|\boldsymbol{\tau}'(t)|}{|p'(t)|}. \quad (1.34)$$

A better formula can be obtained as follows:

$$\begin{aligned} \frac{d\boldsymbol{\tau}}{ds} &= \frac{1}{|p'(t)|} \frac{d\boldsymbol{\tau}}{dt} = \frac{1}{|p'(t)|} \frac{d}{dt} \frac{p'(t)}{|p'(t)|} = \frac{1}{|p'|} \frac{p''|p'| - p' \frac{p'' \cdot p'}{|p'|}}{|p'|^2} = \\ &= \frac{p'' - \boldsymbol{\tau} p'' \cdot \boldsymbol{\tau}}{|p'|^2} = (\mathbf{I} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}) \frac{p''}{|p'|^2}. \end{aligned} \quad (1.35)$$

By consequence,

$$c(s) = \left| \frac{d\boldsymbol{\tau}(s)}{ds} \right| = \frac{1}{|p'|^2} |(\mathbf{I} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}) p''|. \quad (1.36)$$

Now, we use the following general formula expressing a skew tensor  $\mathbf{W}$ :

$$\mathbf{W}\mathbf{W} = -\frac{1}{2} |\mathbf{W}|^2 (\mathbf{I} - \mathbf{w} \otimes \mathbf{w}); \quad (1.37)$$

if we use this formula for  $\boldsymbol{\tau}$ , so that  $\mathbf{W}$  is the axial tensor of  $\boldsymbol{\tau}$ , we get

$$\mathbf{I} - \boldsymbol{\tau} \otimes \boldsymbol{\tau} = -2 \frac{\mathbf{W}\mathbf{W}}{|\mathbf{W}|^2} = -\mathbf{W}\mathbf{W}, \quad (1.38)$$

because if  $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ , then

$$\begin{aligned} |\mathbf{W}|^2 = \mathbf{W} \cdot \mathbf{W} &= \begin{bmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{bmatrix} = \\ &= 2(\tau_1^2 + \tau_2^2 + \tau_3^2) = 2. \end{aligned} \quad (1.39)$$

So, recalling that for any skew tensor  $\mathbf{W}$ ,

$$\mathbf{W} \mathbf{u} = \mathbf{w} \times \mathbf{u} \quad \forall \mathbf{u} \in \mathcal{V}, \quad (1.40)$$

with  $\mathbf{w}$  the axial vector of  $\mathbf{W}$ , we get

$$\begin{aligned} |(\mathbf{I} - \boldsymbol{\tau} \otimes \boldsymbol{\tau}) p''| &= |-\mathbf{W}\mathbf{W} p''| = |-\mathbf{W}(\boldsymbol{\tau} \times p'')| = |-\boldsymbol{\tau} \times (\boldsymbol{\tau} \times p'')| = \\ &= |\boldsymbol{\tau} \times (\boldsymbol{\tau} \times p'')| = |\boldsymbol{\tau} \times p''| = \frac{|p' \times p''|}{|p'|}, \end{aligned} \quad (1.41)$$

so that finally

$$c = \frac{|p' \times p''|}{|p'|^3}. \quad (1.42)$$

Applying this last formula to a plane curve  $p(t) = (x(t), y(t))$ , we get

$$c = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}} \quad (1.43)$$

and if the curve is given in the form  $y = y(x)$ , so that the parameter  $t = x$ , then we obtain

$$c = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}}. \quad (1.44)$$

This last formula shows that if  $|y'| \ll 1$ , like in the infinitesimal theory of strain, then

$$c \simeq |y''|. \quad (1.45)$$

## 1.6 The Frenet-Serret formulae

From eqs. (1.25) for  $t = s$  and (1.34) we get

$$\frac{d\boldsymbol{\tau}}{ds} = c \boldsymbol{\nu} \quad (1.46)$$

which is the *first Frenet-Serret Formula*, giving the variation of  $\boldsymbol{\tau}$  per unit length of the curve. Such a variation is a vector whose norm is the curvature and that has as direction that of  $\boldsymbol{\nu}$ .

Let us now consider the variation of  $\boldsymbol{\beta}$  per unit length of the curve; because  $\boldsymbol{\beta}$  is a unit vector, we have

$$\frac{d\boldsymbol{\beta}}{ds} \cdot \boldsymbol{\beta} = 0, \quad (1.47)$$

and

$$\boldsymbol{\beta} \cdot \boldsymbol{\tau} = 0 \Rightarrow \frac{d(\boldsymbol{\beta} \cdot \boldsymbol{\tau})}{ds} = \frac{d\boldsymbol{\beta}}{ds} \cdot \boldsymbol{\tau} + \boldsymbol{\beta} \cdot \frac{d\boldsymbol{\tau}}{ds} = 0. \quad (1.48)$$

Through eq. (1.46) and because  $\boldsymbol{\beta} \cdot \boldsymbol{\nu} = 0$  we get

$$\frac{d\boldsymbol{\beta}}{ds} \cdot \boldsymbol{\tau} = -c \boldsymbol{\beta} \cdot \boldsymbol{\nu} = 0, \quad (1.49)$$

so that  $\frac{d\boldsymbol{\beta}}{ds}$  is necessarily parallel to  $\boldsymbol{\nu}$ . We then put

$$\frac{d\boldsymbol{\beta}}{ds} = \vartheta \boldsymbol{\nu}, \quad (1.50)$$

which is the *second Frenet-Serret formula*. The scalar  $\vartheta(s)$  is called the *torsion of the curve* in  $p = p(s)$ . So, we see that the variation of  $\boldsymbol{\beta}$  per unit length is a vector parallel to  $\boldsymbol{\nu}$  and proportional to the torsion of the curve.

We can now find the variation of  $\boldsymbol{\nu}$  per unit length of the curve:

$$\frac{d\boldsymbol{\nu}}{ds} = \frac{d(\boldsymbol{\beta} \times \boldsymbol{\tau})}{ds} = \frac{d\boldsymbol{\beta}}{ds} \times \boldsymbol{\tau} + \boldsymbol{\beta} \times \frac{d\boldsymbol{\tau}}{ds} = \vartheta \boldsymbol{\nu} \times \boldsymbol{\tau} + c \boldsymbol{\beta} \times \boldsymbol{\nu}, \quad (1.51)$$

so finally

$$\frac{d\boldsymbol{\nu}}{ds} = -c \boldsymbol{\tau} - \vartheta \boldsymbol{\beta}, \quad (1.52)$$

which is the *third Frenet-Serret formula*: the variation of  $\boldsymbol{\nu}$  per unit length of the curve is a vector of the rectifying plane.

The three formulae of Frenet-Serret (discovered independently by J. F. Frenet in 1847 and by J. A. Serret in 1851) can be condensed in the symbolic matrix product

$$\begin{Bmatrix} \boldsymbol{\tau}' \\ \boldsymbol{\nu}' \\ \boldsymbol{\beta}' \end{Bmatrix} = \begin{bmatrix} 0 & c & 0 \\ -c & 0 & -\vartheta \\ 0 & \vartheta & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\tau} \\ \boldsymbol{\nu} \\ \boldsymbol{\beta} \end{Bmatrix}. \quad (1.53)$$

## 1.7 The torsion of a curve

We have introduced the torsion of a curve in the previous section, with the second formula of Frenet-Serret. The torsion measures the deviation of a curve from flatness: if a curve is planar, it belongs to the osculating plane and  $\boldsymbol{\beta}$ , which is perpendicular to the osculating plane, is hence a constant vector. So, its derivative is null and by the Frenet-Serret second formula  $\vartheta = 0$ .

Conversely, if  $\vartheta = 0$  everywhere,  $\boldsymbol{\beta}$  is a constant vector and hence the osculating plane does not change and the curve is planar. So we have that *a curve is planar if and only if the torsion is null  $\forall p(s)$* .

Using the Frenet-Serret formulae in the expression of  $p'''(s)$  we get a formula for the torsion:

$$\begin{aligned} p'(t) &= |p'| \boldsymbol{\tau} = \frac{dp}{ds} \frac{ds}{dt} = s' \boldsymbol{\tau} \Rightarrow |p'| = s' \rightarrow \\ p''(t) &= s'' \boldsymbol{\tau} + s' \boldsymbol{\tau}' = s'' \boldsymbol{\tau} + s'^2 \frac{d\boldsymbol{\tau}}{ds} = s'' \boldsymbol{\tau} + c s'^2 \boldsymbol{\nu} \rightarrow \\ p'''(t) &= s''' \boldsymbol{\tau} + s'' \boldsymbol{\tau}' + (c s'^2)' \boldsymbol{\nu} + c s'^2 \boldsymbol{\nu}' = \\ &= s''' \boldsymbol{\tau} + s'' s' \frac{d\boldsymbol{\tau}}{ds} + (c s'^2)' \boldsymbol{\nu} + c s'^3 \frac{d\boldsymbol{\nu}}{ds} = \\ &= s''' \boldsymbol{\tau} + s'' s' c \boldsymbol{\nu} + (c s'^2)' \boldsymbol{\nu} - c s'^3 (c \boldsymbol{\tau} + \vartheta \boldsymbol{\beta}) = \\ &= (s''' - c^2 s'^3) \boldsymbol{\tau} + (s'' s' c + c' s'^2 + 2c s' s'') \boldsymbol{\nu} - c s'^3 \vartheta \boldsymbol{\beta}, \end{aligned} \quad (1.54)$$

so that, through eq. (1.42), we get

$$\begin{aligned} p' \times p'' \cdot p''' &= s' \boldsymbol{\tau} \times (s'' \boldsymbol{\tau} + c s'^2 \boldsymbol{\nu}) \cdot [(s''' - c^2 s'^3) \boldsymbol{\tau} + \\ &+ (s'' s' c + c' s'^2 + 2c s' s'') \boldsymbol{\nu} - c s'^3 \vartheta \boldsymbol{\beta}] = \\ &= -c^2 s'^6 \vartheta = -c^2 |p'|^6 \vartheta = -\frac{|p' \times p''|^2}{|p'|^6} |p'|^6 \vartheta, \end{aligned} \quad (1.55)$$

so that, finally,

$$\vartheta = -\frac{p' \times p'' \cdot p'''}{|p' \times p''|^2}. \quad (1.56)$$

To remark that while the curvature is linked to the second derivative of the curve, the torsion is a function also of the third derivative.

Unlike curvature, which is intrinsically positive, the torsion can be negative. In fact, still using the Frenet-Serret formulae,

$$\begin{aligned}
 p(s + \varepsilon) - p(s) &= \varepsilon p' + \frac{1}{2}\varepsilon^2 p'' + \frac{1}{6}\varepsilon^3 p''' + o(\varepsilon^3) = \\
 &= \varepsilon \boldsymbol{\tau} + \frac{1}{2}\varepsilon^2 c \boldsymbol{\nu} + \frac{1}{6}\varepsilon^3 (c \boldsymbol{\nu})' + o(\varepsilon^3) = \\
 &= \varepsilon \boldsymbol{\tau} + \frac{1}{2}\varepsilon^2 c \boldsymbol{\nu} + \frac{1}{6}\varepsilon^3 (c' \boldsymbol{\nu} - c^2 \boldsymbol{\tau} - c \vartheta \boldsymbol{\beta}) + o(\varepsilon^3) \rightarrow \\
 (p(s + \varepsilon) - p(s)) \cdot \boldsymbol{\beta} &= -\frac{1}{6}\varepsilon^3 c \vartheta + o(\varepsilon^3).
 \end{aligned} \tag{1.57}$$

The above dot product determines if the point  $p(s + \varepsilon)$  is located, with respect to the osculating plane, on the side of  $\boldsymbol{\beta}$  or on the opposite one, see Fig. 1.6: if following the curve for increasing values of  $s$ ,  $\varepsilon > 0$ , the point passes into the semi-space of  $\boldsymbol{\beta}$  from the opposite one, because  $1/6 c \varepsilon^3 > 0$ , it will be  $\vartheta < 0$ , while in the opposite case it will be  $\vartheta > 0$ .

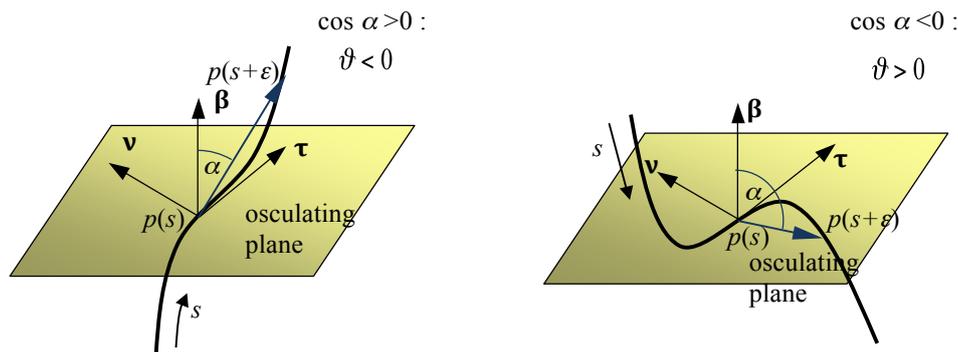


Figure 1.6: Torsion of a curve.

This result is intrinsic, i.e. it does not depend upon the choice of the parameter, hence of the positive orientation of the curve; in fact,  $\boldsymbol{\nu}$  is intrinsic, but changing the orientation of the curve,  $\boldsymbol{\tau}$ , and hence  $\boldsymbol{\beta}$ , change of orientation.

## 1.8 Osculating sphere and circle

The *osculating sphere*<sup>2</sup> to a curve at a point  $p$  is a sphere to which the curve tends to adhere in the neighborhood of  $p$ . Mathematically speaking, if  $q_s$  is the center of the sphere relative to point  $p(s)$ , then

$$|p(s + \varepsilon) - q_s|^2 = |p(s) - q_s|^2 + o(\varepsilon^3). \tag{1.58}$$

<sup>2</sup>The word *osculating* comes from the latin word *osculo* that means to kiss; an osculating sphere or circle or plane is a geometric object that is very close to the curve, as close as two lovers are in a kiss.

Using this definition, discarding the terms of order  $o(\varepsilon^3)$  and using the Frenet-Serret formulae, we get:

$$\begin{aligned}
|p(s + \varepsilon) - q_s|^2 &= |p(s) - q_s + \varepsilon p' + \frac{1}{2}\varepsilon^2 p'' + \frac{1}{6}\varepsilon^3 p''' + o(\varepsilon^3)|^2 = \\
&= |p(s) - q_s + \varepsilon \boldsymbol{\tau} + \frac{1}{2}\varepsilon^2 c \boldsymbol{\nu} + \frac{1}{6}\varepsilon^3 (c\boldsymbol{\nu})' + o(\varepsilon^3)|^2 = \\
&= |p(s) - q_s|^2 + 2\varepsilon(p(s) - q_s) \cdot \boldsymbol{\tau} + \varepsilon^2 + \varepsilon^2 c(p(s) - q_s) \cdot \boldsymbol{\nu} + \\
&\quad \frac{1}{3}\varepsilon^3(p(s) - q_s) \cdot (c'\boldsymbol{\nu} - c^2\boldsymbol{\tau} - c\vartheta\boldsymbol{\beta}) + o(\varepsilon^3),
\end{aligned} \tag{1.59}$$

which gives

$$\begin{aligned}
(p(s) - q_s) \cdot \boldsymbol{\tau} &= 0, \\
(p(s) - q_s) \cdot \boldsymbol{\nu} &= -\frac{1}{c} = -\rho, \\
(p(s) - q_s) \cdot \boldsymbol{\beta} &= -\frac{c'}{c^2\vartheta} = \frac{\rho'}{\vartheta},
\end{aligned} \tag{1.60}$$

and finally

$$q_s = p + \rho \boldsymbol{\nu} - \frac{\rho'}{\vartheta} \boldsymbol{\beta}, \tag{1.61}$$

so the center of the sphere belongs to the normal plane; the sphere is not defined for a plane curve.  $\rho$  is the *radius of curvature* of the curve, defined as

$$\rho = \frac{1}{c}. \tag{1.62}$$

The radius of the osculating sphere is

$$\rho_s = |p - q_s| = \sqrt{\rho^2 + \left(\frac{\rho'}{\vartheta}\right)^2}. \tag{1.63}$$

The intersection between the osculating sphere and the osculating plane at a same point  $p$  is the *osculating circle*. This circle has the property to share the same tangent in  $p$  with the curve and its radius is the radius of curvature,  $\rho$ . From eq. (1.61) we get the position of the osculating circle center  $q$ :

$$q = p + \rho \boldsymbol{\nu}. \tag{1.64}$$

The osculating circle is a diametral circle of the osculating sphere only when  $q = q_s$ , so if and only if

$$\frac{\rho'}{\vartheta} = -\frac{c'}{c^2\vartheta} = 0, \tag{1.65}$$

i.e. when the curvature is constant.

## 1.9 Exercises

1. The curve whose polar equation is

$$r = a \theta, \quad a \in \mathbb{R},$$

is an *Archimede's spiral*. Find its curvature, its length for  $\theta \in [0, 2\pi)$  and prove that any straight line passing by the origin is divided by the spiral in segments of constant length  $2\pi a$  (that is why it is used to record disks).

2. The curve whose polar equation is

$$r = a e^{b\theta}, \quad a, b \in \mathbb{R},$$

is the *logarithmic spiral*. Prove that the origin is an asymptotic point of the curve, find its curvature and the length of the segment in which a straight line by the origin is divided by two consecutive intersections with the spiral. Then prove that the curve is plane and its *equiangular property*:  $(p(\theta) - o) \cdot \boldsymbol{\tau}(\theta) = \text{const.}$

3. The curve whose parametric equation is

$$p(\theta) = a(\cos \theta + \theta \sin \theta)\mathbf{e}_1 + a(\sin \theta - \theta \cos \theta)\mathbf{e}_2$$

with the parameter  $\theta$  the angle formed by  $p(\theta) - o$  with the  $x_1$ -axis is the *involute of the circle*. Find its curvature and length for  $\theta \in [0, 2\pi)$  and prove that the geometrical set of the points  $p(\theta) + \rho(\theta)\boldsymbol{\nu}(\theta)$  is exactly the circle of center  $o$  and radius  $a$  (that is why the involute of the circle is used to profile engrenages).

4. The curve whose parametric equation is

$$p(\theta) = a \cos \omega\theta \mathbf{e}_1 + a \sin \omega\theta \mathbf{e}_2 + b\omega\theta \mathbf{e}_3$$

is a *helix* that winds on a circular cylinder of radius  $a$ . Show that the angle formed by the helix and any generatrix of the cylinder is constant (a property that defines a helix in the general case). Then, find its length for  $\theta \in [0, 2\pi)$ , curvature, torsion and pitch (the distance, on a same generatrix, between two successive intersections with the helix). Prove then the *Bertrand's theorem*: a curve is a cylindrical helix if and only if the ratio  $c/\vartheta = \text{const.}$  Finally, prove that for the above circular helix there are two constants  $A$  and  $B$  such that

$$p' \times p'' = A\mathbf{u}(\theta) + B\mathbf{e}_3,$$

with

$$\mathbf{u} = \sin \omega\theta \mathbf{e}_1 - \cos \omega\theta \mathbf{e}_2;$$

find then  $A$  and  $B$ .

5. For the curve whose cylindrical equation is

$$\begin{cases} r = 1, \\ z = \sin \theta \end{cases}$$

find the highest curvature and determine whether or not it is planar.



# Chapter 2

## Cables

### 2.1 Introduction

We define as a *cable* a special type of continuum body: a curve endowed with a linear mass density. This definition is not sufficient to characterize mechanically a cable (also a curved rod geometrically is a curve): the fundamental relation mechanically defining a cable must be given in terms of internal actions, which is done in the next section.

A cable is called *inextensible* if the distance

$$\Delta s = s_B - s_A \tag{2.1}$$

measured along the cable between two arbitrary points  $p_A$  and  $p_B$  of the curve, determined on it by two curvilinear abscissae  $s_B > s_A$ , is constant at any time  $t$ . If this does not happen, the cable is *extensible*.

### 2.2 Mechanical definition of a cable

We use the *Euler Sections Principle*, valid for any continuum body, to define mechanically a cable. This principle states simply that *a body is in equilibrium if and only if any of its parts is in equilibrium*.

For the whole body, this necessitates the equilibrium of all the external forces applied to the body, actions and reactions. But if one isolates, ideally, a part of the body, then only a part of the whole external forces will be directly applied to the isolated portion of the body, so, generally speaking, the equilibrium of the isolated part will be restored only by admitting the existence of some actions, called the *internal actions*, that are transmitted to the isolated part by the remaining parts of the body, throughout the contact zones. Physically speaking, these internal actions are the resultant and resultant moment of the contact forces, i.e. of the stresses that the different parts of the body exchange at any point of it. The Euler's section in the case of a curve reduces of course to a point, called the *separation point*.

This concept is shared by all the continuum bodies, that differ for the definition of the type of internal actions that they can develop at any point of the continuum.

We give then the following mechanical definition:

*a cable is a curvilinear unidimensional continuum body whose internal actions that two ideally separated parts of it exchange through the separation point are statically equivalent to a single tension force applied exactly in correspondence of the separation point.*

Such a tensile force is currently named *tension in the cable* and will be indicated in the following by  $\boldsymbol{\theta}(s)$ , and its norm, the intensity of the force, by  $\theta(s)$ .

This is the physical property that mechanically defines a cable in our model; it is true for extensible or inextensible cables. Let us see now what are the consequences of this definition.

To this purpose, we write the statics balance equations for the part (1) of a cable of length  $\ell$ , from  $p_0 = p(s = 0)$ , the beginning of the cable, to the point  $p(s)$ , see Fig. 2.1. For the while, we assume that only distributed loads, whose linear intensity is  $\mathbf{f}(s)$ , act upon the cable.  $\boldsymbol{\theta}(s)$  is the tension that the part (2) of the cable, that between  $p(s)$  and  $p_1 = p(s = \ell)$  apply to the part (1):

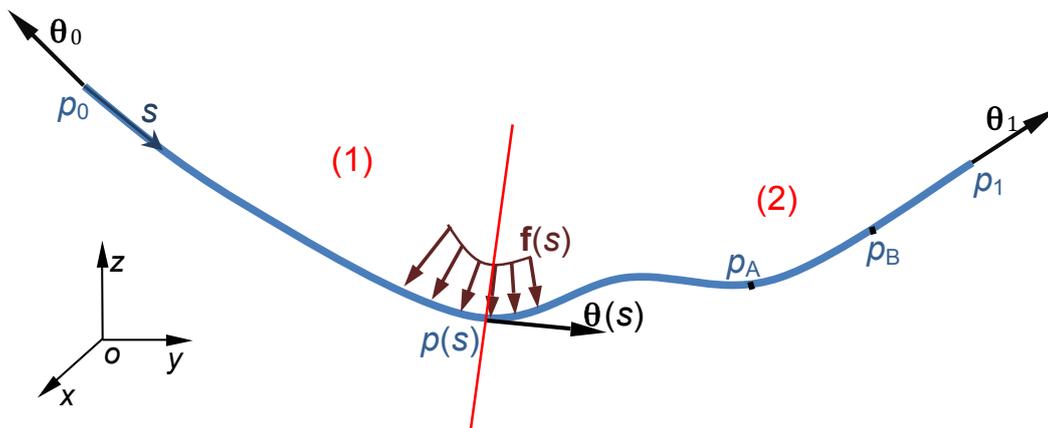


Figure 2.1: General scheme of a cable

- balance of the forces:

$$\boldsymbol{\theta}_0 + \int_0^s \mathbf{f}(s^*) ds^* + \boldsymbol{\theta}(s) = \mathbf{o}, \quad (2.2)$$

- balance of the moments, with respect to a point  $o$ :

$$(p_0 - o) \times \boldsymbol{\theta}_0 + \int_0^s (p(s^*) - o) \times \mathbf{f}(s^*) ds^* + (p(s) - o) \times \boldsymbol{\theta}(s) = \mathbf{o}. \quad (2.3)$$

Differentiating these equations, we get the *local balance equations* for the cable:

$$\begin{aligned} \mathbf{f}(s) + \frac{d\boldsymbol{\theta}(s)}{ds} &= \mathbf{o}, \\ (p(s) - o) \times \mathbf{f}(s) + \frac{d}{ds}[(p(s) - o) \times \boldsymbol{\theta}(s)] &= \mathbf{o}. \end{aligned} \quad (2.4)$$

After differentiating the second term in eq. (2.4)<sub>2</sub> we get

$$\begin{aligned} (p(s) - o) \times \mathbf{f}(s) + \boldsymbol{\tau}(s) \times \boldsymbol{\theta}(s) + (p(s) - o) \times \frac{d\boldsymbol{\theta}(s)}{ds} &= \mathbf{o} \rightarrow \\ (p(s) - o) \times \left( \mathbf{f}(s) + \frac{d\boldsymbol{\theta}(s)}{ds} \right) + \boldsymbol{\tau}(s) \times \boldsymbol{\theta}(s) &= \mathbf{o}, \end{aligned} \quad (2.5)$$

which gives, through the (2.4)<sub>1</sub>,

$$\boldsymbol{\tau}(s) \times \boldsymbol{\theta}(s) = \mathbf{o} \quad \forall s, \quad (2.6)$$

so that

$$\boldsymbol{\theta}(s) = \theta(s) \boldsymbol{\tau}(s) \quad \forall s. \quad (2.7)$$

Then, at the equilibrium the tension is a force which is constantly tangent to the cable. Because the fundamental assumption is that the tension is a tensile force, it is necessarily

$$\theta(s) \geq 0 \quad \forall s. \quad (2.8)$$

Eqs. (2.7) and (2.8) defines mechanically a cable. To remark that a cable cannot transmit couples; in fact, the moment of  $\boldsymbol{\theta}(s)$  with respect to  $p(s)$  vanishes identically  $\forall s$ , as a consequence of the fact that  $\boldsymbol{\theta}(s)$  is applied exactly in  $p(s)$ . Finally, a cable is a unidimensional continuum body without compression, shear and bending stiffness, but with only tension stiffness.

In a dual approach, we could assume eqs. (2.7) and (2.8) as the mathematical conditions defining a cable. Then, the local balance of the moments is always fulfilled because, through eqs. (2.4)<sub>1</sub>, (2.7) and (2.8) we get

$$\begin{aligned} (p(s) - o) \times \mathbf{f}(s) + \frac{d}{ds}[(p(s) - o) \times \boldsymbol{\theta}(s)] &= \mathbf{o} \rightarrow \\ - (p(s) - o) \times \frac{d}{ds}[\theta(s)\boldsymbol{\tau}(s)] + \boldsymbol{\tau}(s) \times \theta(s)\boldsymbol{\tau}(s) + \\ (p(s) - o) \times \frac{d}{ds}[\theta(s)\boldsymbol{\tau}(s)] &= \mathbf{o} \text{ identically.} \end{aligned} \quad (2.9)$$

Finally, for cables it is the balance of the forces, eq. (2.4)<sub>1</sub>, that defines equilibrium. The balance of moments, eq. (2.4)<sub>2</sub>, can be used either to define the tension  $\boldsymbol{\theta}(s)$ , eq. (2.7), or, if this last is assumed *a priori*, it is automatically satisfied  $\forall s$ .

Intrinsically, eq. (2.4)<sub>1</sub> states that the equilibrium of the cable, no matter if it is extensible or not, is possible only thanks to the configuration of this last: in fact, applying the above vector equation to the infinitesimal interval  $[s, s + ds]$ , equilibrium can be satisfied, if  $\mathbf{f}(s)$ ,  $\boldsymbol{\theta}(s)$  and  $\boldsymbol{\theta}(s + ds)$  are not collinear, only thanks to a curvature of the cable: the statics of cables is a *matter of geometry*. So, if the cable is free, it will find a configuration able to ensure the equilibrium under the action of the applied loads, while if it is wrapped on a surface, the configuration of the cable on the surface will develop contact forces equilibrated with the tension of the cable at the ends.

The balance equations of the whole cable are:

$$\begin{aligned} \boldsymbol{\theta}_0 + \int_0^\ell \mathbf{f}(s) ds + \boldsymbol{\theta}_1 &= \mathbf{o}, \\ (p_0 - o) \times \boldsymbol{\theta}_0 + \int_0^\ell (p(s) - o) \times \mathbf{f}(s) ds + (p_1 - o) \times \boldsymbol{\theta}_1 &= \mathbf{o}. \end{aligned} \quad (2.10)$$

In the previous equations, it is  $\boldsymbol{\theta}_0 = \boldsymbol{\theta}(s = 0)$  and  $\boldsymbol{\theta}_1 = \boldsymbol{\theta}(s = \ell)$ . From eq. (2.10), we get immediately that

$$\boldsymbol{\theta}_0 = -\theta(s = 0)\boldsymbol{\tau}(s = 0), \quad \boldsymbol{\theta}_1 = \theta(s = \ell)\boldsymbol{\tau}(s = \ell). \quad (2.11)$$

## 2.3 The intrinsic balance equations

The local force balance equation can be projected onto the intrinsic frame of Frenet-Serret. From the first formula of Frenet-Serret (1.46) we get

$$\mathbf{f}(s) + \frac{d}{ds}[\theta(s) \boldsymbol{\tau}(s)] = \mathbf{f}(s) + \frac{d\theta(s)}{ds}\boldsymbol{\tau}(s) + c(s) \theta(s) \boldsymbol{\nu}(s) = \mathbf{o}. \quad (2.12)$$

If we put, in the Frenet-Serret frame,  $\mathbf{f} = (f_\tau, f_\nu, f_\beta)$ , we obtain the three scalar *intrinsic balance equations* (Jc. Bernoulli, 1698):

$$\begin{aligned} \frac{d\theta}{ds} + f_\tau &= 0, \\ c \theta + f_\nu &= 0, \\ f_\beta &= 0. \end{aligned} \quad (2.13)$$

To remark that  $f_\nu < 0$ . In several problems, however, the equilibrium configuration is not known and constitutes, together with  $\theta(s)$ , the unknown of the problem.

The intrinsic equations state that the cable finds always a configuration of equilibrium where, pointwise, the component of the external load on  $\boldsymbol{\beta}$  is null; i.e. the loads belong to the osculating plane.

The first of (2.13) gives us a general result: whenever  $f_\tau = 0$ ,  $\theta(s) = \text{const}$ . The second one also gives us an interesting result: if  $f_\nu = 0$ , then  $c \theta = 0$  and there are two possible cases: the first one is  $\theta \neq 0 \forall s \Rightarrow c = 0 \forall s$ : the equilibrium configuration is a straight line. So, e.g., all the parts of a cable that are unloaded assume as configuration of equilibrium a straight line. The other possibility is  $\theta = 0 \forall s \Rightarrow c$  can take any value, i.e. the equilibrium configuration is undetermined: in the case of null tension, the cable can take any possible configuration, all of them are equilibrated; this is the case, e.g., of a cable simply lying on a horizontal plane, without tension forces applied to its ends.

## 2.4 Forces depending upon a potential

Be  $\mathbf{f}(s) = \nabla U$ ; then

$$f_\tau = \mathbf{f}(s) \cdot \boldsymbol{\tau}(s) = \nabla U \cdot \boldsymbol{\tau}(s) = \nabla U \cdot \frac{dp(s)}{ds} = \frac{\partial U}{\partial x} \frac{dx}{ds} + \frac{\partial U}{\partial y} \frac{dy}{ds} + \frac{\partial U}{\partial z} \frac{dz}{ds} = \frac{dU}{ds}. \quad (2.14)$$

So

$$f_\tau + \frac{d\theta}{ds} = 0 \rightarrow \frac{d(\theta + U)}{ds} = 0 \Rightarrow \theta + U = \text{const}. \quad (2.15)$$

This means that  $\theta$  is maximum where  $U$  is minimum; that is why in a cable in equilibrium under the action of its own weight the highest tension is in correspondence of its highest point.

## 2.5 Parallel and coplanar forces

Be  $\mathbf{f}(s) = f(s) \mathbf{e}$ , with  $\mathbf{e}$  a constant unit vector. The balance of forces between  $s = 0$  and  $s$  gives then

$$\theta(s)\boldsymbol{\tau}(s) = \theta(0)\boldsymbol{\tau}(0) - \int_0^s f(s^*)ds^* \mathbf{e}. \quad (2.16)$$

By consequence,  $\boldsymbol{\theta}(s) = \theta(s)\boldsymbol{\tau}(s)$  is a linear combination of  $\boldsymbol{\tau}(0)$  and  $\mathbf{e} \forall s$ , i.e. the curve lies in the plane passing through  $p(0)$  and containing  $\mathbf{e}$ .

This result is valid also for the case of coplanar forces, i.e. of forces of the type  $\mathbf{f}(s) = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ , with  $\mathbf{e}_1$  and  $\mathbf{e}_2$  two constant unit vectors. In fact:

$$\theta(s)\boldsymbol{\tau}(s) = \theta(0)\boldsymbol{\tau}(0) - \int_0^s f_1(s^*)ds^* \mathbf{e}_1 - \int_0^s f_2(s^*)ds^* \mathbf{e}_2. \quad (2.17)$$

In addition, this is true also for the case of concentrated forces, see below.

## 2.6 Concentrated forces

The local and intrinsic balance equations are valid uniquely if  $\mathbf{f}(s)$  is regular, i.e. if it has not discontinuities, like in the case of concentrated forces.

We can tackle such a problem in the following way: be  $\mathbf{f}_p(s)$  a distribution of forces in the interval  $(s - \varepsilon, s + \varepsilon)$ ; we define the concentrated load associated to  $\mathbf{f}_p(s)$  as the vector

$$\mathbf{F} = \lim_{\varepsilon \rightarrow 0^+} \int_{s-\varepsilon}^{s+\varepsilon} \mathbf{f}_p(s^*)ds^*. \quad (2.18)$$

In such a way, the concentrated force is treated as a particular distributed load over an interval that tends toward zero. Writing the balance of the forces between  $s - \varepsilon$  and  $s + \varepsilon$  we get

$$\theta(s + \varepsilon)\boldsymbol{\tau}(s + \varepsilon) - \theta(s - \varepsilon)\boldsymbol{\tau}(s - \varepsilon) + \int_{s-\varepsilon}^{s+\varepsilon} \mathbf{f}(s^*)ds^* + \int_{s-\varepsilon}^{s+\varepsilon} \mathbf{f}_p(s^*)ds^* = \mathbf{o}, \quad (2.19)$$

whose limit for  $\varepsilon \rightarrow 0$  gives

$$\mathbf{F} = \theta^- \boldsymbol{\tau}^- - \theta^+ \boldsymbol{\tau}^+, \quad (2.20)$$

where  $\theta^- = \lim_{\varepsilon \rightarrow 0} \theta(s - \varepsilon)$  etc. This law let us see that the presence of a concentrated load produces, generally speaking, a discontinuity of both  $\theta$ , the tension, and  $\boldsymbol{\tau}$ , the direction of the cable.

The above equation is nothing but the rule of parallelogram of the forces, which implies that the three forces are coplanar, see Fig. 2.2.

The discontinuity on  $\theta(s)$  vanishes if and only if  $\mathbf{F}$  acts along the bissectrice of the angle formed by  $\boldsymbol{\tau}^-$  and  $-\boldsymbol{\tau}^+$ .

Equation (2.20) must be written in correspondence of each concentrated load, while in all the other parts, the local or intrinsic balance equations govern the problem.

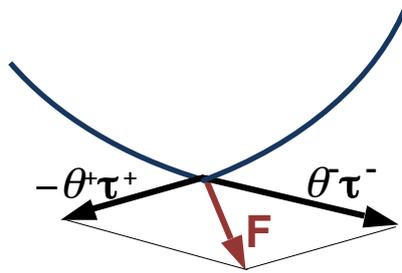


Figure 2.2: Concentrated loads.

## 2.7 Cables on surfaces

Let us consider the case of a cable wrapped on a surface, which gives the reaction of contact  $\boldsymbol{\phi}$ ; we assume that  $\boldsymbol{\phi}$  is much greater than all the other distributed forces, like the weight of the cable, so that all of them are negligible with respect to  $\boldsymbol{\phi}$ . In such a situation, the intrinsic balance equations become

$$\begin{aligned} \frac{d\theta}{ds} + \phi_\tau &= 0, \\ c\theta + \phi_\nu &= 0, \\ \phi_\beta &= 0. \end{aligned} \tag{2.21}$$

In the case of a frictionless contact between the cable and the surface, if this last is regular and of unit normal  $\mathbf{N}$ , it is

$$\begin{aligned} \boldsymbol{\phi}(s) \times \mathbf{N}(s) &= \mathbf{o}, \\ \boldsymbol{\tau}(s) \cdot \mathbf{N}(s) &= 0, \end{aligned} \Rightarrow \phi_\tau = 0 \quad \forall s. \tag{2.22}$$

Hence, by the (2.21)<sub>1</sub>, we get

$$\frac{d\theta}{ds} = 0, \tag{2.23}$$

i.e. the tension is constant everywhere in the cable, and in particular it is equal to the tensile loads applied to the ends of the cable. Moreover, because  $\phi_\tau = 0$ ,

$$\boldsymbol{\phi}(s) = \phi(s) \boldsymbol{\nu}(s) = -\phi(s)\mathbf{N}(s) \iff \boldsymbol{\nu}(s) \times \mathbf{N}(s) = \mathbf{o} \quad \forall s. \tag{2.24}$$

Then, by the *Theorem of Bernstein*, the cable wraps the surface along one of its geodesic lines.

Let us now suppose that there is friction in the contact between the cable and the surface, and let assume the Coulomb's non-slipping condition:

$$\sigma |\phi_N(s)| \geq |\phi_\tau(s)| \quad \forall s, \quad (2.25)$$

with  $\sigma$  the *friction coefficient*. If the cable is wrapped on the surface along a geodetic line, then  $\boldsymbol{\nu}(s) = \mathbf{N}(s)$ , so we get

$$\sigma |\phi_\nu(s)| \geq |\phi_\tau(s)| \quad \forall s, \quad (2.26)$$

and by the intrinsic equations (2.21) we get

$$\sigma c \theta \geq \left| \frac{d\theta}{ds} \right| \rightarrow \sigma c \geq \left| \frac{1}{\theta} \frac{d\theta}{ds} \right| = \left| \frac{d \log \theta}{ds} \right|. \quad (2.27)$$

For the whole cable we get hence the overall non slipping condition

$$\int_0^\ell \sigma c(s) ds \geq \int_0^\ell \left| \frac{d \log \theta(s)}{ds} \right| ds \geq \left| \int_0^\ell \frac{d \log \theta(s)}{ds} ds \right| = \left| \log \frac{\theta(\ell)}{\theta(0)} \right|, \quad (2.28)$$

with  $\ell$  the winding length of the cable on the surface. If now we assume  $\theta(\ell) > \theta(0)$ , then we obtain the non-slipping condition that links the tension at  $s = \ell$  with that at  $s = 0$ :

$$\theta(\ell) \leq \theta(0) e^{\int_0^\ell \sigma c(s) ds}. \quad (2.29)$$

This condition depends upon the friction coefficient, the winding length and the curvature of the cable.

## 2.8 Applications

### 2.8.1 The catenary

The *catenary* is the equilibrium curve of an inextensible, homogeneous cable that is supported at the ends and that is acted upon uniquely by its own weight<sup>1</sup>, see Fig. 2.3.

Be  $\ell$  the length of the catenary, which is fixed in  $p_0 = (x_0, y_0)$  and  $p_1 = (x_1, y_1)$ ,  $|p_0 - p_1| < \ell$ . The only force is

$$\mathbf{f}(s) = -\mu g \mathbf{e}_2, \quad (2.30)$$

with  $\mu$  the mass per unit length of the cable; because  $\mathbf{e}_2$  is a constant vector, the equilibrium curve is contained in a vertical plane, the one passing by  $p_0$  and  $p_1$ . Looking for the

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<sup>1</sup>The problem of the catenary has been very important in the history of mechanics and mathematics, because it is one of the problems at the origin of both the differential calculus and of the calculus of variations. It was proposed by Jc. Bernoulli to scientists in 1690, and besides his solution, he obtained different methods of solution by his brother, Jh. Bernoulli, Leibniz and Huygens.

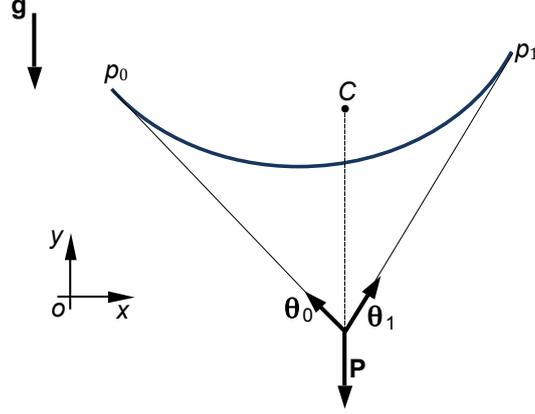


Figure 2.3: The catenary.

equilibrium curve in the form  $y = y(x)$  we have

$$\begin{aligned}
 p - o &= x\mathbf{e}_1 + y(x)\mathbf{e}_2, \\
 \boldsymbol{\tau} &= (\mathbf{e}_1 + y'\mathbf{e}_2) \frac{1}{\sqrt{1 + y'^2}}, \\
 \boldsymbol{\nu} &= (-y'\mathbf{e}_1 + \mathbf{e}_2) \frac{1}{\sqrt{1 + y'^2}}, \\
 c &= \frac{y''}{(1 + y'^2)^{\frac{3}{2}}}.
 \end{aligned} \tag{2.31}$$

Then,

$$\begin{aligned}
 f_{\boldsymbol{\tau}} &= \mathbf{f} \cdot \boldsymbol{\tau} = -\mu g \frac{y'}{\sqrt{1 + y'^2}}, \\
 f_{\boldsymbol{\nu}} &= \mathbf{f} \cdot \boldsymbol{\nu} = -\frac{\mu g}{\sqrt{1 + y'^2}},
 \end{aligned} \tag{2.32}$$

so the intrinsic balance equations are

$$\begin{aligned}
 \frac{d\theta}{ds} &= \mu g \frac{y'}{\sqrt{1 + y'^2}}, \\
 \theta \frac{y''}{(1 + y'^2)^{\frac{3}{2}}} &= \frac{\mu g}{\sqrt{1 + y'^2}}.
 \end{aligned} \tag{2.33}$$

To obtain  $y(x)$  we project the local balance equation onto  $\mathbf{e}_1$ , and because  $\mathbf{e}_1$  is a constant vector we get

$$\frac{d\theta}{ds} \cdot \mathbf{e}_1 + \mathbf{f} \cdot \mathbf{e}_1 = 0 \rightarrow \frac{d\theta}{ds} \cdot \mathbf{e}_1 - \mu g \mathbf{e}_2 \cdot \mathbf{e}_1 = 0 \tag{2.34}$$

which gives the following first integral of the problem:

$$\boldsymbol{\theta} \cdot \mathbf{e}_1 = \text{const} := \theta_0, \tag{2.35}$$

where the scalar  $\theta_0$  is the horizontal component of the tension in the cable.  $\theta_0$  is hence a constant of the problem; in particular, it is the horizontal component of the reactions in  $p_0$  and  $p_1$  and the tension in the cable in its lowest point. Moreover,

$$\boldsymbol{\theta} \cdot \mathbf{e}_1 = \frac{\theta}{\sqrt{1+y'^2}} = \theta_0 \Rightarrow \theta = \theta_0 \sqrt{1+y'^2}, \quad (2.36)$$

that injected into eq. (2.33)<sub>2</sub> gives the differential equation of the equilibrium curve:

$$\frac{y''}{\sqrt{1+y'^2}} = \frac{\mu g}{\theta_0}. \quad (2.37)$$

To solve the above equation, we put

$$y'' = \frac{dy'}{dx} \quad (2.38)$$

and write it as

$$\frac{dy'}{\sqrt{1+y'^2}} = \frac{\mu g}{\theta_0} dx, \quad (2.39)$$

whose solution is

$$\operatorname{arcsinh} y' = \frac{\mu g}{\theta_0} x + c_1 \rightarrow y' = \sinh \left( \frac{\mu g}{\theta_0} x + c_1 \right), \quad (2.40)$$

and integrating again

$$y = \frac{\theta_0}{\mu g} \cosh \left( \frac{\mu g}{\theta_0} x + c_1 \right) + c_2. \quad (2.41)$$

This is the equation of the catenary. The three constant  $c_1, c_2$  and  $\theta_0$  are determined using the two boundary conditions

$$y(x = x_0) = y_0, \quad y(x = x_1) = y_1, \quad (2.42)$$

and the condition that the length of the equilibrium curve is  $\ell$ , because the cable is inextensible:

$$\begin{aligned} \ell &= \int_{p_0}^{p_1} ds = \int_{x_0}^{x_1} \sqrt{1+y'^2} dx = \int_{x_0}^{x_1} \sqrt{1 + \sinh^2 \left( \frac{\mu g}{\theta_0} x + c_1 \right)} dx = \\ &= \int_{x_0}^{x_1} \cosh \left( \frac{\mu g}{\theta_0} x + c_1 \right) dx = \frac{\theta_0}{\mu g} \left[ \sinh \left( \frac{\mu g}{\theta_0} x_1 + c_1 \right) - \sinh \left( \frac{\mu g}{\theta_0} x_0 + c_1 \right) \right], \end{aligned} \quad (2.43)$$

We can now find the value of the tension  $\theta(y)$ : injecting the expression (2.40) of  $y'$  into eq. (2.36) we get

$$\theta = \theta_0 \sqrt{1 + \sinh^2 \left( \frac{\mu g}{\theta_0} x + c_1 \right)} = \theta_0 \cosh \left( \frac{\mu g}{\theta_0} x + c_1 \right), \quad (2.44)$$

but for the (2.41), it is

$$\cosh \left( \frac{\mu g}{\theta_0} x + c_1 \right) = \frac{\mu g}{\theta_0} (y - c_2), \quad (2.45)$$

and finally

$$\theta = \mu g(y - c_2). \quad (2.46)$$

The tension in the cable is hence a linear function of the vertical position of the cable, so it is minimum at the lowest point, where it is equal to  $\theta_0$ , and maximum for  $\max\{y_0, y_1\}$ , which could be predicted because  $\mathbf{f}$  depends upon a potential.

As a particular case, we consider a cable where  $x_0 = 0, x_1 = \alpha\ell, 0 < \alpha < 1, y_0 = y_1 = 0$ . If we put  $|p_1 - p_0| = x_1 - x_0 = L$ , then  $\alpha = L/\ell$ . In such a situation, the equations that determine  $c_1$  and  $c_2$  become

$$\begin{aligned} c_2 &= -\frac{\theta_0}{\mu g} \cosh c_1, & c_1 &= -\frac{\mu g \alpha \ell}{2\theta_0}, \\ c_2 &= -\frac{\theta_0}{\mu g} \cosh\left(\frac{\mu g}{\theta_0} \alpha \ell + c_1\right), & c_2 &= -\frac{\theta_0}{\mu g} \cosh\left(-\frac{\mu g \alpha \ell}{2\theta_0}\right). \end{aligned} \quad (2.47)$$

To determine  $\theta_0$ , we inject these results in eq. (2.43):

$$\begin{aligned} \frac{\theta_0}{\mu g} \left[ \sinh\left(\frac{\mu g}{\theta_0} \alpha \ell + c_1\right) - \sinh c_1 \right] &= \ell \rightarrow \\ \sinh\left(\frac{\mu g \alpha \ell}{2\theta_0}\right) - \sinh\left(-\frac{\mu g \alpha \ell}{2\theta_0}\right) &= \frac{\mu g \ell}{\theta_0}. \end{aligned} \quad (2.48)$$

Because  $\sinh$  is an odd function, i.e.  $\sinh(-x) = -\sinh x \forall x$ , we get the equation

$$\sinh \alpha k = k, \quad k = \frac{\mu g \ell}{2\theta_0}. \quad (2.49)$$

The parameter  $k > 0$  is half the ratio of the total weight of the cable to the minimal tension  $\theta_0$ . Putting

$$\begin{aligned} \xi_1 &= \sinh \alpha k, \\ \xi_2 &= k, \end{aligned} \quad (2.50)$$

the solutions to eq. (2.49) are the intersections of  $\xi_1(k)$  and  $\xi_2(k)$ , see Fig. 2.4.

We observe that  $k$ , the abscissa of the intersection point, increases if  $\alpha$  decreases, i.e. for a looser cable; because  $\mu g \ell / 2 = \text{const.}$ , this implies that  $\theta_0$  increases with  $\alpha$ : the more stretched the cable, the higher  $\theta_0$ . In particular, because for  $\alpha = 1$

$$\left. \frac{d \sinh k}{dk} \right|_0 = 1, \quad (2.51)$$

the tangent in  $k = 0$  to the curve  $\xi_1(k)$  is just  $\xi_2 = k$ ; no other intersections between  $\xi_1$  and  $\xi_2$  are possible for  $\alpha = 1$ , because  $\xi_1(k)$  is always increasing, so for  $\alpha = 1$ , i.e. for a straight equilibrium configuration of the cable, the only possible solution is  $k = 0$ , which means that  $\theta \rightarrow \infty$ . Because this is physically impossible, *a cable can never take a rectilinear equilibrium configuration under the only action of its own weight.*

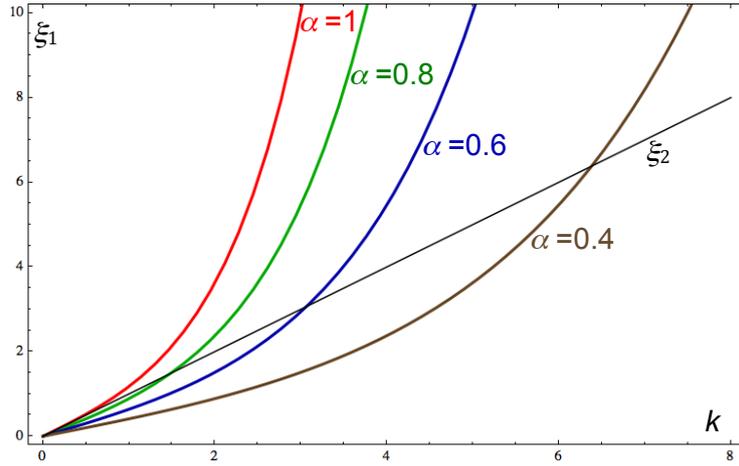


Figure 2.4: The tension in the cable as function of  $\alpha$ .

The Taylor's expansion of the catenary's equation (2.41) around a point  $\bar{x}$  is

$$y = \frac{\theta_0}{\mu g} \cosh\left(\frac{\mu g}{\theta_0}\bar{x} + c_1\right) + c_2 + \sinh\left(\frac{\mu g}{\theta_0}\bar{x} + c_1\right)(x - \bar{x}) + \frac{1}{2} \frac{\mu g}{\theta_0} \cosh\left(\frac{\mu g}{\theta_0}\bar{x} + c_1\right)(x - \bar{x})^2 + o(|x - \bar{x}|^3). \quad (2.52)$$

The terms of order  $o(|x - \bar{x}|^3)$  decrease with the ratio  $\mu g/\theta_0$ , so, for the same unit mass  $\mu$ , with the stretch of the cable. Finally, the above equation states that for a stretched cable, the catenary can be well approximated by a parabola.

From the overall balance of the forces on the cable, it is immediate to see that the tangents to the catenary in  $p_0$  and  $p_1$  cross exactly in correspondence of the vertical drawn from the barycenter  $C$  of the curve, see Fig. 2.3. The weight  $\mathbf{P}$  of the cable is in fact decomposed by the two tensions  $-\boldsymbol{\theta}_0$  and  $\boldsymbol{\theta}_1$ ; this property was discovered by I. G. Pardies in 1673, well before the finding of the catenary's equation.

If the catenary is put upside down, it becomes the equilibrium curve of a body able to transmit only normal compressive forces: the inverse catenary is hence the equilibrium curve of a masonry or stone or concret arch of constant section submitted uniquely to its own weight. This property was observed by Hooke, that published it under the form of an anagram in 1675<sup>2</sup>, and later confirmed analytically by Jc. Bernoulli in 1704.

In 1740, L. Euler discovered that the catenary is also the *catenoid*, i.e. the curve joining two points at different distances from an axis that, when turned about this axis, produces the surface of revolution of least area.

There is a more elegant way to find the equation of the catenary, that using the calculus of variation. In fact, the equilibrium configuration can be found applying the *Torricelli's principle*: a body acted upon only by its weight is in a stable configuration of equilibrium when its barycenter occupies the lowest position.

<sup>2</sup>Translated from Latin, the anagram gives the statement: as hangs the flexible line, so but inverted will stand the rigid arch (J. Heyman, *The stone skeleton*, Cambridge University Press, 1995).

The position of the barycenter  $C$  is

$$y_C = \frac{\int_{p_0}^{p_1} y ds}{\int_{p_0}^{p_1} ds}, \quad (2.53)$$

and by eq. (1.20) and because the cable is inextensible, so that  $\ell = \int_{p_0}^{p_1} ds = \text{const.}$ , the problem can be reduced to

$$\min J = \int_{p_0}^{p_1} y \sqrt{1 + y'^2} dx \quad (2.54)$$

with the isoperimetric constraint

$$\int_{p_0}^{p_1} \sqrt{1 + y'^2} dx = \ell. \quad (2.55)$$

Applying the Lagrange multiplier's method, we minimize the functional

$$\min J^* = \int_{p_0}^{p_1} f(y, y'; \psi) dx; \quad (2.56)$$

with

$$f(y, y'; \psi) = (y + \psi) \sqrt{1 + y'^2}. \quad (2.57)$$

The Euler-Lagrange equation with respect to  $y$  (the one relative to  $\psi$  gives the isoperimetric constraint),

$$\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0, \quad (2.58)$$

because  $f$  does not depend explicitly upon  $x$ , gives the condition<sup>3</sup>

$$\frac{d}{dx} \left( f - y' \frac{\partial f}{\partial y'} \right) = 0 \Rightarrow f - y' \frac{\partial f}{\partial y'} = \gamma, \quad \gamma \in \mathbb{R}. \quad (2.59)$$

From eq. (2.57) we obtain

$$\frac{\partial f}{\partial y'} = (y + \psi) \frac{y'}{\sqrt{1 + y'^2}}, \quad (2.60)$$

that injected into eq. (2.59) gives

$$y + \psi = \gamma \sqrt{1 + y'^2} \rightarrow \frac{dy}{\sqrt{(y + \psi)^2 - \gamma^2}} = \frac{dx}{\gamma}. \quad (2.61)$$

---

<sup>3</sup> This is a classical result in analytical mechanics and can be shown introducing the expression of  $\frac{\partial f}{\partial y}$  obtained by the Euler-Lagrange equation into  $\frac{df(y, y')}{dx}$ :

$$\frac{\partial f}{\partial y} = \frac{d}{dt} \frac{\partial f}{\partial y'} \rightarrow \frac{df}{dt} = \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial y'} \frac{dy'}{dt} = \frac{d}{dt} \frac{\partial f}{\partial y'} \frac{dy}{dt} + \frac{\partial f}{\partial y'} \frac{dy'}{dt} = \frac{d}{dt} \left( y' \frac{\partial f}{\partial y'} \right) \rightarrow \frac{d}{dt} \left( f - y' \frac{\partial f}{\partial y'} \right) = 0.$$

The quantity in the brackets of the last expression is  $-H$ , the opposite of the integral of Jacobi of the problem. In the end, the constant  $\gamma$  is the opposite of  $H$ .

Putting

$$y + \psi = \gamma \cosh z \rightarrow dy = \gamma \sinh z dz \quad (2.62)$$

we get

$$\frac{\gamma \sinh z}{\gamma \sqrt{\cosh^2 z - 1}} dz = \frac{dx}{\gamma} \rightarrow dz = \frac{dx}{\gamma} \rightarrow z = \frac{x}{\gamma} + \beta, \beta \in \mathbb{R}. \quad (2.63)$$

Because

$$z = \operatorname{arccosh} \frac{y + \psi}{\gamma}, \quad (2.64)$$

we obtain finally

$$y = \gamma \cosh \left( \frac{x}{\gamma} + \beta \right) - \psi. \quad (2.65)$$

This equation coincides with the equation (2.41) of the catenary, with  $\gamma = \frac{\theta_0}{\mu g}$ ,  $\beta = c_1$  and  $\psi = -c_2$ .

## 2.8.2 The suspension bridge

The problem of the suspended bridge (solved by Beeckman en 1615) is similar to that of the catenary, the difference is that the load is uniformly distributed not along the cable, but along its horizontal projection<sup>4</sup>:

$$\mathbf{q}(x) = -q\mathbf{e}_2, \quad q > 0. \quad (2.66)$$

So, now the load to be used into the intrinsic equations

$$\mathbf{f}(s) = -f(s)\mathbf{e}_2, \quad f(s) > 0 \quad \forall s, \quad (2.67)$$

is unknown. It can be deduced as follows:

$$\begin{aligned} \int_{s_1}^{s_2} f(s) ds &= \int_{x_1}^{x_2} f(s) \sqrt{1 + y'^2} dx = \\ \int_{x_1}^{x_2} q dx &= q(x_2 - x_1) \rightarrow f(s) = \frac{q}{\sqrt{1 + y'^2}}. \end{aligned} \quad (2.68)$$

Now, we can proceed just like for the catenary: the intrinsic balance equations are

$$\begin{aligned} \frac{d\theta}{ds} &= q \frac{y'}{1 + y'^2}, \\ \theta \frac{y''}{(1 + y'^2)^{\frac{3}{2}}} &= \frac{q}{1 + y'^2}. \end{aligned} \quad (2.69)$$

From the first equation, we obtain once more the same first integral (2.35), having exactly the same mechanical meaning, and by consequence also eq. (2.36), that injected into eq. (2.69)<sub>2</sub> gives

$$y'' = \frac{q}{\theta_0} \rightarrow y = \frac{q}{2\theta_0} x^2 + c_1 x + c_2. \quad (2.70)$$

---

<sup>4</sup>This is justified by the fact that the deck of a suspended bridge normally has a weight per unit length  $q$  which is far greater than that of the cable,  $\lambda$ , so that to a first approximation this last can be neglected.

The equilibrium curve is hence a parabola; once more, the constants  $c_1, c_2$  and  $\theta_0$  are determined using the conditions

$$\begin{aligned} y(x = x_0) &= y_0, \\ y(x = x_1) &= y_1, \\ \ell &= \int_{p_0}^{p_1} ds = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx. \end{aligned} \tag{2.71}$$

This last, once solved, gives the explicit condition

$$\begin{aligned} \frac{\theta_0}{2q} \left[ \left( \frac{q}{\theta_0} x_1 + c_1 \right) \sqrt{1 + \left( \frac{q}{\theta_0} x_1 + c_1 \right)^2} - \left( \frac{q}{\theta_0} x_0 + c_1 \right) \sqrt{1 + \left( \frac{q}{\theta_0} x_0 + c_1 \right)^2} + \right. \\ \left. \log \frac{\frac{q}{\theta_0} x_1 + \sqrt{1 + \left( \frac{q}{\theta_0} x_1 + c_1 \right)^2}}{\frac{q}{\theta_0} x_0 + \sqrt{1 + \left( \frac{q}{\theta_0} x_0 + c_1 \right)^2}} \right] = \ell. \end{aligned} \tag{2.72}$$

The tension can still be recovered using eq. (2.36), that in this case gives

$$\theta(x) = \theta_0 \sqrt{1 + \left( \frac{q}{\theta_0} x + c_1 \right)^2}. \tag{2.73}$$

If, like in the case of the catenary, we consider a cable where  $x_0 = 0, x_1 = \alpha\ell, 0 < \alpha < 1, y_0 = y_1 = 0$ , then it is easy to check that now

$$\begin{aligned} c_1 &= -\alpha \zeta, \\ c_2 &= 0, \\ 2\alpha \zeta \sqrt{1 + \alpha^2 \zeta^2} + \log \left( \frac{2\alpha \zeta}{\sqrt{1 + \alpha^2 \zeta^2}} + 1 \right) &= 4\zeta, \end{aligned} \tag{2.74}$$

where

$$\zeta = \frac{q\ell}{2\theta_0}. \tag{2.75}$$

### 2.8.3 The curve of uniform vertical load

We consider a cable wrapped on a rigid plane profile described by the function  $y(x)$ ; the cable is stretched by a tension  $\theta$  applied at its ends, so that the cable is everywhere in contact with the profile. The weight of the cable is negligible and the contact between the cable and the profile is frictionless. We want to determine  $y(x)$  so that the vertical component of the contact force between the cable and the profile is constant everywhere.

Because there is no friction, the reaction on the cable is of the type

$$\boldsymbol{\varphi} = -\varphi \boldsymbol{\nu}, \quad \varphi > 0; \tag{2.76}$$

so,  $f_\tau = 0$ ,  $f_\nu = -\varphi$  and by the first intrinsic balance equation (2.13) we get  $\theta = \text{const.}$  and of course equal to the tension applied at the ends of the cable. Hence, this problem cannot be equal to the previous one of the suspension bridge, though at a first sight it could seem to be.

The second intrinsic equation gives now

$$\varphi = c \theta, \quad (2.77)$$

while  $\nu$  has components

$$\nu = \frac{1}{\sqrt{1+y'^2}}(-y', 1). \quad (2.78)$$

The condition that determines the shape  $y(x)$  of the curve is

$$-\varphi \cdot \mathbf{e}_2 = p, \quad p > 0, \quad (2.79)$$

so by eq. (2.76)

$$\varphi \nu \cdot \mathbf{e}_2 = p \rightarrow \frac{\varphi}{\sqrt{1+y'^2}} = p, \quad (2.80)$$

that inserted into eq. (2.77) gives the differential equation

$$\frac{y''}{(1+y'^2)^2} = \frac{p}{\theta}. \quad (2.81)$$

This differential equation has not a solution in closed form, and it should be resolved numerically; nevertheless, if the curve is slowly changing, i.e. if  $|y'| \ll 1$ , then

$$y'' = \frac{p}{\theta} \rightarrow y = \frac{p}{2\theta}x^2 + c_1x + c_2. \quad (2.82)$$

So, in such an approximation, the solution is a parabola. The two constants  $c_1$  and  $c_2$  can be determined using the boundary conditions:  $y(x_0) = y_0$ ,  $y(x_1) = y_1$ . The value of the vertical load  $p$  can be obtained easily: for a parabola whose equation is  $y = a x^2 + c_1x + c_2$ , because in the above approximation  $c \simeq y'' = 2a$ , we get comparing with eq. (2.82),

$$p = 2a\theta, \quad (2.83)$$

so it depends linearly upon the tension that stretches the cable.

## 2.8.4 The cable coiled on a rough cylinder

We consider the case of a cable coiled on a rough cylinder, whose radius is  $R$ ; the friction coefficient is  $\sigma$  and we assume that the cable is coiled on a helix of pitch  $2\pi b$ , the helix being a geodesic of the cylinder. The cable is pulled by a tension  $\theta_0$  at the end  $s = 0$ ; we want to know what is the highest tension  $\theta_1$  that can be applied to the other end before slipping of the cable on the cylinder.

The equation of the helix is

$$p(\alpha) - o = R \cos \alpha \mathbf{e}_1 + R \sin \alpha \mathbf{e}_2 + b\alpha \mathbf{e}_3, \quad (2.84)$$

$\alpha$  being the winding angle of the cable on the cylinder. The curvature of the helix is constant and equal to

$$c = \frac{R}{R^2 + b^2}, \quad (2.85)$$

so that the non-slipping condition (2.29) becomes

$$\theta_1 \leq \theta_0 e^{\sigma \int_0^\ell \frac{R}{R^2 + b^2} ds} = \theta_0 e^{\sigma \frac{R}{R^2 + b^2} \ell}, \quad (2.86)$$

$\ell$  being the length of the cable in contact with the cylinder. For a circular helix,

$$s = \sqrt{R^2 + b^2} \alpha, \quad (2.87)$$

so we get the non-slipping condition

$$\theta_1 \leq \theta_0 e^{\sigma \frac{R}{\sqrt{R^2 + b^2}} \alpha \ell}, \quad (2.88)$$

with

$$\alpha_\ell = \frac{\ell}{\sqrt{R^2 + b^2}} \quad (2.89)$$

the winding angle. The friction force that the cable can exert depends hence upon the exponential of  $\alpha_\ell$ . For the case of  $b \ll R$ , we get

$$\theta_1 < \sim \theta_0 e^{\sigma \alpha_\ell}. \quad (2.90)$$

Just as an exemple, if  $\sigma = 1/2$  and  $\alpha_\ell = 2\pi$ , we get  $\theta_1 < \sim 23.14\theta_0$ : winding a cable on a rough cylinder is a very effective way to anchor it!

## 2.9 Elastic cables

The results in the previous sections concern inextensible cables; we consider now what happens if the cable is elastic. To fix the ideas, we consider the equation of an elastic cable whose unstretched length is  $\ell$  and whose points are determined by the vector function

$$\mathbf{r}(s) = p(s) - o, \quad s \in [0, \ell], \quad (2.91)$$

and with end points

$$p_0 = p(s = 0) = (0, 0), \quad p_1 = p(s = \ell) = (x_1, y_1). \quad (2.92)$$

Once more the tangent vector is still defined by eq. (1.22), but now, because the cable is extensible,  $|p'(s)| \neq 1$ :

$$\boldsymbol{\tau}(s) = \frac{p'(s)}{\lambda(s)}, \quad (2.93)$$

with

$$\lambda(s) = |p'(s)| \quad (2.94)$$

the *stretch* or *elongation* of the elastic cable. If the cable is inextensible,  $\lambda(s) = 1$ , while generally speaking,

$$\lambda(s) = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}. \quad (2.95)$$

The unique balance equation for an elastic cable is still eq. (2.4)<sub>1</sub>, as it can be easily recognized:

$$\boldsymbol{\theta}'(s) + \mathbf{f}(s) = \mathbf{o}. \quad (2.96)$$

About the constitutive law, we restrict our attention to the case of *linearly elastic cables*. In such a case, the constitutive law, generalizing to elastic cables the classical Hooke's law, is

$$\theta(s) = \kappa(\lambda(s) - 1) \quad (2.97)$$

The parameter

$$\kappa = EA \quad (2.98)$$

is the *stiffness* of the cable,  $E$  being the Young's modulus of the material and  $A$  the area of the cross section of the cable.

### 2.9.1 The catenary of an elastic cable

As an application, we search for the catenary of an elastic cable (problem solved by Routh in 1891); in this case, we can have also  $\ell < \sqrt{x_1^2 + y_1^2}$ , because the cable can be stretched to join two points whose distance is  $> \ell$ <sup>5</sup>. The unique load is still given by eq. (2.30). Projecting eq. (2.96) onto the two axes and taking into account that the directions of the axes are fixed, gives

$$\begin{aligned} (\boldsymbol{\theta}'(s) + \mathbf{f}(s)) \cdot \mathbf{e}_1 = 0 &\rightarrow \boldsymbol{\theta}'(s) \cdot \mathbf{e}_1 = 0 \rightarrow \frac{d}{ds}(\boldsymbol{\theta}(s) \cdot \mathbf{e}_1) = 0 \\ (\boldsymbol{\theta}'(s) + \mathbf{f}(s)) \cdot \mathbf{e}_2 = 0 &\rightarrow \boldsymbol{\theta}'(s) \cdot \mathbf{e}_2 = \mu g \rightarrow \frac{d}{ds}(\boldsymbol{\theta}(s) \cdot \mathbf{e}_2) = \mu g \end{aligned} \quad (2.99)$$

The first of the above equations gives once more the first integral of the problem of the catenary, eq. (2.35): the horizontal component of the tension is still a constant, also for an elastic cable. We can transform eq. (2.99)<sub>1</sub> as follows:

$$\begin{aligned} \frac{d}{ds}(\boldsymbol{\theta} \cdot \mathbf{e}_1) &= \frac{d}{ds}(\theta \boldsymbol{\tau} \cdot \mathbf{e}_1) = \frac{d}{ds} \left( \theta \frac{dx}{ds} \right) = \frac{d}{ds} \left( \frac{\theta dx}{\sqrt{dx^2 + dy^2}} \right) = \\ &= \frac{d}{ds} \left( \frac{\theta \frac{dx}{ds}}{\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}} \right) = \frac{d}{ds} \left( \frac{\theta \frac{dx}{ds}}{\lambda} \right) = 0. \end{aligned} \quad (2.100)$$

In the same way, eq. (2.99)<sub>2</sub> can be transformed to

$$\frac{d}{ds} \left( \frac{\theta \frac{dy}{ds}}{\lambda} \right) = \mu g. \quad (2.101)$$

---

<sup>5</sup>However,  $\ell$  cannot be much less than  $\sqrt{x_1^2 + y_1^2}$  for applying the Hooke's law.

The first integrals of eqs. (2.100) and (2.101) are

$$\begin{aligned}\frac{dx}{ds} &= \lambda \frac{\theta_0}{\theta}, \\ \frac{dy}{ds} &= \frac{V + \mu g s}{\theta} \lambda,\end{aligned}\tag{2.102}$$

with  $\theta_0$  still the horizontal component of the tension and  $V$  the vertical component of the tension at the end  $s = 0$ . To find a solution, we square and add the two equations above, to obtain

$$\theta^2 \left[ \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 \right] = \lambda^2 [\theta_0^2 + (V + \mu g s)^2],\tag{2.103}$$

and through eq. (2.95) finally

$$\theta = \sqrt{\theta_0^2 + (V + \mu g s)^2}.\tag{2.104}$$

Inverting the constitutive law (2.97) we get

$$\lambda = \frac{\theta}{\kappa} + 1,\tag{2.105}$$

that injected into eq. (2.102) gives, through eq. (2.104),

$$\begin{aligned}\frac{dx}{ds} &= \theta_0 \left( \frac{1}{\kappa} + \frac{1}{\theta} \right) = \theta_0 \left( \frac{1}{\kappa} + \frac{1}{\sqrt{\theta_0^2 + (V + \mu g s)^2}} \right), \\ \frac{dy}{ds} &= (V + \mu g s) \left( \frac{1}{\kappa} + \frac{1}{\theta} \right) = (V + \mu g s) \left( \frac{1}{\kappa} + \frac{1}{\sqrt{\theta_0^2 + (V + \mu g s)^2}} \right),\end{aligned}\tag{2.106}$$

These are the differential equations that describe the equilibrium curve; once integrated, they give

$$\begin{aligned}x &= \frac{\theta_0 s}{\kappa} + \frac{\theta_0}{\mu g} \left( \operatorname{arcsinh} \frac{V + \mu g s}{\theta_0} - \operatorname{arcsinh} \frac{V}{\theta_0} \right), \\ y &= \frac{\mu g s}{\kappa} \left( \frac{V}{\mu g} + \frac{s}{2} \right) + \frac{\theta_0}{\mu g} \left( \sqrt{1 + \left( \frac{V + \mu g s}{\theta_0} \right)^2} - \sqrt{1 + \left( \frac{V}{\theta_0} \right)^2} \right).\end{aligned}\tag{2.107}$$

The two still unknown constants  $\theta_0$  and  $V$  can be determined imposing the boundary conditions for  $s = \ell$ :  $x(s = \ell) = x_1$ ,  $y(s = \ell) = y_1$ , which gives the two conditions

$$\begin{aligned}x_1 &= \frac{\theta_0 \ell}{\kappa} + \frac{\theta_0}{\mu g} \left( \operatorname{arcsinh} \frac{V + \mu g \ell}{\theta_0} - \operatorname{arcsinh} \frac{V}{\theta_0} \right), \\ y_1 &= \frac{\mu g \ell}{\kappa} \left( \frac{V}{\mu g} + \frac{\ell}{2} \right) + \frac{\theta_0}{\mu g} \left( \sqrt{1 + \left( \frac{V + \mu g \ell}{\theta_0} \right)^2} - \sqrt{1 + \left( \frac{V}{\theta_0} \right)^2} \right).\end{aligned}\tag{2.108}$$

These two conditions must usually be solved numerically; in the case of  $y_1 = 0$ , i.e. with the two end supports at the same level, we get, from eq. (2.108)<sub>2</sub>, the expected condition

$$V = -\frac{1}{2} \mu g \ell,\tag{2.109}$$

while eq. (2.108)<sub>1</sub> becomes

$$\frac{\mu g \ell}{2\theta_0} = \sinh \left( \frac{\mu g x_1}{2\theta_0} - \frac{\mu g \ell}{2\kappa} \right), \quad (2.110)$$

equation that for  $\kappa \rightarrow \infty$  coincides with eq. (2.49), being  $x_1 = \alpha \ell$ .

In Fig. 2.5 we show the catenaries of different cables: all the cables are hung between the points  $(0,0)$  and  $(0,1)$ , and the unstretched length of the cables is always  $\ell = 1.2$ , while  $\mu g = 1$  (all the dimensions are in appropriate units), while the cables differ for the value of  $\kappa$ . As expected, the more the  $\kappa$ , the less the depth of the cable; the black curve corresponds to the catenary of an inextensible cable.

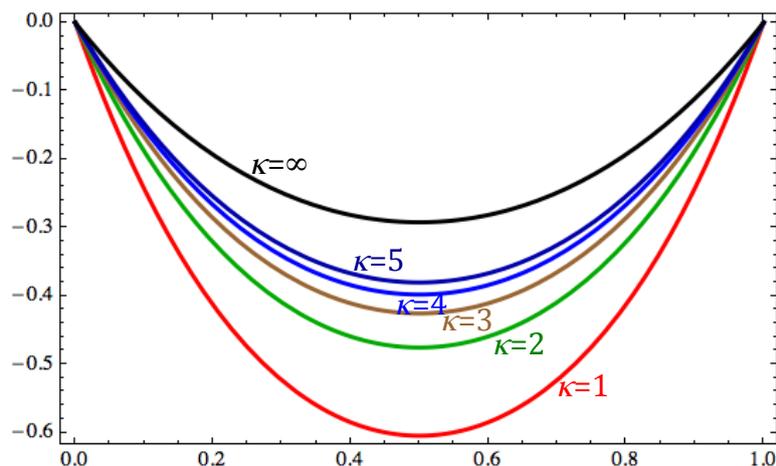


Figure 2.5: Elastic catenaries.

## 2.10 Exercises

1. An inextensible cable of length  $2\ell$  is fixed at two points, e.g. two pitons, at the same height and at a distance  $2d < 2\ell$ . The cable is taut by a concentrated force  $\mathbf{f} = f(\cos \alpha, \sin \alpha)$  at mid distance from the ends, see Fig. 2.6. Determine, as a function of  $\alpha$ , the equilibrium configuration and the tension in each one of the two parts of the cable. Such a system is currently used by alpinists to realize a relais during a climbing; could you say which is the minimum length of the rope of the relais to make it safer than a relais using a single piton (consider a vertical force)?
2. Make the same exercise but this time the cable is linearly elastic with elastic constant  $\kappa$ . Trace the curve describing the dependence of the force upon the displacement of its point (consider a vertical force).
3. The *velaria* is the equilibrium configuration of a cable acted upon uniquely by wind; if the action of the wind can be represented by a uniform load  $p$  acting orthogonally to the cable (a characteristic of all the actions of inviscid fluids), show that the *velaria* is an arc of circle (this problem was solved by Jh. Bernoulli).
4. A variant of the above problem is that of a parallel stream; in such a case, show that the equilibrium configuration of the *velaria* is actually the catenary.

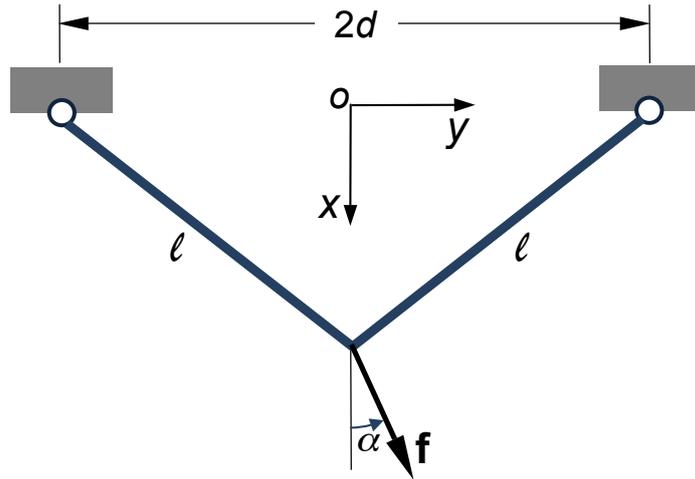


Figure 2.6: Scheme of exercise 1.

5. A cable is coiled on a rough cylinder and the friction coefficient is  $\sigma$ . The cylinder is restrained to rotation by an angular spring whose elastic constant is  $\mu$  and the cable must bear, at the free end, a mass  $m$ , while at the other end it is fixed to the cylinder by a device whose resistance to tension is  $\varphi$ . If the radius of the cylinder is  $R$  and the cable is coiled three times on the cylinder along a helix whose pitch is  $2\pi b \ll R$ , which is the greatest mass that the cable can support?

# Chapter 3

## Arches

### 3.1 Introduction

Arches are plane curved rods. They are currently used in a lot of situations; in civil engineering, the most impressive realizations are some bridges, see Fig. 3.1. Arches are used



Figure 3.1: Some examples of arch bridges.

since long time (probably the most ancient arches in the world are two arches in Italy, at Velia and Volterra, both from the IVth century b.C.), though in the form of arches composed of carved stone voussoirs or masonry bricks. Arches, in fact, have a great advantage with respect to rectilinear rods: while rods are mainly subjected to bending, internal axial forces are dominating in arches: strain energy is stored in the form of bending energy in rods, while mainly, though not exclusively, in the form of extension energy in arches. This is rather advantageous, because axial stiffness and strength are normally much greater than the bending ones: arches bear greater loads than rectilinear loads. A simple scheme gives account of this: let us consider a rectilinear, simply supported rod of span  $\ell$  submitted to a uniform weight  $p$ . The highest bending moment, at mid-span, is  $M_{max} = \frac{p\ell^2}{8}$ . For equilibrium, such a couple must be balanced by an internal couple given by internal stresses. This internal couple has a small lever arm, less than the thickness of the cross section. As a consequence, the cross section must have a rather great thickness, in order

on one side to increase the internal lever arm, on the other side to decrease stresses and bring them under the admissible value.

If now we consider an arch hinged at the ends and submitted to the same load, the horizontal reactions give a couple at mid-span that balances the couple given by  $p$ ; in the limit, the bending moment at mid-span can be null, and the section is submitted exclusively to an axial force equal to the horizontal reactions. This is a much more favorable structural situation, that allows to the structure, for the same cross section and material, to carry more important loads or to cover much longer spans.

This simple case allows also for understating a basic fact: in a simply supported or clamped arch vertical loads produce also horizontal reactions. If the arch is simply supported at one of the two ends, then these horizontal reactions do not exist and the arch is actually just a curved beam, it works almost exclusively in bending: it is the presence of the horizontal reactions that makes a curved rod an effective arch.

In the following, we study the statics of elastic curved rods and then we apply the theory to the statics of arches.

## 3.2 Balance equations

We consider a curvilinear plane rod like in Fig. 3.2;  $s$  is a curvilinear abscissa, that we will put, conventionally, equal to zero at the left end of the rod. We then consider a part of the rod between two sections at the abscissae  $s$  and  $s+ds$  infinitesimally close together, Fig. 3.3, so that we can consider as constant the curvature of the rod in the part  $ds$ , and be  $d\theta$  the infinitesimal angle subtended by the two normals to the rod at  $s$  and  $ds$ , that meet together in  $o$ , the centre of the local osculating circle.

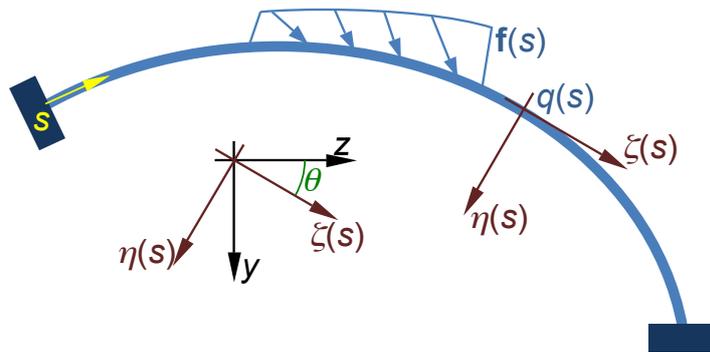


Figure 3.2: General scheme of an arch.

We remain in the framework of small strain and displacements and write the balance equations in the undeformed configuration. Then, referring to Fig. 3.3, it is easy to write the balance equations:

- horizontal equilibrium:

$$T \sin \frac{d\theta}{2} + N \cos \frac{d\theta}{2} - r ds + (T + dT) \sin \frac{d\theta}{2} - (N + dN) \cos \frac{d\theta}{2} = 0; \quad (3.1)$$

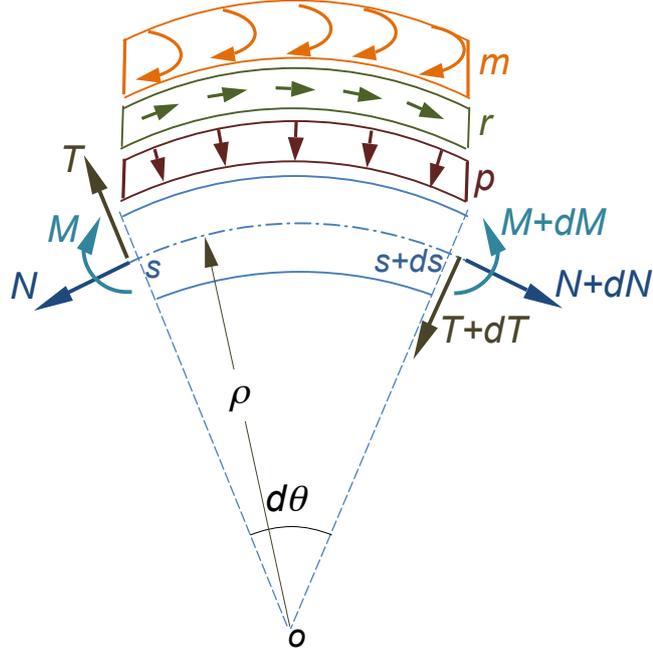


Figure 3.3: General scheme for the balance equations.

- vertical equilibrium:

$$T \cos \frac{d\theta}{2} - N \sin \frac{d\theta}{2} - p ds - (T + dT) \cos \frac{d\theta}{2} - (N + dN) \sin \frac{d\theta}{2} = 0; \quad (3.2)$$

- rotation equilibrium (written here with respect to  $o$ ):

$$M + m ds + (N + dN)\rho + r\rho ds - N\rho - (M + dM) = 0, \quad (3.3)$$

with  $\rho$  the radius of curvature. Because  $ds$  is infinitesimal,  $d\theta \rightarrow 0$  so that  $\cos \frac{d\theta}{2} \simeq 1$  while  $\sin \frac{d\theta}{2} \simeq \frac{d\theta}{2}$ . By consequence, once simplified the balance equations transform to

$$\begin{aligned} dN &= T d\theta - r ds, \\ dT &= -N d\theta - p ds, \\ dM &= \rho dN + \rho r ds + m ds. \end{aligned} \quad (3.4)$$

Injecting eq. (3.4)<sub>1</sub> into eq. (3.4)<sub>3</sub> and considering that  $ds = \rho d\theta$ , we finally get

$$\begin{aligned} \frac{dN}{ds} &= \frac{T}{\rho} - r, \\ \frac{dT}{ds} &= -\frac{N}{\rho} - p, \\ \frac{dM}{ds} &= T + m. \end{aligned} \quad (3.5)$$

Unlike the case of rectilinear rods, now all the balance equations are coupled; to remark that for  $\rho \rightarrow \infty$ , i.e. for a curvilinear rod that tends to become rectilinear, the above equations tend to those of the straight rods.

Introducing now the symbolic matrix

$$\mathbf{D}_1 = \begin{bmatrix} \frac{d}{ds} & -\frac{1}{\rho} & 0 \\ \frac{1}{\rho} & \frac{d}{ds} & 0 \\ 0 & -1 & \frac{d}{ds} \end{bmatrix} \quad (3.6)$$

along with the vectors of the internal actions,  $\mathbf{S}$ , and of the applied loads,  $\mathbf{f}$ ,

$$\mathbf{S} = \begin{Bmatrix} N \\ T \\ M \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} r \\ p \\ m \end{Bmatrix}, \quad (3.7)$$

eq. (3.5) can be given in matrix form:

$$\mathbf{D}_1 \mathbf{S} + \mathbf{f} = \mathbf{o}. \quad (3.8)$$

Implicitly, the above equations have been written in the local frame  $\{\zeta(s), \eta(s)\}$ , see Fig. 3.2. They can be written in a fixed frame  $\{z, y\}$  simply writing that:

$$\mathbf{f} = \mathbf{Q} \mathbf{f}^* : = \begin{Bmatrix} r \\ p \\ m \end{Bmatrix} \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} r^* \\ p^* \\ m^* \end{Bmatrix}, \quad (3.9)$$

where the symbol  $*$  denotes a vector written in the fixed frame  $\{z, y\}$  and  $\mathbf{Q}$  the rotation tensor operating the change of frame from  $\{z, y\}$  to  $\{\zeta(s), \eta(s)\}$ . Of course,  $\mathbf{Q} = \mathbf{Q}(s)$ , because  $\theta = \theta(s)$ .

So, finally, the matrix equation (3.8) written in the fixed frame  $\{z, y\}$  is simply

$$\mathbf{D}_1 \mathbf{S} + \mathbf{Q} \mathbf{f}^* = \mathbf{o}. \quad (3.10)$$

### 3.3 Compatibility equations

Just as for straight rods, we need to link the displacement of the arch to the kinematical quantities defining its deformation. We write such relations in the local frame  $\{\zeta(s), \eta(s)\}$ , see Fig. 3.4.

A point  $q(s)$  of the arch centerline becomes, after deformation, the point  $q'(s)$ , so the displacement vector is

$$\mathbf{u}(s) = q'(s) - q(s) = u_\tau(s) \boldsymbol{\tau}(s) + u_\nu(s) \boldsymbol{\nu}(s). \quad (3.11)$$

We want express the link between the internal kinematical quantities defining the deformation of the arch and the following geometrical quantities, defining the transformation of the cross section of the arch in correspondence of  $q(s)$ :

- $v(s)$ , the displacement along  $\eta(s)$ , i.e. parallel to the principal normal  $\boldsymbol{\nu}(s)$ ;

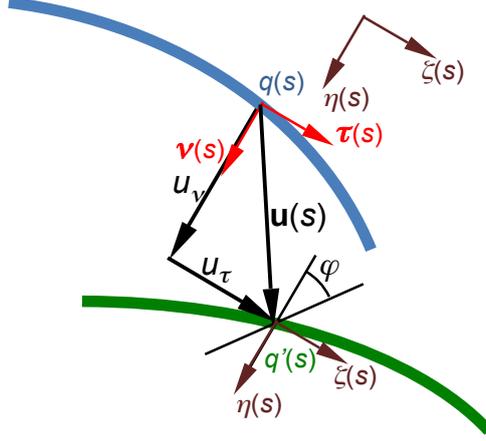


Figure 3.4: General scheme for the compatibility equations.

- $w(s)$ , the displacement along  $\zeta(s)$ , i.e. parallel to the tangent  $\boldsymbol{\tau}(s)$ ;
- $\varphi(s)$ , the rotation of the normal  $\boldsymbol{\nu}(s)$ .

We introduce the following kinematical quantities for describing the deformation of the arch:

- $\varepsilon$ : the extension of the arch mid-line;
- $\kappa$ : the bending curvature of the arch centerline (not to be confused with the *geometric* curvature  $c$  of the arch centerline, i.e. the curvature of the undeformed arch);
- $\gamma$ : the shear of the cross section.

To define and determine these quantities we need to analyze how a generic point of the cross section is transformed. Remembering that the arch is a plane curve and that the problem is planar, i.e. the arch bends in its plane, it is sufficient to study the transformation of a generic point  $\hat{q}(s, \eta)$  of the cross section at the distance  $\eta$  from the centerline (so, actually,  $q(s) = \hat{q}(s, 0)$ ).

We then introduce the displacement of  $\hat{q}(s, \eta)$ :

$$\hat{\mathbf{u}}(s, \eta) = \hat{u}_\tau(s, \eta)\boldsymbol{\tau}(s) + \hat{u}_\nu(s, \eta)\boldsymbol{\nu}(s), \quad (3.12)$$

and considering the scheme of Fig. 3.4 we recognize immediately that, thanks to the assumptions of small deformations and displacements (namely,  $\varphi \rightarrow 0 \Rightarrow \sin \varphi \simeq \varphi$ ,  $\cos \varphi \simeq 1$ ), we can write

$$\begin{aligned} \hat{u}_\tau(s, \eta) &= w(s) - \eta \sin \varphi \simeq w(s) - \eta \varphi, \\ \hat{u}_\nu(s, \eta) &= v(s) - \eta(1 - \cos \varphi) \simeq v(s). \end{aligned} \quad (3.13)$$

We calculate now the derivatives of  $\hat{\mathbf{u}}(s, \eta)$  with respect to  $\eta$  and to the curvilinear abscissa of  $\hat{q}(s, \eta)$ ; this is  $\hat{s} \neq s$ , due to the geometrical curvature of the arch. Nevertheless, see

Fig. 3.5, it is easy to find the relation between  $s$  and  $\hat{s}$ :

$$d\theta = \frac{ds}{\rho} = \frac{d\hat{s}}{\rho - \eta} \rightarrow \frac{ds}{d\hat{s}} = \frac{\rho}{\rho - \eta}, \quad (3.14)$$

relation that gives also

$$\frac{\partial \cdot}{\partial \hat{s}} = \frac{\partial \cdot}{\partial s} \frac{ds}{d\hat{s}} = \frac{\partial \cdot}{\partial s} \frac{\rho}{\rho - \eta}. \quad (3.15)$$

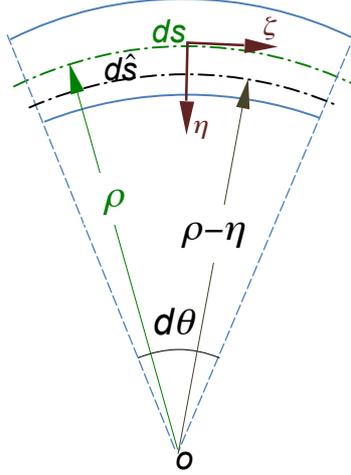


Figure 3.5: Scheme for the relation between  $ds$  and  $d\hat{s}$ .

The derivatives are hence:

$$\begin{aligned} \frac{\partial \hat{\mathbf{u}}}{\partial \hat{s}} &= \frac{\partial v(s)}{\partial \hat{s}} \boldsymbol{\nu}(s) + v(s) \frac{\partial \boldsymbol{\nu}(s)}{\partial \hat{s}} + \frac{\partial (w(s) - \eta \varphi(s))}{\partial \hat{s}} \boldsymbol{\tau}(s) + \\ &\quad (w(s) - \eta \varphi(s)) \frac{\partial \boldsymbol{\tau}(s)}{\partial \hat{s}}, \quad (3.16) \\ \frac{\partial \hat{\mathbf{u}}}{\partial \eta} &= -\varphi(s) \boldsymbol{\tau}(s). \end{aligned}$$

Applying eq. (3.15) to  $\frac{\partial \boldsymbol{\tau}(s)}{\partial \hat{s}}$  and  $\frac{\partial \boldsymbol{\nu}(s)}{\partial \hat{s}}$ , using the Frenet-Serret formulae and remembering that  $\vartheta(s) = 0 \forall s$  because the arch is a plane curve, gives finally (we do not write explicitly the dependence on  $s$  for the sake of conciseness)

$$\frac{\partial \hat{\mathbf{u}}}{\partial \hat{s}} = \frac{\rho}{\rho - \eta} \left( w' - \frac{v}{\rho} - \eta \varphi' \right) \boldsymbol{\tau} + \frac{\rho}{\rho - \eta} \left( v' + \frac{w}{\rho} - \eta \frac{\varphi}{\rho} \right) \boldsymbol{\nu}, \quad (3.17)$$

where the prime denotes differentiation with respect to  $s$ :  $w' = \frac{\partial w}{\partial s}$  etc.

We define now mathematically the kinematical quantities introduced above to describe

the deformation of the arch:

$$\begin{aligned}
\varepsilon_\eta(s, \eta) &:= \left( \frac{\partial \hat{\mathbf{u}}}{\partial \hat{s}} \right)_\tau = \frac{\rho}{\rho - \eta} \left( w' - \frac{v}{\rho} - \eta \varphi' \right), \\
\kappa(s) &:= -\varphi', \\
\gamma_\eta(s, \eta) &:= \left( \frac{\partial \hat{\mathbf{u}}}{\partial \hat{s}} \right)_\nu + \left( \frac{\partial \hat{\mathbf{u}}}{\partial \eta} \right)_\tau = \frac{\rho}{\rho - \eta} \left( v' + \frac{w}{\rho} - \eta \frac{\varphi}{\rho} \right) - \varphi,
\end{aligned} \tag{3.18}$$

that can be rearranged to give finally

$$\begin{aligned}
\varepsilon_\eta(s, \eta) &= \frac{\rho}{\rho - \eta} \left( w' - \frac{v}{\rho} + \eta \kappa \right), \\
\kappa(s) &= -\varphi', \\
\gamma_\eta(s, \eta) &= \frac{\rho}{\rho - \eta} \left( v' + \frac{w}{\rho} - \varphi \right).
\end{aligned} \tag{3.19}$$

For *small curvature arches*, i.e. when the radius of curvature  $\rho$  is far greater than the thickness of the cross section:  $\frac{\rho}{\max \eta} \gg 1 \forall s \Rightarrow \frac{\rho}{\rho - \eta} \rightarrow 1$ , the above equations become

$$\begin{aligned}
\varepsilon_\eta(s, \eta) &= w' - \frac{v}{\rho} + \eta \kappa, \\
\kappa(s) &= -\varphi', \\
\gamma(s) &= v' + \frac{w}{\rho} - \varphi.
\end{aligned} \tag{3.20}$$

The term  $\varepsilon_\eta(s, \eta)$  is the extension of the fibers at a distance  $\eta$  from the centerline; we can write it as

$$\varepsilon_\eta(s, \eta) = \varepsilon(s) + \eta \kappa, \tag{3.21}$$

where

$$\varepsilon(s) = \varepsilon_\eta(s, \eta = 0) = w' - \frac{v}{\rho}. \tag{3.22}$$

Finally, for small curvature arches, the case which is by far the most interesting in applications and that we will consider in the following, the kinematical quantities describing its deformation can be reduced to quantities referring to the deformation of the centerline, and are

$$\begin{aligned}
\varepsilon(s) &= w' - \frac{v}{\rho}, \\
\gamma(s) &= v' + \frac{w}{\rho} - \varphi, \\
\kappa(s) &= -\varphi'.
\end{aligned} \tag{3.23}$$

These are the compatibility equations for small curvature arches. We remark that in the limit case of  $\rho \rightarrow \infty$  the above quantities become

$$\begin{aligned}
\varepsilon(s) &= w', \\
\gamma(s) &= v' - \varphi, \\
\kappa(s) &= -\varphi',
\end{aligned} \tag{3.24}$$

i.e. they coincide, as it must be, with those of the straight rods.

If now we introduce the symbolic matrix

$$\mathbf{D}_2 = \begin{bmatrix} \frac{d}{ds} & -\frac{1}{\rho} & 0 \\ \frac{1}{\rho} & \frac{d}{ds} & -1 \\ 0 & 0 & -\frac{d}{ds} \end{bmatrix} \quad (3.25)$$

along with the vectors of the displacement,  $\boldsymbol{\xi}$ , and of the deformation,  $\boldsymbol{\delta}$ ,

$$\boldsymbol{\xi} = \begin{Bmatrix} w \\ v \\ \varphi \end{Bmatrix}, \quad \boldsymbol{\delta} = \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix}, \quad (3.26)$$

eq. (3.23) can be written in matrix form:

$$\mathbf{D}_2 \boldsymbol{\xi} = \boldsymbol{\delta}. \quad (3.27)$$

The same equation can be written in the fixed frame  $\{z, y\}$ :

$$\mathbf{D}_2 \mathbf{Q} \boldsymbol{\xi}^* = \boldsymbol{\delta}, \quad (3.28)$$

where  $\boldsymbol{\xi}^*$  is the displacement vector written in the frame  $\{z, y\}$  and  $\mathbf{Q}$  is the rotation matrix (3.9).

### 3.4 Constitutive equations

In the framework of linear elasticity, we assume as constitutive equations for plane arches the same of the straight rods:

$$\begin{aligned} N &= EA\varepsilon, \\ T &= \frac{\mu A}{\chi} \gamma, \\ M &= EJ\kappa, \end{aligned} \quad (3.29)$$

with  $E$  the Young's modulus and  $\mu$  the shear modulus of the material,  $A$  the area of the cross section,  $j$  the moment of inertia of the cross section about an horizontal axis passing by the barycenter and  $\chi$  the shear factor of the cross section.

Introducing the matrix

$$\mathbf{C} = \begin{bmatrix} EA & 0 & 0 \\ 0 & \frac{\mu A}{\chi} & 0 \\ 0 & 0 & EJ \end{bmatrix}, \quad (3.30)$$

we can write eq. (3.29) in matrix form:

$$\mathbf{C} \boldsymbol{\delta} = \mathbf{S}. \quad (3.31)$$

## 3.5 The problem of elastic equilibrium for the arches

We can now put together the constitutive, compatibility and equilibrium equations found above to write the equations of the elastic equilibrium for plane arches. Some simple calculations give

$$\begin{aligned} \left[ EA \left( w' - \frac{v}{\rho} \right) \right]' &= \frac{\mu A}{\rho \chi} \left( v' + \frac{w}{\rho} - \varphi \right) - r, \\ \left[ \frac{\mu A}{\chi} \left( v' + \frac{w}{\rho} - \varphi \right) \right]' &= -\frac{EA}{\rho} \left( w' - \frac{v}{\rho} \right) - p, \\ (-EJ\varphi')' &= \frac{\mu A}{\chi} \left( v' + \frac{w}{\rho} - \varphi \right) + m. \end{aligned} \quad (3.32)$$

The above equations show that all the equations are coupled; in particular, unlike the case of straight rods, as an effect of the geometry extension and bending are coupled.

We can put the above equations in a matrix form; to this purpose, we inject successively eqs. (3.31) and (3.27) into eq. (3.8) to get easily

$$\mathbf{D}_1 \mathbf{C} \mathbf{D}_2 \boldsymbol{\xi} + \mathbf{f} = \mathbf{o}. \quad (3.33)$$

To write the same equation in the fixed frame  $\{z, y\}$ , it is sufficient to do the same but with eqs. (3.10), (3.28) and (3.31), and left-multiply by  $\mathbf{Q}^\top$ :

$$\mathbf{Q}^\top \mathbf{D}_1 \mathbf{C} \mathbf{D}_2 \mathbf{Q} \boldsymbol{\xi}^* + \mathbf{f}^* = \mathbf{o}. \quad (3.34)$$

No matter of the form given to the above equations, they remain a system of three second-order coupled linear differential equations, that need 6 boundary conditions, three for each end of the arch. Their solution is, normally, impossible analytically and numerical methods must be used.

## 3.6 Transforming the equations of the arches

The equations found in the previous Sections can be transformed, so as to obtain equations that can be more easily solved; some assumptions on the geometry or the kinematics of the arch can also be introduced, with the same purpose. We consider first the case of the balance equations (3.5), then that of the elastic equilibrium equations (3.32).

### 3.6.1 Transforming the balance equations

For an isostatic arch the balance equations are sufficient to determine the internal actions  $N, T$  and  $M \forall s$ . A typical example, very used in the applications, especially in bridge constructions, is that of a three-hinged arch, see Fig. 3.6.

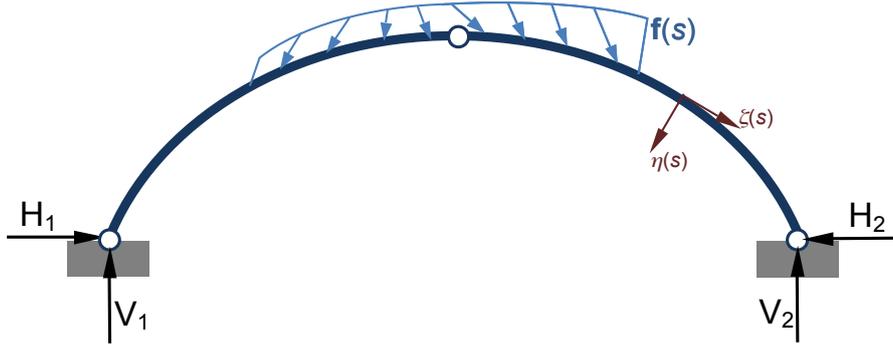


Figure 3.6: Scheme of a three-hinged arch.

We can rearrange the balance equations (3.5) in order to eliminate the shear from eqs. (3.5)<sub>1,3</sub> so as to obtain a system of two linear differential equations. This can be done as follows: from eq. (3.5)<sub>3</sub> we get

$$T = \frac{dM}{ds} - m, \quad (3.35)$$

that injected into eq. (3.5)<sub>1</sub> gives

$$\frac{dN}{ds} = \frac{1}{\rho} \frac{dM}{ds} - \frac{m}{\rho} - r. \quad (3.36)$$

Then, we differentiate eq. (3.5)<sub>3</sub> to obtain

$$\frac{d^2 M}{ds^2} = \frac{dT}{ds} + \frac{dm}{ds} \quad (3.37)$$

and using eq. (3.5)<sub>2</sub> we get

$$\frac{d^2 M}{ds^2} = -\frac{N}{\rho} - p + \frac{dm}{ds}. \quad (3.38)$$

Equations (3.36) and (3.38) constitute a system of two coupled differential equations where the unknowns are the functions  $N(s)$  and  $M(s)$ . The shear force  $T(s)$  can be calculated by eq. (3.35) once  $M(s)$  is known.

In the end, we need 3 boundary conditions for the solution of the equilibrium problem of isostatic arches. Once eqs. (3.36) and (3.38) are solved,  $T(s)$  can be recovered using eq. (3.5)<sub>3</sub>.

Of course, no information about the deformation of the arch is given by the solution of the balance equations; once the problem is solved, one can introduce  $N$ ,  $T$  and  $M$  into the constitutive laws, eqs. (3.29), to obtain  $\varepsilon$ ,  $\gamma$  and  $\kappa$ , that introduced in the compatibility equations (3.23), once integrated, give the components of displacement  $w$ ,  $v$  and  $\varphi$ . Nevertheless, it is normally preferable, at least to find the components of displacement at a specific point, to employ other methods, like the dummy load method.

### 3.6.2 Transforming the elastic equilibrium equations

First of all, for the sake of conciseness, we call

$$\alpha := EA, \quad \beta := \frac{\mu A}{\chi}, \quad \lambda := EJ. \quad (3.39)$$

Then, eqs. (3.32) become

$$\begin{aligned} \left[ \alpha \left( w' - \frac{v}{\rho} \right) \right]' &= \frac{\beta}{\rho} \left( v' + \frac{w}{\rho} - \varphi \right) - r, \\ \left[ \beta \left( v' + \frac{w}{\rho} - \varphi \right) \right]' &= -\frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) - p, \\ (-\lambda \varphi')' &= \beta \left( v' + \frac{w}{\rho} - \varphi \right) + m. \end{aligned} \quad (3.40)$$

We remark that in the limit case of an arch that tends to be a straight rod, i.e. for  $\rho \rightarrow \infty$ , the above equations become

$$\begin{aligned} (\alpha w')' &= -r, \\ [\beta (v' - \varphi)]' &= -p, \\ (-\lambda \varphi')' &= \beta (v' - \varphi) + m, \end{aligned} \quad (3.41)$$

that are just the equations of the Timoshenko's rod model.

We can obtain an equivalent expression of eqs. (3.40), where  $p$  appears directly into the third equation and where the first and third equations contain exclusively the extension and bending stiffnesses,  $\alpha$  and  $\lambda$ . To this purpose, it is sufficient to replace successively eqs. (3.29) and (3.23) into eqs. (3.36) and (3.38) to obtain

$$\begin{aligned} \left[ \alpha \left( w' - \frac{v}{\rho} \right) \right]' &= -\frac{1}{\rho} (\lambda \varphi')' - r - \frac{m}{\rho}, \\ \left[ \beta \left( v' + \frac{w}{\rho} - \varphi \right) \right]' &= -\frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) - p, \\ (\lambda \varphi')'' &= \frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) + p - m'. \end{aligned} \quad (3.42)$$

Unlike the case of the balance equations, it is not possible to uncouple the elastic equilibrium equations. Six boundary conditions complete eqs. (3.42), they specify either the values of  $v$ ,  $w$  and  $\varphi$  or of their derivatives, at the ends of the arch.

### 3.6.3 The Euler-Bernoulli model for arches

Let us now generalize the Euler-Bernoulli rod model to arches, assuming that the cross section remains plane and orthogonal to the deformed centerline of the arch, i.e.

$$\varphi = v'. \quad (3.43)$$

Then, eqs. (3.42) become

$$\begin{aligned} \left[ \alpha \left( w' - \frac{v}{\rho} \right) \right]' &= -\frac{1}{\rho} (\lambda v'')' - r - \frac{m}{\rho}, \\ (\lambda v'')'' &= \frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) + p - m'. \end{aligned} \quad (3.44)$$

The third equation, where the shear stiffness  $\beta$  appears, is now meaningless; the shear  $T$  can be recovered, once solved the above equations, which still needs six boundary conditions, using eq. (3.35).

It is easily checked that for  $\rho \rightarrow \infty$  we get the Euler-Bernoulli rod equations.

### 3.6.4 Arches of constant section

A particular case is that of an arch with a constant section; in such a case  $\alpha' = \beta' = \lambda' = 0$ , so eqs. (3.42) become

$$\begin{aligned} \alpha \left( w' - \frac{v}{\rho} \right)' &= -\frac{1}{\rho} \lambda \varphi'' - r - \frac{m}{\rho}, \\ \beta \left( v' + \frac{w}{\rho} - \varphi \right)' &= -\frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) - p, \\ \lambda \varphi''' &= \frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) + p - m'. \end{aligned} \quad (3.45)$$

and eqs. (3.44)

$$\begin{aligned} \alpha \left( w' - \frac{v}{\rho} \right)' &= -\frac{1}{\rho} \lambda v''' - r - \frac{m}{\rho}, \\ \lambda v^{iv} &= \frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) + p - m'. \end{aligned} \quad (3.46)$$

### 3.6.5 Circular arches

A particularly important case of arches is that of circular arches; in such a case,  $\rho = \text{const.} \Rightarrow \rho' = 0$ , and actually  $\rho$  is just the radius of the circle of which the arch is only a circular segment.

It is interesting to see what happens to the equations for the case of constant section: eqs. (3.45) become

$$\begin{aligned} \alpha \left( w'' - \frac{v'}{\rho} \right) &= -\frac{1}{\rho} \lambda \varphi'' - r - \frac{m}{\rho}, \\ \beta \left( v'' + \frac{w'}{\rho} - \varphi' \right) &= -\frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) - p, \\ \lambda \varphi''' &= \frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) + p - m', \end{aligned} \quad (3.47)$$

while eqs. (3.46) become

$$\begin{aligned}\alpha \left( w'' - \frac{v'}{\rho} \right) &= -\frac{1}{\rho} \lambda v''' - r - \frac{m}{\rho}, \\ \lambda v^{iv} &= \frac{\alpha}{\rho} \left( w' - \frac{v}{\rho} \right) + p - m'.\end{aligned}\tag{3.48}$$

## 3.7 Examples

We give here some examples of the use of the theory developed in this Chapter. All the cases concern circular arches with a constant section. The first two examples have been treated using eqs. (3.47), i.e. the general theory. Then, they have been also treated using eqs. (3.48) of the Euler-Bernoulli model; the results coincide almost perfectly with those reported below, that is why they have not been reported here. The third example concerns an isostatic case, hence the balance equations are sufficient. All the results have been found numerically, using a standard commercial code (Mathematica). All the diagrams in the figures have been normalized, and the deformed shape exaggerated, hence the scale of the diagrams is not real.

### 3.7.1 Example 1

As a first example, we consider the case of a circular arch of constant section loaded by a uniform load  $q = 10$  t/m, with  $\rho = 10$  m, spanning a chord  $\ell = 17.1$  m, see Fig. 3.7. The cross section is rectangular, with the base  $b = 1$  m and the thickness  $h = 2$  m; the Young's modulus is  $E = 200$  MPa, the Poisson's ration  $\nu = 0.2$ , the shear factor  $\chi = 1.2$ . Hence, we get  $\alpha = 4 \times 10^7$  kN,  $\beta = \alpha/2.88 = 1.39 \times 10^8$  kN and  $\lambda = \alpha/3 = 1.33 \times 10^8$  kN  $m^2$ . The ends of the arch are clamped. Hence, the appropriate boundary conditions are  $w = v = \varphi = 0$  at both the ends.

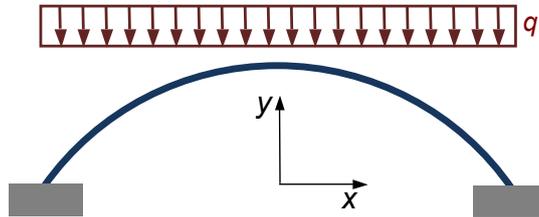


Figure 3.7: Example 1: a uniformly loaded circular arch.

In the following Figures, we report the diagrams, referred to an horizontal axis or to the arch axis, in grey, of  $w$ ,  $v$ ,  $\varphi$ ,  $N$ ,  $T$ ,  $M$  and finally of the deformed shape.  $N$  is negative everywhere, i.e. the arch is compressed all along  $s$ , as it must be, while  $M$  changes of sign, as a consequence of the clamped edges.

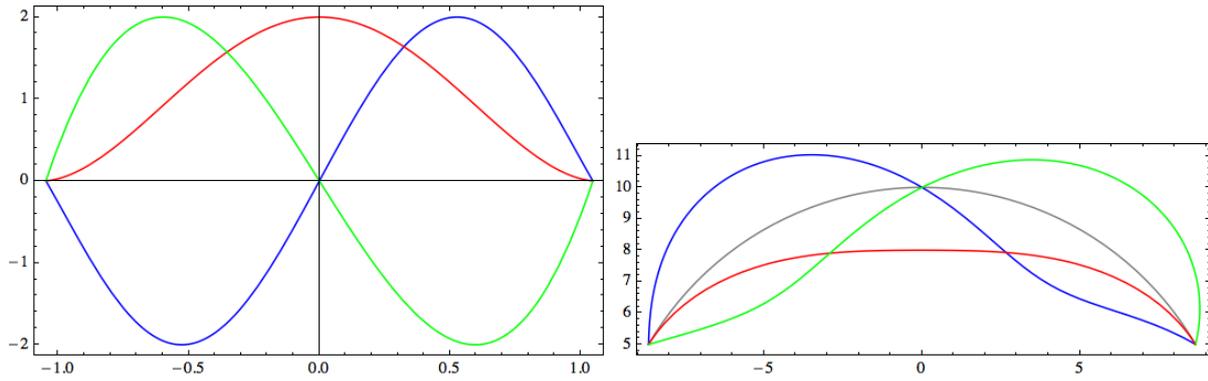


Figure 3.8: Diagrams of  $w$ , blue,  $v$ , red, and  $\varphi$ , green, example 1.

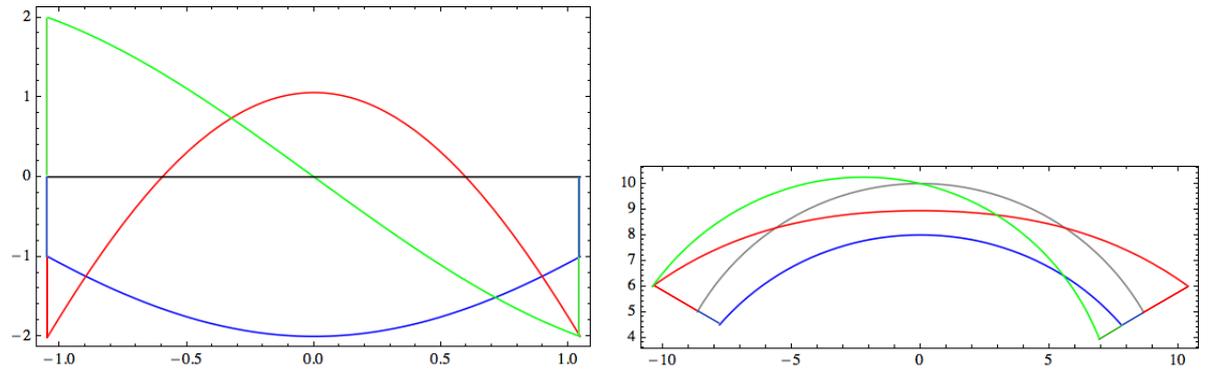


Figure 3.9: Diagrams of  $N$ , blue,  $M$ , red, and  $T$ , green, example 1.

### 3.7.2 Example 2

A second example is that of a semicircular arch, with constant section, acted upon by a vertical force  $F = 5$  kN on the top, see Fig. 3.11. In this case,  $\rho = 0.865$  m,  $\alpha = 1.5 \times 10^7$  MPa,  $\beta = \alpha/2.88$  and  $\lambda = \alpha/12 \times 10^4$ .

Thanks to the symmetry of the problem, we can study just one half of the structure, acted upon by a force equal to  $F/2$  and with a slide as a constraint on the top of the arch. The appropriate boundary conditions are hence: at the left end,  $w = \varphi = 0$ ,  $T = -F/2 \Rightarrow \beta(v' + w/\rho - \varphi) = -F/2$ , while at the right end it is  $w = v = \varphi = 0$ . The following diagrams, referring to only half of the arch, show the results for this example.

### 3.7.3 Example 3

This last example concerns a three-hinge arch, hence an isostatic case. For this reason, we can simply consider the equilibrium equations, either in the form (3.5) or in the form of eqs. (3.36) and (3.38).

The problem is analogous at those of Example 2 and in particular the data are the same, but now the arch has an hinge also at the top, in correspondence of the force  $F$ . Again, we can study just one half of the structure, see Fig. 3.15. The boundary conditions now concern exclusively  $T$  and  $M$ . In fact, no information is given about the value of  $N$  at

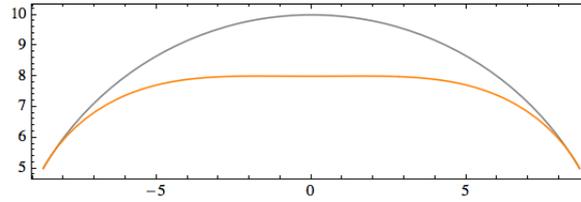


Figure 3.10: Deformed shape of the arch, example 1.

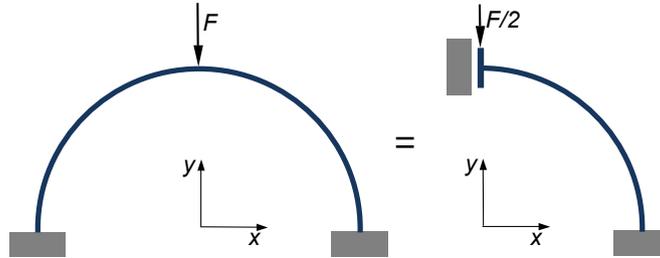


Figure 3.11: Example 2: a circular arch with a concentrated vertical load.

the edges of the arch. In particular, the boundary conditions are: at the left edge,  $M = 0$  and  $T = -F/2$ , while at the right edge  $M = 0$ . If the system of the two differential equations (3.36) and (3.38) is used, then the condition of the shear force must be written in terms of the derivative of  $M$ , eq. (3.5)<sub>3</sub>:  $T = -F/2 \Rightarrow M' - m = -F/2$ . In this case, we have just solved numerically eqs. (3.36) and (3.38), and then obtained  $T$  from (3.5)<sub>3</sub>. The displacement components  $w$ ,  $v$  and  $\varphi$  have been obtained by the procedure described in Sect. 3.6.1; in particular, the three boundary conditions for integrating the compatibility equations (3.23) are  $w = 0$  at the left end and  $w = v = 0$  at the right one. The results are presented in Figs. 3.16 to 3.18. To remark that the rotation  $\varphi$  is almost constant.

We remark that in this particular case the balance equations have an analytical solution, that can be easily found using either eqs. (3.5) or eqs. (3.36) and (3.38):

$$\begin{aligned}
 N &= -\frac{F}{2}(\sin \theta + \cos \theta), \\
 T &= \frac{F}{2}(\sin \theta - \cos \theta), \\
 M &= \frac{F}{2}\rho(1 - \sin \theta - \cos \theta).
 \end{aligned}
 \tag{3.49}$$

### 3.8 Exercices

1. Write the equations of the elastic equilibrium of arches using as variable the angle  $\theta$  of a polar frame instead of  $s$ .
2. Write and solve analytically the balance equations for the circular arch of constant section in Fig. 3.19. Try also to find the functions  $N$ ,  $T$  and  $M$  using basic equilibrium considerations.

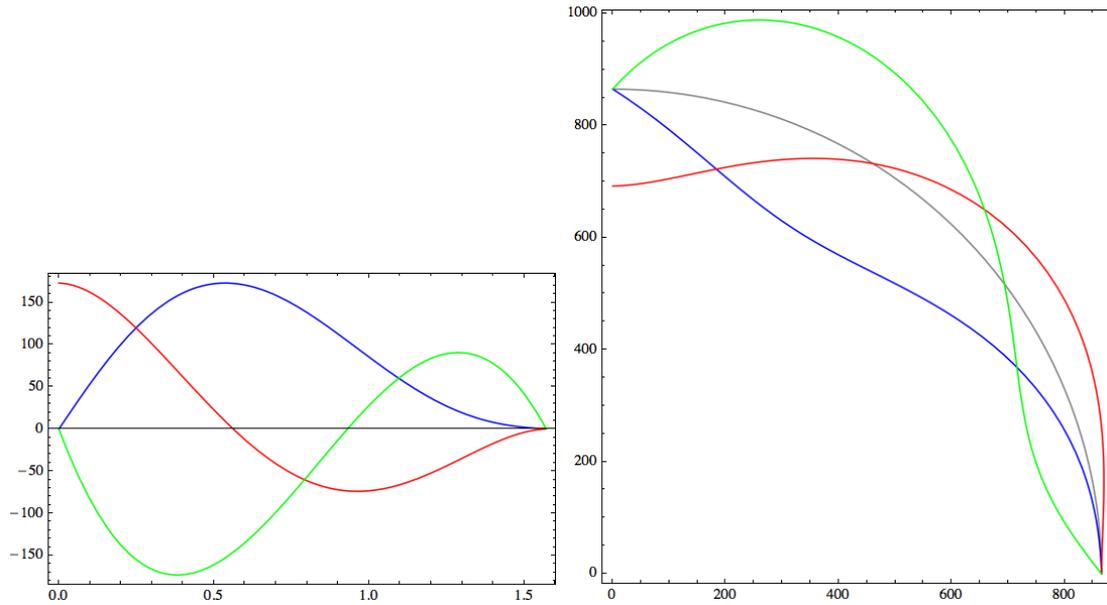


Figure 3.12: Diagrams of  $w$ , blue,  $v$ , red, and  $\varphi$ , example 2.

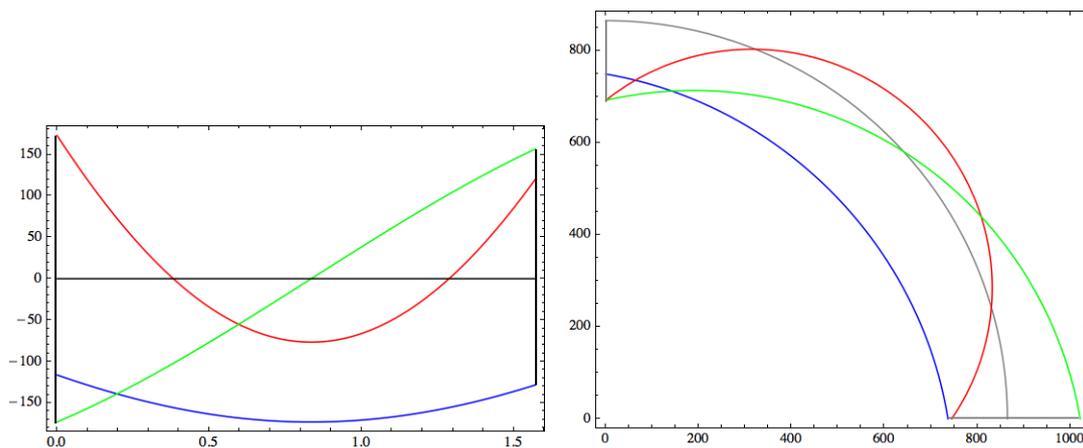


Figure 3.13: Diagrams of  $N$ , blue,  $M$ , red, and  $T$ , green, example 2.

3. For the previous exercise, find the horizontal displacement of the loaded node using the dummy load method ( $A, J, E, \nu$  and  $\chi$  are given).
4. Consider again the Example 1, Sect. 3.7.1; in place of the clamped edges, consider a hinge at the left end and a simple horizontal support at the right one. Using the dummy load method find the displacement of the right edge (neglect the effects of  $N$  and  $T$ ).
5. Still for the Example 1 of Sect. 3.7.1, be  $A = A_0 \cos \theta$ ,  $J = J_0 \cos \theta$  the variation of the cross section area and moment of inertia, respectively, with the angle  $\theta$ , the angle of the polar coordinates in a frame centered in the center of the arch.  $A_0$  and  $J_0$  are the values for  $\theta = 0$  and they coincide with those specified in the Example 1. Using  $\theta$  as independent variable, write the equations of the elastic equilibrium for the two cases of the general theory and of the Euler-Bernoulli model.

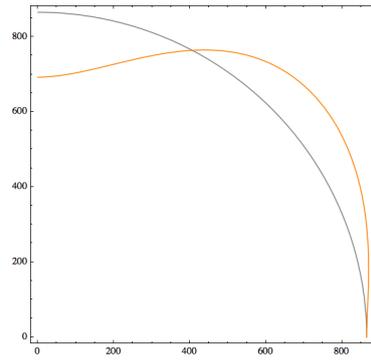


Figure 3.14: Deformed shape of the arch, example 2.

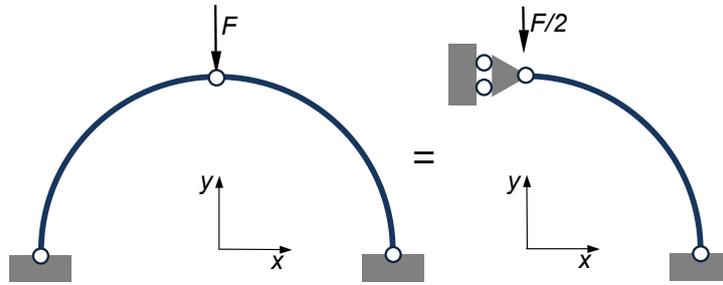


Figure 3.15: Example 3: three-hinged arch.

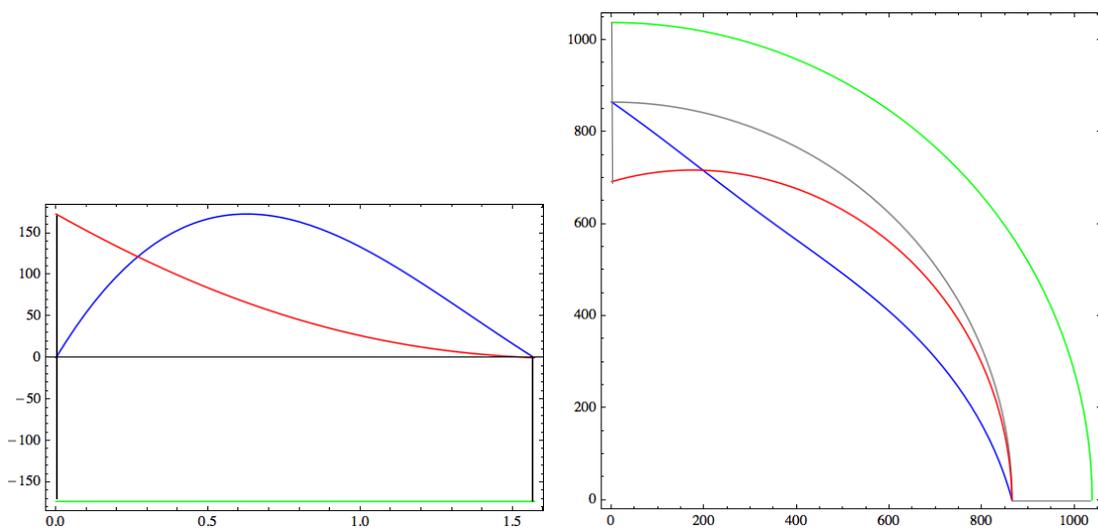


Figure 3.16: Diagrams of  $w$ , blue,  $v$ , red, and  $\varphi$ , example 3.

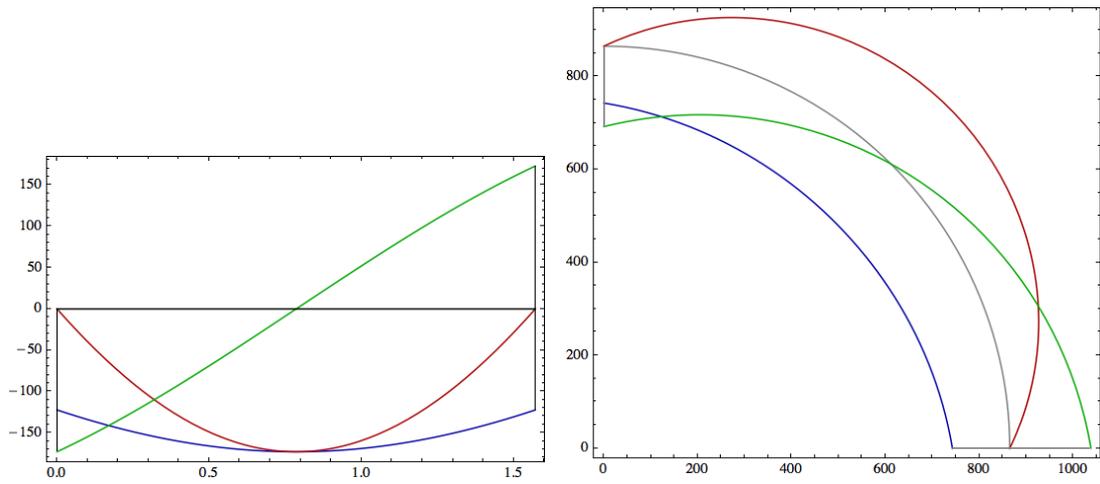


Figure 3.17: Diagrams of  $N$ , blue,  $M$ , red, and  $T$ , green, example 3.

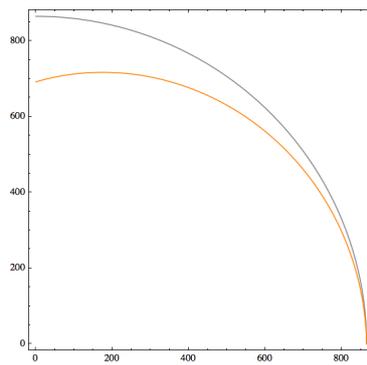


Figure 3.18: Deformed shape of the arch, example 3.

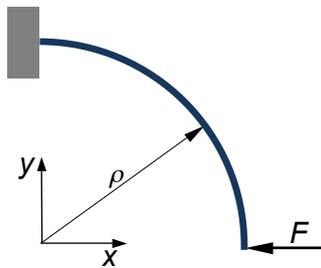


Figure 3.19: Arch of exercise 2.

# Chapter 4

## Plates

### 4.1 Problem definition: basic assumptions

A plate, see Fig. 4.1, is a solid  $\Omega$  bounded by two parallel planes, the upper and lower faces  $S_T$  and  $S_B$ , and by a lateral surface  $S_L$ , orthogonal to the two faces, so that, finally, a plate is just a *flat cylinder*. The *mid-plane* is the plane at equal distance from the two faces. We introduce the orthonormal Cartesian frame  $\{o; x, y, z\}$ , with the axes  $x$  and  $y$  that belong to the mid-plane. Be  $h$  the *thickness* of the plate; for a solid to be really a plate, it must be  $h \ll d$ , with  $d$  the mean chord of the mid-plane. The *contour* of the plate is the line  $\gamma$  intersection of  $S_L$  with the mid-plane. Usually, plates are classified as follows:

- thin plates:  $\frac{h}{d} \lesssim \frac{1}{10}$ ;
- moderately thick plates:  $\frac{1}{10} \lesssim \frac{h}{d} \lesssim \frac{1}{5}$ ;
- thick plates:  $\frac{h}{d} \gtrsim \frac{1}{5}$ .

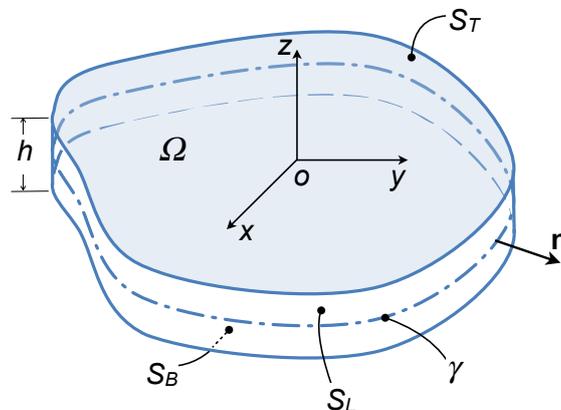


Figure 4.1: General sketch of a plate.

We assume that displacements, rotations and strains are small. As a consequence, the small strain tensor  $\boldsymbol{\varepsilon}$  can be taken as an appropriate measure of deformation and the equilibrium equations can be written in the undeformed configuration  $\Omega$ .

We consider elastic plates, i.e. plates composed by linearly elastic materials whose constitutive law is specified by the Lamé's equations:

$$\boldsymbol{\sigma} = \frac{E}{1 + \nu} \left( \boldsymbol{\varepsilon} + \frac{\nu}{1 - 2\nu} \text{tr} \boldsymbol{\varepsilon} \mathbf{I} \right). \quad (4.1)$$

Finally, the problem so defined is linear.

Concerning the loading, we consider:

- distributed body forces  $\mathbf{b}(p)$  on  $\Omega$ ;
- distributed surface tractions  $\boldsymbol{\tau} \left( x, y, \pm \frac{h}{2} \right)$  on  $S_T$  and  $S_B$ ;
- distributed line forces  $\hat{\mathbf{F}}$  and couples  $\hat{\mathbf{M}}$  on  $S_L$ ;
- distributed couples inside  $\Omega$  are excluded.

In addition, we consider *even distribution of the forces*, with respect to the mid-plane, i.e.

$$\mathbf{b}(x, y, z) = \mathbf{b}(x, y, -z), \quad \boldsymbol{\tau}(x, y, z) = \boldsymbol{\tau}(x, y, -z), \quad (4.2)$$

and in particular

$$\boldsymbol{\tau} \left( x, y, \frac{h}{2} \right) = \boldsymbol{\tau} \left( x, y, -\frac{h}{2} \right). \quad (4.3)$$

There are different possible approaches to the construction of a plate theory; we will follow here a classical *axiomatic approach*, i.e. an approach based upon the *a-priori* (i.e. axiomatic) choice of a displacement field. Such choice corresponds to a kinematical model, characterizing the theory.

Putting  $\mathbf{u} = (u, v, w)$  the *displacement vector*, we need to specify how the three components  $u = u(x, y, z)$ ,  $v = v(x, y, z)$ ,  $w = w(x, y, z)$  of  $\mathbf{u}$  vary, and in particular how they change across the thickness, i.e. along  $z$ . This choice is suggested, of course, by the particular geometry of the plate. In particular, being  $h/d$  a fundamental parameter defining the plate, kinematical models differ from each other as a function of  $h/d$ .

## 4.2 The Kirchhoff model for thin plates

### 4.2.1 The displacement field

In Fig. 4.2 it is schematically represented a cross-section of the plate, along the plane  $x - z$ , before and after the deformation. Let us consider the segment  $AC$ , orthogonal to the mid-plane, with  $B$  its intersection with the mid-plane and  $p$  a point of  $AC$  with the distance  $z$  from the mid-plane. After the deformation each material point of the plate

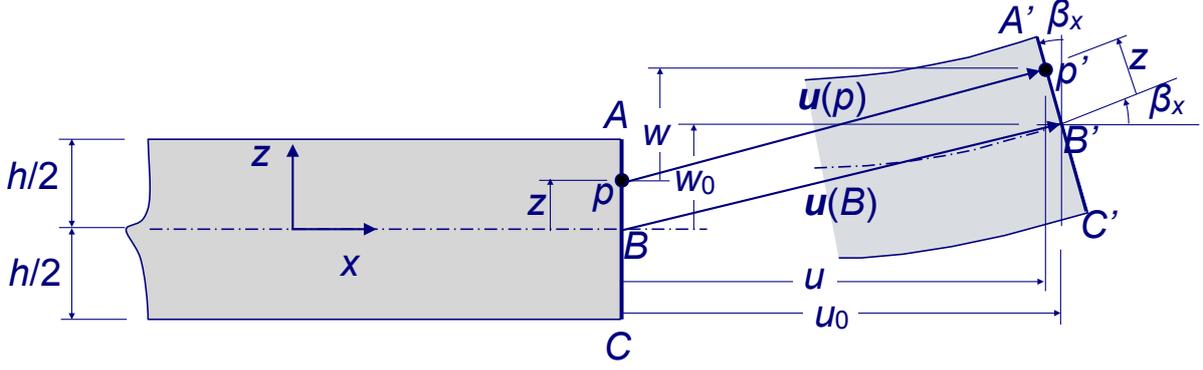


Figure 4.2: Kirchhoff's kinematics in the  $\{x, z\}$  plane.

moves to a new position indicated by a prime:  $p$  goes into  $p'$ ,  $B$  into  $B'$  and so on. We can express the displacement  $\mathbf{u}(p)$  as

$$\mathbf{u}(p) = p' - p = p' - B' + B' - B + B - p = \mathbf{u}(B) - z\mathbf{e}_3 + p' - B', \quad (4.4)$$

i.e. we can decompose the displacement of any point of  $\Omega$  as the sum of the displacement of its projection onto the mid-plane plus a position vector,  $B - p$ , and a local displacement,  $p' - B'$ , which is the displacement of  $p'$  relative to  $B'$ . Because  $B = (x, y, 0)$ ,  $\mathbf{u}(B) = \mathbf{u}(x, y, 0)$ , so we put

$$\mathbf{u}_0 := \mathbf{u}(x, y, 0) = (u_0(x, y), v_0(x, y), w_0(x, y)); \quad (4.5)$$

$\mathbf{u}_0(x, y)$  is a bi-dimensional vector field describing the displacement of the points of the mid-plane. We need now to define the local displacement  $p' - B'$ . It is exactly at this point that a kinematical model is needed.

The *Kirchhoff model* is based upon three assumptions concerning the deformation of any segment  $AC$ . According to the Kirchhoff model, any material segment orthogonal to the mid-plane remains:

- i. a straight segment;
- ii. orthogonal to the deformed mid-surface;
- iii. of the same length.

The third assumption of Kirchhoff states that

$$|p - B| = |p' - B'| = z; \quad (4.6)$$

if  $\beta$  is the angle that the deformed segment  $A'C'$  forms with the axis  $z$ , then, because of the first Kirchhoff assumption,

$$p' - B' = (-z \sin \beta_x, -z \sin \beta_y, z \cos \beta), \quad (4.7)$$

where  $\beta_x$  and  $\beta_y$  are the angles that  $z$  forms with the projections of  $A'C'$  onto the planes  $x - z$  and  $y - z$ , respectively. It is an easy task to show that  $\beta_x \leq \beta, \beta_y \leq \beta$ , Fig. 4.3. The second Kirchhoff assumption lets now affirm that the angle formed by the line intersection of the plane containing axis  $z$  and  $A'C'$  with the tangent plane, in  $B'$ , to the

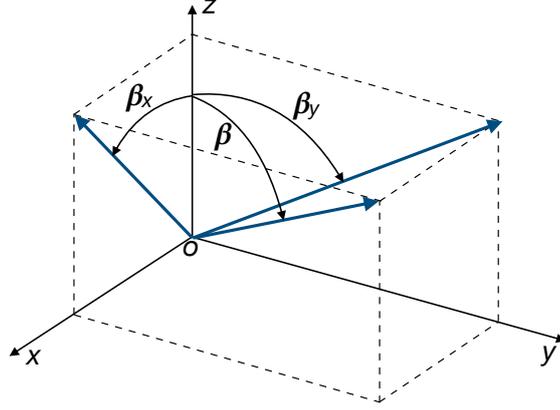


Figure 4.3: Angles scheme.

bent mid-surface and by the projection of  $A'C'$  onto the plane  $x-y$  is exactly  $\beta$ ; the same is valid also for the projection angles  $\beta_x$  and  $\beta_y$ . If now we use the assumption of small rotations, then

$$\beta \rightarrow 0 \Rightarrow \beta_x \rightarrow 0, \beta_y \rightarrow 0 \Rightarrow \begin{cases} \beta \simeq \sin \beta \simeq \tan \beta, & \cos \beta \simeq 1, \\ \beta_x \simeq \sin \beta_x \simeq \tan \beta_x = \frac{\partial w_0}{\partial x}, \\ \beta_y \simeq \sin \beta_y \simeq \tan \beta_y = \frac{\partial w_0}{\partial y}. \end{cases} \quad (4.8)$$

Hence,

$$p' - B' \simeq \left( -z \frac{\partial w_0}{\partial x}, -z \frac{\partial w_0}{\partial y}, z \right), \quad (4.9)$$

i.e. within the Kirchhoff model, the local displacement is a linear function of  $z$ . Finally

$$\mathbf{u}(p) = \begin{pmatrix} u_0(x, y) - z \frac{\partial w_0(x, y)}{\partial x} \\ v_0(x, y) - z \frac{\partial w_0(x, y)}{\partial y} \\ w_0(x, y) \end{pmatrix}. \quad (4.10)$$

Some remarks about this result:

- $\mathbf{u}(p)$  depends linearly upon  $z$ , so all the problem is reduced to find the planar vector field  $\mathbf{u}_0(x, y)$  describing the displacement of the points of the mid-plane; the 3D problem has hence reduced to a 2D one, but this passage, that simplifies remarkably the problem, has heavy consequences, as we will see in the following;
- the kinematics assumptions of the Kirchhoff model are rather heavy; namely, the second one eliminates out-of-plane shear deformations, giving on one hand an increased stiffness to the plate, that cannot deform for shear, and, on the other hand, causing difficulties in the computation of the transversal shear stresses;
- the Kirchhoff assumptions are plausible only for thin plates; when this is not true, they should be removed, especially the first and second ones, passing in this way to higher order theories.

## 4.2.2 Strain field

The strain field  $\boldsymbol{\varepsilon}(p)$  can be easily calculated using the displacement field  $\mathbf{u}(p)$ :

$$\boldsymbol{\varepsilon}(p) = \frac{\nabla \mathbf{u}(p) + \nabla \mathbf{u}(p)^\top}{2} = \begin{bmatrix} \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} & \frac{1}{2} \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) - z \frac{\partial^2 w_0}{\partial x \partial y} & 0 \\ \frac{1}{2} \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) - z \frac{\partial^2 w_0}{\partial x \partial y} & \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (4.11)$$

So,  $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$ : as a consequence of the Kirchhoff model,  $\boldsymbol{\varepsilon}$  is planar. Contrarily to what often said, however, this is *not* a plane strain state, because  $w$  is not identically null and  $\mathbf{u}$  and  $\boldsymbol{\varepsilon}$  depend upon  $z$  too. Actually, a plane strain state is typical of long cylinders acted upon by forces that do not vary along the axis of the cylinder; here, the only thing that can be said is that in the Kirchhoff model  $\boldsymbol{\varepsilon}$  is planar.

## 4.2.3 Stress field

About the stress field  $\boldsymbol{\sigma}(p)$  a supplementary assumption is made:  $\forall p \in \Omega$ ,

$$\sigma_{zz} = 0. \quad (4.12)$$

This assumption, common also to other plates theories, can be justified as follows:

- from the inverse Lamé's equation,

$$\boldsymbol{\varepsilon} = \frac{1 + \nu}{E} \boldsymbol{\sigma} - \frac{\nu}{E} \text{tr} \boldsymbol{\sigma} \mathbf{I}, \quad (4.13)$$

we get

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E}, \quad (4.14)$$

and because in the Kirchhoff model  $\varepsilon_{zz}$ , we get

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}); \quad (4.15)$$

- for common materials,  $0 < \nu < 1/2$ , so  $\sigma_{zz}$  is at most of the same order of magnitude of  $\sigma_{xx}$  and  $\sigma_{yy}$ ;
- for the assumptions on the loads,  $\sigma_{zz}$  is a smooth function of  $z$ ; so, because  $\sigma_{zz}$  is regular and the plate thin, its value cannot increase very much inside the plate with respect to the values that  $\sigma_{zz}$  has on the faces of the plate;
- because the value of  $\sigma_{zz}$  on  $S_B$  and  $S_T$  is just the value of the contact forces applied to the faces, i.e.  $\tau_z$ , it has not, in usual situations, a great value compared to the design values of  $\sigma_{xx}$  and  $\sigma_{yy}$ . For this reason, we can neglect  $\sigma_{zz}$  and finally consider that it is null.

These arguments are merely empirical; of course, they are not valid in some situations, i.e. in correspondence of concentrated loads, e.g. impact forces, or of supporting parts. In such cases the stress state is properly a 3D one and in such zones the behavior is, locally, far from that of a plate.

Because  $\varepsilon_{xz} = \varepsilon_{yz}$ , from eq. (4.1) we get also

$$\sigma_{xz} = \frac{E}{1+\nu}\varepsilon_{xz} = 0, \quad \sigma_{yz} = \frac{E}{1+\nu}\varepsilon_{yz} = 0, \quad (4.16)$$

so finally also the stress field is planar:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (4.17)$$

This is a strange situation, because in elasticity, a plane stress state is incompatible with a plane strain one and vice-versa. However, because  $\boldsymbol{\varepsilon}(p)$  depends upon  $x, y$  and  $z$ , this is the same also for the stress field:  $\boldsymbol{\sigma}(p)$  is not a plane field.

A dramatic consequence of eq. (4.16) is that the plate cannot be equilibrated under the action of loads that have a component orthogonal to the mid-plane: strictly speaking, the Kirchhoff model concerns plates that are loaded in their plane or by couples on the boundary. This problem will be solved making use directly of the equilibrium equations, cf. Sec. 4.2.9.

We remark also that eq. (4.16) is a consequence not only of the kinematical model, but also of the constitutive law: for generally oriented anisotropic plates,  $\sigma_{xz}$  and  $\sigma_{yz}$  are in general not null, also for the Kirchhoff model.

We can now obtain the constitutive law for a Kirchhoff plate: from eq. (4.13) we get

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}), \\ \varepsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}), \\ \varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}, \end{aligned} \quad (4.18)$$

that once inverted give

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}), \\ \sigma_{yy} &= \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}), \\ \sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} = 2G\varepsilon_{xy}. \end{aligned} \quad (4.19)$$

We can put the last result in matrix form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}, \quad (4.20)$$

or synthetically

$$\boldsymbol{\sigma} = \mathbb{D} \boldsymbol{\varepsilon}, \quad (4.21)$$

with  $\mathbb{D}$  the matrix in eq. (4.20). It can be checked that  $\mathbb{D}$  corresponds to the so-called *reduced stiffness matrix*, typical of a plane stress state: finally, though the assumptions of the Kirchhoff model do not coincide with those typical of a plane stress state ( $\boldsymbol{\sigma}$  is not a plane field), the plate's constitutive law, eq. (4.20) is just like that of a plane stress state.

#### 4.2.4 Internal actions

The internal actions, forces and couples, are obtained integrating through the thickness the stress field. In particular, we introduce:

- the *extension tensor*  $\mathbf{N}$ , defined as

$$\mathbf{N} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma} dz, \quad (4.22)$$

whose components

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xx} dz \\ \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yy} dz \\ \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy} dz \end{Bmatrix} \quad (4.23)$$

are represented in Fig. 4.4;

- the *transverse shear forces*

$$T_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xz} dz, \quad T_y = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yz} dz, \quad (4.24)$$

also represented in Fig. 4.4,

- the *bending tensor*  $\mathbf{M}$ , defined as

$$\mathbf{M} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \boldsymbol{\sigma} dz, \quad (4.25)$$

whose components

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \sigma_{xx} dz \\ \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \sigma_{yy} dz \\ \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \sigma_{xy} dz \end{Bmatrix} \quad (4.26)$$

are still represented in Fig. 4.4.

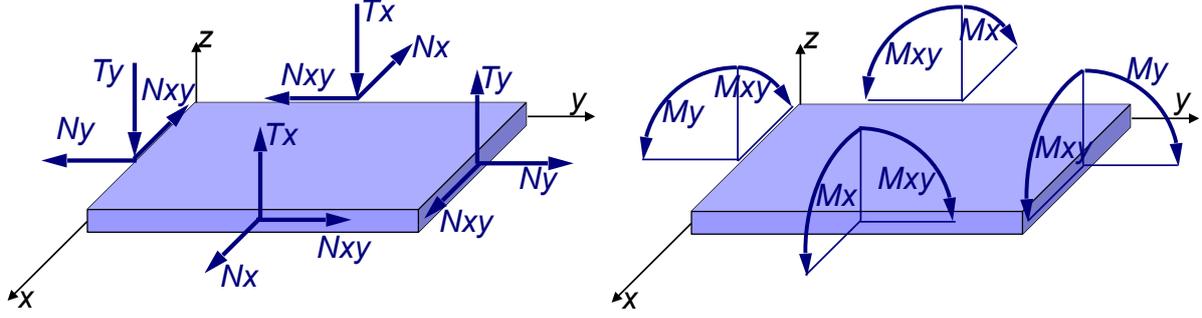


Figure 4.4: Internal actions.

Being integrals of a planar symmetric second-rank tensor, both  $\mathbf{N}$  and  $\mathbf{M}$  are planar symmetric second-rank tensors too; in addition, they are plane fields too:  $\mathbf{N} = \mathbf{N}(x, y)$ ,  $\mathbf{M} = \mathbf{M}(x, y)$ . To remark that the units of  $\mathbf{N}$  and of the shear forces  $T_x$  and  $T_y$  are a force per unit length, while those of  $\mathbf{M}$  are a force.

#### 4.2.5 Uncoupling bending and extension

The expression of  $\boldsymbol{\varepsilon}$  can be written in the form

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + z \boldsymbol{\kappa}, \quad (4.27)$$

where

$$\boldsymbol{\varepsilon}^0 = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ 2\varepsilon_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \quad (4.28)$$

is the *in-plane deformation tensor*, describing the strain of the mid-plane, while

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \quad (4.29)$$

is the *curvatures tensor*, describing the bending of the mid-plane. Using this decomposition of  $\boldsymbol{\varepsilon}$ , we can write

$$\boldsymbol{\sigma} = \mathbb{D}\boldsymbol{\varepsilon} = \mathbb{D}\boldsymbol{\varepsilon}^0 + z\mathbb{D}\boldsymbol{\kappa} \quad (4.30)$$

and because  $\mathbb{D}$ ,  $\boldsymbol{\varepsilon}^0$  and  $\boldsymbol{\kappa}$  do not depend upon  $z$ , we get

$$\begin{aligned} \mathbf{N} &= \mathbb{D}\boldsymbol{\varepsilon}^0 \int_{-\frac{h}{2}}^{+\frac{h}{2}} dz + \mathbb{D}\boldsymbol{\kappa} \int_{-\frac{h}{2}}^{+\frac{h}{2}} z dz = h \mathbb{D}\boldsymbol{\varepsilon}^0, \\ \mathbf{M} &= \mathbb{D}\boldsymbol{\varepsilon}^0 \int_{-\frac{h}{2}}^{+\frac{h}{2}} z dz + \mathbb{D}\boldsymbol{\kappa} \int_{-\frac{h}{2}}^{+\frac{h}{2}} z^2 dz = \frac{h^3}{12} \mathbb{D}\boldsymbol{\kappa}, \end{aligned} \quad (4.31)$$

i.e.  $\mathbf{N}$  depends only upon in-plane strains and  $\mathbf{M}$  only upon curvatures: there is not any no coupling between extension and curvatures nor between bending and in-plane deformation. Actually, the absence of any coupling effect is due to two facts: on one hand, geometry (the flatness of the plate) and on the other hand homogeneity (the plate is composed by a unique layer): generally speaking for shells, i.e. "curved plates", and laminates, i.e. plates obtained superposing layers, coupling effects are present. Finally, extension and bending of the plate can be examined separately, they do not interact.

A plate that is subjected only to in-plane forces, so that  $\boldsymbol{\kappa} = \mathbf{o} \Rightarrow \mathbf{M} = \mathbf{o}$ , is called a (flat) *membrane* (curved membranes exist too, e.g. an inflated balloon). The membrane regime, however, is compatible only with a distribution of the forces which is point-wise symmetric with respect to the mid-plane, assumption that we make in the following.

## 4.2.6 Balance equations

There are different ways for writing local balance equations. A standard approach is to write the equilibrium to translation and rotation of a small part of the plate, and then discard higher order terms. We prefer here a more concise approach: starting from the 3D equilibrium equations,

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = \mathbf{o}, \quad (4.32)$$

where  $\mathbf{b}(p)$  is the field of body forces (forces per unit volume) on  $\Omega$ , we obtain, after integration on  $z$ ,

$$\begin{aligned} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x \right) dz &= 0, \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y \right) dz &= 0, \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z \right) dz &= 0. \end{aligned} \quad (4.33)$$

Because the integration bounds are fixed, we can bring the differential operators outside the integration symbol:

$$\begin{aligned} \frac{\partial}{\partial x} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dz + \frac{\partial}{\partial y} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{xz}}{\partial z} + b_x \right) dz &= 0, \\ \frac{\partial}{\partial x} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} dz + \frac{\partial}{\partial y} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{yz}}{\partial z} + b_y \right) dz &= 0, \\ \frac{\partial}{\partial x} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} dz + \frac{\partial}{\partial y} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yz} dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{zz}}{\partial z} + b_z \right) dz &= 0, \end{aligned} \quad (4.34)$$

so that, introducing eqs. (4.23) and (4.24), we get

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x &= 0, \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + f_y &= 0, \end{aligned} \quad (4.35)$$

and

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + f_z = 0, \quad (4.36)$$

with<sup>1</sup>

$$\begin{aligned} f_x &= [\sigma_{xz}]_{-\frac{h}{2}}^{\frac{h}{2}} + \int_{-\frac{h}{2}}^{\frac{h}{2}} b_x dz = 2\tau_x + \int_{-\frac{h}{2}}^{\frac{h}{2}} b_x dz \\ f_y &= [\sigma_{yz}]_{-\frac{h}{2}}^{\frac{h}{2}} + \int_{-\frac{h}{2}}^{\frac{h}{2}} b_y dz = 2\tau_y + \int_{-\frac{h}{2}}^{\frac{h}{2}} b_y dz \\ f_z &= [\sigma_{zz}]_{-\frac{h}{2}}^{\frac{h}{2}} + \int_{-\frac{h}{2}}^{\frac{h}{2}} b_z dz = 2\tau_z + \int_{-\frac{h}{2}}^{\frac{h}{2}} b_z dz. \end{aligned} \quad (4.37)$$

Equations (4.35) are the *extension, or membrane, equilibrium equations* of the plate, while eq. (4.36) is the *transverse shear equilibrium equation*.

The bending equilibrium can be obtained in a similar way, now integrating eq. (4.32) once multiplied by  $z$ , the distance of each component  $\sigma_{ij}(p)$  from the mid-plane:

$$\begin{aligned} \int_{-\frac{h}{2}}^{\frac{h}{2}} z \left( \frac{\partial \sigma_{xx}}{\partial x} + z \frac{\partial \sigma_{xy}}{\partial y} + z \frac{\partial \sigma_{xz}}{\partial z} + z b_x \right) dz &= 0, \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( z \frac{\partial \sigma_{xy}}{\partial x} + z \frac{\partial \sigma_{yy}}{\partial y} + z \frac{\partial \sigma_{yz}}{\partial z} + z b_y \right) dz &= 0, \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( z \frac{\partial \sigma_{xz}}{\partial x} + z \frac{\partial \sigma_{yz}}{\partial y} + z \frac{\partial \sigma_{zz}}{\partial z} + z b_z \right) dz &= 0. \end{aligned} \quad (4.38)$$

Because we are interested in the bending of the plate, the third equation of (4.41) can be discarded, as it concerns the equilibrium to rotation about the axis  $z$ . Observing that

$$z \frac{\partial \sigma_{\alpha\beta}}{\partial \alpha} = \frac{\partial z \sigma_{\alpha\beta}}{\partial \alpha} \quad \alpha, \beta \in \{1, 2\}, \quad (4.39)$$

and that, because  $\mathbf{b}(p)$  is an even function,  $z\mathbf{b}(p)$  is an odd one so that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} z b_x dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} z b_y dz = 0, \quad (4.40)$$

we get

$$\begin{aligned} \frac{\partial}{\partial x} \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xx} dz + \frac{\partial}{\partial y} \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xy} dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} z \frac{\partial \sigma_{xz}}{\partial z} dz &= 0, \\ \frac{\partial}{\partial x} \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xy} dz + \frac{\partial}{\partial y} \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{yy} dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} z \frac{\partial \sigma_{yz}}{\partial z} dz &= 0. \end{aligned} \quad (4.41)$$

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<sup>1</sup>Thanks to the assumption of symmetrical distribution of the actions,  $\boldsymbol{\tau}(x, y, -\frac{h}{2}) = \boldsymbol{\tau}(x, y, \frac{h}{2}) = (\tau_x, \tau_y, \tau_z)$ ; but  $\boldsymbol{\tau}(x, y, \pm \frac{h}{2}) = \boldsymbol{\sigma}(x, y, \pm \frac{h}{2}) \mathbf{n}(x, y, \pm \frac{h}{2})$  and  $\mathbf{n}(x, y, -\frac{h}{2}) = -\mathbf{n}(x, y, \frac{h}{2}) = (0, 0, -1)$ , so that  $\boldsymbol{\tau}(x, y, -\frac{h}{2}) = (\tau_x, \tau_y, \tau_z) = (-\sigma_{xz}, -\sigma_{yz}, -\sigma_{zz})_{-\frac{h}{2}}$  and  $\boldsymbol{\tau}(x, y, \frac{h}{2}) = (\tau_x, \tau_y, \tau_z) = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})_{\frac{h}{2}} \Rightarrow [\sigma_{xz}]_{-\frac{h}{2}}^{\frac{h}{2}} = 2\tau_x$ ,  $[\sigma_{yz}]_{-\frac{h}{2}}^{\frac{h}{2}} = 2\tau_y$ ,  $[\sigma_{zz}]_{-\frac{h}{2}}^{\frac{h}{2}} = 2\tau_z$ .

Integration by parts gives, cf. Note 1 and eq. (4.24),

$$\begin{aligned}\int_{-\frac{h}{2}}^{\frac{h}{2}} z \frac{\partial \sigma_{xz}}{\partial z} dz &= [z \sigma_{xz}]_{-\frac{h}{2}}^{\frac{h}{2}} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} dz = -T_x, \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} z \frac{\partial \sigma_{yz}}{\partial z} dz &= [z \sigma_{yz}]_{-\frac{h}{2}}^{\frac{h}{2}} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yz} dz = -T_y.\end{aligned}\tag{4.42}$$

If now we introduce eq. (4.26) we obtain

$$\begin{aligned}\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} &= T_x, \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial z} &= T_y,\end{aligned}\tag{4.43}$$

and introducing this result into eq. (4.36) we get the unique second-order equation including bending and shear equilibrium:

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + f_z = 0.\tag{4.44}$$

## 4.2.7 Elastic equilibrium equations

Injecting into eq. (4.31)<sub>1</sub> the expressions of  $\mathbb{D}$  and  $\boldsymbol{\varepsilon}^0$  we get

$$\begin{aligned}N_x &= \frac{E h}{1 - \nu^2} \left( \frac{\partial u_0}{\partial x} + \nu \frac{\partial v_0}{\partial y} \right), \\ N_y &= \frac{E h}{1 - \nu^2} \left( \nu \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \right), \\ N_{xy} &= G \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right),\end{aligned}\tag{4.45}$$

which injected into eq. (4.35) gives ( $G = E/2(1 + \nu)$ )

$$\begin{aligned}\frac{E h}{1 + \nu} \left[ \frac{1}{1 - \nu} \left( \frac{\partial^2 u_0}{\partial x^2} + \nu \frac{\partial^2 v_0}{\partial x \partial y} \right) + \frac{1}{2} \left( \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y} \right) \right] + f_x &= 0, \\ \frac{E h}{1 + \nu} \left[ \frac{1}{2} \left( \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial^2 v_0}{\partial x^2} \right) + \frac{1}{1 - \nu} \left( \nu \frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial^2 v_0}{\partial y^2} \right) \right] + f_y &= 0.\end{aligned}\tag{4.46}$$

These are the *elastic membrane equilibrium equations* for isotropic homogeneous plates; they are the 2D equivalent of the rod extension equilibrium equation  $EA w'' + p_z = 0$ .

Similarly, if now we inject into eq. (4.31)<sub>2</sub> the expressions of  $\mathbb{D}$  and  $\boldsymbol{\kappa}$ , we obtain

$$\begin{aligned}M_x &= -\frac{h^3}{12} \frac{E}{1 - \nu^2} \left( \frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2} \right), \\ M_y &= -\frac{h^3}{12} \frac{E}{1 - \nu^2} \left( \frac{\partial^2 w_0}{\partial y^2} + \nu \frac{\partial^2 w_0}{\partial x^2} \right), \\ M_{xy} &= -\frac{h^3}{12} \frac{E}{1 + \nu} \frac{\partial^2 w_0}{\partial x \partial y},\end{aligned}\tag{4.47}$$

that inserted into eq. (4.44) gives

$$\frac{h^3}{12} \frac{E}{1 - \nu^2} \left( \frac{\partial^4 w_0}{\partial x^4} + 2 \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4} \right) = f_z. \quad (4.48)$$

Usually, the symbol  $D$  is used to denote the bending stiffness of the plate per unit of length:

$$D = \frac{h^3}{12} \frac{E}{1 - \nu^2}, \quad (4.49)$$

so that, once introduced the classical *double laplacian* differential operator,

$$\Delta^2 = \frac{\partial^4 \bullet}{\partial x^4} + 2 \frac{\partial^4 \bullet}{\partial x^2 \partial y^2} + \frac{\partial^4 \bullet}{\partial y^4}, \quad (4.50)$$

eq. (4.48) can be rewritten in the form

$$D \Delta^2 w_0 = f_z. \quad (4.51)$$

This is the celebrated *Germain-Lagrange equation* (1816); it is the 2D corresponding of the elastic bending equilibrium equation  $EJ v^{iv} = p_y$  for rods: the Kirchhoff model is, for plates, the equivalent of the Euler-Bernoulli model for the bending of rods. Actually, because  $h^3/12$  is the moment of inertia, about axes  $x$  or  $y$ , of a unit length cross section of the plate, the true difference between the 1D case, rods, and the 2D one, plates, is in the term  $1 - \nu^2$ , which is a consequence of the reduced stiffness  $\mathbb{D}$ , hence of the fact that  $\boldsymbol{\sigma}(p)$  is planar: plates are intrinsically stiffer than rods.

Equations (4.46) and (4.51) are the *elastic equilibrium equations of plates*, respectively for membrane and for bending.

In the same way, from the equilibrium equations for bending, eq. (4.43), we get:

$$\begin{aligned} T_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -\frac{h^3}{12} \frac{E}{1 - \nu^2} [w_{0,xxx} + \nu w_{0,xyy} + (1 - \nu)w_{0,xyy}], \\ T_y &= \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial z} = -\frac{h^3}{12} \frac{E}{1 - \nu^2} [w_{0,xyy} + \nu w_{0,xyy} + (1 - \nu)w_{0,yyy}], \end{aligned} \quad (4.52)$$

i.e.

$$\begin{aligned} T_x &= -\frac{h^3}{12} \frac{E}{1 - \nu^2} \Delta w_{0,x}, \\ T_y &= -\frac{h^3}{12} \frac{E}{1 - \nu^2} \Delta w_{0,y}. \end{aligned} \quad (4.53)$$

## 4.2.8 Expressions for stresses

Inverting eqs. (4.31) gives

$$\mathbb{D}\boldsymbol{\varepsilon}^0 = \frac{\mathbf{N}}{h}, \quad \mathbb{D}\boldsymbol{\kappa} = \frac{12}{h^3}\mathbf{M}, \quad (4.54)$$

that injected into eq. (4.30) gives

$$\boldsymbol{\sigma} = \frac{1}{h}\mathbf{N} + \frac{12}{h^3}\mathbf{M} z. \quad (4.55)$$

This formula generalizes to plates the *Navier's formula*, giving the normal stress distribution in beams subjected to bending and extension.

## 4.2.9 Transverse shear stresses

We can now go back to the problem of determining the *transverse shear stresses*  $\sigma_{xz}$  and  $\sigma_{yz}$ , that, as we have seen, are incompatible with the Kirchhoff's assumptions. Actually, we can find them using the equilibrium equations: from eqs. (4.33)<sub>1,2</sub>, we get

$$\begin{aligned}\sigma_{xz} &= - \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x \right) dz, \\ \sigma_{yz} &= - \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y \right) dz.\end{aligned}\tag{4.56}$$

Let us transform these equations: from (4.55) we get

$$\begin{aligned}\sigma_{xx} &= \frac{N_x}{h} + \frac{12}{h^3} M_x z, \\ \sigma_{yy} &= \frac{N_y}{h} + \frac{12}{h^3} M_y z, \\ \sigma_{xy} &= \frac{N_{xy}}{h} + \frac{12}{h^3} M_{xy} z,\end{aligned}\tag{4.57}$$

that injected into eq. (4.56) gives

$$\begin{aligned}\sigma_{xz} &= - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{h} \left( \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) + b_x dz - \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) z dz, \\ \sigma_{yz} &= - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{h} \left( \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) + b_y dz - \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \right) z dz.\end{aligned}\tag{4.58}$$

If now we use eqs. (4.35), (4.37) and (4.43) we get

$$\begin{aligned}\sigma_{xz} &= -2\tau_x - \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} z T_x dz, \\ \sigma_{yz} &= -2\tau_y - \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} z T_y dz,\end{aligned}\tag{4.59}$$

and finally

$$\begin{aligned}\sigma_{xz} &= -2\tau_x - \frac{6}{h^3} T_x z^2 + c_x, \\ \sigma_{yz} &= -2\tau_y - \frac{6}{h^3} T_y z^2 + c_y,\end{aligned}\tag{4.60}$$

with  $c_x$  and  $c_y$  two integration constants that can be determined through the boundary conditions. Because we have integrated two first-order equilibrium equations, we only have one boundary condition for each stress equation, that allows for determining just one integration constant, while there are two boundaries,  $S_B$  and  $S_T$ , on which the stresses  $\sigma_{xz}$  or  $\sigma_{yz}$  must match the value of the applied surface tractions. Because  $\sigma_{xz}$  and  $\sigma_{yz}$  are quadratic functions of  $z$ , this is possible only if the boundary conditions on  $S_B$  and

$S_T$  are the same, i.e. if the plate is loaded symmetrically with respect to the mid-plane, assumption that we have made.

If we put that

$$\sigma_{xz} \left( \pm \frac{h}{2} \right) = \tau_x, \quad \sigma_{yz} \left( \pm \frac{h}{2} \right) = \tau_y, \quad (4.61)$$

we get

$$c_x = 3\tau_x + \frac{3}{2h}T_x, \quad c_y = 3\tau_y + \frac{3}{2h}T_y, \quad (4.62)$$

and finally

$$\begin{aligned} \sigma_{xz} &= \tau_x + \frac{6}{h}T_x \left( \frac{1}{4} - \frac{z^2}{h^2} \right), \\ \sigma_{yz} &= \tau_y + \frac{6}{h}T_y \left( \frac{1}{4} - \frac{z^2}{h^2} \right). \end{aligned} \quad (4.63)$$

In the most common case of null tangential loads on  $S_B$  and  $S_T$ , i.e. for  $\tau_x = \tau_y = 0$ , we obtain the classical result

$$\begin{aligned} \sigma_{xz} &= \frac{6}{h}T_x \left( \frac{1}{4} - \frac{z^2}{h^2} \right), \\ \sigma_{yz} &= \frac{6}{h}T_y \left( \frac{1}{4} - \frac{z^2}{h^2} \right) \end{aligned} \quad (4.64)$$

for plates acted upon uniquely by transversal forces; these formulae generalize to plates the classical result of the *Jourawski's formula* for the shear stresses in beams. They state that the variation of the transverse shear stresses  $\sigma_{xz}$  and  $\sigma_{yz}$  through the thickness of the plate is parabolic and that the maximum is get in correspondence of the mid-plane, where they attain 1.5 times their average value:

$$\sigma_{xz}^{max} = \frac{3}{2} \frac{T_x}{h}, \quad \sigma_{yz}^{max} = \frac{3}{2} \frac{T_y}{h}. \quad (4.65)$$

#### 4.2.10 Boundary conditions

The equilibrium equations (4.35) and (4.44) are not sufficient to determine all the static unknowns: plates are *intrinsically hyperstatic bodies*. We need hence to introduce in the computation not only the balance equations, but also the constitutive law, i.e. we need to make use of the *elastic* equilibrium equations (4.46) and (4.51). These are partial differential equations of the second and fourth order respectively. The question of how many and which boundary conditions are to be associated to eqs. (4.46) and (4.51) has needed a long period to be solved and the arguments that lead to the solution are detailed below. In order to write the boundary conditions, we refer to a generic situation for the boundary of the plate, the lateral surface  $S_L$  whose trace in the mid-plane is the curve  $\gamma$ , see Fig. 4.5; at a point  $p \in \gamma$ ,  $\mathbf{n} = (n_x, n_y)$  and  $\mathbf{t} = (-n_y, n_x)$  are the unit vectors of the mid-plane respectively orthogonal and tangent to  $\gamma$ .

On an infinitesimal surface element  $d\omega \in S_L$  of width  $ds$ , there are the actions  $\mathbf{N}_n = (N_n, N_t)$  for extension,  $T_n$ , for shear, and  $\mathbf{M}_n = (M_n, M_t)$  for bending, see again Fig. 4.5.

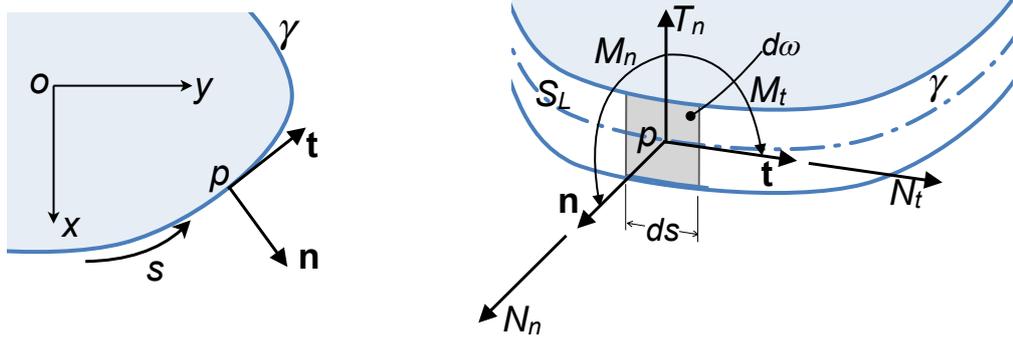


Figure 4.5: Boundary actions.

Because  $\mathbf{N}$  and  $\mathbf{M}$  are second-rank tensors, we have that

$$\mathbf{N}_{\mathbf{n}} = \mathbf{N} \mathbf{n} \rightarrow \begin{cases} N_n = (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{N} = N_x n_x^2 + 2N_{xy} n_x n_y + N_y n_y^2, \\ N_t = (\mathbf{t} \otimes \mathbf{n}) \cdot \mathbf{N} = N_{xy}(n_x^2 - n_y^2) + n_x n_y (N_y - N_x), \end{cases} \quad (4.66)$$

and in the same way

$$\mathbf{M}_{\mathbf{n}} = \mathbf{M} \mathbf{n} \rightarrow \begin{cases} M_n = (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{M} = M_x n_x^2 + 2M_{xy} n_x n_y + M_y n_y^2, \\ M_t = (\mathbf{t} \otimes \mathbf{n}) \cdot \mathbf{M} = M_{xy}(n_x^2 - n_y^2) + n_x n_y (M_y - M_x). \end{cases} \quad (4.67)$$

For what concerns  $T_n$ , it is obtained by simple projection on  $\mathbf{n}$  of the shear forces vector  $\mathbf{T} = (T_x, T_y)$ :

$$T_n = \mathbf{T} \cdot \mathbf{n} = T_x n_x + T_y n_y. \quad (4.68)$$

In the same way, we can proceed for the displacement vector of the mid-plane,  $\mathbf{u}_0 = (u_0, v_0, w_0)$ . In particular, we need to project  $\mathbf{u}_0$  on  $\mathbf{n}$ ,  $\mathbf{t}$  and  $\mathbf{e}_z$ , the unit vector of the axis  $z$ :

$$\begin{aligned} u_n &= \mathbf{u}_0 \cdot \mathbf{n} = (u_0, v_0, w_0) \cdot (n_x, n_y, 0) = u_0 n_x + v_0 n_y, \\ u_t &= \mathbf{u}_0 \cdot \mathbf{t} = (u_0, v_0, w_0) \cdot (-n_y, n_x, 0) = -u_0 n_y + v_0 n_x, \\ u_z &= \mathbf{u}_0 \cdot \mathbf{e}_z = (u_0, v_0, w_0) \cdot (0, 0, 1) = w_0. \end{aligned} \quad (4.69)$$

We can now write the boundary conditions for any point of the border and let us begin with the boundary conditions for the membrane equations:

- *kinematical* (or *geometrical*) conditions:

$$u_n = \hat{u}_n, \quad u_t = \hat{u}_t; \quad (4.70)$$

- *natural* conditions:

$$N_n = \hat{F}_n, \quad N_t = \hat{F}_t, \quad (4.71)$$

where  $\hat{u}_n, \hat{u}_t, \hat{F}_n$  and  $\hat{F}_t$  are known values. A list, not exhaustive but comprehending the more important and common cases of boundary conditions for the membrane equations, is given below:

- supported border:  $u_n = 0, u_t = 0$ ;

- supported border free to slide along  $\mathbf{t}$  :  $u_n = 0$ ;
- free unloaded border:  $N_n = 0, N_t = 0$ ;
- free loaded border:  $N_n = \hat{F}_n, N_t = \hat{F}_t$ .

Let us now consider the boundary conditions for the bending equation; now, a point  $p \in \gamma$  can have:

- a displacement along  $z$ :  $w_0$ ;
- a rotation about  $\mathbf{t}$ :  $\frac{\partial w_0}{\partial \mathbf{n}}$ ;
- a rotation about  $\mathbf{n}$ :  $\frac{\partial w_0}{\partial \mathbf{t}}$ .

Statically, the possible actions are:

- a shear:  $T_n$ ;
- a bending moment:  $M_n$ ;
- a twisting moment:  $M_t$ .

The conditions to be written must be specified for each case, let us see the most important ones.

#### 4.2.10.1 Simply supported edges

In this case the boundary conditions are

$$w_0 = 0, \quad M_n = 0. \quad (4.72)$$

For the case of a border orthogonal to the  $x$ -axis, i.e. for  $\mathbf{n} = \mathbf{e}_x$ ,

$$M_n = M_x = -\frac{h^3}{12} \frac{E}{1 - \nu^2} \left( \frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2} \right), \quad (4.73)$$

so the natural condition becomes

$$\frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2} = 0. \quad (4.74)$$

Because on such a border  $x = \text{const.}$  and  $w_0 = 0$ ,

$$\frac{\partial^2 w_0}{\partial y^2} = 0, \quad (4.75)$$

and finally the natural boundary condition can be written

$$\frac{\partial^2 w_0}{\partial x^2} = 0. \quad (4.76)$$

For this particular case, originally, Navier proposed the boundary conditions

$$w_0 = 0, \quad \Delta w_0 = 0, \quad (4.77)$$

that actually are equivalent to the above ones.

#### 4.2.10.2 Clamped edges

For such a case, the boundary conditions are

$$w_0 = 0, \quad \frac{\partial w_0}{\partial \mathbf{n}} = 0. \quad (4.78)$$

To recall that

$$\frac{\partial w_0}{\partial \mathbf{n}} = \nabla w_0 \cdot \mathbf{n} = \left( \frac{\partial w_0}{\partial x}, \frac{\partial w_0}{\partial y}, \frac{\partial w_0}{\partial z} \right) \cdot (n_x, n_y, 0) = \frac{\partial w_0}{\partial x} n_x + \frac{\partial w_0}{\partial y} n_y. \quad (4.79)$$

#### 4.2.10.3 Free edges

According to the approach of Poisson (1829), there are three boundary conditions to be written for a free edge:

$$T_n = \hat{F}_z, \quad M_n = \hat{M}_n, \quad M_t = \hat{M}_t. \quad (4.80)$$

Actually, this is true also for the two previous cases, where one could write:

- simply supported edge:

$$w_0 = 0 \Rightarrow T_n \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{t}} = 0 \Rightarrow M_t \neq 0, \quad M_n = 0 \Rightarrow \frac{\partial w_0}{\partial \mathbf{n}} \neq 0;$$

- clamped edge:

$$w_0 = 0 \Rightarrow T_n \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{t}} = 0 \Rightarrow M_t \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{n}} = 0 \Rightarrow M_n \neq 0.$$

However, this approach is *false*: in fact,  $\frac{\partial w_0}{\partial \mathbf{n}}$  and  $\frac{\partial w_0}{\partial \mathbf{t}}$  are not independent, so the number of boundary conditions cannot be three. The reduction of the number of boundary conditions from 3 to 2, and their correct expression, can be illustrated by a classical procedure due to Kelvin, detailed in the next Section<sup>2</sup>.

#### 4.2.10.4 The Kelvin's reduction of the boundary conditions

Let us imagine of subdividing the plate's edge into a series of parallel stripes of infinitesimal width  $ds$ , see Fig. 4.6. On each stripe acts a shear

$$T_n ds \quad (4.81)$$

and a twisting moment

$$M_t ds. \quad (4.82)$$

We add, ideally, two equal and opposite couples to each stripe: the first one is the couple

$$-M_t' ds, \quad (4.83)$$

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<sup>2</sup>The method proposed by Kelvin (1867) to introduce the correct boundary conditions was first established by Kirchhoff through a variational approach (1850).

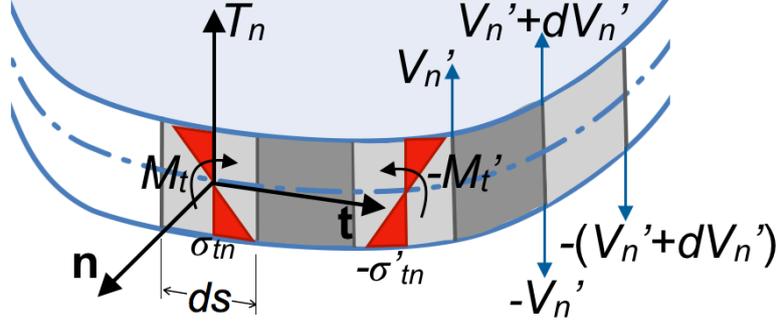


Figure 4.6: Kelvin's reduction of the boundary conditions.

result of the shear stress  $-\sigma'_{tn}$ , opposite of the actual one  $\sigma_{tn}$ . The second couple is composed by two vertical forces  $\pm V'_n$  applied on the edges of the stripe, so as that

$$V'_n ds = M'_t ds \Rightarrow V'_n = M'_t. \quad (4.84)$$

On the following stripe, in the sense of  $\mathbf{t}$ , we add the couple  $-(M'_t + dM'_t)ds$  and the opposite one  $(V'_n + dV'_n)ds$  and so on for all the stripes. On the edge shared by two neighbouring stripes, the forces  $-V'_n$  and  $(V'_n + dV'_n)$  give the net force, per unit length,

$$\frac{dV'_n}{ds} = \frac{dM'_t}{ds}, \quad (4.85)$$

while the twist moment, per unit length, becomes  $M_t - M'_t$ . For the Saint Venant's Principle, because the moments  $-M'_t ds$  and  $V'_n ds$  are equilibrated, the static and elastic regime of the plate is not affected sufficiently far from the edge (for a distance, say, of the order of the plate's thickness). If now we chose  $M'_t = M_t$ , the twisting moment is null on the boundary and it is substituted by a distribution of vertical forces, per unit length, equal to

$$\frac{dV_n}{ds} = \frac{dM_t}{ds}. \quad (4.86)$$

These forces, called *substitution forces*, are statically equivalent to the twist moment  $M_t$ . The vertical reaction of a support is hence not  $T_n$ , but

$$T_n^* = T_n + \frac{dM_t}{ds}. \quad (4.87)$$

The force  $T_n^*$  is called the *Kirchhoff's shear*. Mechanically, this result shows that in the Kirchhoff's model the action of  $M_t$  on an edge does not give rise to a deformation, but only to a reaction, in the form of a vertical force.

Finally, the correct boundary conditions are just two and namely:

- for a free edge:

$$T_n^* = \hat{F}_z, \quad M_n = \hat{M}_n \rightarrow w_0 \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{n}} \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{t}} \neq 0; \quad (4.88)$$

- for a simply supported edge:

$$w_0 = 0, \quad M_n = 0 \rightarrow T_n^* \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{n}} \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{t}} = 0; \quad (4.89)$$

- for a clamped edge:

$$w_0 = 0, \quad \frac{\partial w_0}{\partial \mathbf{n}} = 0 \rightarrow T_n^* \neq 0, \quad M_n \neq 0, \quad \frac{\partial w_0}{\partial \mathbf{t}} = 0. \quad (4.90)$$

We can also obtain an expression of  $T_n^*$  as function of  $w_0$ : if, e.g.,  $\mathbf{n} = \mathbf{e}_x$ , then

$$T_n^* = T_x^* = T_x + \frac{\partial M_{xy}}{\partial y} = -\frac{h^3}{12} \frac{E}{1-\nu^2} \left[ \frac{\partial^3 w_0}{\partial x^3} + (2-\nu) \frac{\partial^3 w_0}{\partial x \partial y^2} \right]; \quad (4.91)$$

in the same way, if  $\mathbf{n} = \mathbf{e}_y$ , we get

$$T_n^* = T_y^* = T_y + \frac{\partial M_{xy}}{\partial x} = -\frac{h^3}{12} \frac{E}{1-\nu^2} \left[ \frac{\partial^3 w_0}{\partial y^3} + (2-\nu) \frac{\partial^3 w_0}{\partial x^2 \partial y} \right]. \quad (4.92)$$

Then,

$$T_n^* = T_x^* n_x + T_y^* n_y. \quad (4.93)$$

As a last point, we need to consider the case of a discontinuous  $M_t$  at some  $s = s_0$ . In such a case, the substitution force is

$$R = \lim_{\varepsilon \rightarrow 0} \int_{s_0-\varepsilon}^{s_0+\varepsilon} \frac{dM_t}{ds} ds = M_t^+ - M_t^-, \quad (4.94)$$

i.e.,  $R$  is exactly the difference of the twist moment immediately after and before  $s_0$ . This situation is typical of corners; in particular, for a right angle,

$$M_t^+ = M_t^- \Rightarrow R = -2M_t^-. \quad (4.95)$$

For a corner formed by two orthogonal sides parallel to the axes  $x$  and  $y$ , we get, see Fig. 4.7

$$R = -2|M_{xy}| = -\frac{h^3}{6} \frac{E}{1+\nu} \left| \frac{\partial^2 w_0}{\partial x \partial y} \right|. \quad (4.96)$$

This is a vertical downward reaction that the supports must give near the corners of a rectangular bent plate.

## 4.2.11 Some exact solutions

### 4.2.11.1 Elliptic plate

For an elliptic plate loaded by a uniformly distributed load  $q$  and clamped on the edge, the deflection is

$$w_0(x, y) = \frac{q}{D} \frac{\left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)^2}{\left( \frac{24}{a^4} + \frac{24}{b^4} + \frac{16}{a^2 b^2} \right)}, \quad (4.97)$$

where  $a$  and  $b$  are the semi axes along  $x$  and  $y$ , respectively; the maximum deflection is

$$w_0^{max} = w_0(0, 0) = \frac{q}{D} \frac{1}{\left( \frac{24}{a^4} + \frac{24}{b^4} + \frac{16}{a^2 b^2} \right)}. \quad (4.98)$$

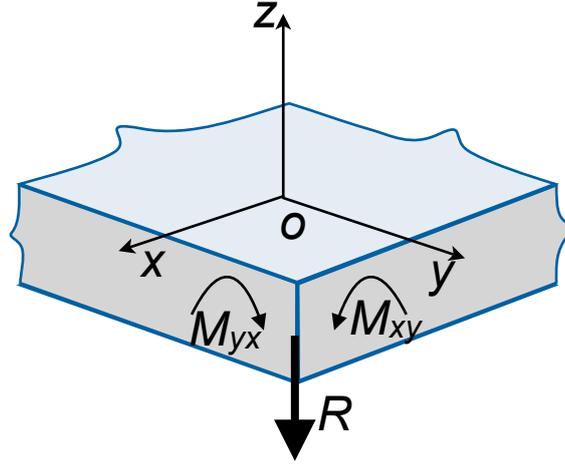


Figure 4.7: The substitution force near a right angle corner.

#### 4.2.11.2 Circular plate

A clamped, uniformly loaded, circular plate of radius  $R$  is a particular case of the previous elliptic plate, where  $a = b = R$ :

$$w_0(x, y) = \frac{q}{64 D} (x^2 + y^2 - R^2)^2, \quad (4.99)$$

with a maximum deflection in the plate's center:

$$w_0^{max} = w(0, 0) = \frac{q R^4}{64 D}. \quad (4.100)$$

In polar coordinates it is

$$w_0(r, \theta) = \frac{q}{64 D} (R^2 - r^2)^2, \quad 0 \leq r \leq R : \quad (4.101)$$

the deflection depends uniquely on the distance  $r$  from the plate's center, while it is independent from the angle  $\theta$ .

#### 4.2.11.3 Equilateral triangular plate

For a plate like that in Fig. 4.8, uniformly loaded by a load  $q$  and simply supported on the edges, it is

$$w_0(x, y) = \frac{q}{64 a D} \left[ x^3 - 3x^2y - a(x^2 + y^2) + \frac{4}{27}a^3 \right] \left( \frac{4}{9}a^2 - x^2 - y^2 \right). \quad (4.102)$$

#### 4.2.11.4 Anticlastic bending

Let us consider the square plate ABCD in Fig. 4.9; we want to examine to which load and boundary conditions corresponds the solution

$$w_0(x, y) = -\frac{M(x^2 - y^2)}{2D(1 - \nu)}. \quad (4.103)$$

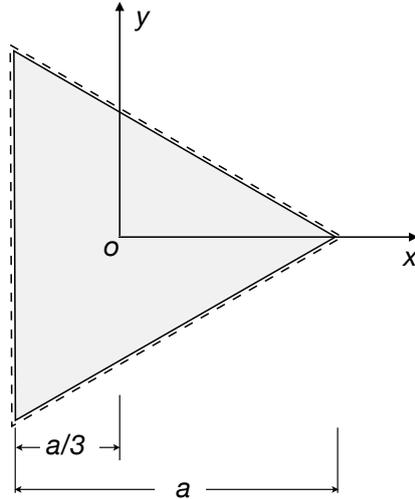


Figure 4.8: Triangular simply supported plate.

Because

$$\frac{\partial^2 w_0}{\partial x^2} = -\frac{\partial^2 w_0}{\partial y^2} = -\frac{M}{D(1-\nu)}, \quad (4.104)$$

then

$$\Delta w_0 = 0 \Rightarrow \Delta^2 w_0 = 0 \rightarrow p = 0, \quad (4.105)$$

i.e. there is not a distributed load on the plate. We remark that for  $y = \pm x$ , i.e. along the square's diagonals,  $w_0 = 0$ : the diagonals do not move nor deform. Also,

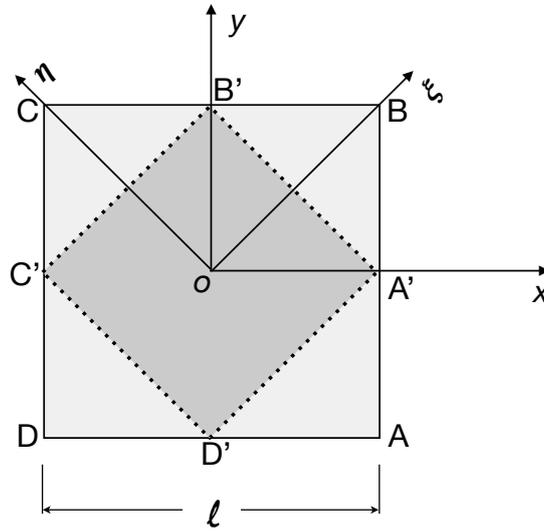


Figure 4.9: Anticlastic bending.

$$\frac{\partial w_0}{\partial \mathbf{e}} = \nabla w_0 \cdot \mathbf{e} = -\frac{M}{D(1-\nu)}(x, -y) \cdot (e_x, e_y). \quad (4.106)$$

So, if  $\mathbf{e}$  is parallel to one of the diagonals,  $\mathbf{e} = \frac{1}{\sqrt{2}}(1, \pm 1) \Rightarrow$

$$\frac{\partial w_0}{\partial \mathbf{e}} = \nabla w_0 \cdot \mathbf{e} = -\frac{M}{\sqrt{2}D(1-\nu)}(x \mp y), \quad (4.107)$$

and on a line parallel to  $\mathbf{e}$ ,  $y = \pm x + a$ ,  $a = \text{const}$ , we get

$$\frac{\partial w_0}{\partial \mathbf{e}} = \nabla w_0 \cdot \mathbf{e} = \pm \frac{M a}{\sqrt{2D(1-\nu)}} : \quad (4.108)$$

the slope along the segments parallel to the diagonals is constant: such segments remain straight lines and the final bent surface is a ruled surface (i.e. through any point of the plate pass two orthogonal lines entirely belonging to the deformed surface). Then,

$$\begin{aligned} M_x &= -D \left( \frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2} \right) = M, \\ M_y &= -D \left( \frac{\partial^2 w_0}{\partial y^2} + \nu \frac{\partial^2 w_0}{\partial x^2} \right) = -M, \\ M_{xy} &= -D(1-\nu) \frac{\partial^2 w_0}{\partial x \partial y} = 0, \end{aligned} \quad (4.109)$$

From eq. (4.103) we see that lines parallel to axis  $x$  deforms into a concave parabola, while those parallel to axis  $y$  into a convex one. In addition, the edges of the plate are not fixed, as it is easily checked:

$$\begin{aligned} w_0 \left( \pm \frac{\ell}{2}, y \right) &\neq 0, \quad w_0 \left( x, \pm \frac{\ell}{2} \right) \neq 0, \\ \frac{\partial w_0}{\partial \mathbf{n}} \Big|_{x=\pm \frac{\ell}{2}} &= \pm \frac{\partial w_0}{\partial x} \Big|_{x=\pm \frac{\ell}{2}} \neq 0, \\ \frac{\partial w_0}{\partial \mathbf{n}} \Big|_{y=\pm \frac{\ell}{2}} &= \pm \frac{\partial w_0}{\partial y} \Big|_{y=\pm \frac{\ell}{2}} \neq 0, \end{aligned} \quad (4.110)$$

and

$$\begin{aligned} T_x^* &= T_x + \frac{\partial M_{xy}}{\partial y} = T_x = -D\Delta w_{0,x} = 0, \\ T_y^* &= T_y + \frac{\partial M_{xy}}{\partial x} = T_y = -D\Delta w_{0,y} = 0. \end{aligned} \quad (4.111)$$

So, finally, the only way to deform the plate is to act upon it with distributed bending couples on the edges:  $-M$  on  $x = \pm \frac{\ell}{2}$ ,  $M$  on  $y = \pm \frac{\ell}{2}$ , like in Fig. 4.10. Now, let us

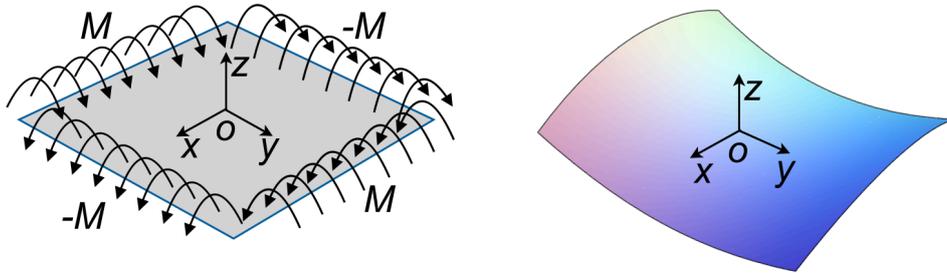


Figure 4.10: Anticlastic bending: actions on the boundary and bent plate.

consider the part A'B'C'D' of the plate shaded in Fig. 4.9;  $w_0$  is still the same, but now

the boundary is different. Nevertheless, because the boundary is parallel to the diagonals, we already know that it remains straight. As a consequence, bending moments are null along these lines, but not necessarily the twisting moments. In fact, because the unit normal to the boundary of the shaded part is  $\mathbf{n} = \frac{1}{\sqrt{2}}(1, \pm 1)$ , we get

$$\begin{aligned} M_n &= M_x n_x^2 + 2M_{xy} n_x n_y + M_y n_y^2 = M(n_x^2 - n_y^2) = 0, \\ M_t &= M_{xy}(n_x^2 - n_y^2) + n_x n_y (M_y - M_x) = -2M n_x n_y = \pm M, \\ T_n &= T_x n_x + T_y n_y = 0. \end{aligned} \quad (4.112)$$

So, in A' and C' we have a net vertical force  $R = -2M$  while in B' and D'  $R = 2M$ ; finally, the plate A'B'C'D' is like in Fig. 4.11. For ending, we remark that the same problem can be studied in the frame  $\{\xi, \eta\}$ , Fig. 4.9. Because  $x = \xi \cos \theta - \eta \sin \theta$ ,  $y = \xi \sin \theta + \eta \cos \theta$ , we get

$$w_0(\xi, \eta) = \frac{M \xi \eta}{D(1 - \eta)}. \quad (4.113)$$

This problem is called the *anticlastic bending* of plates. It is currently used in laboratory tests, because, on one hand, it is rather easy to be realized, on the other hand, the stress and strain fields are homogeneous, so allowing reliable measures.

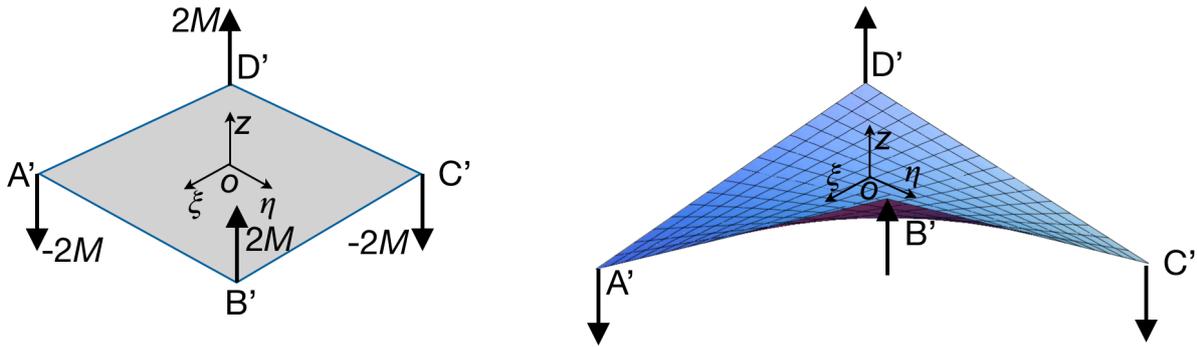


Figure 4.11: Anticlastic bending: actions on the corners and bent plate.

#### 4.2.11.5 Navier's method for rectangular plates

Let us consider a rectangular simply supported plate,  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ . If the plate is loaded by a distributed load of the type

$$q(x, y) = q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (4.114)$$

then it is easy to check that the solution  $w_0(x, y)$  that satisfies both the field equation (4.51) and the boundary conditions (4.72) is

$$w_0(x, y) = w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (4.115)$$

Because

$$\Delta^2 w_0 = \pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (4.116)$$

we get that  $\forall m, n$  it must be

$$\pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 w_{mn} = \frac{q_{mn}}{D} \Rightarrow w_{mn} = \frac{q_{mn}}{\pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 D}. \quad (4.117)$$

As any load function can be expressed by a double Fourier series,

$$q(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (4.118)$$

with

$$q_{mn} = \frac{4}{ab} \int_0^a \int_0^b q(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy, \quad (4.119)$$

a solution for a generic load can be obtained superposing the solutions  $w_{mn}$  of each harmonic. In particular, for  $q(x, y) = q = \text{const}$ , we get

$$q_{mn} = \frac{16 q}{\pi^2 mn}, \quad m, n = 1, 3, 5, \dots \quad (4.120)$$

the double series converges quickly for  $w_0$  but slowly for  $\mathbf{M}$ . Also, it is worth noting that to represent correctly a load, it is often needed to use a large number of terms in the series describing  $q(x, y)$ .

#### 4.2.11.6 Levy's method for rectangular plates

This method for simply supported plates  $0 \leq x \leq a$ ,  $-\frac{b}{2} \leq y \leq \frac{b}{2}$ , makes use of simple series: the deflection  $w_0$  is expressed as

$$w_0(x, y) = w_g(x, y) + w_p(x, y), \quad (4.121)$$

where  $w_g(x, y)$  is the general integral of the homogeneous equation

$$\Delta^2 w_0 = 0, \quad (4.122)$$

while  $w_p(x, y)$  is a particular integral. Generally speaking,

$$w_g(x, y) = \sum_{n=1}^{\infty} Y_n(y) \sin a_n x, \quad a_n = \frac{n\pi}{a}. \quad (4.123)$$

This expression is a solution of (4.122) if and only if

$$Y_n^{iv}(y) - 2a_n^2 Y_n''(y) + a_n^4 Y_n(y) = 0 \quad \forall n. \quad (4.124)$$

The general integral of this equation is

$$Y_n(y) = A_n \sinh a_n y + B_n \cosh a_n y + C_n a_n y \sinh a_n y + D_n a_n y \cosh a_n y. \quad (4.125)$$

The particular integral  $w_p(x, y)$  depends on the load; for a uniformly distributed load  $q$ ,

$$w_p(x, y) = \frac{q}{24D} (x^4 - 2ax^3 + a^3x) = \frac{4qa^4}{\pi^5 D} \sum_{n=1}^{\infty} \frac{1}{n^5} \sin a_n x, \quad n = 1, 3, 5, \dots \quad (4.126)$$

It can be easily checked that  $w_g(x, y)$  and  $w_p(x, y)$  satisfy the boundary conditions for  $x = 0$  and  $x = a$ . In addition, for a uniform load only the second and third terms are not null, because for symmetry it must be  $Y_n(y) = Y_n(-y)$ . In such a case we get

$$w_0(x, y) = \frac{qa^4}{D} \sum_{n=1}^{\infty} \left( \frac{4}{n^5\pi^5} + \xi_n \cosh a_n y + \eta_n \frac{n\pi y}{a} \sinh a_n y \right) \sin \frac{n\pi x}{a}, \quad n = 1, 3, 5, \dots \quad (4.127)$$

The coefficients  $\xi_n$  and  $\eta_n$  are determined by the boundary conditions:

$$w_0(x, \pm \frac{b}{2}) = 0, \quad \frac{\partial^2 w_0}{\partial y^2} \Big|_{y=\pm \frac{b}{2}} = 0, \quad (4.128)$$

that give

$$\xi_n = -\frac{4 + 2\varphi_n \tanh \varphi_n}{n^5\pi^5 \cosh \varphi_n}, \quad \eta_n = \frac{2}{n^5\pi^5 \cosh \varphi_n}, \quad \varphi_n = \frac{n\pi b}{2a}. \quad (4.129)$$

This series is very quickly convergent, practically a very good estimation for  $w_0$  is obtained just with the first term; for  $\mathbf{M}$  more terms are needed, but the convergence is still rapid.

### 4.3 The Reissner-Mindlin theory

The main problem with the Kirchhoff's theory is the absence of transverse shear strains, due to the second kinematical assumption of the model, see Sect. 4.2.1, which engenders the vanishing of the transverse shear stresses<sup>3</sup>. It seems hence obvious, in order to let appear transverse stresses  $\sigma_{xz}$  and  $\sigma_{yz}$ , to remove the second kinematical assumption of conservation of the orthogonality of the normal segments. This has been done independently, and following two different approaches, by Reissner (1945) and Mindlin (1951).

Removing the second Kirchhoff's kinematical assumption means that a segment originally orthogonal to the mid-plane will be no more orthogonal to the deformed mid-surface. However, we preserve the first and third kinematical assumptions of Kirchhoff: any orthogonal segment remains straight and of the same length.

Geometrically speaking, this means that in the plane  $\{x, z\}$ , see Fig. 4.12, it will be

$$\psi_x = \frac{\pi}{2} + \varphi_x - \beta_x. \quad (4.130)$$

The shear strain is hence (recall that for the small strain assumption,  $\beta_x \simeq \tan \beta_x = \frac{\partial w_0}{\partial x}$ )

$$\gamma_x = \frac{\pi}{2} - \psi_x = \beta_x - \varphi_x \simeq \frac{\partial w_0}{\partial x} - \varphi_x. \quad (4.131)$$

Similarly,

$$\gamma_y = \frac{\pi}{2} - \psi_y = \beta_y - \varphi_y \simeq \frac{\partial w_0}{\partial y} - \varphi_y, \quad (4.132)$$

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<sup>3</sup>We have seen that, with the assumption of symmetric loads distribution, these can be calculated using the equilibrium equations

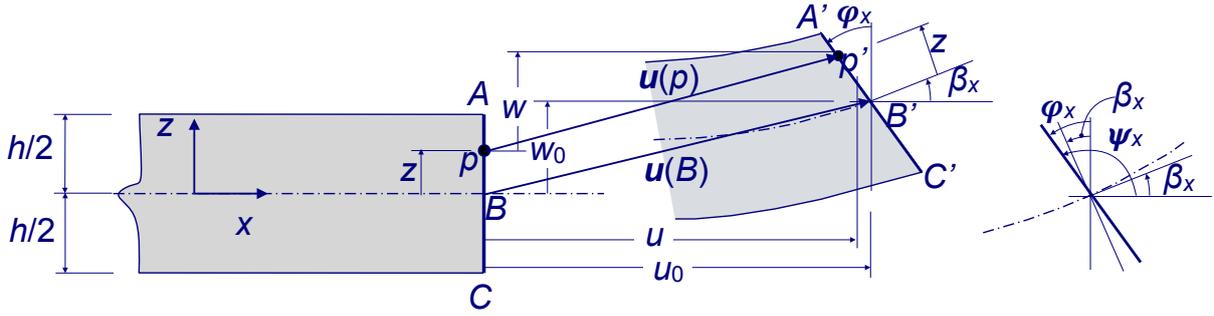


Figure 4.12: Reissner-Mindlin's kinematics in the  $\{x, z\}$  plane.

and the displacement field is then (the small strain assumption is still used)

$$\mathbf{u}(x, y, z) = \begin{pmatrix} u_0(x, y) - z\varphi_x(x, y) \\ v_0(x, y) - z\varphi_y(x, y) \\ w_0(x, y) \end{pmatrix} \quad (4.133)$$

so that now the strain tensor becomes

$$\boldsymbol{\varepsilon}(p) = \frac{\nabla \mathbf{u}(p) + \nabla \mathbf{u}(p)^\top}{2} = \begin{bmatrix} \frac{\partial u_0}{\partial x} - z\frac{\partial \varphi_x}{\partial x} & \frac{1}{2} \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) - \frac{1}{2} z \left( \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial w_0}{\partial x} - \varphi_x \right) \\ \frac{1}{2} \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) - \frac{1}{2} z \left( \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right) & \frac{\partial v_0}{\partial y} - z\frac{\partial \varphi_y}{\partial y} & \frac{1}{2} \left( \frac{\partial w_0}{\partial y} - \varphi_y \right) \\ \frac{1}{2} \left( \frac{\partial w_0}{\partial x} - \varphi_x \right) & \frac{1}{2} \left( \frac{\partial w_0}{\partial y} - \varphi_y \right) & 0 \end{bmatrix}. \quad (4.134)$$

Hence, we still can write eq. (4.30), but now the definition of the curvatures changes:

$$\boldsymbol{\kappa} = \begin{pmatrix} -\frac{\partial \varphi_x}{\partial x} \\ -\frac{\partial \varphi_y}{\partial y} \\ -\left( \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right) \end{pmatrix}. \quad (4.135)$$

Also, now  $\varepsilon_{xz}$  and  $\varepsilon_{yz}$  are different from zero. For the planar part, we still can write eq. (4.30), but now in addition we have also

$$\boldsymbol{\tau} = \mathbb{G}\boldsymbol{\gamma}, \quad (4.136)$$

i.e.

$$\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{pmatrix} 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{pmatrix}. \quad (4.137)$$

Hence, with the Reissner-Mindlin theory, we can obtain the transverse shear stresses directly from the constitutive law. However, we find here a problem of the Reissner-Mindlin theory:  $\sigma_{xz}$  and  $\sigma_{yz}$  are constant through the thickness, because  $\varepsilon_{xz}$  and  $\varepsilon_{yz}$  are,

as consequence of the assumption that any orthogonal segment remains straight also in the bent plate. This result is mechanically inconsistent: on one hand, we obtain tangential stresses  $\sigma_{zx} = \sigma_{xz}$  and  $\sigma_{zy} = \sigma_{yz}$  different from zero on  $S_T$  and  $S_B$ , which is correct only if an equal external tangential load is applied. On the other hand, we know, see Sect. 4.2.9, that the correct variation with  $z$  of  $\sigma_{xz}$  and  $\sigma_{yz}$  is parabolic.

To solve this inconsistency, one should pass to higher order theories (namely, to the Third Order Shear Deformation Theory of Reddy). In the framework of the Reissner-Mindlin theory, however, a *correction function*  $\xi(z)$  is then introduced to have a parabolic variation of  $\sigma_{xz}$  and  $\sigma_{yz}$  through the thickness:

$$\xi(z) = \tau \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right]. \quad (4.138)$$

The constant  $\tau$  can be determined in different ways. A first one, is to conserve the value of the shear stress in correspondance of  $z = 0$ , which is equivalent to fix as fundamental parameter the shear deformation  $\gamma(z = 0)$ ; then

$$\tau = \frac{T}{h}, \quad (4.139)$$

i.e.  $\tau$  corresponds to the mean value of the shear stress produced in the Reissner-Mindlin model without corrections by the shear action  $T$ . If, instead, we conserve the shear action  $T$ , then

$$\tau = \frac{3T}{2h}. \quad (4.140)$$

Finally, the choice commonly done is to put

$$\tau = \frac{5T}{4h}, \quad (4.141)$$

i.e. a mean value: neither the shear stress nor the shear action are conserved. With such a choice for  $\tau$  we get

$$\begin{aligned} \sigma_{xz} &= 2G\varepsilon_{xz}(x, y)\xi(z) = \frac{5}{4} \frac{E}{2(1+\nu)} \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right] \left( \frac{\partial w_0}{\partial x} - \varphi_x \right), \\ \sigma_{yz} &= 2G\varepsilon_{yz}(x, y)\xi(z) = \frac{5}{4} \frac{E}{2(1+\nu)} \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right] \left( \frac{\partial w_0}{\partial y} - \varphi_y \right), \end{aligned} \quad (4.142)$$

which gives

$$\begin{aligned} T_x &= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xz} dz = \frac{5}{6} \frac{E h}{2(1+\nu)} \left( \frac{\partial w_0}{\partial x} - \varphi_x \right), \\ T_y &= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yz} dz = \frac{5}{6} \frac{E h}{2(1+\nu)} \left( \frac{\partial w_0}{\partial y} - \varphi_y \right). \end{aligned} \quad (4.143)$$

The correction factor is hence  $\lambda = \frac{5}{6}$  for the shear forces, while  $\lambda = \frac{5}{4}$  for the shear stress.

The expression of  $\mathbf{N}$  does not change with respect to the Kirchhoff's theory, while for  $\mathbf{M}$  we get

$$\mathbf{M} = \frac{h^3}{12} \mathbb{D}\boldsymbol{\kappa} \Rightarrow \begin{cases} M_x = -\frac{h^3}{12} \frac{E}{1-\nu^2} \left( \frac{\partial \varphi_x}{\partial x} + \nu \frac{\partial \varphi_y}{\partial y} \right), \\ M_y = -\frac{h^3}{12} \frac{E}{1-\nu^2} \left( \frac{\partial \varphi_y}{\partial y} + \nu \frac{\partial \varphi_x}{\partial x} \right), \\ M_{xy} = -\frac{h^3}{12} \frac{E}{1+\nu} \left( \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right). \end{cases} \quad (4.144)$$

The equilibrium equations (4.35), (4.36) and (4.43) do not change, of course; passing to a unique, higher order equation for bending equilibrium, like eq. (4.44), is now meaningless, because we cannot obtain an equation of elastic equilibrium depending on a unique unknown,  $w_0(x, y)$ . In fact, the elastic equilibrium equations can be obtained, as usual, injecting the expressions of  $T_x, T_y, M_x, M_y$  and  $M_{xy}$  into the shear and bending equilibrium equations (the extension equilibrium equations (4.46) do not change), to obtain

$$\begin{aligned} \frac{h^2}{6(1-\nu)} \left[ \frac{\partial^2 \varphi_x}{\partial x^2} + (1-\nu) \frac{\partial^2 \varphi_x}{\partial y^2} + \frac{\partial^2 \varphi_y}{\partial x \partial y} \right] + \lambda \left( \frac{\partial w_0}{\partial x} - \varphi_x \right) &= 0, \\ \frac{h^2}{6(1-\nu)} \left[ (1-\nu) \frac{\partial^2 \varphi_y}{\partial x^2} + \frac{\partial^2 \varphi_y}{\partial y^2} + \frac{\partial^2 \varphi_x}{\partial x \partial y} \right] + \lambda \left( \frac{\partial w_0}{\partial y} - \varphi_y \right) &= 0, \\ \lambda \frac{E h}{2(1+\nu)} \left( \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} - \frac{\partial \varphi_x}{\partial x} - \frac{\partial \varphi_y}{\partial y} \right) + f_z &= 0. \end{aligned} \quad (4.145)$$

These are three second-order coupled partial differential equations; the unknowns are  $w_0(x, y), \varphi_x(x, y)$  and  $\varphi_y(x, y)$ . Three boundary conditions are hence needed for each point of the boundary; they must prescribe either

- kinematical conditions: the value of  $w_0, \varphi_x, \varphi_y$  (or, more generally,  $\varphi_n$  and  $\varphi_t$ ), or:
- natural conditions: the value of  $T_n, M_n, M_t$ .

Finally, with the Reissner-Mindlin theory there is no more the problem of the boundary conditions found in the Kirchhoff's theory. Actually, this was a consequence of the kinematical constraint imposed by the second Kirchhoff's assumption so that, once removed such assumption, the problem disappears.

## 4.4 The Von Karman theory

The assumption of linear strains, done in the two previous theories, fails to account for the so-called *membrane effect*, by which the tensions in a deflected plate help to react the applied *lateral loads*, i.e. the surface loads orthogonal to the mid-plane.

In order to take into account of such an effect, we introduce a measure of the strain suitable for geometric non-linearities: the *Green-Lagrange strain tensor*  $\mathbf{L}$ :

$$\mathbf{L} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^\top + \nabla \mathbf{u}^\top \nabla \mathbf{u}); \quad (4.146)$$

if, for the sake of convenience, we still indicate  $\mathbf{L}$  by  $\boldsymbol{\varepsilon}$ , then the components  $\varepsilon_{ij}$  now are<sup>4</sup>:

$$\begin{aligned}
\varepsilon_{xx} &= u_{,x} + \frac{1}{2} (u_{,x}^2 + v_{,x}^2 + w_{,x}^2), \\
\varepsilon_{yy} &= v_{,y} + \frac{1}{2} (u_{,y}^2 + v_{,y}^2 + w_{,y}^2), \\
\varepsilon_{zz} &= w_{,z} + \frac{1}{2} (u_{,z}^2 + v_{,z}^2 + w_{,z}^2), \\
\varepsilon_{xy} &= \frac{1}{2} (u_{,y} + v_{,x} + u_{,x}u_{,y} + v_{,x}v_{,y} + w_{,x}w_{,y}), \\
\varepsilon_{xz} &= \frac{1}{2} (u_{,z} + w_{,x} + u_{,x}u_{,z} + v_{,x}v_{,z} + w_{,x}w_{,z}), \\
\varepsilon_{yz} &= \frac{1}{2} (v_{,z} + w_{,y} + u_{,y}u_{,z} + v_{,y}v_{,z} + w_{,y}w_{,z}).
\end{aligned} \tag{4.147}$$

In the theory of plates of Von Karman (1910), only some of the quadratic terms are retained, namely: the quadratic terms in  $w_{,x}$ ,  $w_{,y}$  are retained, the other ones, discarded. Von Karman justifies this choice as follows:

- $u_{,x}^2$  is negligible with respect to  $u_{,x}$ ; the same is true for  $v_{,y}^2$  and  $w_{,z}^2$ ;
- a similar argument is used also for the terms  $v_{,y}v_{,z}$ ,  $w_{,y}w_{,z}$ ,  $u_{,z}u_{,x}$ ,  $w_{,z}w_{,x}$ ,  $u_{,x}u_{,y}$ ,  $v_{,x}v_{,y}$ ;
- the terms  $v_{,x}^2$ ,  $u_{,y}^2$ ,  $w_{,y}^2$ ,  $u_{,z}^2$ ,  $v_{,z}^2$  are of the same order of the dropped terms;
- $w_{,x}$  and  $w_{,y}$  are the slopes of cross sections of the deformed mid-plane: they can be large compared to the strain components.

So, finally:

$$\begin{aligned}
\varepsilon_{xx} &= u_{,x} + \frac{1}{2}w_{,x}^2, \\
\varepsilon_{yy} &= v_{,y} + \frac{1}{2}w_{,y}^2, \\
\varepsilon_{zz} &= w_{,z}, \\
\varepsilon_{xy} &= \frac{1}{2} (u_{,y} + v_{,x} + w_{,x}w_{,y}), \\
\varepsilon_{xz} &= \frac{1}{2} (u_{,z} + w_{,x}), \\
\varepsilon_{yz} &= \frac{1}{2} (v_{,z} + w_{,y}).
\end{aligned} \tag{4.148}$$

We introduce again the vector  $\mathbf{u}_0(x, y)$ , displacement of the points of the mid-plane and we need to specify how  $u, v, w$  vary through the thickness. To this purpose, the Von Karman's theory makes the same kinematical assumptions of the Kirchhoff's theory, so

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<sup>4</sup>For the sake of shortness, we indicate partial derivatives by a comma followed by the differentiation variable, e.g.  $\frac{\partial u}{\partial x} = u_{,x}$  etc.

that eq. (4.10) still holds and

$$\begin{aligned}
\varepsilon_{xx} &= e_x - z w_{0,xx}, \\
\varepsilon_{yy} &= e_y - z w_{0,yy}, \\
\gamma_{xy} &= 2\varepsilon_{xy} = e_{xy} - 2z w_{0,xy}, \\
\varepsilon_{xz} &= \varepsilon_{yz} = \varepsilon_{zz} = 0,
\end{aligned} \tag{4.149}$$

where we have put

$$\begin{aligned}
e_x &= u_{0,x} + \frac{1}{2} w_{0,x}^2, \\
e_y &= v_{0,y} + \frac{1}{2} w_{0,y}^2, \\
e_{xy} &= u_{0,y} + v_{0,x} + w_{0,x} w_{0,y}.
\end{aligned} \tag{4.150}$$

Still like in the Kirchhoff's theory, we assume  $\sigma_{zz} = 0$ , while  $\sigma_{xz} = \sigma_{yz} = 0$  as consequence of the constitutive law and of the kinematical assumptions, that lead, as in the Kirchhoff's theory, to  $\varepsilon_{xz} = \varepsilon_{yz} = 0$ . Finally, for  $\boldsymbol{\sigma}$  we obtain once more the results of the Kirchhoff's theory, eq. (4.19), but now the definition of the strain is different.

Let us now follow a variational approach for finding the equilibrium equations (the same approach can be used also for the classical theory of Kirchhoff<sup>5</sup>).

The strain energy density is

$$\begin{aligned}
U &= \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \frac{1}{2} \frac{E}{1 - \nu^2} \left[ (\varepsilon_{xx} + \nu \varepsilon_{yy}) \varepsilon_{xx} + (\varepsilon_{yy} + \nu \varepsilon_{xx}) \varepsilon_{yy} + \frac{1 - \nu}{2} \gamma_{xy}^2 \right] \\
&= \frac{G}{1 - \nu} \left( \varepsilon_{xx}^2 + \varepsilon_{yy}^2 + 2\nu \varepsilon_{xx} \varepsilon_{yy} + \frac{1 - \nu}{2} \gamma_{xy}^2 \right),
\end{aligned} \tag{4.151}$$

by consequence, the total strain energy of the plate is

$$U_p = \int_{\Omega} U \, d\omega = \int_S \left( \int_{-\frac{h}{2}}^{\frac{h}{2}} U \, dz \right) dx \, dy, \tag{4.152}$$

where  $S$  denotes the mid-surface. Once the integration over  $z$  done, we get

$$U_p = U_m + U_b, \tag{4.153}$$

where

- $U_m$ : membrane strain energy, linear in  $h$ :

$$\begin{aligned}
U_m &= \frac{G h}{1 - \nu} \int_S \left[ \left( u_{0,x} + \frac{w_{0,x}^2}{2} \right)^2 + \left( v_{0,y} + \frac{w_{0,y}^2}{2} \right)^2 \right. \\
&\quad + 2\nu \left( u_{0,x} + \frac{w_{0,x}^2}{2} \right) \left( v_{0,y} + \frac{w_{0,y}^2}{2} \right) \\
&\quad \left. + \frac{1 - \nu}{2} (u_{0,x} + v_{0,y} + w_{0,x} w_{0,y})^2 \right] dx;
\end{aligned} \tag{4.154}$$

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<sup>5</sup>To this purpose, the reader is addressed to the classical text of Langhaar, see the suggested references. In such a text, a presentation of the the variational approach to the boundary conditions is also given.

- $U_b$ : bending strain energy, cubic in  $h$ :

$$U_b = \frac{G h^3}{12(1-\nu)} \int_S [w_{0,xx}^2 + w_{0,yy}^2 + 2\nu w_{0,xx} w_{0,yy} + 2(1-\nu)w_{0,xy}^2] dx dy. \quad (4.155)$$

The potential energy of a lateral load  $q(x, y)$  acting on the plate is

$$U_q = - \int_S q(x, y) w_0(x, y) dx dy, \quad (4.156)$$

so finally the total potential energy is the functional

$$V = U_M + U_b + U_q. \quad (4.157)$$

The principle of the minimum total potential energy is then used: the equilibrium configuration is that giving the least value of  $V$ . We need hence to write the conditions giving the minimum of the quadratic functional  $V$  of the two independent variables  $x$  and  $y$ . These conditions are the Euler-Lagrange equations for  $V$ : they are the equilibrium equations for the plate. To this purpose, we notice that

$$V = \int_S F(u_0, u_{0,x}, u_{0,y}; v_0, v_{0,x}, v_{0,y}; w_0, w_{0,x}, w_{0,y}, w_{0,xx}, w_{0,xy}, w_{0,yy}; x, y) dx dy, \quad (4.158)$$

i.e.  $F$  is a functional of three independent functions,  $u_0, v_0$  and  $w_0$ , so we need to write three Euler-Lagrange equations:

$$\frac{\partial F}{\partial \xi} - \frac{\partial}{\partial x} \frac{\partial F}{\partial \xi_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial \xi_y} + \frac{\partial^2}{\partial x^2} \frac{\partial F}{\partial \xi_{xx}} + \frac{\partial^2}{\partial x \partial y} \frac{\partial F}{\partial \xi_{xy}} + \frac{\partial^2}{\partial y^2} \frac{\partial F}{\partial \xi_{yy}} = 0, \quad (4.159)$$

where  $\xi$  can be either  $u_0, v_0$  or  $w_0$ . Because

$$\begin{aligned} F = & \frac{G h}{1-\nu} \left[ \left( u_{0,x} + \frac{w_{0,x}^2}{2} \right)^2 + \left( v_{0,y} + \frac{w_{0,y}^2}{2} \right)^2 \right. \\ & \left. + 2\nu \left( u_{0,x} + \frac{w_{0,x}^2}{2} \right) \left( v_{0,y} + \frac{w_{0,y}^2}{2} \right) + \frac{1-\nu}{2} (u_{0,x} + v_{0,y} + w_{0,x} w_{0,y})^2 \right] \\ & + \frac{G h^3}{12(1-\nu)} [w_{0,xx}^2 + w_{0,yy}^2 + 2\nu w_{0,xx} w_{0,yy} + 2(1-\nu)w_{0,xy}^2] - q w_0, \end{aligned} \quad (4.160)$$

we get finally the three equilibrium equations of the plate (the third one is simplified using the first two equations):

$$\begin{aligned} (e_x + \nu e_y)_{,x} + \frac{1-\nu}{2} e_{xy,y} &= 0, \\ \frac{1-\nu}{2} e_{xy,x} + (e_y + \nu e_x)_{,y} &= 0, \\ \Delta^2 w_0 &= \frac{q}{D} + \frac{12}{h^2} [(e_x + \nu e_y)w_{0,xx} + (e_y + \nu e_x)w_{0,yy} + (1-\nu)e_{xy}w_{0,xy}]. \end{aligned} \quad (4.161)$$

The internal actions as defined in eq. (4.22) become now, through eqs. (4.19) and (4.149),

$$\begin{aligned} N_x &= \frac{E h}{1-\nu^2} (e_x + \nu e_y), \\ N_y &= \frac{E h}{1-\nu^2} (e_y + \nu e_x), \\ N_{xy} &= G h e_{xy} = \frac{E h}{2(1+\nu)} e_{xy}. \end{aligned} \quad (4.162)$$

By consequence, eqs. (4.161)<sub>1,2</sub> can be written in terms of components of membrane internal actions; some simple calculations give

$$\begin{aligned} N_{x,x} + N_{xy,y} &= 0, \\ N_{xy,x} + N_{y,y} &= 0. \end{aligned} \quad (4.163)$$

We find again eqs. (4.35) for the case  $f_x = f_y = 0$ , i.e. when loads parallel to the mid-plane are null (an assumption tacitly done in this case, where the only action is the lateral load  $q$ ). The above two extension equilibrium equations can be solved using the technique of the *Airy's stress function*  $\chi(x, y)$ :

$$N_x = \chi_{,yy}, \quad N_y = \chi_{,xx}, \quad N_{xy} = -\chi_{,xy}. \quad (4.164)$$

If  $\mathbf{N}$  is related to  $\chi(x, y)$  by the previous relations, than eqs. (4.163) are automatically satisfied: the problem of determining  $\mathbf{N}$  is reduced to the that of finding a unique scalar function,  $\chi(x, y)$ . For what concerns eq. (4.161)<sub>3</sub>, it becomes

$$D\Delta^2 w_0 = q + N_x w_{0,xx} + N_y w_{0,yy} + 2N_{xy} w_{0,xy}, \quad (4.165)$$

and introducing  $\chi$ ,

$$D\Delta^2 w_0 = q + \chi_{,yy} w_{0,xx} + \chi_{,xx} w_{0,yy} - 2\chi_{,xy} w_{0,xy}. \quad (4.166)$$

A second relation between  $w_0(x, y)$  and  $\chi(x, y)$  can be easily obtained: from the equations of  $N_x, N_y$  and  $N_{xy}$  we get

$$\begin{aligned} h e_x &= \frac{1}{E}(N_x - \nu N_y) = \frac{1}{E}(\chi_{,yy} - \nu\chi_{,xx}), \\ h e_y &= \frac{1}{E}(N_y - \nu N_x) = \frac{1}{E}(\chi_{,xx} - \nu\chi_{,yy}), \\ h e_{xy} &= \frac{N_{xy}}{G} = -\frac{\chi_{,xy}}{G}, \end{aligned} \quad (4.167)$$

while

$$e_{x,yy} + e_{y,xx} - e_{xy,xy} = w_{0,xy}^2 - w_{0,xx} w_{0,yy}. \quad (4.168)$$

Eliminating  $e_x, e_y$  and  $e_{xy}$  by means of the previous equations gives

$$\Delta^2 \chi = E h (w_{0,xy}^2 - w_{0,xx} w_{0,yy}). \quad (4.169)$$

Equations (4.166) and (4.169) are the fundamental relations in the Von Karman theory of plates. They reduce the problem to the search of two bi-dimensional functions,  $w_0(x, y)$  and  $\chi(x, y)$ . The use of  $\chi$  automatically ensures the in-plane equilibrium and  $\mathbf{N}$  is found by derivation, once  $\chi$  known.

## 4.5 Exercises

1. Check that the solution for a circular clamped plate of radius  $R$ , loaded by a uniform load  $q$  is, in polar coordinates,

$$w(r, \theta) = \frac{q}{64D}(R^2 - r^2)^2.$$

Calculate then the deflection in the center and the couples on the boundary.

2. Using the inverse method, study the case of a rectangular plate, whose sides  $2a$  and  $2b$  are parallel to the axes  $x$  and  $y$  respectively, with origin at the plate's center, and whose solution is the field of vertical displacements

$$w = c[(x^2 - a^2)^2 + (y^2 - b^2)^2], \quad c < 0.$$

In particular, find the conditions on the boundary (displacements and forces) and calculate the deflection and the moments in the plate's center.

3. Study a simply supported rectangular plate of sides  $a$  and  $b$  subjected to the sinusoidal load

$$p(x, y) = p_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}.$$

4. Consider a simply supported square plate of side  $\ell$ , acted upon by a uniform load  $p$ ; compare the results for the deflection and moments in the plate's center using first the Navier and then the Levy approach.



# Suggested texts

1. A. E. H. Love: *A treatise on the mathematical theory of elasticity*. Fourth edition. Dover, 1944.
2. I. S. Sokolnikoff: *Mathematical theory of elasticity*. McGraw-Hill, 1946.
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6. P. Villaggio: *Mathematical models for elastic structures*, Cambridge University Press, 1997.
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8. J.-J. Marigo: *Mécanique des milieux continus I*. Ecole Polytechnique, 2014.  
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9. P. M. Mariano, L. Galano: *Fundamentals of the Mechanics of Solids*, Birkhäuser, 2016.