Basics elements on linear elastic fracture mechanics and crack growth modeling

Sylvie Pommier

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Fail Safe

Damage Tolerant Design

- **Consider the eventuality of damage or of the presence of defects,**
- **predict if these defects or damage may lead to fracture,**
- **and, in the event of failure, predicts the consequences (size, velocity and trajectory of the fragments)**

Foundations of fracture mechanics : The Liberty Ships

- **2700** Liberty Ships were built between 1942 and the end of WWII
- The production rate was of **70 ships / day**
- duration of construction: 5 days
- **30%** of ships built in **1941** have suffered catastrophic failures
- **362** lost ships

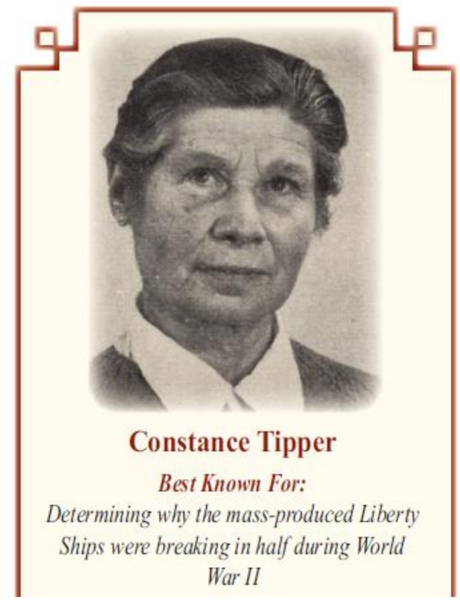
Liberty ships – hiver 1941

The fracture mechanics concepts were still unknown

Causes of fracture:

- **Welded** Structure rather than bolted, offering a substantial assembly time gain but with a **continuous path** offered for cracks to propagate through the structure.
- **Low quality** of the **welds** (presence of cracks and internal stresses)
- **Low quality steel**, ductile/brittle transition around **0°C**

Liberty Ships, WWII, 1941, Brittle fracture

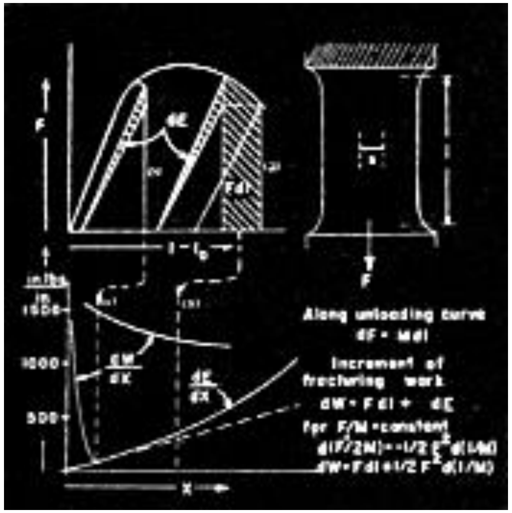


LEFM - Linear elastic fracture mechanics

Georges Rankine Irwin "*the godfather of fracture mechanics »*

- Stress intensity factor **K**
- Introduction of the concept of fracture toughness **K_{IC}**
- Irwin's plastic zone (monotonic and cyclic)
- Energy release rate **G** and **Gc**

  (G in reference to Griffith)



Georges Rankine Irwin

Historical context

Previous authors

Griffith A. A. - 1920 –"*The phenomenon of rupture and flow in solids", 1920, Philosophical Transactions of the Royal Society, Vol. A221 pp.163-98*

Westergaard H. M. – 1939 - *Bearing Pressures and Cracks, Journal of Applied Mechanics* ***6****: 49-53.*

Muskhelishvili N. – 1954 - *Ali Kheiralla, A. Muskhelishvili, N.I. Some Basic problems of the mathematical theory of elasticity. Third revis. and augmented. Moscow,* ***1949****, J.Appl. Mech.,21* ***(1954),*** *No 4, 417-418.*

n.b. Joseph Staline died in 1953

Fatigue crack growth: De Havilland Comet

3 accidents

26/10/1952, departing from Rome Ciampino

March 1953, departing from Karachi Pakistan

10/01/1954, Crash on the Rome-London flight (with passengers)

Paris & Erdogan 1961

They correlated the cyclic fatigue crack growth rate **da / dN** with the stress intensity factor amplitude **ΔK**

Introduction of the **Paris' law** for modeling fatigue crack growth

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Fatigue remains a topical issue

8 Mai 1842 - Meudon (France)
Fracture of an axle by fatigue

3 Juin 1998 - Eschede (Allemagne)
Fracture of a wheel by Fatigue

Development of rules for the EASA certification

Aloha April, 28th 1988,

Los Angeles, June, 2nd 2006,

Rotor Integrity Sub-Committee (RISC)

UAL 232, July 19, 1989 Sioux City, Iowa

- DC10-10 crashed on landing
- In-Flight separation of Stage 1 Fan Disk
- **Failed from cracks out of material anomaly**
  - Hard Alpha produced during melting
- Life Limit: 18,000 cycles. Failure: 15,503 cycles.
- 111 fatalities
- FAA Review Team Report (1991) recommended:
  - **Changes in Ti melt practices, quality controls**
  - Improved mfg and **in-service inspections**
  - **Lifing Practices based on damage tolerance**

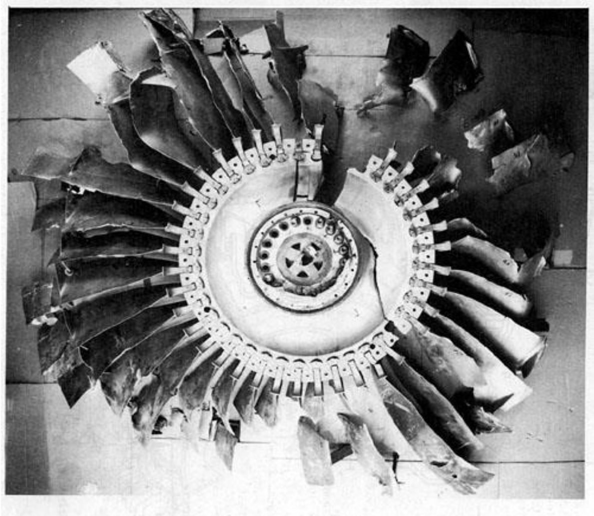

Figure 18.--No. 2 engine stage 1 fan disk (reconstructed with blades).

AIA Rotor Integrity Sub-Committee (RISC) : <u>Elaboration of AC 33.14-1</u>

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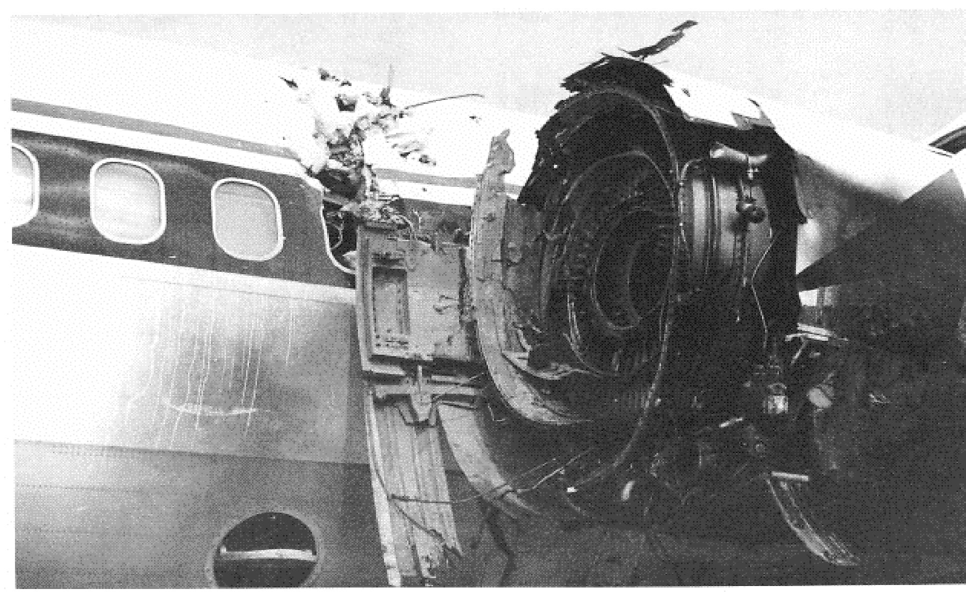
DL 1288, July 6, 1996 , Pensacola, Florida

- MD-88 engine failure on take-off roll
- Pilot aborted take-off
- Stage 1 Fan Disk separated; impacted cabin
- Failure from abusively machined bolthole
- Life Limit: 20,000 cycles. Failure: 13,835 cycles.
- 2 fatalities
- NTSB Report recommended ...
  - Changes in inspection methods, shop practices
  - Fracture mechanics based damage tolerance



<u>Elaboration of AC 33.70-2</u>

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Why ?

- To prevent fatalities and disaster

Where ?

- Public transportation (trains, aircraft, ships…)
- Energy production (nuclear power plant, oil extraction and transportation …)
- Any areas of risk to public health and environment

How ?

- Critical components are designed to be damage tolerant / fail safe
- Rare events (defects and cracks) are assumed to be certain (deterministic approach) and are introduced on purpose for lab. tests and certification

Damage tolerance

Fracture mechanics

One basic assumption :

The structure contains a singularity (ususally a geometric discontinuity, for example: a crack)

Two main questions :

What are the **relevant variables** to characterize the risk of fracture and to be used in fracture criteria ?

What are the **suitable criteria** to determine if the crack may propagate or remain arrested, the crack growth rate and the crack path ?

Classes of material behaviour : relevant variables

Linear elastic behaviour: linear elastic fracture mechanics (K)

Nonlinear behavior: non-linear fracture mechanics

Hypoelasticity : Hutchinson Rice & Rosengren, (J)

Ideally plastic material : Irwin, Dugdale, Barrenblatt etc.

Time dependent material behaviours: viscoelasticity, viscoplasticity (C*)

Complex non linear material behaviours :

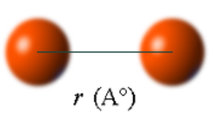
Various local and non local approaches of failure, J. Besson, A. Pineau, G. Rousselier, A. Needleman, Tvergaard , S. Pommier etc.

Classes of fracture mechanisms : criteria

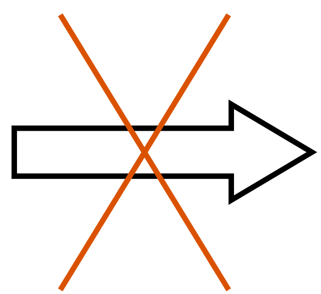
- Brittle fracture
- Ductile fracture
- Dynamic fracture
- Fatigue crack growth
- Creep crack growth
- Crack growth by corrosion, oxydation, ageing
- Coupling between damage mechanisms

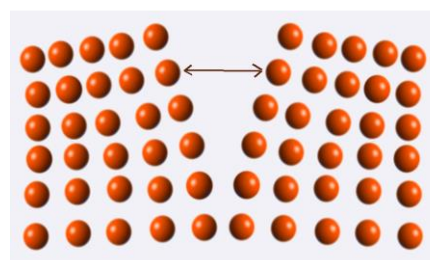
Mechanisms acting at very different scales of time and space, an assumption of scales separation

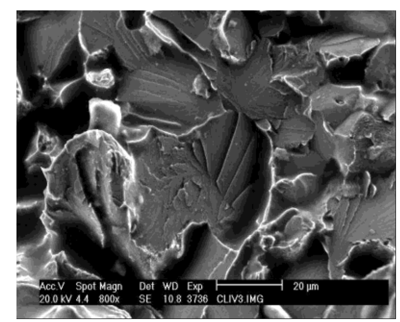
- Atomic scale (surface oxydation, ageing, …)
- Microstructural scale (grain boundary corrosion, creep, oxydation, persistent slip band in fatigue etc…)
- Plastic zone scale or damaged zone (material hardening or softening, continuum damage, ductile damage...)
- Scale of the structure (wave propagation …)



Atomic cohesion energy
<u>10 J/m²</u>







Brittle fracture energy
<u>10 000 J/m²</u>

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Classes of relevant assumptions : application of criteria

Long cracks (2D problem, planar crack with a straight crack)

Curved cracks, branched cracks, merging cracks (3D problem, non-planar cracks, curved crack fronts)

Short cracks (3D problem, influence of free surfaces, scale and gradients effects)

Other discontinuities and singularities:

- Interfaces / free surfaces,
- Contact front in partial slip conditions,
- acute angle ending on a edge,

Griffith' theory

Threshold for unsteady crack growth (brittle or ductile)

Relevant variable : energy release rate **G**

Criteria : An unsteady crack growth occurs if the cohesion energy released by the structure because of the creation of new cracked surfaces reaches the energy required to create these new cracked surfaces

$$G = G_c$$

Data : critical energy release rate **Gc**

Griffith' theory

W_{ext} : work of external forces

$\Delta U_{elastic}$: variation of the elastic energy of the structure

$\Delta U_{surface}$: variation of the surface energy of the structure

$$\boxed{\begin{aligned}&\Delta U = \Delta U_{elastic} + \Delta U_{surface} = W_{ext} \\ &\Rightarrow \Delta U_{surface} = 2\gamma\, da = W_{ext} - \Delta U_{elastic}\end{aligned}}$$

Criteria :

$$\boxed{\begin{aligned}&G = 2\gamma \\ &where \\ &G = -\frac{\Delta U_{elastic} - W_{ext}}{da}\end{aligned}}$$

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Evolution by Bui, Erlacher & Son

$$dU = W_{ext} + Q \quad where \quad TdS - Q \geq 0$$

$$dU = dF + TdS + SdT$$

$$in \quad isothermal \quad conditions \quad dT = 0$$

$$TdS - Q = -\left(dF - W_{ext}\right) \geq 0$$

$$-\left(dF_{volume} + dF_{surface} - W_{ext}\right) \geq 0$$

$$\Rightarrow \left(G - G_c\right) da \geq 0$$

$$where$$

$$G_c = \frac{\Delta F_{surface}}{da} = 2\gamma$$

$$G = -\frac{\left(\Delta F_{volume} - W_{ext}\right)}{da}$$

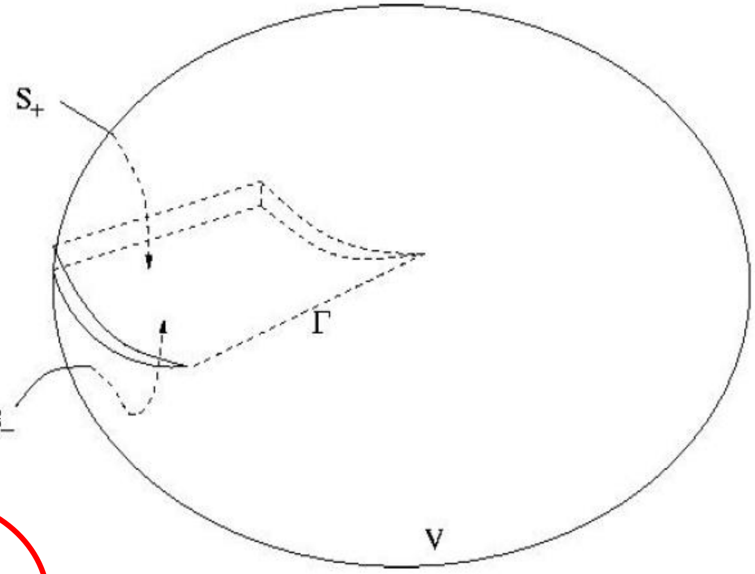
Free energy instead of internal energy

Isothermal conditions instead of adiabatic

Second principle

J Integral (Rice)

$$G = -\frac{(\Delta F_{volume} - W_{ext})}{da}$$



Eschelby tensor : energy density

$$P_{ij} = \frac{1}{2}\sigma_{kl}\epsilon_{kl}\delta_{ij} - \sigma_{kj}u_{k,i}$$

J integral , (Rice's integral if q is coplanar)

$$J = -\int_V q_{i,j}P_{ij}dV + \int_{S_+\cup S_-} q_i P_{ij} n_j dS$$

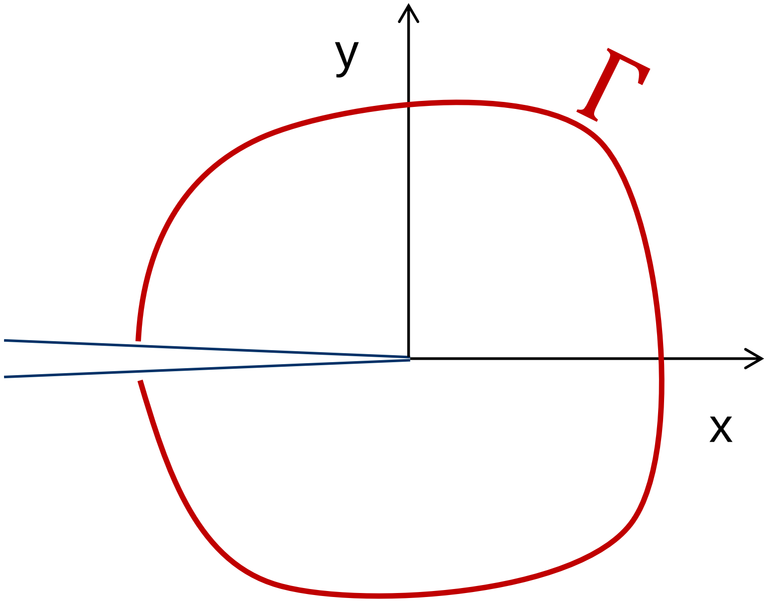
- q vector: the crack front motion

J contour integral

If the crack faces are free surfaces (no friction, no fluid pressure …),

If volume forces can be neglected (inertia, electric field...)

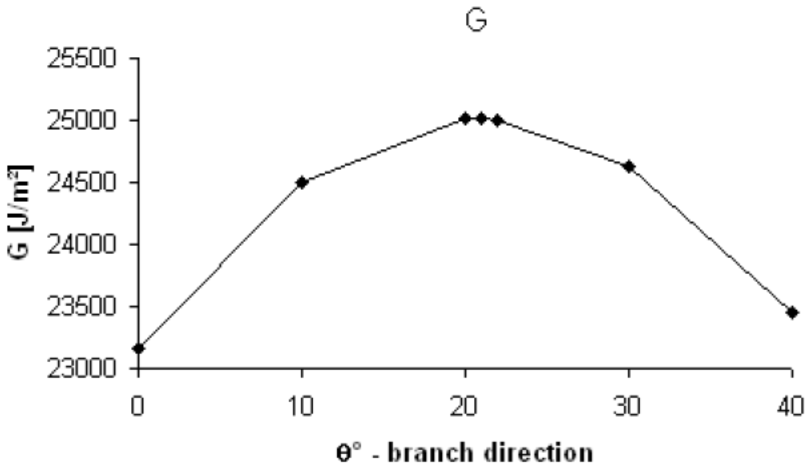
Then the J integral is shown to be independent of the choice of the selected integration contour

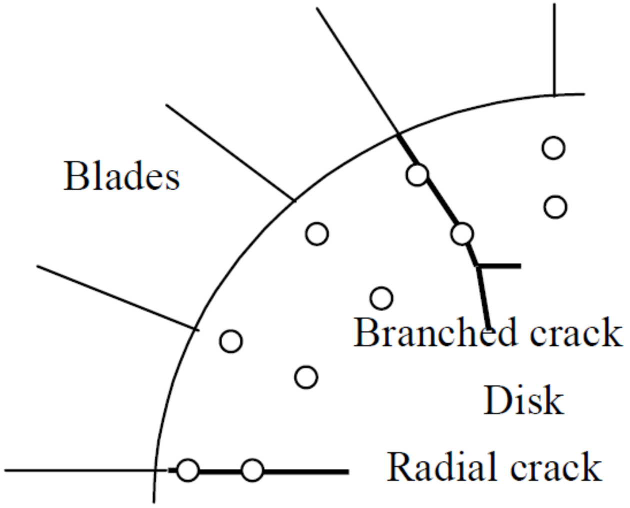


$$G = J{=}\int_{\Gamma} \left(\varphi_{free\ energy\ density}dy - \underline{\underline{\sigma}}\,\underline{n}.\frac{\partial \underline{u}}{\partial x}\right)$$

Applications

C. Stoisser, I. Boutemy and F. Hasnaoui





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Limitations

- The crack faces must be free surfaces (no friction, no fluid pressure)
- **Gc** is a material constant (single mechanism, surfacic mechanism only)
- What if non isothermal conditions are considered ?
- Unsteady crack growth criteria, non applicable to steady crack propagation,
- The surfacic energy 2γ may be negligible compared with the energy dissipated in plastic work or continuum damage / localization process

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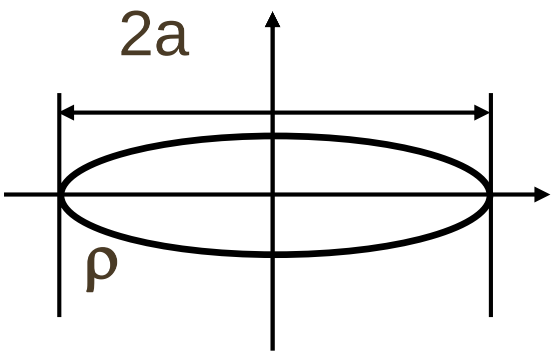
Linear Elastic Fracture Mechanics (LEFM)

Characterize the state of the structure where useful (near the crack front where damage occurs) for a linear elastic behavior of the material

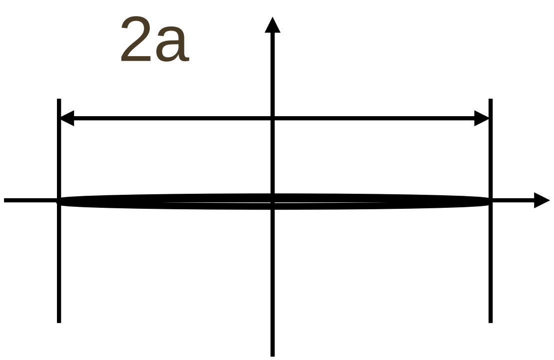
Preliminary remarks:
From the discontinuity to the singularity

Stress concentration factor **Kt** of an elliptical hole,

With a length **2a** and a curvature radius ρ



$$K_t = \frac{\sigma_{loc}}{\sigma^{\infty}} = 1 + 2\sqrt{\frac{a}{\rho}}$$

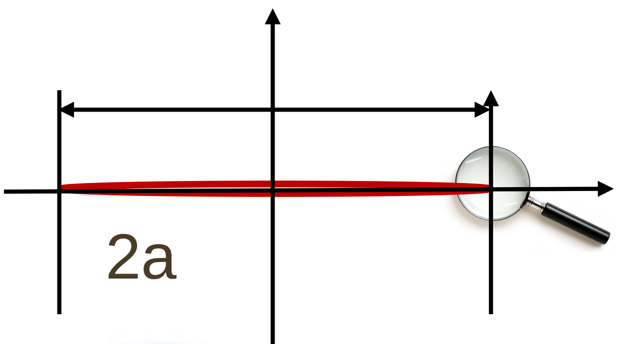


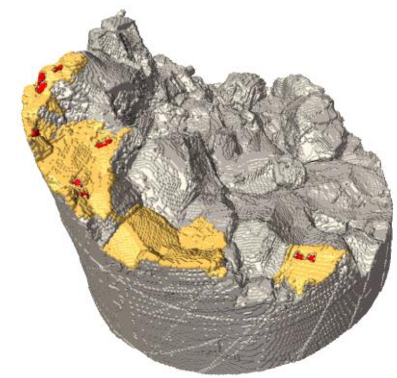
$$\rho \rightarrow 0 \Rightarrow \frac{\sigma_{loc}}{\sigma^{\infty}} \rightarrow 2\sigma_{loc}\sqrt{\frac{a}{\rho}} \rightarrow \infty$$

Singularity

Remarks: existence of a singularity

Geometry locally-self-similar → self-similar solution → principle of simulitude



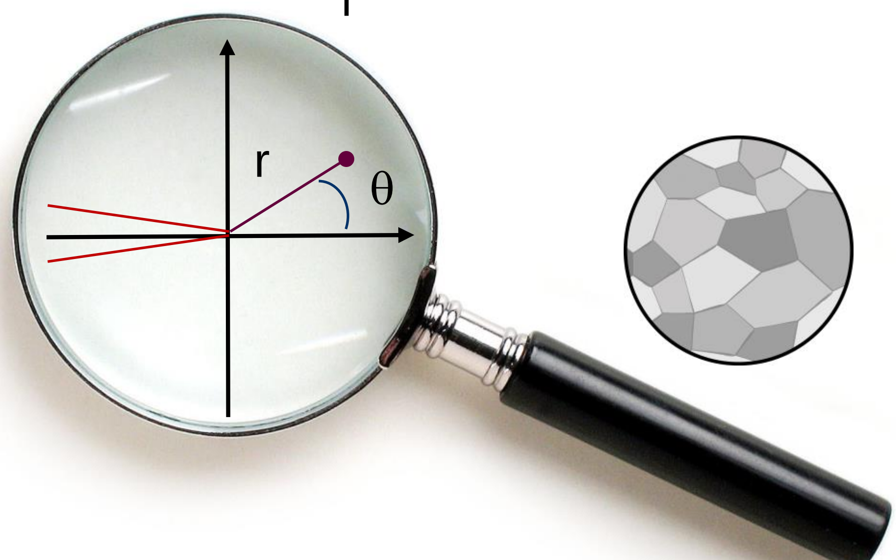


$$\rho \to 0 \Rightarrow$$

$$\underline{\underline{\sigma}}(r,\theta) = f(r)\underline{\underline{g}}(\theta)$$

$$r^* = \alpha r$$

$$f(r^*) = q(\alpha) f(r)$$



r : distance to the discontinuity

Warning: implicit choice of scale

Order of this singularity

Linear elasticity:

$$\varepsilon(r) \underset{r\to 0}{\to} Br^{\lambda}$$

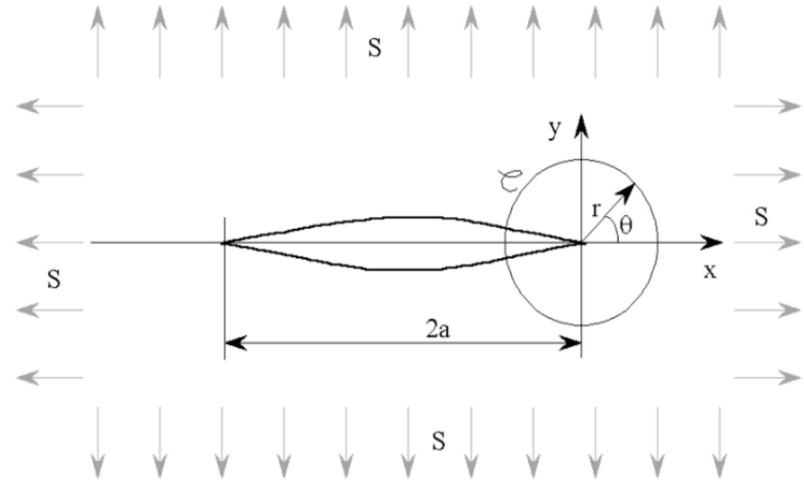
$$\sigma(r) \underset{r\to 0}{\to} Cr^{\lambda}$$

$$E_{elast} \underset{r\to 0}{\to} A \int_{\theta=-\pi}^{\theta=\pi} \int_{r=0}^{r=r_{\max}} r^{\lambda} r^{\lambda} (rd\theta) dr$$

$$E_{elast} \underset{r\to 0}{\to} 2\pi A \int_{r=0}^{r=r_{\max}} r^{2\lambda+1} dr$$

$$E_{elast} \underset{r\to 0}{\to} 2\pi A \frac{r_{\max}^{2\lambda+2}}{2\lambda+2}$$

For a crack : λ=-0.5



$$\Rightarrow \lambda < 0$$

$$E_{elast} \underset{r\to 0}{\to} 2\pi A \frac{r_{\max}^{2\lambda+2}}{2\lambda+2}$$

$$\left.\begin{array}{l} 2\lambda+2>0 \Rightarrow \lambda > -1 \\ 2\lambda+2 \neq 0 \Rightarrow \lambda \neq -1 \end{array}\right\} \Rightarrow -1 < \lambda < 0$$

Non linear material behaviour ?

$$\varepsilon = \varepsilon_o \left(\frac{\sigma}{\sigma_o} \right)^n \qquad (n = 1 \; elastic)$$

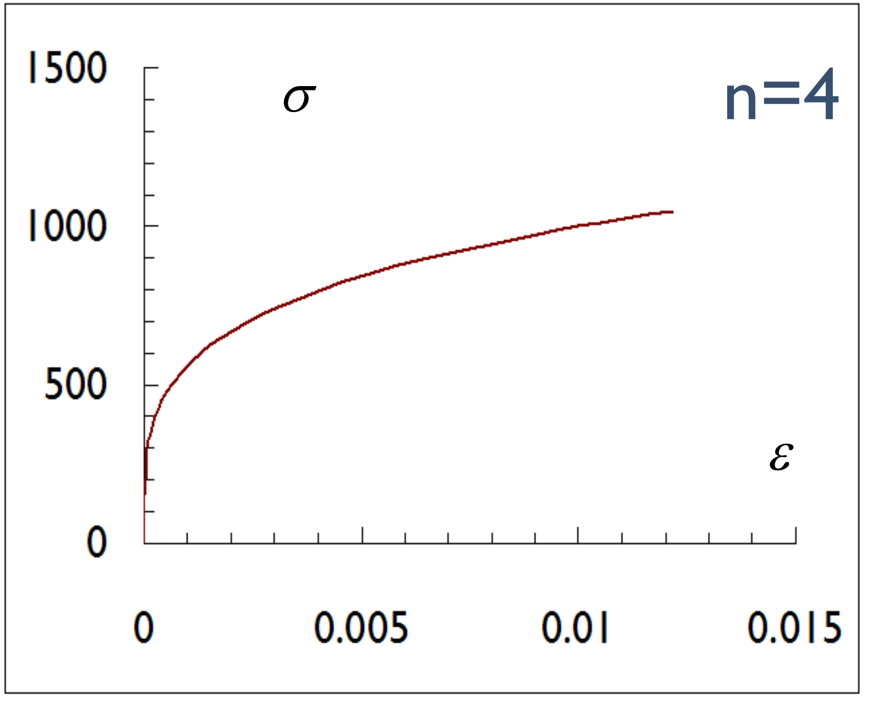
$$\sigma(r) \underset{r \to 0}{\to} A r^{\lambda}$$

$$\varepsilon(r) \underset{r \to 0}{\to} B r^{n\lambda}$$

$$E_{elast} \underset{r \to 0}{\to} C \int_{\theta=-\pi}^{\theta=\pi} \int_{r=0}^{r=r_{\max}} r^{\lambda} r^{n\lambda} (r d\theta) dr$$

$$E_{elast} \underset{r \to 0}{\to} 2\pi C \int_{r=0}^{r=r_{\max}} r^{(1+n)\lambda+1} dr$$

$$E_{elast} \underset{r \to 0}{\to} 2\pi C \frac{r_{\max}^{(1+n)\lambda+2}}{(1+n)\lambda + 2}$$



$$(1+n)\lambda + 2 > 0$$

$$\Rightarrow \lambda > -\frac{2}{1+n}$$

A. Modes

B. Airy stress functions

C. Westergaard's solution

D. Irwin's asymptotic development

E. Stress intensity factor

F. Williams analysis

G. Fracture Toughness

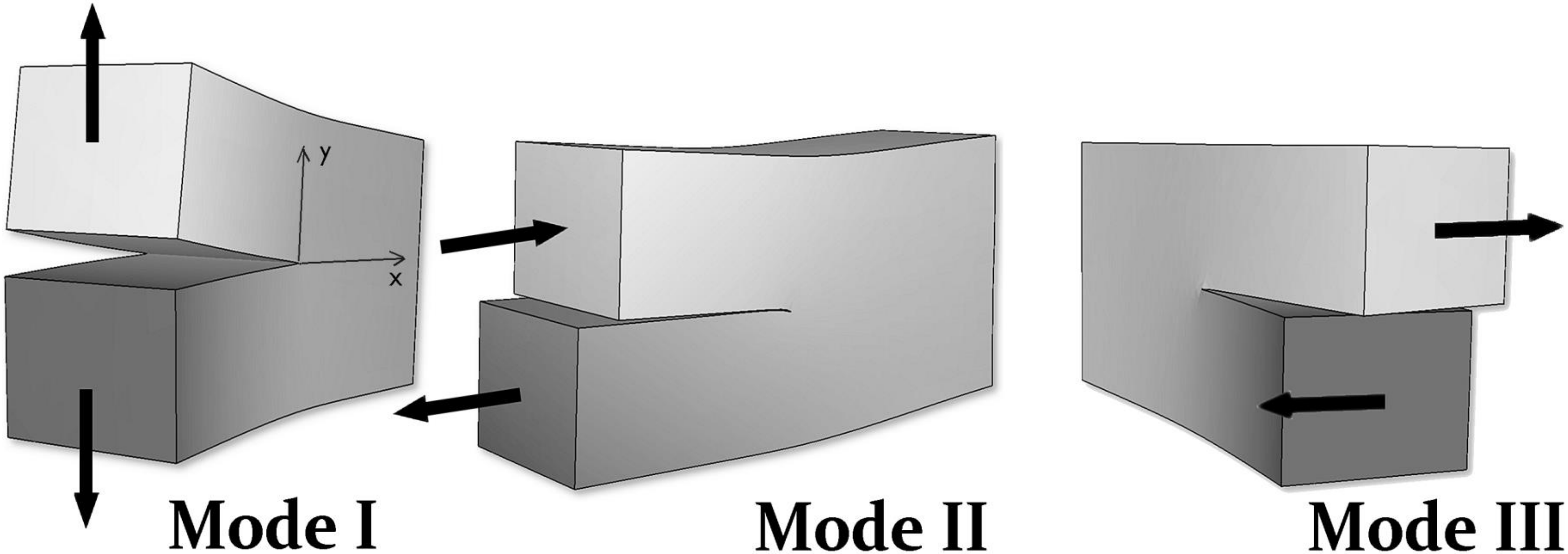
H. Irwin's plastic zones

LEFM

KI, KII, KIII

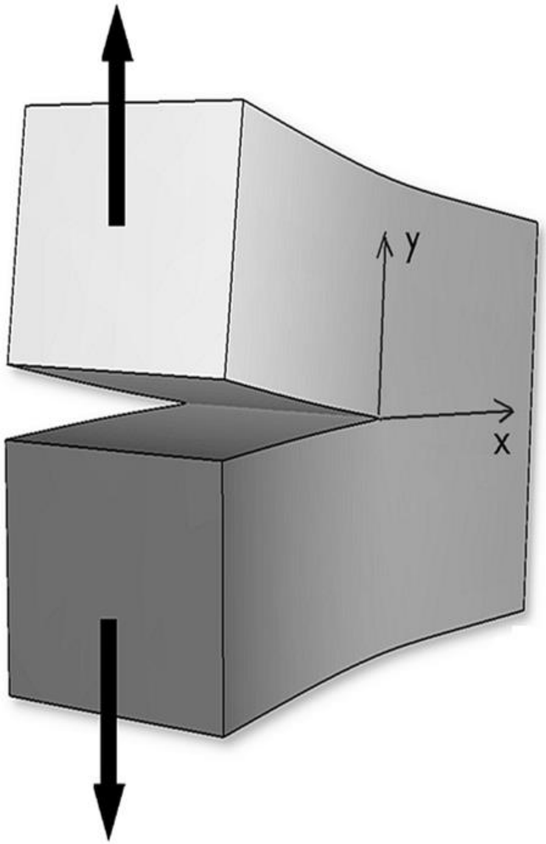
T, Tz, Γ

Fracture modes



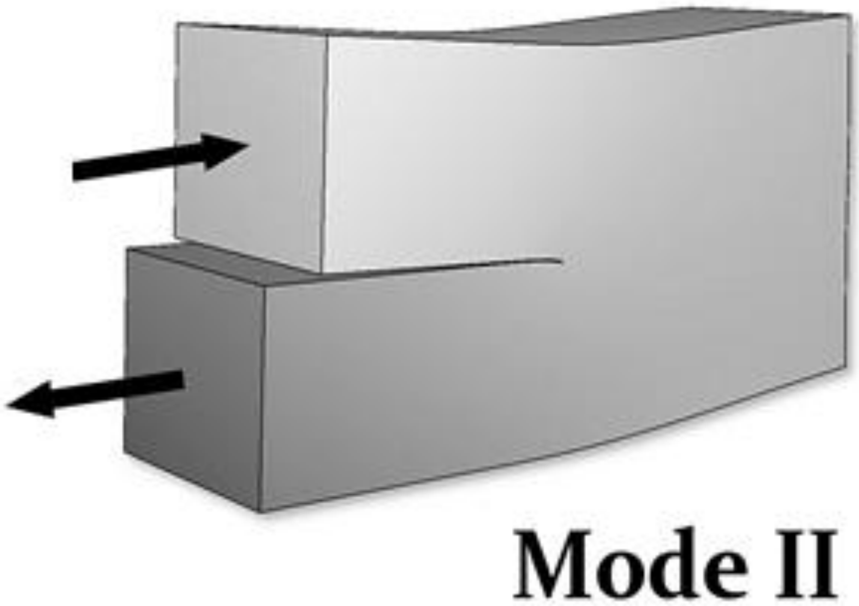
Planar symmetric **Planar anti-symmetric** **Anti-planar**

Fracture modes

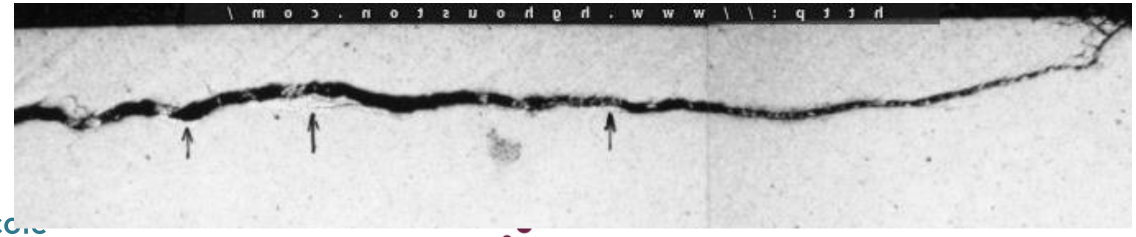


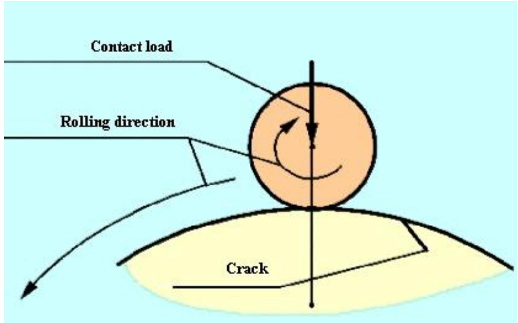
Tubes (pipe line)

Fracture modes



Various fractures in compression





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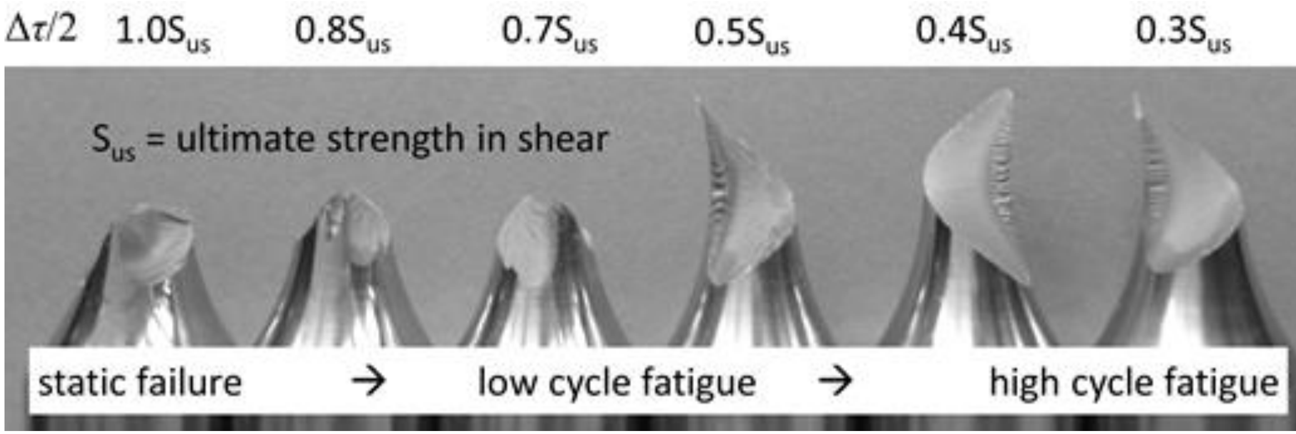
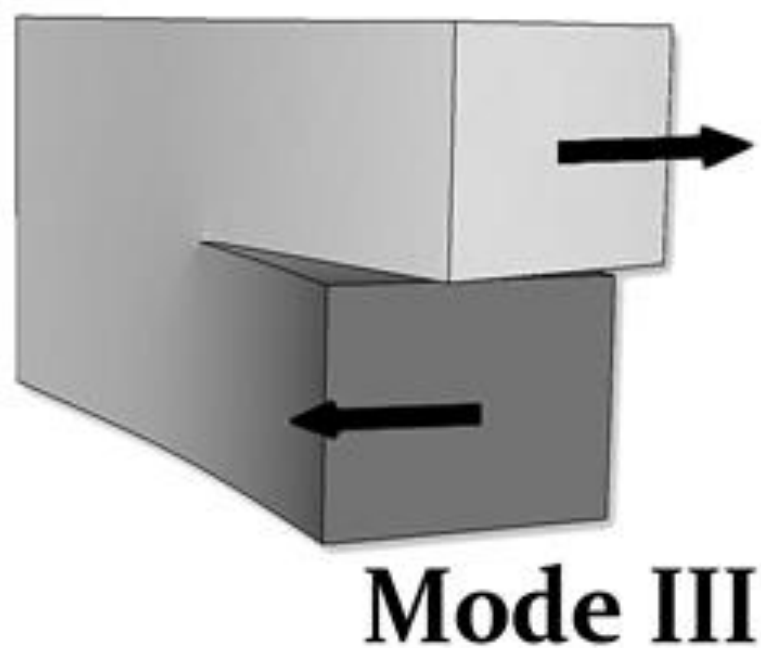
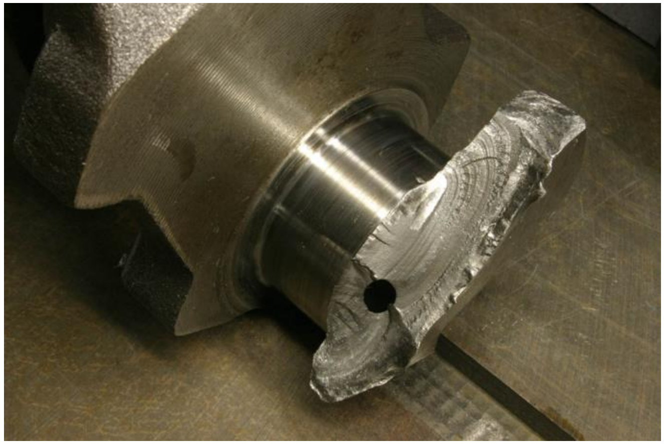
Fracture modes





Various fractures in torsion



A. Modes

B. Airy stress functions

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E. Stress intensity factor

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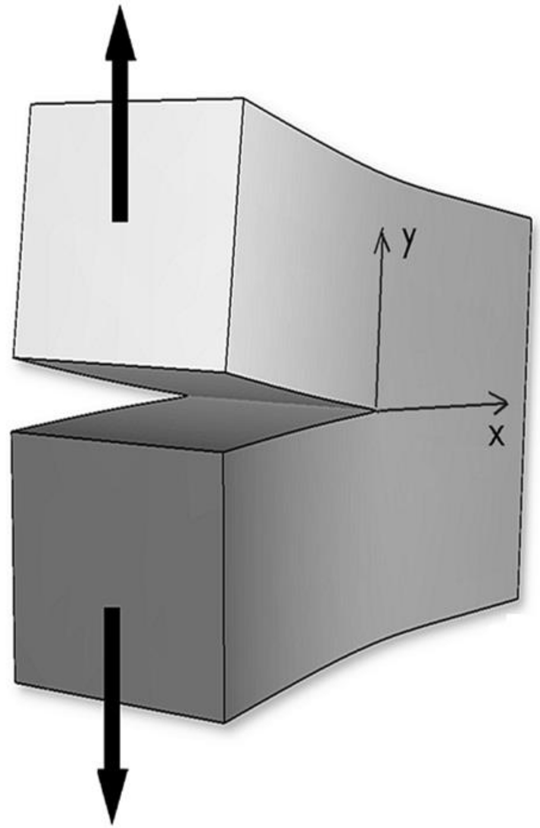
LEFM

KI, KII, KIII

T, Tz, Γ

Case of mode I

Analysis of Irwin based on Westergaard's analysis and Williams expansions



Planar Symmetric

Balance equation

$$Div\underline{\underline{\sigma}} + \cancel{\underline{f}_v} = \cancel{\rho\underline{a}}$$

2D problem, quasi-static, no volume force

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \cancel{\frac{\partial\sigma_{xz}}{\partial z}} = 0$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \cancel{\frac{\partial\sigma_{yz}}{\partial z}} = 0$$

$$\cancel{\frac{\partial\sigma_{xz}}{\partial x}} + \cancel{\frac{\partial\sigma_{yz}}{\partial y}} + \cancel{\frac{\partial\sigma_{zz}}{\partial z}} = 0$$

Linear isotropic elasticity : E, ν

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E}Tr\left(\underline{\underline{\sigma}}\right)\underline{\underline{1}}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right)$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

Compatibility equations

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \longrightarrow \frac{\partial^2 \varepsilon_{xx}}{\partial^2 y} = \frac{\partial^3 u_x}{\partial x \partial^2 y}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \longrightarrow \frac{\partial^2 \varepsilon_{yy}}{\partial^2 x} = \frac{\partial^3 u_y}{\partial y \partial^2 x}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \longrightarrow 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^3 u_y}{\partial y \partial^2 x} + \frac{\partial^3 u_x}{\partial x \partial^2 y}$$

$$\boxed{2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{yy}}{\partial^2 x} + \frac{\partial^2 \varepsilon_{xx}}{\partial^2 y}}$$

Combination

Compatibility **+** **Linear elasticity**

$$2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{yy}}{\partial^2 x} + \frac{\partial^2 \varepsilon_{xx}}{\partial^2 y}$$

$$\boxed{2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = \frac{\partial^2 \sigma_{yy}}{\partial^2 x} + \frac{\partial^2 \sigma_{xx}}{\partial^2 y}}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu \varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu \varepsilon_{xx}\right)$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

+

Balance equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

= 3 Equations, 3 unknowns

Airy function F(x,y) -1862-

Balance equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Compatibility

$$2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = \frac{\partial^2 \sigma_{yy}}{\partial^2 x} + \frac{\partial^2 \sigma_{xx}}{\partial^2 y}$$

Assuming

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 F}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

$$\frac{\partial^4 F}{\partial^4 x} + 2\frac{\partial^4 F}{\partial^2 x \partial^2 y} + \frac{\partial^4 F}{\partial^4 y} = 0$$

1 equation, 1 unknow F(x,y)

Z(z) , z complex,

$$\boxed{\frac{\partial^4 F}{\partial^4 x} + 2\frac{\partial^4 F}{\partial^2 x \partial^2 y} + \frac{\partial^4 F}{\partial^4 y} = 0}$$

F=F(x,y)

A point in the plane is defined by a complex number z = x + i y
Z a function of z : Z(z)=F(x,y)

$$\frac{\partial^4 Z}{\partial^4 x} = \frac{\partial^4 Z}{\partial^4 z}$$

$$\frac{\partial^4 Z}{\partial^2 x \partial^2 y} = -\frac{\partial^4 Z}{\partial^4 z}$$

$$\frac{\partial^4 Z}{\partial^4 y} = \frac{\partial^4 Z}{\partial^4 z}$$

Z (z) always fulfill all the equations of the problem

Z(z) must verify the symmetry and the boundary conditions

A. Modes

B. Airy stress functions

C. Westergaard's solution

D. Irwin's asymptotic development

E. Stress intensity factor

F. Williams analysis

G. Fracture Toughness

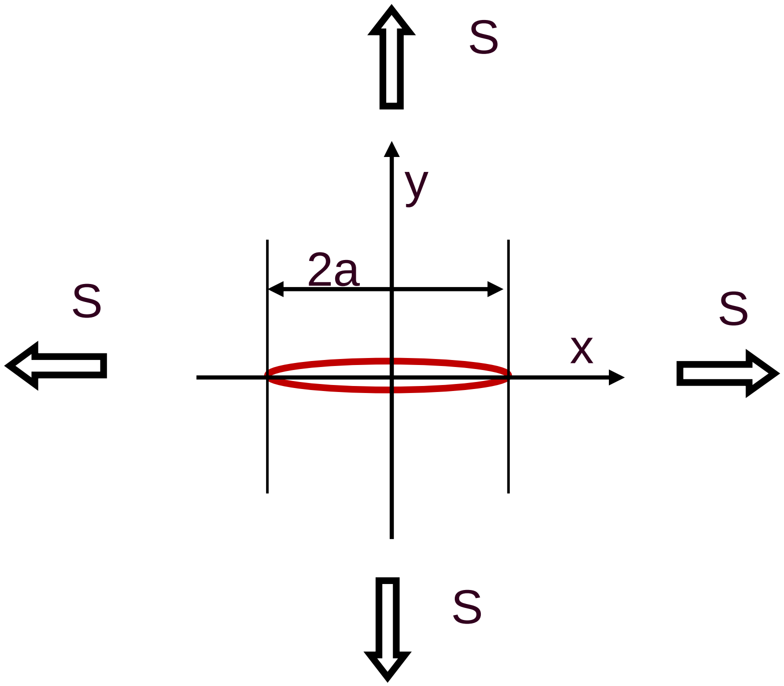
H. Irwin's plastic zones

LEFM

KI, KII, KIII

T, Tz, Γ

Irwin's or Westergaard's analyses



6 boundary or symmetry conditions
2 singularities,
0 boundary conditions along the crack faces
Exact solution
Taylor's development with respect to the distance to the crack front
Separated variables
Similitude principle

2D problem, plane (x,y) : $\mathbf{S_{zz}=\nu(S_{xx}+S_{yy})}$

Symmetric with respect to **y=0 & x=0**

Away from the crack (x & y >> a) : $\mathbf{s_{xx}= S}$ $\mathbf{s_{yy}= S}$ **&** $\mathbf{s_{xy}= 0}$

Singularities in **y=0 x=+a** & **y=0 x=-a**

Boundary conditions & Symmetries

$$\sigma_{xx}^{\infty} = \sigma_{yy}^{\infty} = S, \quad \sigma_{xy}^{\infty} = 0$$

&

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 F}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

$$\rightarrow \quad F^{\infty} = \frac{S}{2}\left(y^2 + x^2\right) + a_2 x + a_3 y + a_4$$

symmetries $\rightarrow$

$$F^{\infty} = \frac{S}{2}\left(y^2 + x^2\right) + a_4$$

Construction of Z(z)

$$F^{\infty} = \frac{S}{2}\left(y^2 + x^2\right) + a_4$$

$$Z^{\infty} = \frac{S}{2} z^2 + a_4$$

Relation

$$F = R_e(Z) - yR_e\left(\frac{\partial Z}{\partial y}\right) = R_e(Z) + yI_m\left(\frac{\partial Z}{\partial z}\right)$$

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 F}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

$$\Rightarrow$$

$$\sigma_{xx} = R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) - yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

$$\sigma_{yy} = R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) + yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

$$\sigma_{xy} = -yR_e\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

Solution

At infinity

$$\boxed{\sigma_{xx}^{\infty} = \sigma_{yy}^{\infty} = S, \quad \sigma_{xy}^{\infty} = 0}$$

At infinity

$$\boxed{Z^{\infty} = \frac{S}{2} z^{2} + a_{4}}$$

Solution:

$$\boxed{\begin{aligned}
\sigma_{xx} &= R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) - yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right) \\
\sigma_{yy} &= R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) + yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right) \\
\sigma_{xy} &= -yR_e\left(\frac{\partial^3 Z}{\partial z^3}\right)
\end{aligned}}$$

Valid for any 2D problem, with symmetries along the planes y=0 & x=0, and biaxial BCs

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Exact solution for a crack

$$\sigma_{xx} = R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) - yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

$$\sigma_{yy} = R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) + yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

$$\sigma_{xy} = -yR_e\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

Singularities
in y=0 x=+a
& y=0 x=-a

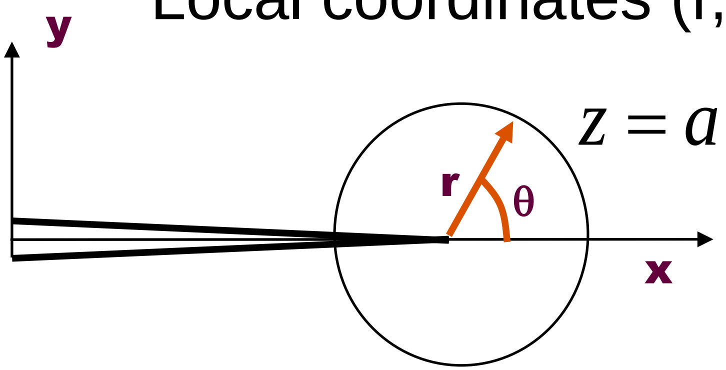
$$+ \boxed{\frac{1}{z-a}} \quad \boxed{\frac{1}{z+a}} \quad + \boxed{Z^{\infty} = \frac{S}{2} z^2 + a_4 \qquad \frac{\partial Z^{\infty}}{\partial z} = Sz}$$

$$\boxed{\frac{\partial Z^{\infty}}{\partial z} = S\left(z^2 - a^2\right)^{1/2}}$$

Exact solution

Asymptotic solution - Irwin-

Local coordinates (r,θ), $r \to 0$



$$z = a + re^{i\theta}$$

Exact Solution

$$\boxed{\frac{\partial Z^{\infty}}{\partial z} = S\left(z^2 - a^2\right)^{1/2}}$$

$$\frac{\partial^2 Z}{\partial z^2} = \frac{Sz}{\left(z^2 - a^2\right)^{1/2}} \quad \Longrightarrow \quad \frac{\partial^2 Z}{\partial z^2} = \frac{Sa}{\left(2are^{i\theta}\right)^{1/2}} \to \frac{S\sqrt{a}}{\sqrt{2r}} e^{-i\frac{\theta}{2}}$$

$$\frac{\partial^3 Z}{\partial z^3} = \frac{-Sa^2}{\left(z^2 - a^2\right)^{3/2}} \quad \Longrightarrow \quad \frac{\partial^3 Z}{\partial z^3} = \frac{-Sa^2}{\left(2are^{i\theta}\right)^{3/2}} \to -\frac{1}{r}\frac{S\sqrt{a}}{\sqrt{2r}} e^{-i\frac{3\theta}{2}}$$

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Asymptotic solution - Irwin-

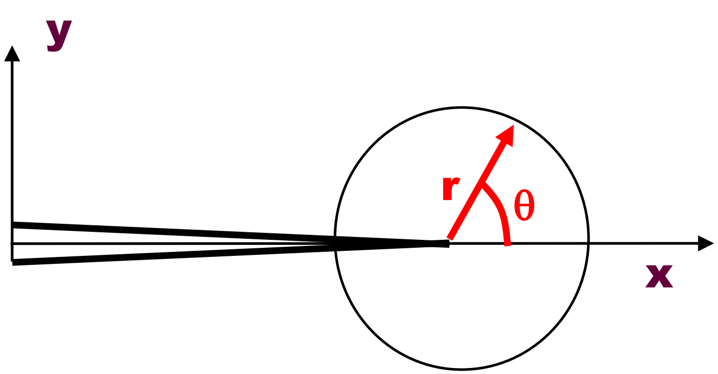
$$\sigma_{xx} = R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) - yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

$$\sigma_{yy} = R_e\left(\frac{\partial^2 Z}{\partial z^2}\right) + yI_m\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

$$\sigma_{xy} = -yR_e\left(\frac{\partial^3 Z}{\partial z^3}\right)$$

$$\frac{\partial^2 Z}{\partial z^2} \rightarrow \frac{S\sqrt{a}}{\sqrt{2r}}e^{-i\frac{\theta}{2}}$$

$$\frac{\partial^3 Z}{\partial z^3} \rightarrow -\frac{1}{r}\frac{S\sqrt{a}}{\sqrt{2r}}e^{-i\frac{3\theta}{2}}$$



Westergaard's stress function :

$$\sigma_{xx} = \frac{S\sqrt{\pi a}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{yy} = \frac{S\sqrt{\pi a}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xy} = \frac{S\sqrt{\pi a}}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

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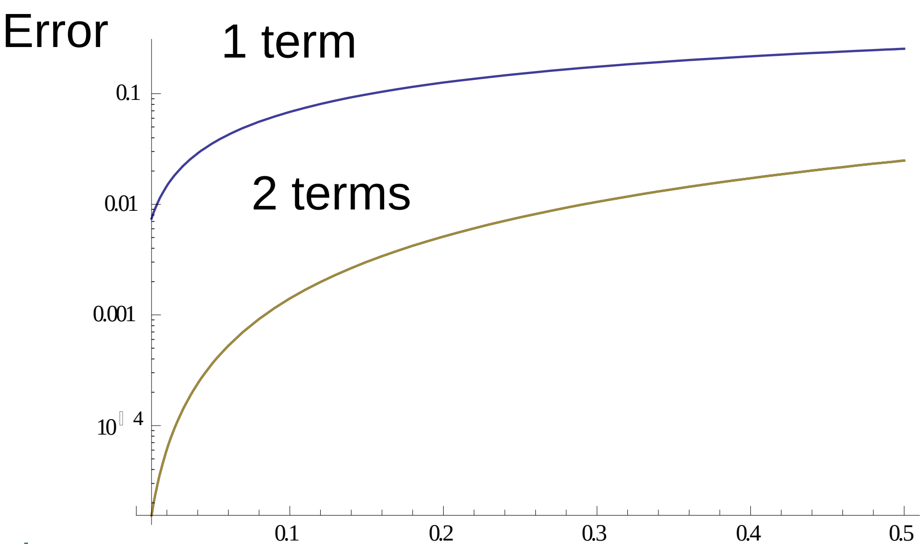
Error associated to this Taylor development along θ=0

Exact solution $$\sigma_{yy}(r,\theta=0)=\frac{S_{yy}(a+r)}{\sqrt{r(2a+r)}}=\frac{K_I(a+r)}{\sqrt{\pi a r(2a+r)}}$$

Asymptotic solution

$$\sigma_{yy}(r,\theta=0)=\frac{K_I}{\sqrt{2\pi r}}\left(1+\frac{3}{4}\left(\frac{r}{a}\right)+\frac{5}{32}\left(\frac{r}{a}\right)^2\right)+o\left(r^{\frac{5}{2}}\right)$$

$$error \sim \frac{3}{4}\frac{r}{a}$$



Erreur = 1%

1 term	$\frac{r}{a}=\mathbf{0.013}$
2 terms	$\frac{r}{a}=\mathbf{0.29}$
3 terms	$\frac{r}{a}=\mathbf{0.69}$

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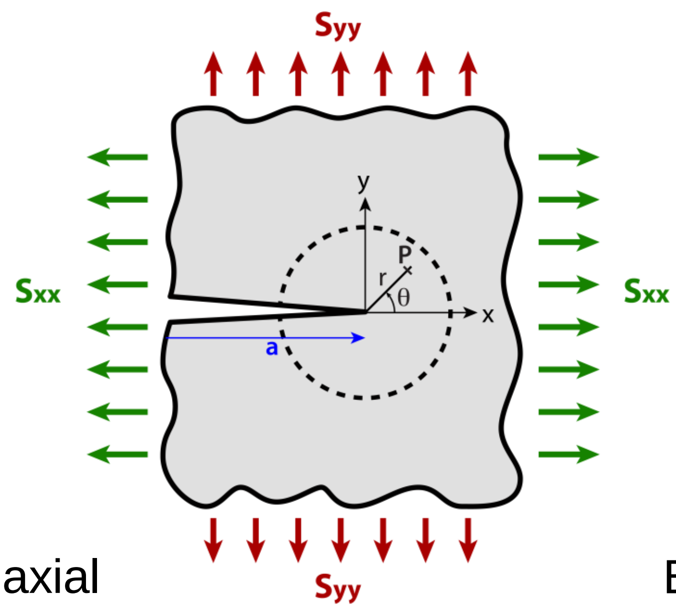
Mode I, non equi-biaxial conditions



$$\sigma_{xx}^{\infty} = S_{xx} = S_{yy} + S_{xx} - S_{yy}$$

$$\sigma_{yy}^{\infty} = S_{yy}$$

$$\sigma_{xy}^{\infty} = 0$$

Equibiaxial

$$\sigma_{xx} = \frac{S\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{yy} = \frac{S\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{zz} = \frac{S\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Biaxial (Superposition)

$$\sigma_{xx} = \frac{S_{yy}\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \overbrace{S_{xx} - S_{yy}}^{\underbrace{T = S_{xx} - S_{yy}}_{\Downarrow}}$$

$$\sigma_{yy} = \frac{S_{yy}\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{zz} = \frac{S_{yy}\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\overbrace{K_I = S_{yy}\sqrt{\pi a}}^{\Uparrow}$$

Stress intensity factors

Similitude principle

(geometry locally planar, with a straigth crack front, self-similar, singularity)

Same **KI & T** → Same local field

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + T$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

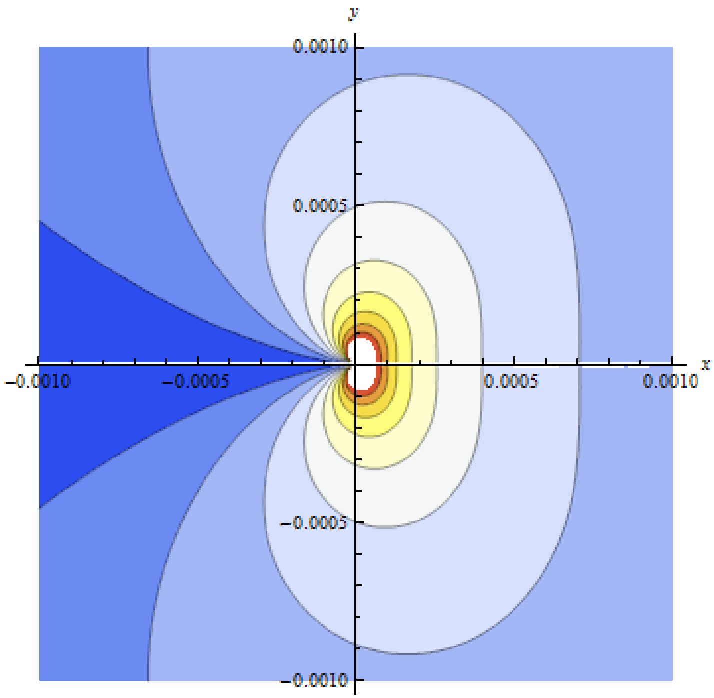
KI &T
Crack geometry and boundary conditions

Spatial distribution, given once for all, in the crack front region
$g_{ij}(\theta)$ **$f(r)=r^{\lambda}$**

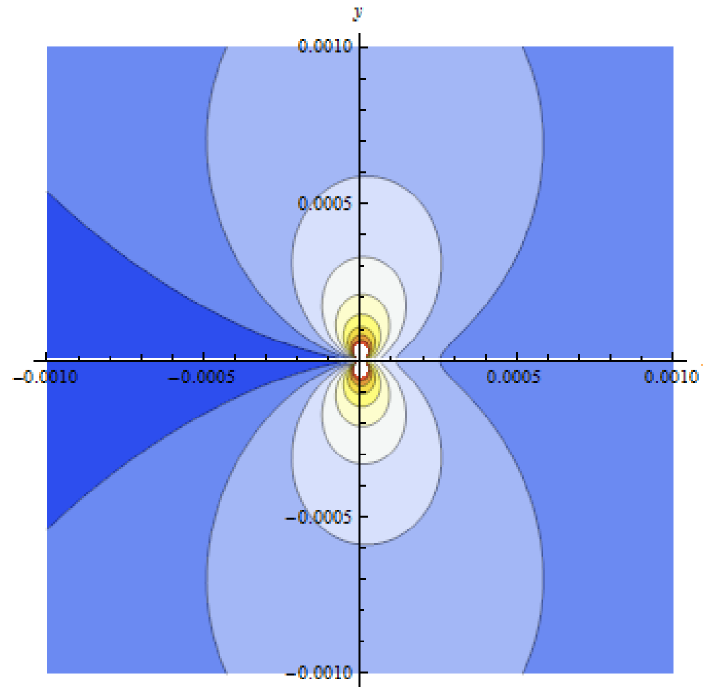
von Mises stress field

$$\underline{\underline{\sigma}}^D(r,\theta)=\underline{\underline{\sigma}}(r,\theta)-\frac{Tr\left(\underline{\underline{\sigma}}(r,\theta)\right)}{3}\underline{\underline{1}} \qquad \sigma_{eq}(r,\theta)=\sqrt{\frac{3}{2}\underline{\underline{\sigma}}^D(r,\theta):\underline{\underline{\sigma}}^D(r,\theta)}$$

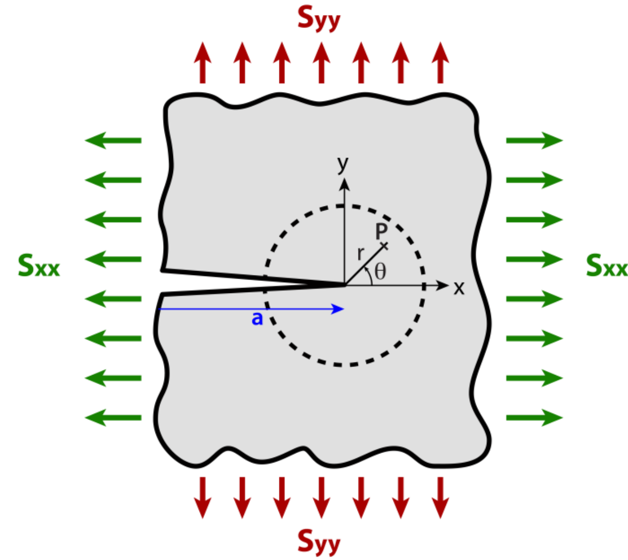
Plane stress, Mode I, T=0



Plane strain, Mode I, T=0



von Mises stress field

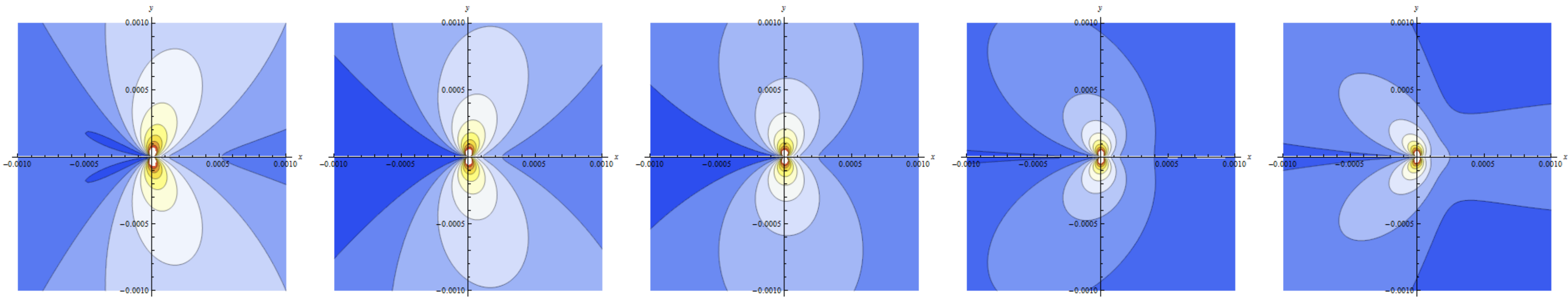


Mechanisms controlled by shear

- **Plasticity,**
- **Visco-plasticity**
- **Fatigue**

$$T = S_{xx} - S_{yy}$$

T / K = -10 m$^{-1/2}$ **T / K = -5 m$^{-1/2}$** **T / K = 0 m$^{-1/2}$** **T / K = 5 m$^{-1/2}$** **T / K = 10 m$^{-1/2}$**



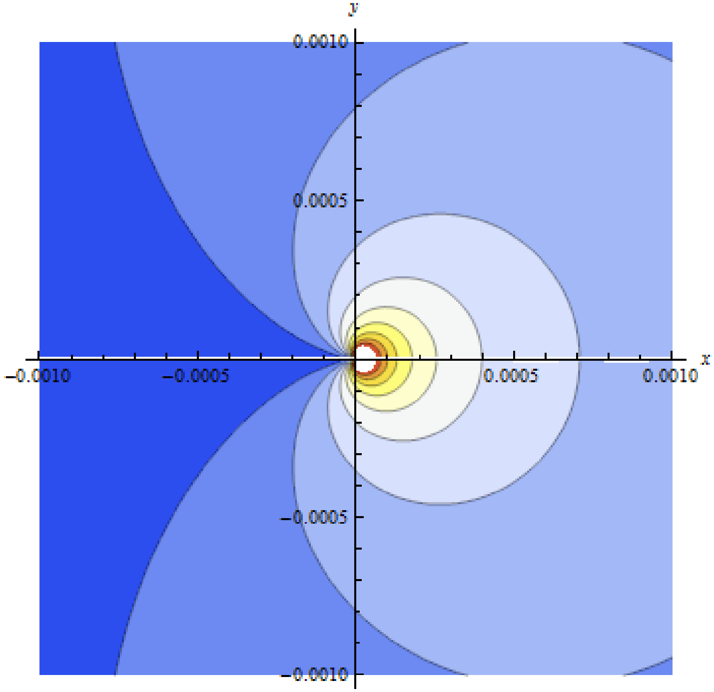
Plane strain, Mode I

Hydrostatic pressure

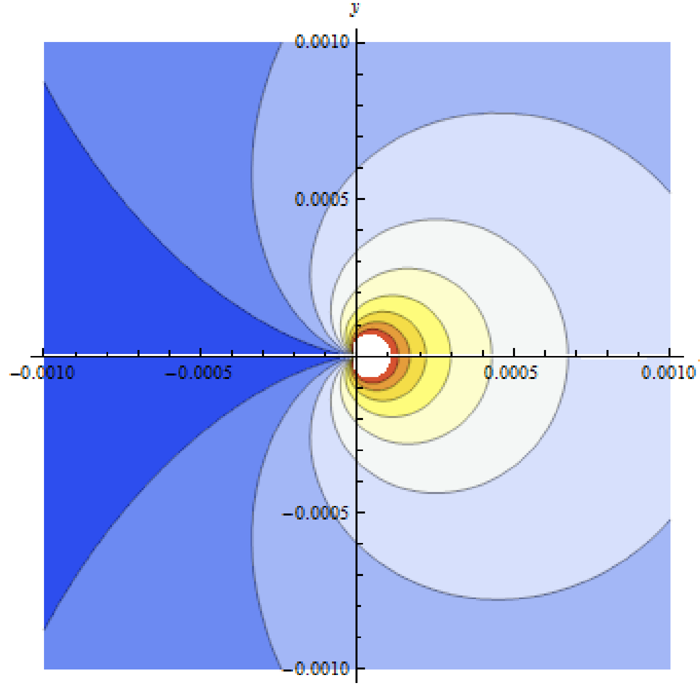
$$Tr\left(\underline{\underline{\sigma}}(r,\theta)\right)$$

Fluid diffusion (Navier Stokes),
Diffusion creep (Nabarro-Herring)
Chemical diffusion

Plane stress, Mode I, T=0



Plane strain, Mode I, T=0



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Hydrostatic pressure

$$Tr\left(\underline{\underline{\sigma}}(r,\theta)\right)$$

Fluid diffusion (Navier Stokes),
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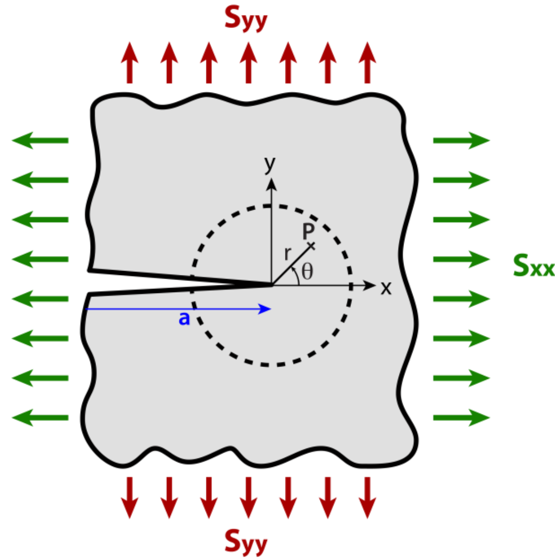
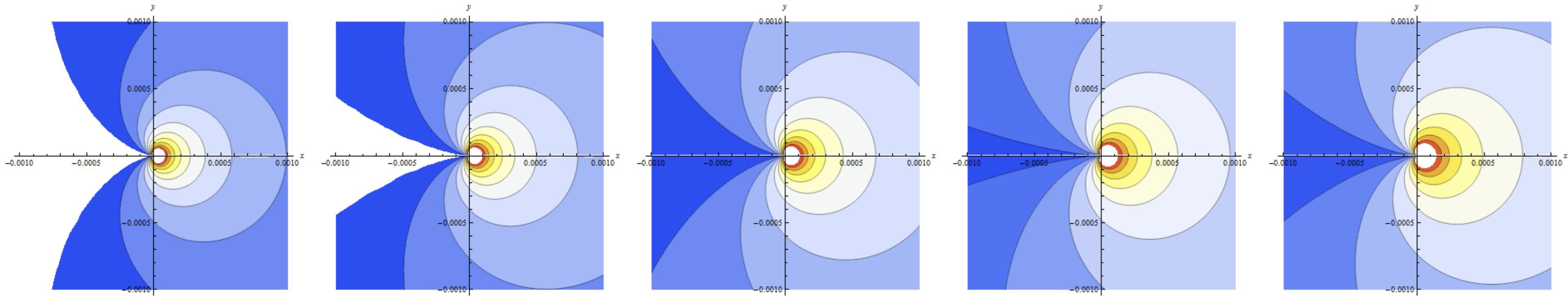


$$T = S_{xx} - S_{yy}$$

T / K = -10 $m^{-1/2}$ **T / K = -5 $m^{-1/2}$** **T / K = 0 $m^{-1/2}$** **T / K = 5 $m^{-1/2}$** **T / K = 10 $m^{-1/2}$**



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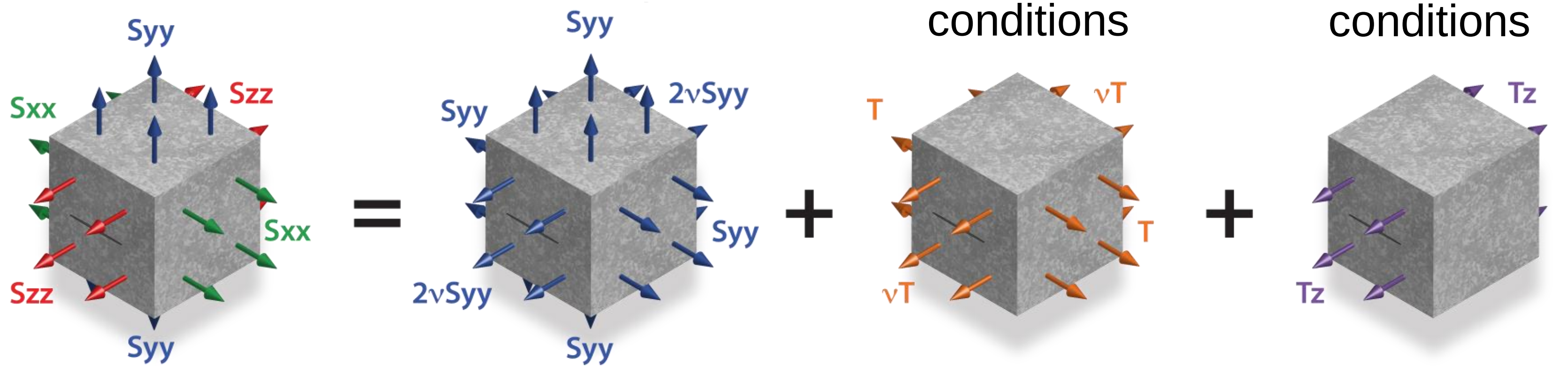
Plane strain, Mode I

Other T components, in Mode I

General triaxial loading | Equibiaxial plane strain | Superposition non equibiaxial conditions | Superposition non plane strain conditions



$$K_I = S_{yy}\sqrt{\pi a}$$

$$T = S_{xx} - S_{yy}$$

$$T_z = S_{zz} - 2\nu S_{yy} - \nu T$$

Full solutions KI, KII, KIII, T, Tz & Γ

Mode I

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)+T$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$u_x = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}(\kappa-\cos\theta)$$

$$u_y = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}(\kappa-\cos\theta)$$

Mode II

$$\sigma_{xx} = \frac{-K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$u_x = \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}(2+\kappa+\cos\theta)$$

$$u_y = \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}(2+\kappa-\cos\theta)$$

Mode III

$$\sigma_{xz} = \frac{-K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}+\Gamma$$

$$\sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$u_z = \frac{4K_{III}}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}$$

Déformation plane

$$\sigma_{zz} = \nu\left(\sigma_{xx}+\sigma_{yy}\right)+T_z$$

$$\kappa = (3-4\nu)$$

Contrainte plane

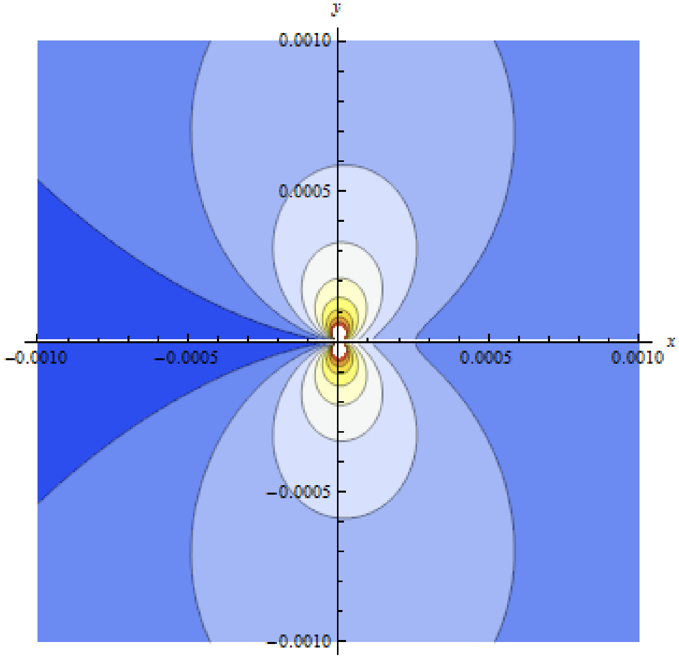
$$\kappa = \frac{(3-\nu)}{(1+\nu)}$$

von Mises stress field

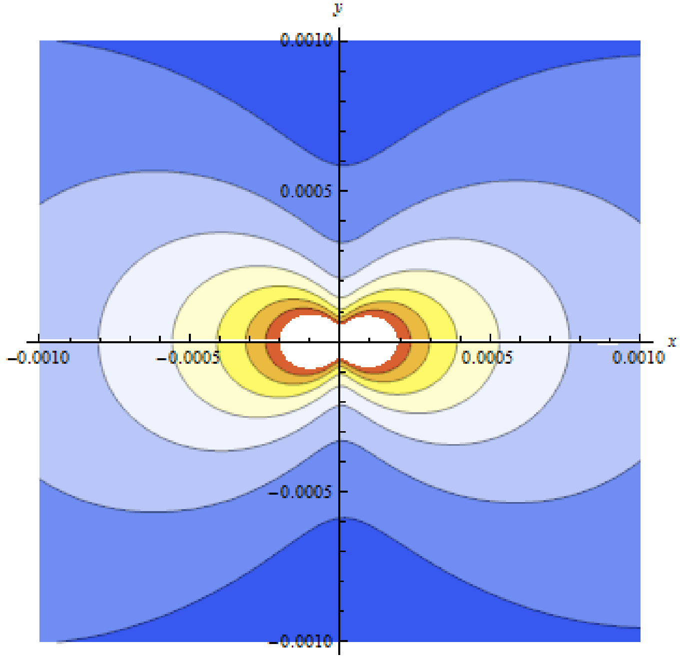
$$\underline{\underline{\sigma}}^D(r,\theta)=\underline{\underline{\sigma}}(r,\theta)-\frac{Tr\left(\underline{\underline{\sigma}}(r,\theta)\right)}{3}\underline{\underline{1}}$$

$$\sigma_{eq}(r,\theta)=\sqrt{\frac{3}{2}\underline{\underline{\sigma}}^D(r,\theta):\underline{\underline{\sigma}}^D(r,\theta)}$$

Mode I



Mode II



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Summary

- Exact solutions for the 3 modes, determined for one specific geometry
- Taylor development, 1st order → asymptotic solution generalized to any other cracks
- First order
  - Solution expressed with separate variables f (r) g (θ) and f (r) self-similar
  - Solution : f (r) a power function, r^{λ}, with $\lambda = - 1/2$
- Higher Orders
  - A unique stress intensity factor for all terms
  - The exponent of (r/a) increasing with the order of the Taylor's development
- Boundary conditions
  - Singularity along the crack front, symmetries, planar crack and straight front
  - no prescribed BCs along the crack faces,
  - Boundary conditions defined at infinity

6 independent components of the stress tensor at infinity → 6 degrees of freedoms in MLER: KI, KII, KIII and T, Tz, and Γ

A. Modes

B. Airy stress functions

C. Westergaard's solution

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LEFM

KI, KII, KIII

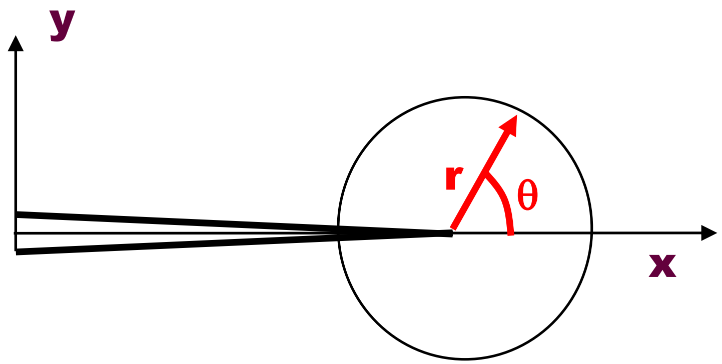
T, Tz, Γ

Williams expansion

$$\boxed{\frac{\partial^4 F}{\partial^4 x}+2\frac{\partial^4 F}{\partial^2 x\partial^2 y}+\frac{\partial^4 F}{\partial^4 y}=\nabla^4 F=0}$$

A self-similar solution in the form is sought directly as follows :

$$F(r,\theta)=r^{\lambda+2}g(\theta)$$

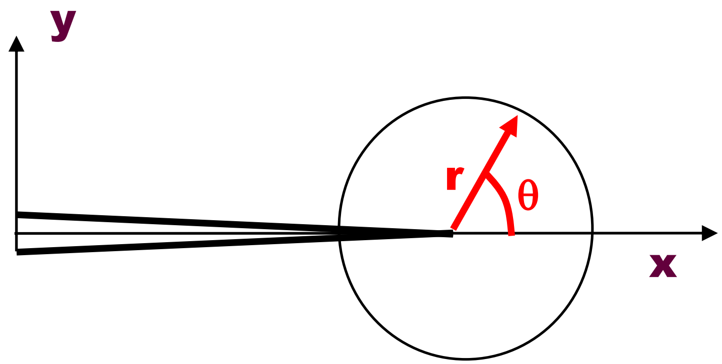


$$\nabla^2 F=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}=(\lambda+2)^2 r^{\lambda}g(\theta)+r^{\lambda}\frac{\partial^2 g(\theta)}{\partial\theta^2}$$

$$\nabla^4 F=(\lambda+2)^2\lambda^2 r^{\lambda-2}g(\theta)+\lambda^2 r^{\lambda-2}\frac{\partial^2 g(\theta)}{\partial\theta^2}(\lambda+2)^2 r^{\lambda-2}\frac{\partial^2 g(\theta)}{\partial\theta^2}+r^{\lambda-2}\frac{\partial^4 g(\theta)}{\partial\theta^4}$$

$$\nabla^4 F=r^{\lambda-2}\left((\lambda+2)^2\lambda^2 g(\theta)+\left((\lambda+2)^2+\lambda^2\right)\frac{\partial^2 g(\theta)}{\partial\theta^2}+\frac{\partial^4 g(\theta)}{\partial\theta^4}\right)$$

Williams expansion



A self-similar solution in the form is sought directly as follows :

$$F(r,\theta)=r^{\lambda+2}g(\theta)$$

$$\boxed{\nabla^4 F = r^{\lambda-2}\left((\lambda+2)^2\lambda^2 g(\theta)+\left((\lambda+2)^2+\lambda^2\right)\frac{\partial^2 g(\theta)}{\partial\theta^2}+\frac{\partial^4 g(\theta)}{\partial\theta^4}\right)=0}$$

Dans ce cas g(θ) doit vérifier

$$\frac{d^4 g(\theta)}{d\theta^4}+\left[\lambda^2+(\lambda+2)^2\right]\frac{d^2 g(\theta)}{d\theta^2}+\lambda^2(\lambda+2)^2 g(\theta)=0$$

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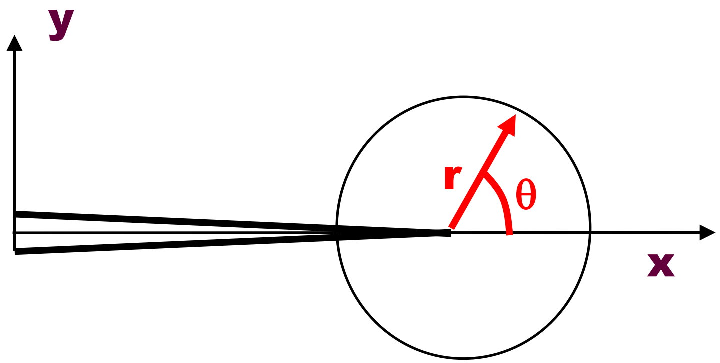
Williams expansion



$$\frac{d^4 g(\theta)}{d\theta^4} + \left[\lambda^2 + (\lambda+2)^2\right]\frac{d^2 g(\theta)}{d\theta^2} + \lambda^2(\lambda+2)^2 g(\theta) = 0$$

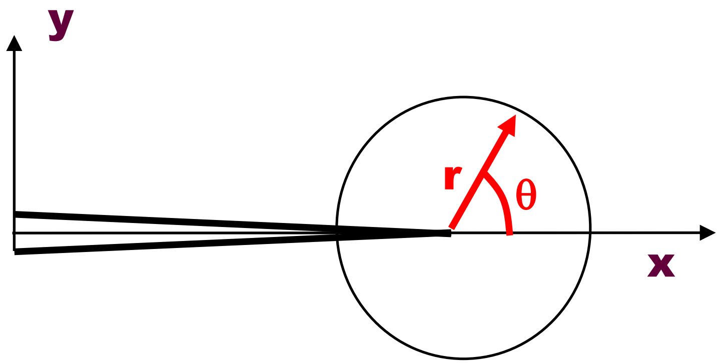
The solution is sought as follows : $g(\theta) = Ae^{ip\theta}$

$$\Rightarrow p^4 - \left[\lambda^2 + (\lambda+2)^2\right]p^2 + \lambda^2(\lambda+2)^2 = 0 = \left(p^2 - \lambda^2\right)\left(p^2 - (\lambda+2)^2\right)$$

$$p = \pm\lambda$$

$$p = \pm(\lambda+2)$$

Williams expansion



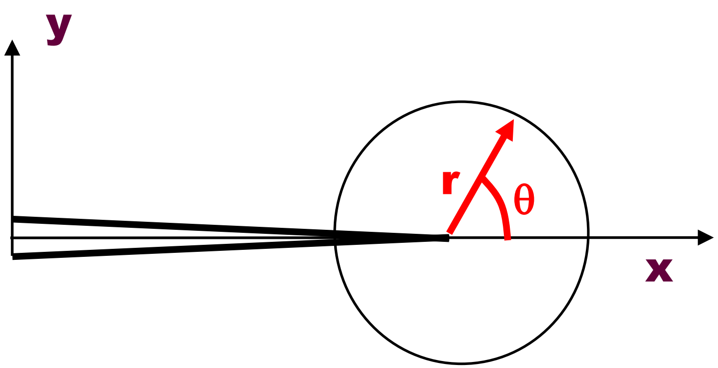
$$F(r,\theta) = \mathrm{Re}\left[r^{\lambda+2}\left(Ae^{i\lambda\theta} + Be^{-i\lambda\theta} + Ce^{i(\lambda+2)\theta} + De^{-i(\lambda+2)\theta}\right)\right]$$

Boundary conditions are defined along the crack faces which are defined as free surface (fluid pressure & friction between faces are excluded)

$$\sigma_{\theta\theta}(r,\theta=\pm\pi) = \frac{\partial^2 F}{\partial r^2}(r,\theta=\pm\pi) = 0$$

$$\sigma_{r\theta}(r,\theta=\pm\pi) = -\frac{\partial}{\partial r}\left(\frac{\partial F}{r\partial\theta}\right)(r,\theta=\pm\pi) = 0$$

Williams expansion

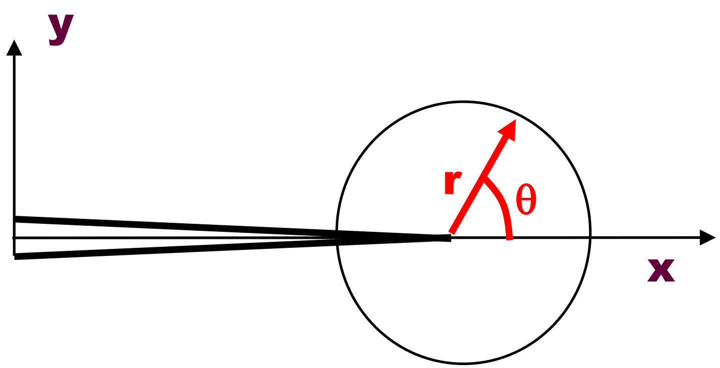


$$F(r,\theta)=\mathrm{Re}\left[r^{\lambda+2}\left(Ae^{i\lambda\theta}+Be^{-i\lambda\theta}+Ce^{i(\lambda+2)\theta}+De^{-i(\lambda+2)\theta}\right)\right]$$

$$\sigma_{\theta\theta}(r,\theta=\pm\pi)=0\Rightarrow\begin{array}{l}\mathrm{Re}\left[Ae^{i\lambda\pi}+Be^{-i\lambda\pi}+Ce^{i(\lambda+2)\pi}+De^{-i(\lambda+2)\pi}\right]=0\\ \mathrm{Re}\left[Ae^{-i\lambda\pi}+Be^{i\lambda\pi}+Ce^{-i(\lambda+2)\pi}+De^{i(\lambda+2)\pi}\right]=0\end{array}$$

$$\sigma_{r\theta}(r,\theta=\pm\pi)=0\Rightarrow\begin{array}{l}\mathrm{Re}\left[\lambda Ae^{-i\lambda\pi}-\lambda Be^{i\lambda\pi}+(\lambda+2)Ce^{-i(\lambda+2)\pi}-(\lambda+2)De^{i(\lambda+2)\pi}\right]=0\\ \mathrm{Re}\left[\lambda Ae^{i\lambda\pi}-\lambda Be^{-i\lambda\pi}+(\lambda+2)Ce^{i(\lambda+2)\pi}-(\lambda+2)De^{-i(\lambda+2)\pi}\right]=0\end{array}$$

Williams expansion



A sery of eligible solutions is obtained :

$$2(\lambda+1)=n$$

n even $$g(\theta)=B\cos\left(\left(\frac{n}{2}-1\right)\theta\right)+D\cos\left(\left(\frac{n}{2}+1\right)\theta\right)$$

n odd $$g(\theta)=A\sin\left(\left(\frac{n}{2}-1\right)\theta\right)+C\sin\left(\left(\frac{n}{2}+1\right)\theta\right)$$

$$F(r,\theta)=r^{\frac{n}{2}+1}g(\theta)$$

La solution en contrainte s’exprime alors à partir des dérivées d’ordre 2 de F, toutes les valeurs de n sont possibles, tous les modes apparaissent

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Williams versus Westergaard

- The boundary conditions are free surface conditions along the crack faces (apply on 3 components of the stress tensor), no boundary condition at infinity $\rightarrow$ absence of T, Tz, and Γ

- Super Singular terms $\rightarrow$ missing BCs

- The first singular term of the Williams expansion is identical to the first term of the Taylor expansion of the exact solution of Westergaard

- The stress intensity factors of the higher order terms are not forced to be the same as the one of the first term,
  - advantage, leaves some flexibility to ensure the compatibility of the solution with a distant, non-uniform field
  - drawbacks, it replaces the absence of boundary conditions at infinity by condition of free surface on the crack, and it lacks 3 BCs, it is obliged to add constraints T, Tz, and G arbitraitement

A. Modes

B. Airy stress functions

C. Westergaard's solution

D. Irwin's asymptotic development

E. Stress intensity factor

F. Williams analysis

G. Fracture Toughness

H. Irwin's plastic zones

LEFM

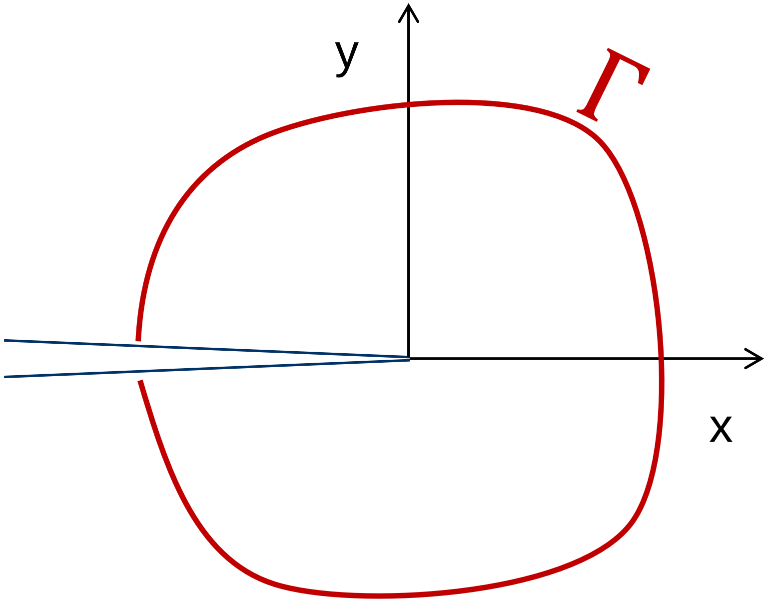
KI, KII, KIII

T, Tz, Γ

J contour integral

The J integral is shown to be independent of the choice of the selected integration contour

$$G = J = \int_{\Gamma} \left(\varphi dy - \underline{\underline{\sigma}} \underline{n} . \frac{\partial \underline{u}}{\partial x} \right)$$



The integration contour G can be chosen inside the domain of validity of the Westergaard's stress functions to get G in linear elastic conditions

Energy release rate $$G = \left(\frac{1-\nu^2}{E} \right) (K_I^2 + K_{II}^2) + \left(\frac{1+\nu}{E} \right) K_{III}^2$$

$$G_c = \left(\frac{1-\nu^2}{E} \right) K_{Ic}^2$$ **Fracture toughness**

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A. Modes

B. Airy stress functions

C. Westergaard's solution

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E. Stress intensity factor

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G. Fracture Toughness

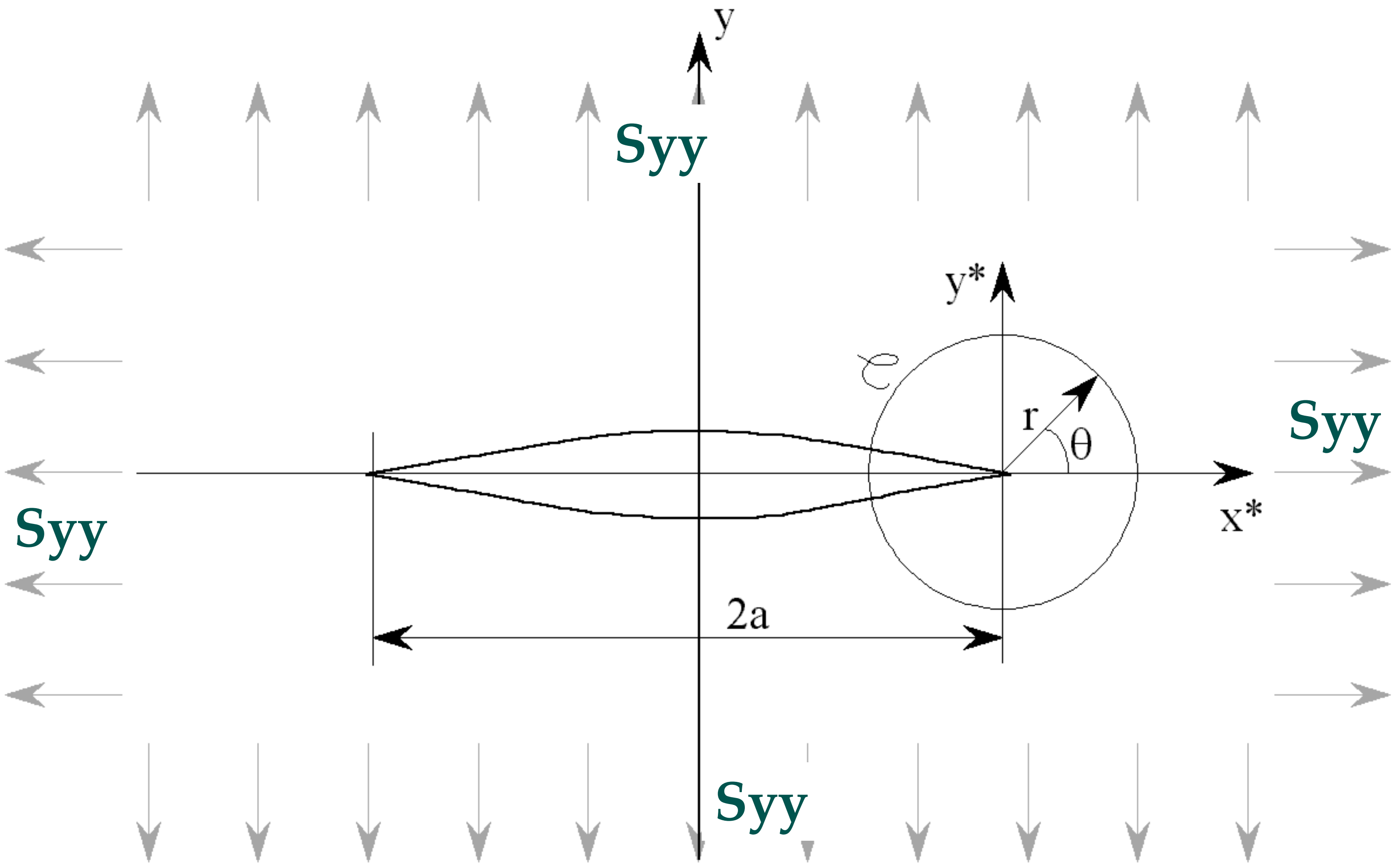
H. Irwin's plastic zones

LEFM

KI, KII, KIII

T, Tz, Γ

Mode I, LEFM, T=0



LEFM stress field (Mode I)

$$\sigma^I_{xx}(r,\theta) = K_I \frac{1}{\sqrt{2\pi r}} cos(\frac{\theta}{2})(1 - sin(\frac{\theta}{2})sin(\frac{3\theta}{2})) + T$$
$$\sigma^I_{yy}(r,\theta) = K_I \frac{1}{\sqrt{2\pi r}} cos(\frac{\theta}{2})(1 + sin(\frac{\theta}{2})sin(\frac{3\theta}{2}))$$
$$\sigma^I_{xy}(r,\theta) = K_I \frac{1}{\sqrt{2\pi r}} cos(\frac{\theta}{2})sin(\frac{\theta}{2})cos(\frac{3\theta}{2})$$
$$\sigma^I_{zz}(r,\theta) = \nu(\sigma^I_{xx} + \sigma^I_{yy})$$

Von Mises equivalent deviatoric stress

$$\boldsymbol{\sigma}^{'}(r,\theta) = \boldsymbol{\sigma}(r,\theta) - Tr[\frac{\boldsymbol{\sigma}(r,\theta)}{3}]\boldsymbol{I}$$
$$\sigma^I_{eq}(r,\theta) = \sqrt{\frac{3}{2}Tr[\boldsymbol{\sigma}^{'}.\boldsymbol{\sigma}^{'}]}$$

Irwin's plastic zones size, step 1: r_Y

Along the crack plane, θ=0

$$\sigma_{xx}(r,\theta=0)=\sigma_{yy}(r,\theta=0)=\frac{K_I}{\sqrt{2\pi r}},\ \sigma_{zz}(r,\theta=0)=2\nu\frac{K_I}{\sqrt{2\pi r}},\ \sigma_{xy}(r,\theta=0)=0$$

$$\boxed{p_H(r,\theta=0)=\frac{K_I}{\sqrt{2\pi r}}\frac{2}{3}(1+\nu)}$$

$$\boxed{\sigma_{eq}(r,\theta=0)=\frac{K_I(1-2\nu)}{\sqrt{2\pi r}}}$$

Yield criterion : $\sigma_{eq}(r_Y,\theta=0)=\sigma_Y$

$$\boxed{r_Y=\frac{(1-2\nu)^2}{2\pi}\frac{K_I^2}{\sigma_Y^2}}$$

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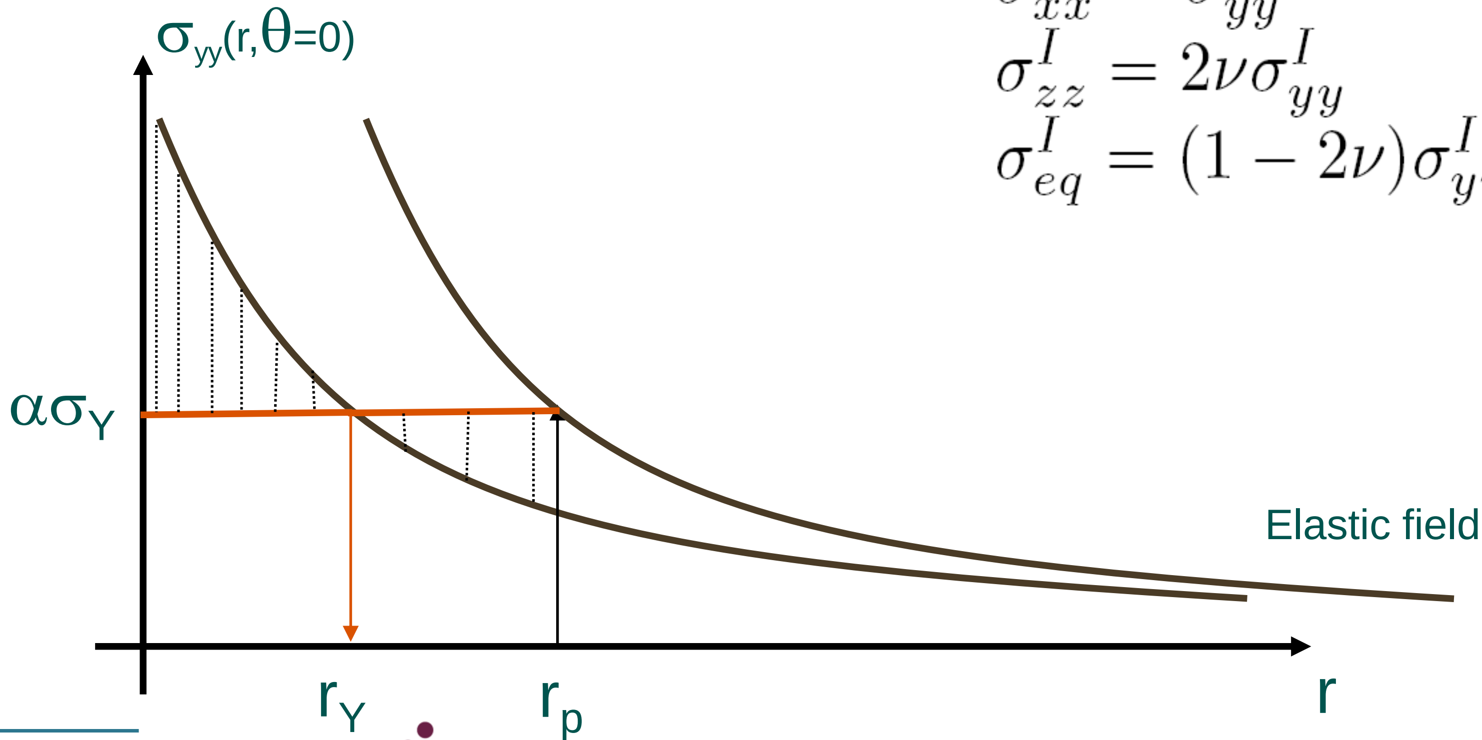
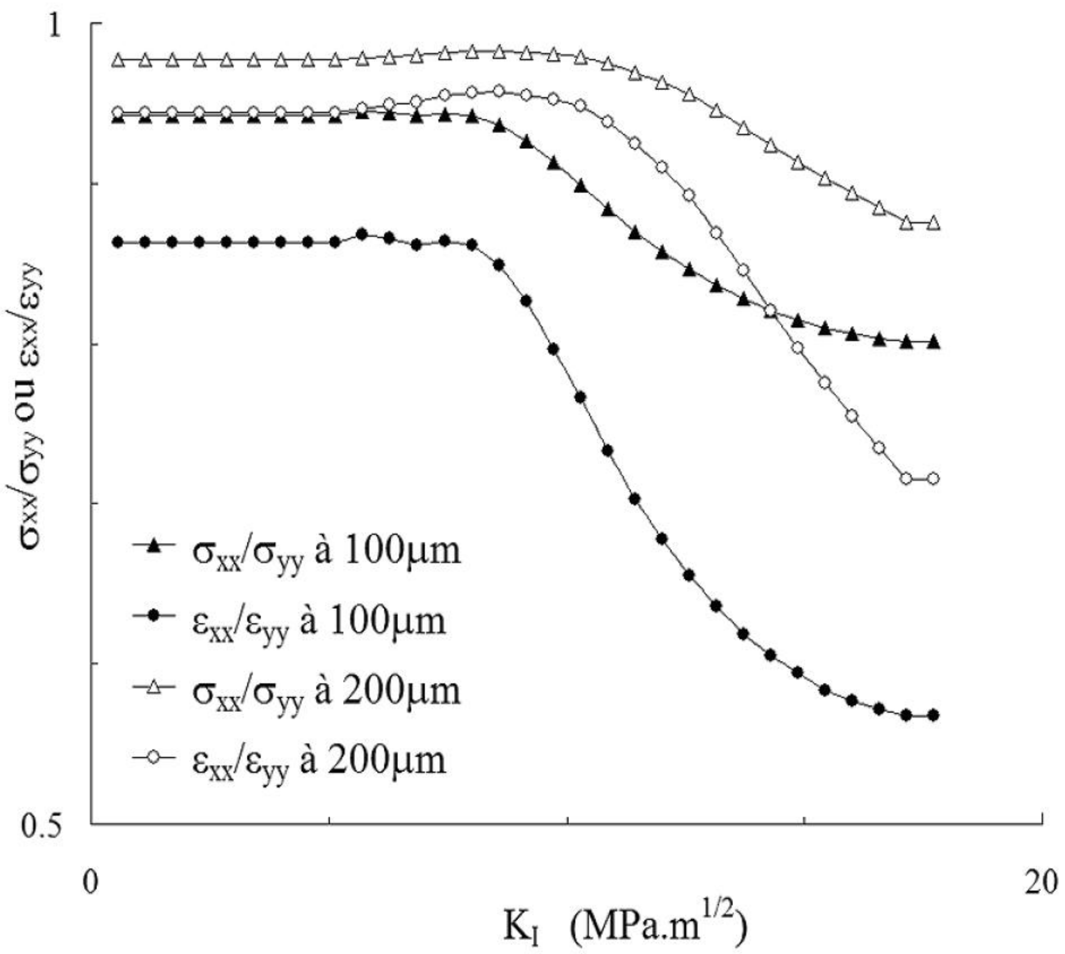
Irwin's plastic zones size, step 2: balance

Hypothesis: when plastic deformation occurs, the stress tensor remains proportionnal to the LEFM one

$$\sigma^I_{xx} = \sigma^I_{yy}$$
$$\sigma^I_{zz} = 2\nu\sigma^I_{yy}$$
$$\sigma^I_{eq} = (1-2\nu)\sigma^I_{yy}$$



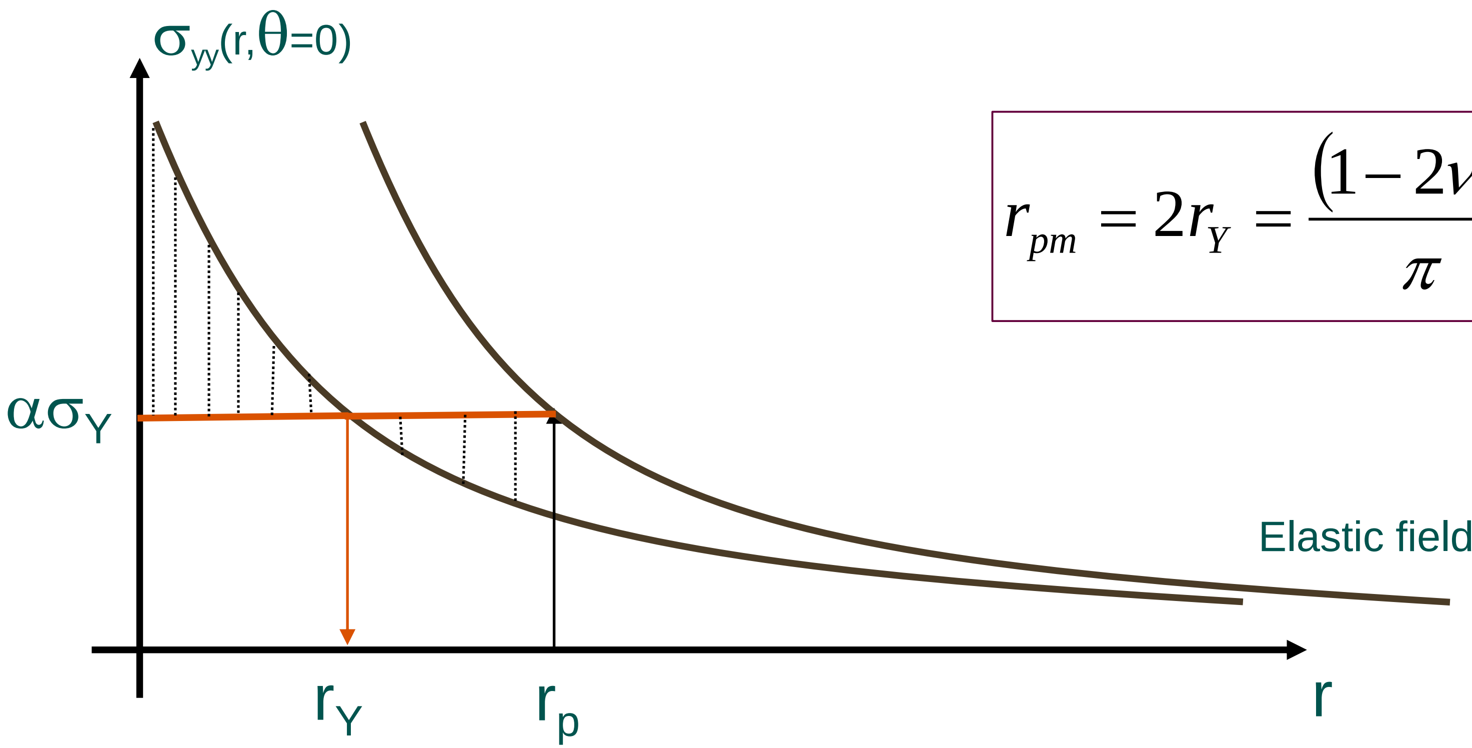
Limitations



Crack tip blunting modifies the proportionnality ratio between the components of the stress and strain tensors

FE results, Mesh size 10 micrometers, Re=350 MPa, Rm=700 MPa, along the crack plane

Irwin's plastic zones size, step 2: balance



$$r_{pm} = 2r_Y = \frac{(1-2\nu)^2}{\pi}\frac{K_I^2}{\sigma_Y^2}$$

$$\int_{r=0}^{r=\infty} \frac{K_I^{\max}}{\sqrt{2\pi r}}\,dr = \int_{r=0}^{r=r_{pm}} \frac{\sigma_Y}{(1-2\nu)}\,dr + \int_{r=r_{pm}}^{r=\infty} \frac{K_I^{\max}}{\sqrt{2\pi(r-r_Y)}}\,dr$$

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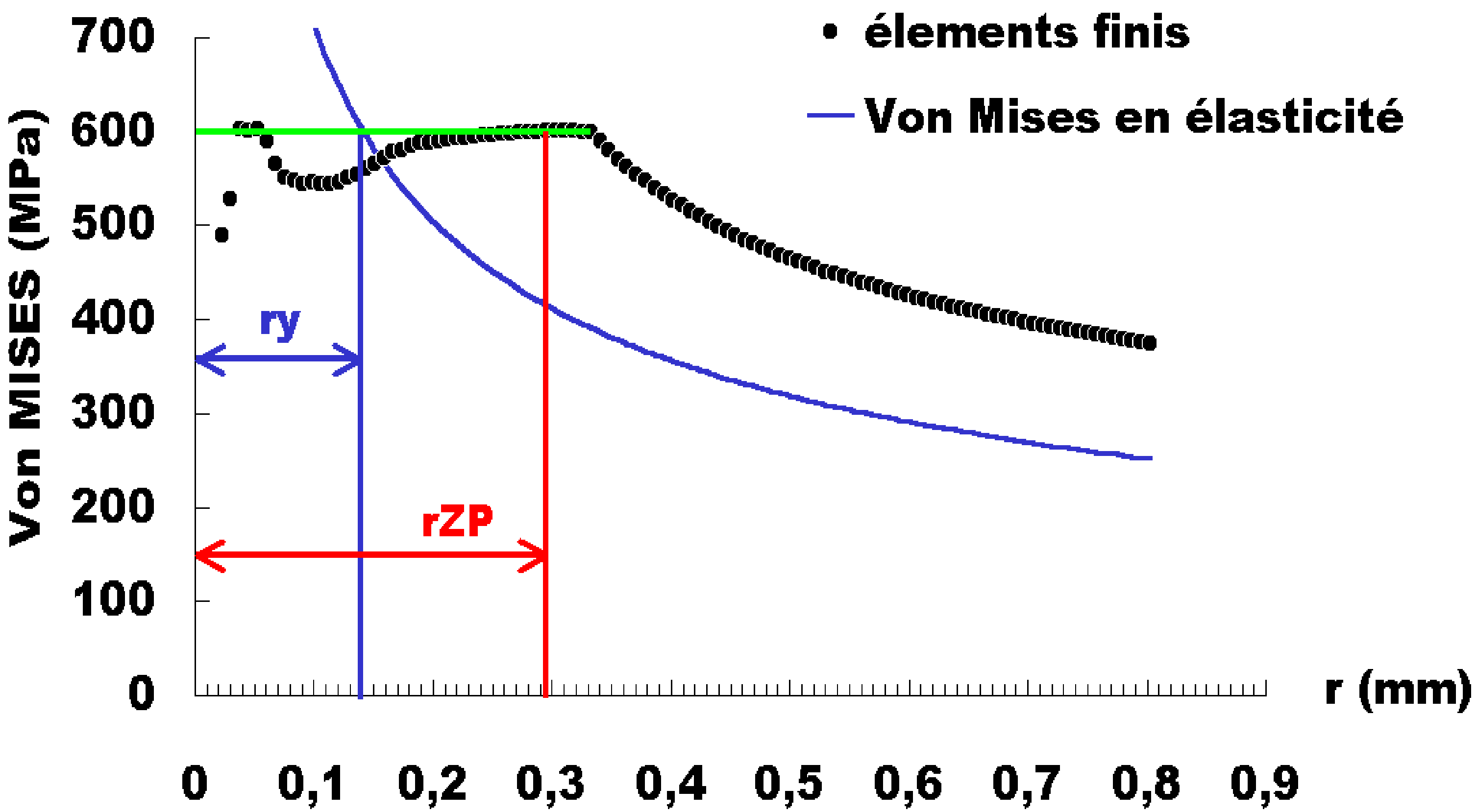
Irwin's plastic zone versus FE computations

Ideally elastic-plastic material σ_Y=600 MPa, E=200 GPa, ν=0.3
plane strain, along the plane θ=0



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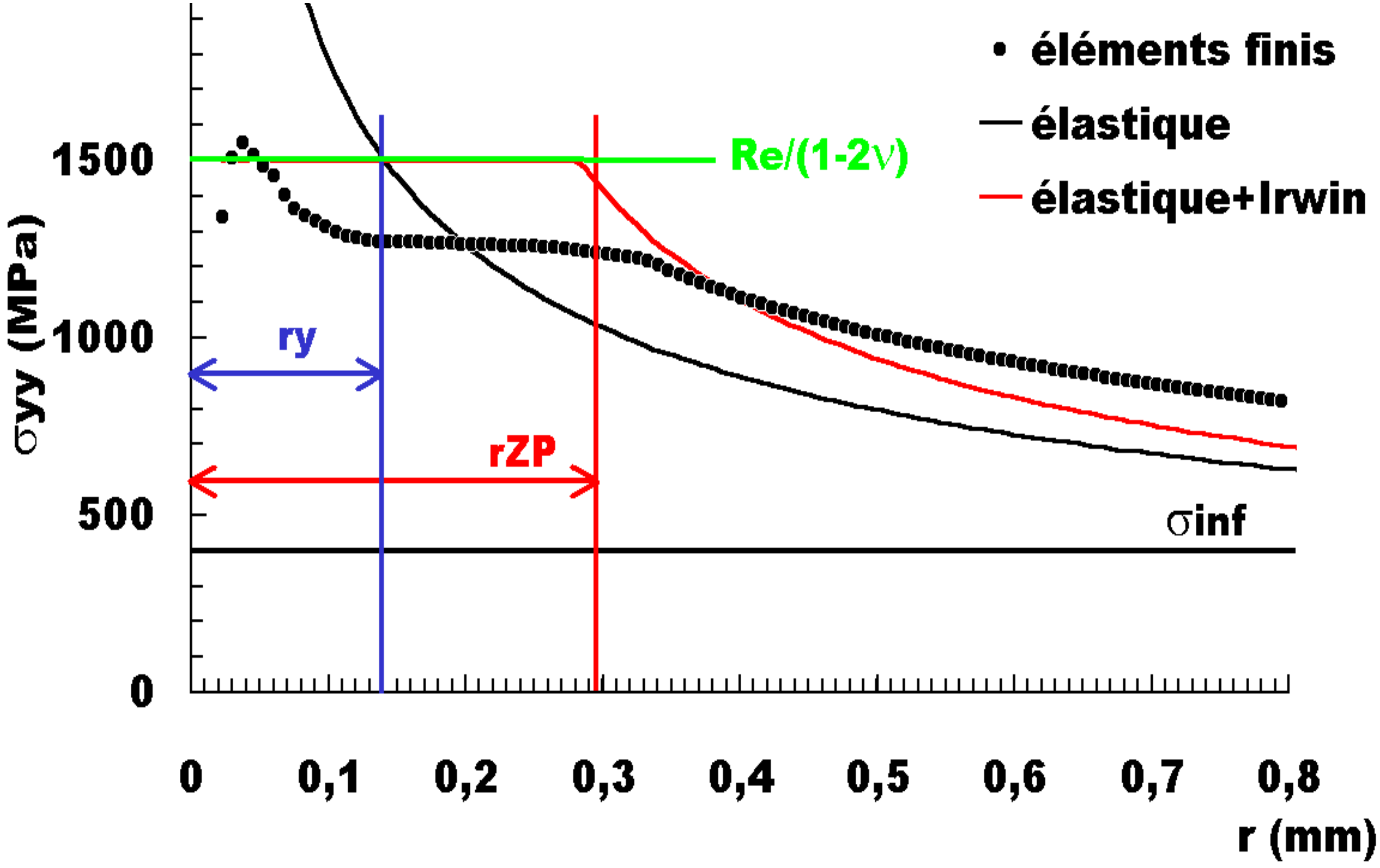

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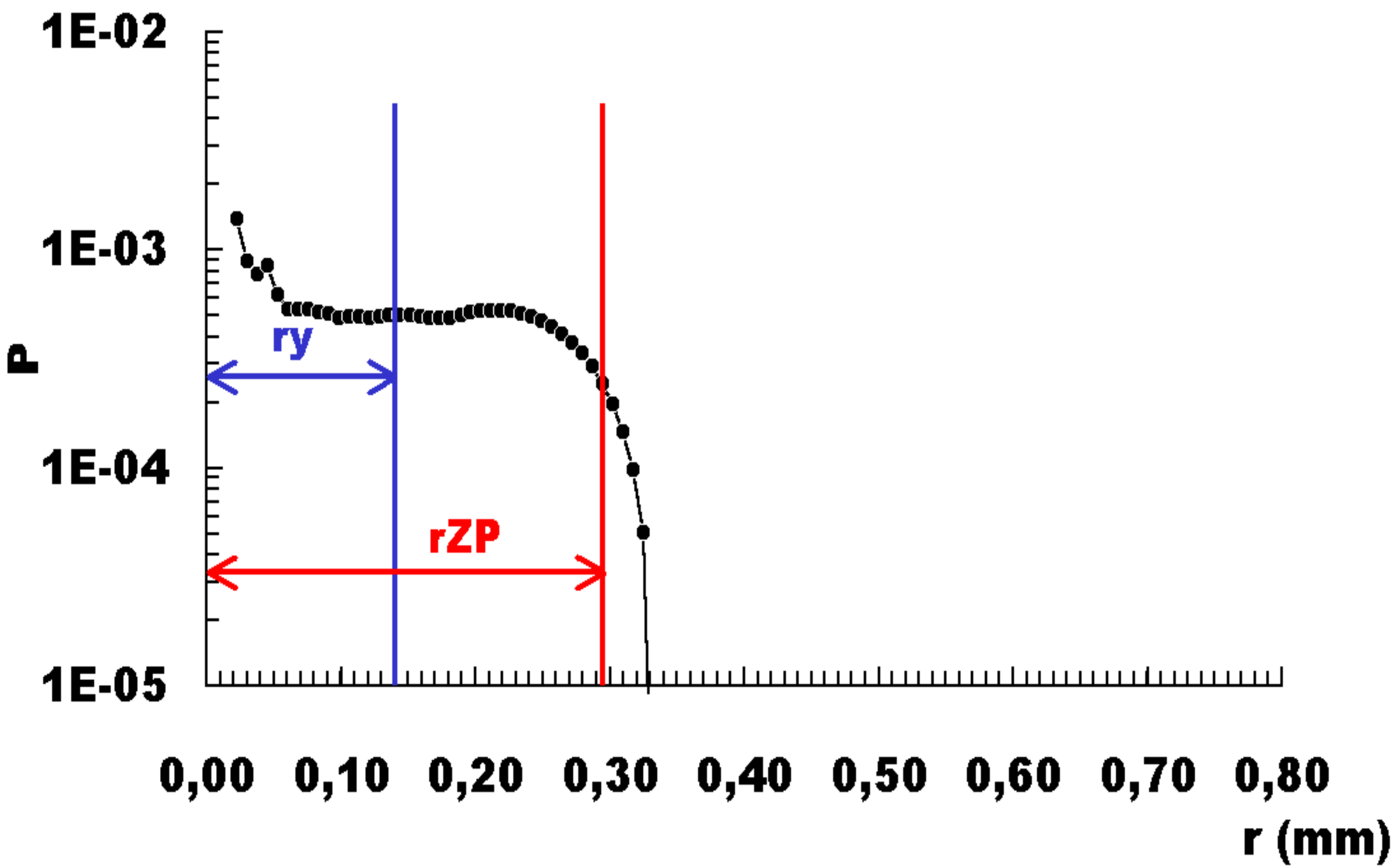
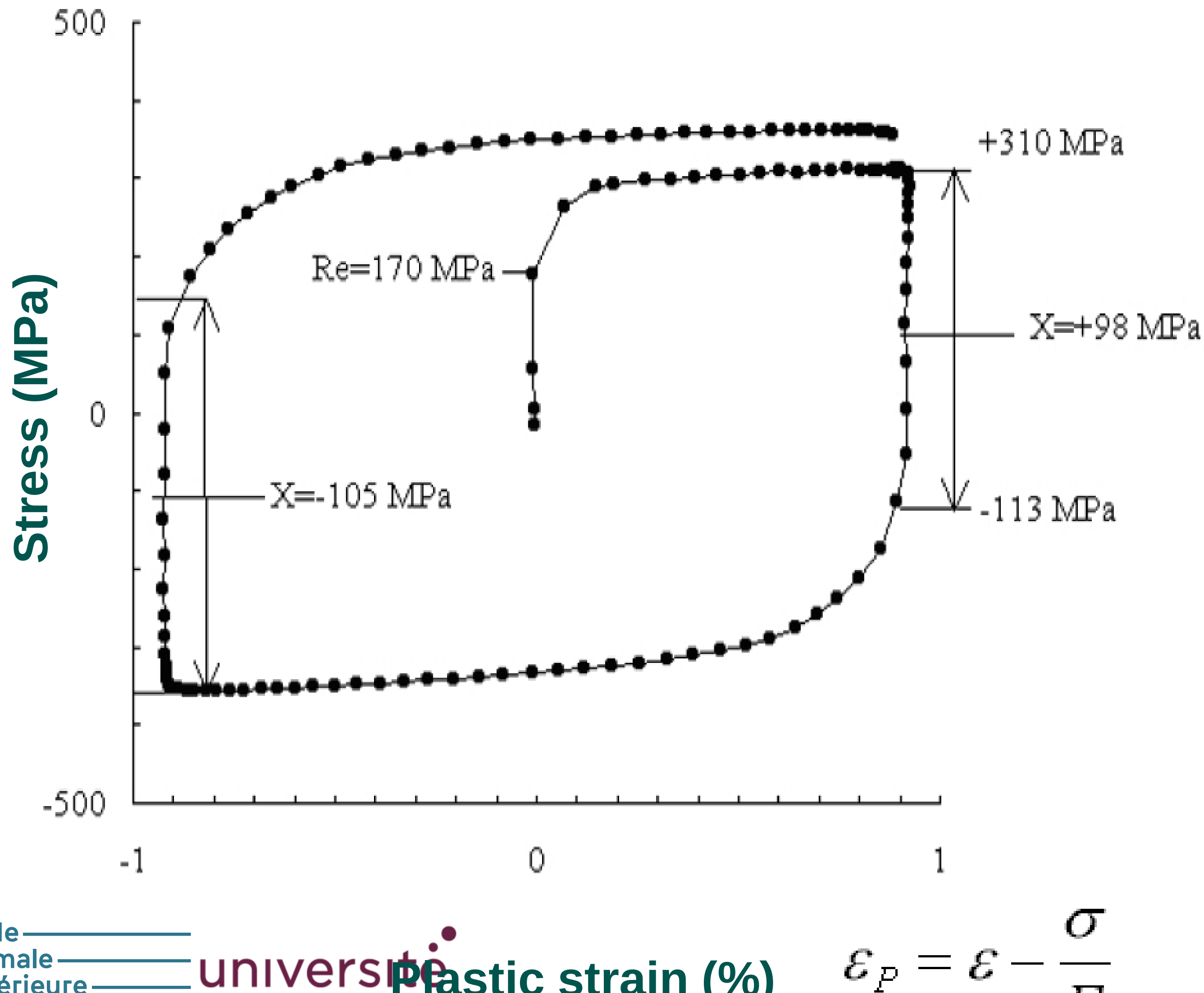
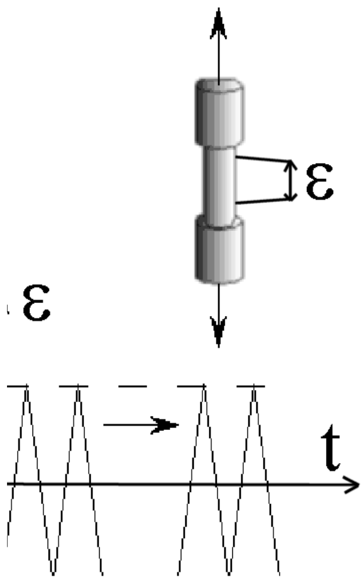


Mode I, Monotonic and cyclic plastic zones





$$\varepsilon_P = \varepsilon - \frac{\sigma}{E}$$

Mode I, Monotonic and cyclic plastic zones

Monotonic plastic zone

$$\boxed{r_{mpz} = \frac{(1-2\nu)^2}{\pi} \frac{K_{I\,\max}^2}{\sigma_Y^2}}$$

Cyclic plastic zone

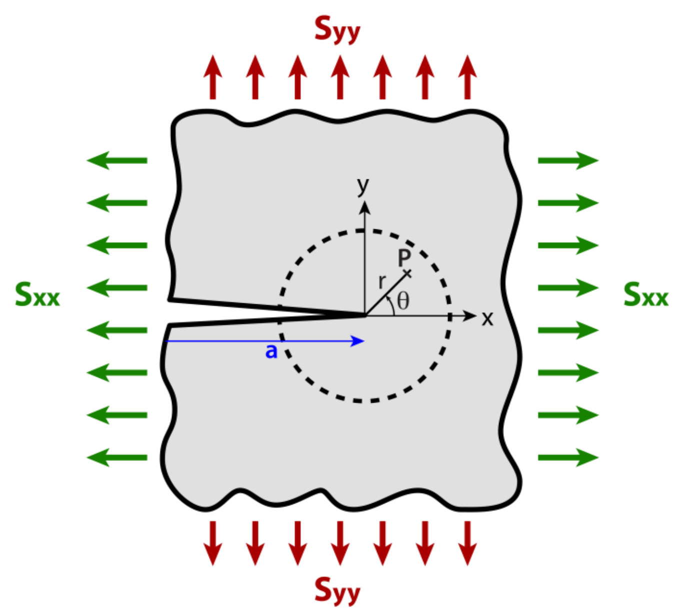
$$\boxed{r_{cpz} = \frac{(1-2\nu)^2}{\pi} \frac{\Delta K_I^2}{4\sigma_Y^2}}$$

T-Stress effect

$$\sigma_{xx} = \frac{S_{yy}\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \overbrace{S_{xx} - S_{yy}}^{T = S_{xx} - S_{yy}}$$

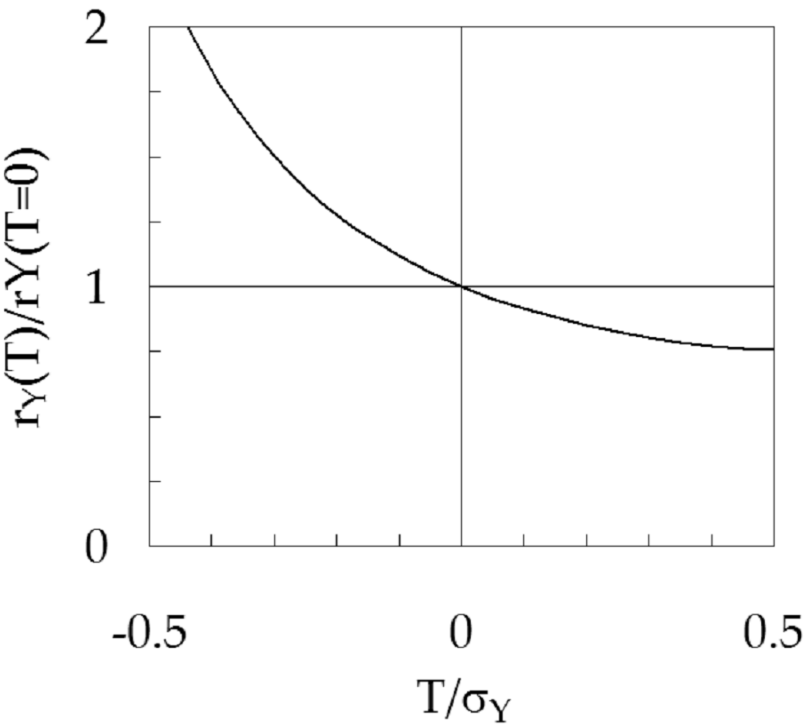
$$\sigma_{yy} = \frac{S_{yy}\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

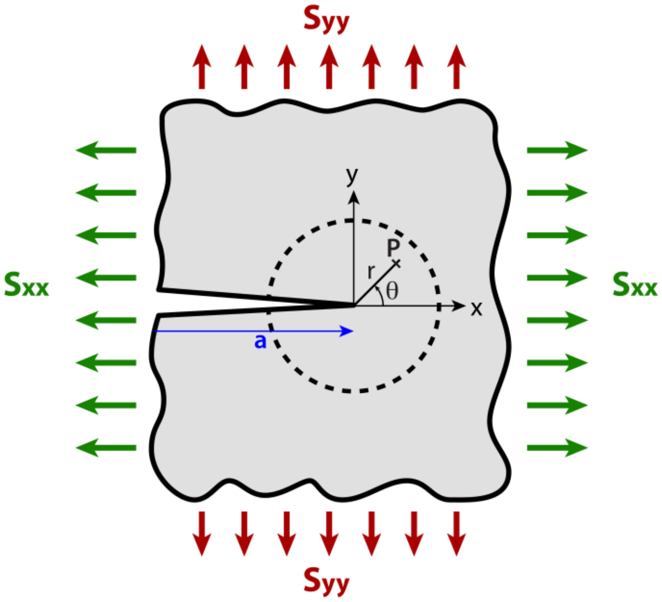
$$\sigma_{zz} = \frac{\underbrace{S_{yy}\sqrt{\pi a}}_{K_I = S_{yy}\sqrt{\pi a}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$



T-Stress effect

$$\frac{r_Y(T)}{r_Y(T=0)} = \frac{2}{2 - \left(\frac{T}{\sigma_Y}\right)^2 + \left(\frac{T}{\sigma_Y}\right)\sqrt{4 - 3\left(\frac{T}{\sigma_Y}\right)^2}}$$

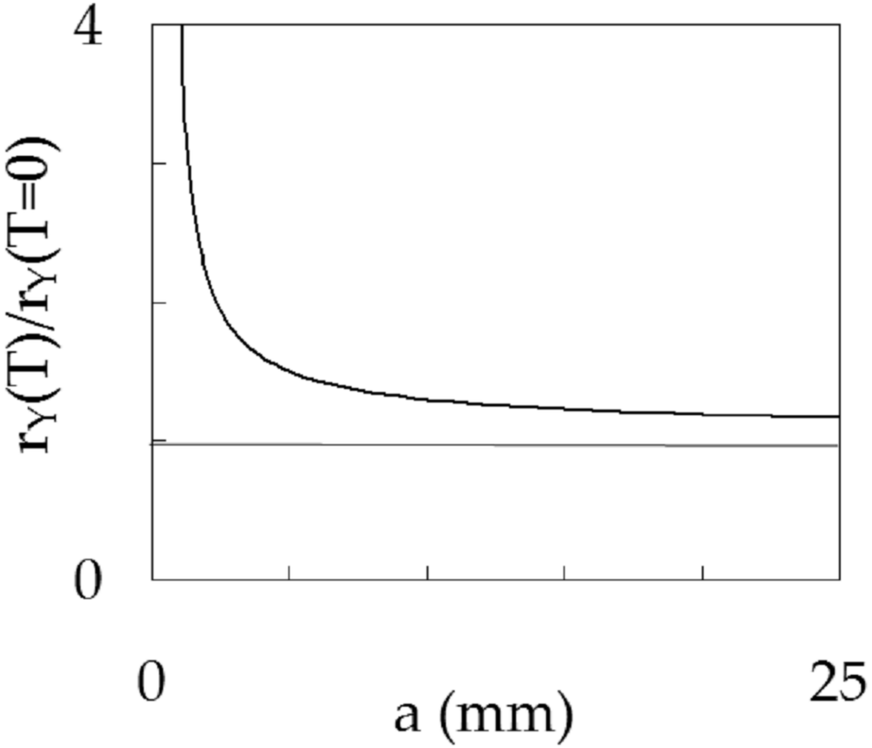


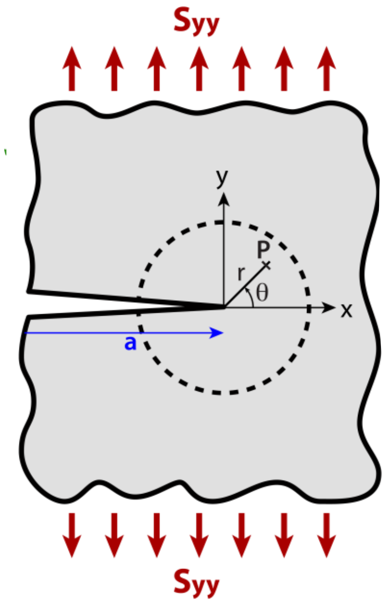


$$T = S_{xx} - S_{yy}$$

$$K_I = S_{yy}\sqrt{\pi a}$$

T-Stress effect





$$K_I = S_{yy}\sqrt{\pi a}$$

$$T = -S_{yy}$$

Irwin's plastic zone, σ_Y=400 MPa, K_I=15MPa.m$^{1/2}$

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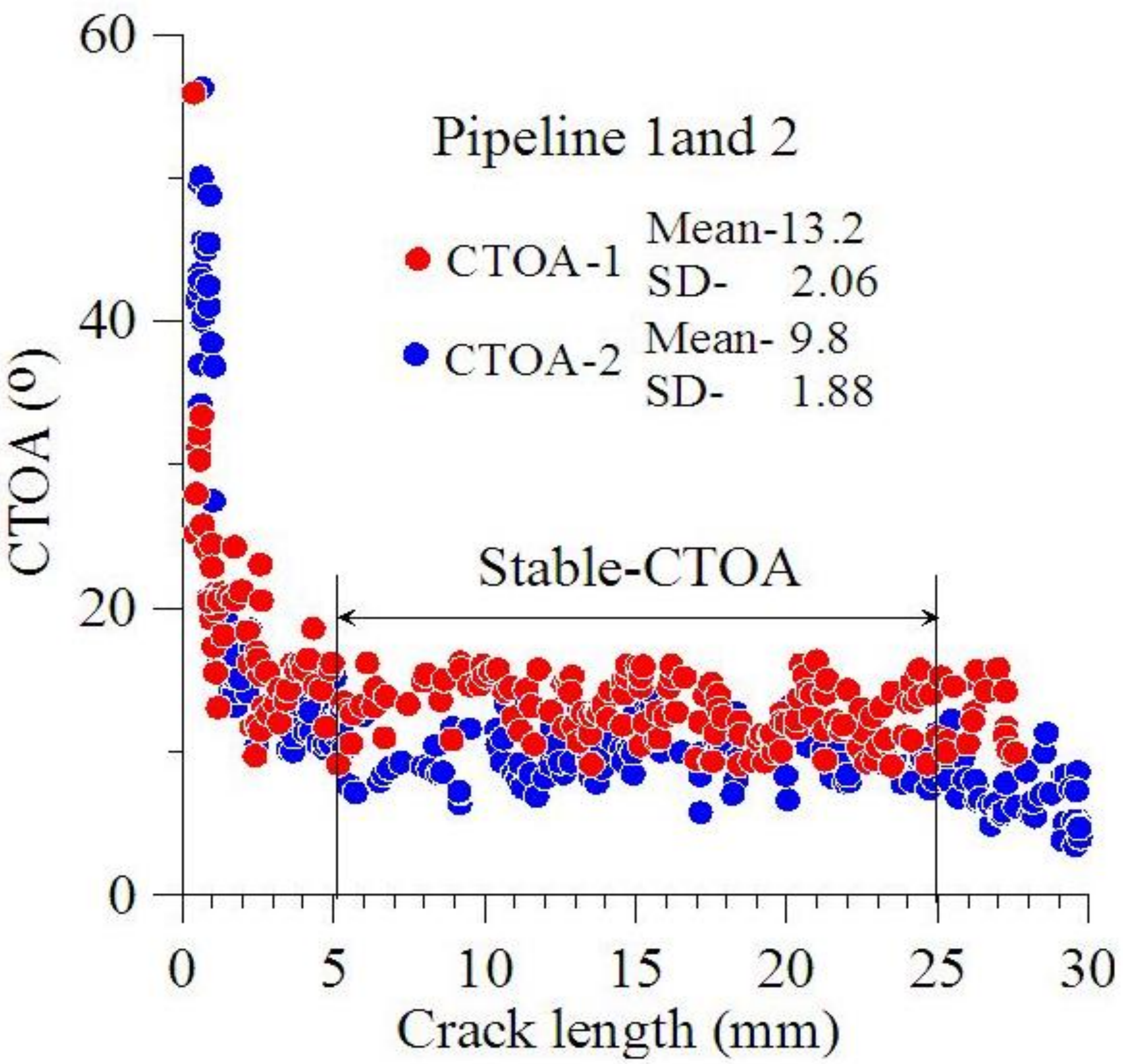
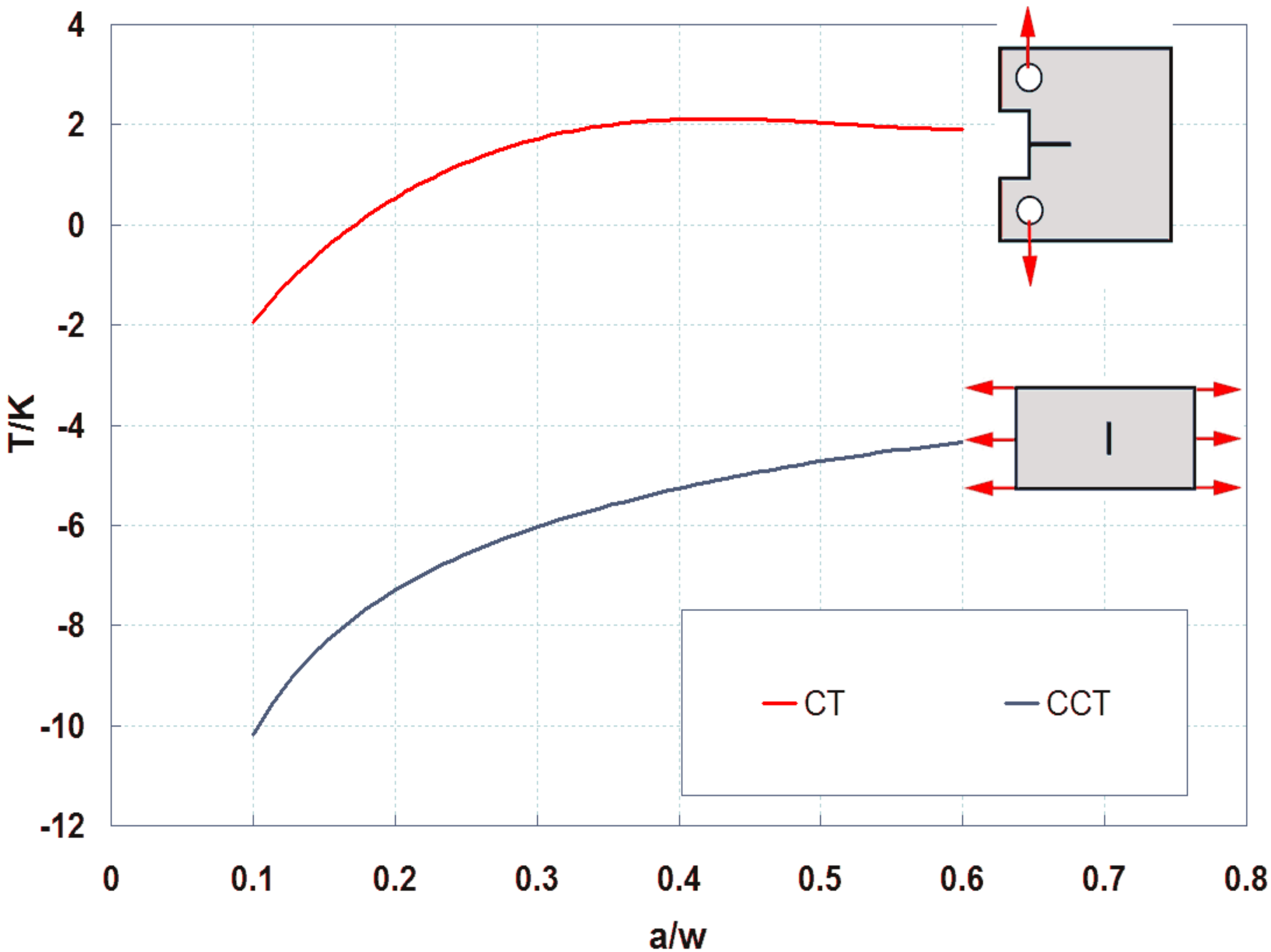
Ductile fracture

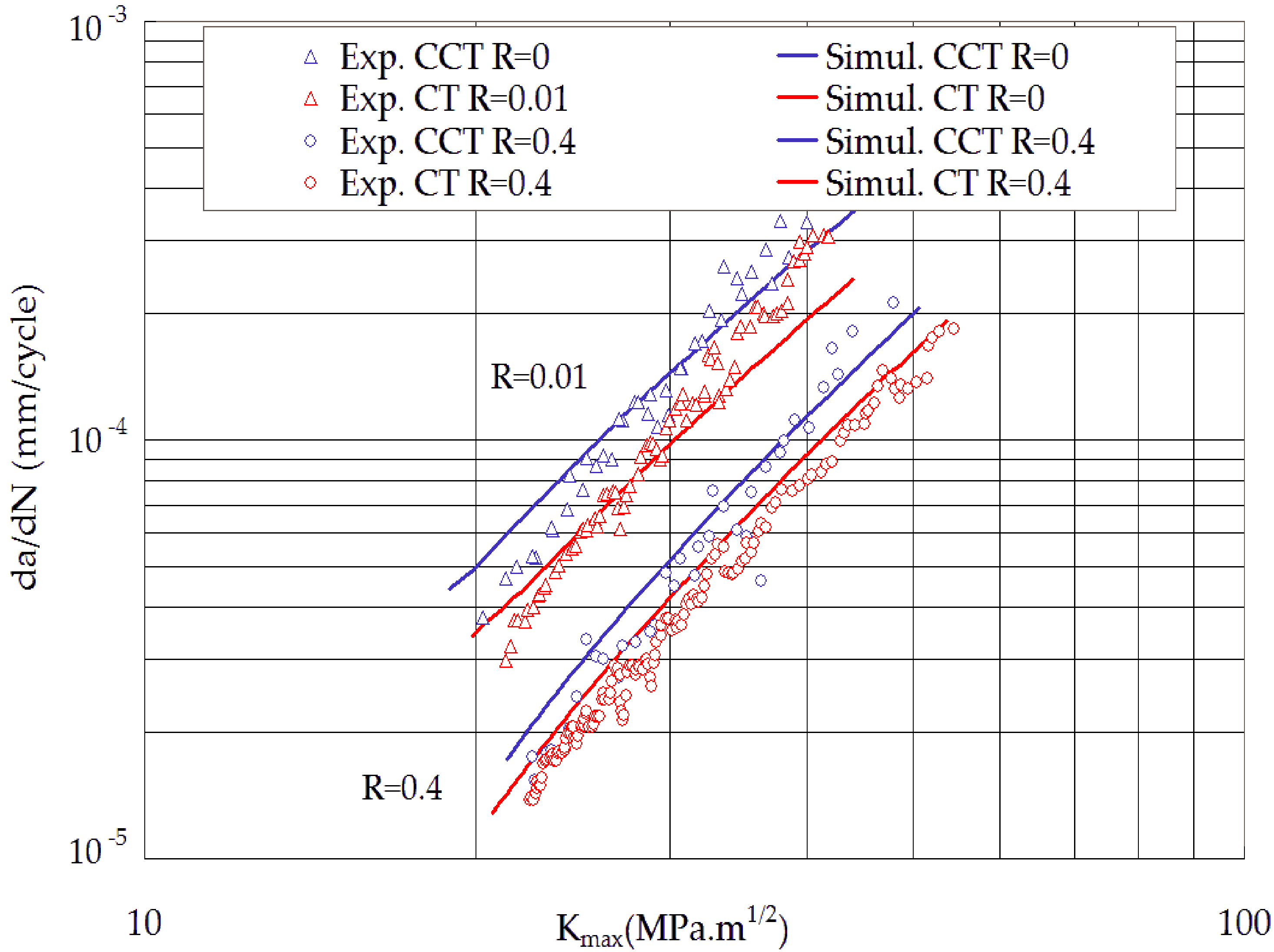
Measurement of the crack tip opening angle at the onset of fracture



Example of the effect of a T-Stress for long cracks



Example of the effect of a T-Stress for long cracks



0.48 % Carbon Steel [Hamam,2007]

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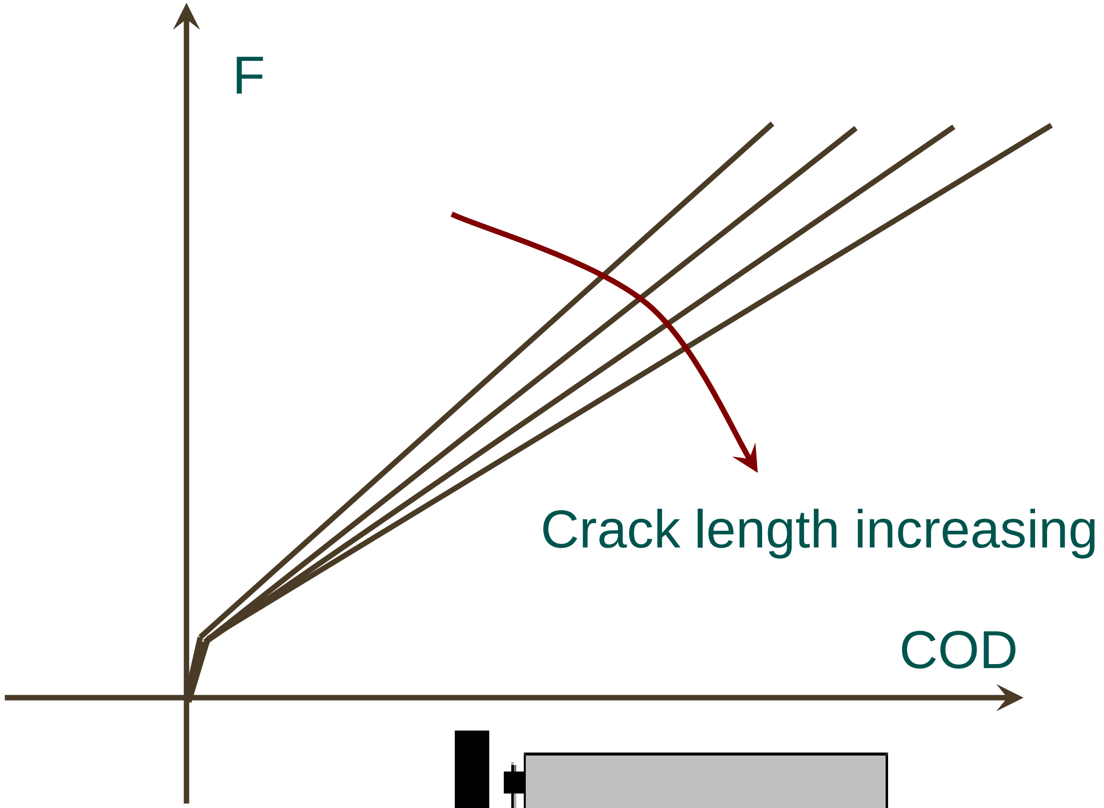
Fatigue, and crack growth modeling

ECF21 European Conference on Fracture — Fracture and Safety

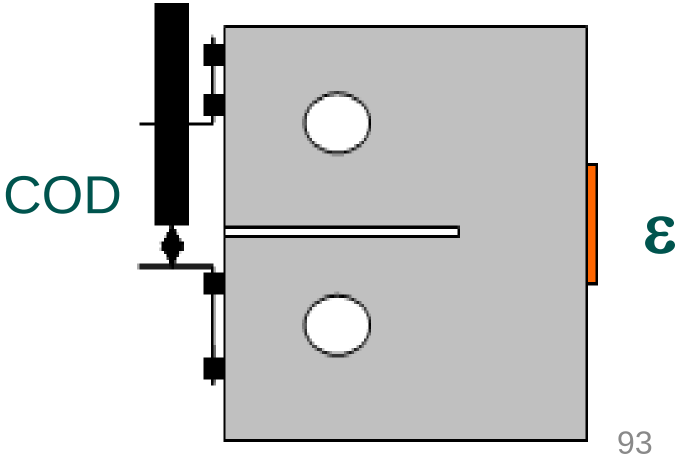
ENS CACHAN — université PARIS-SACLAY

Measurements

J.Petit



Potential drop
Direct optical measurements
Digital image correllation



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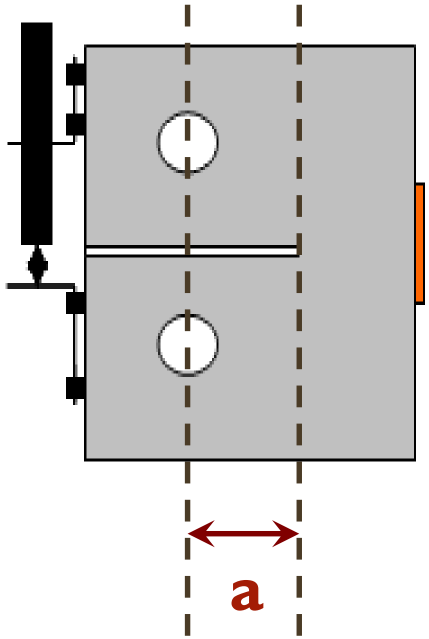
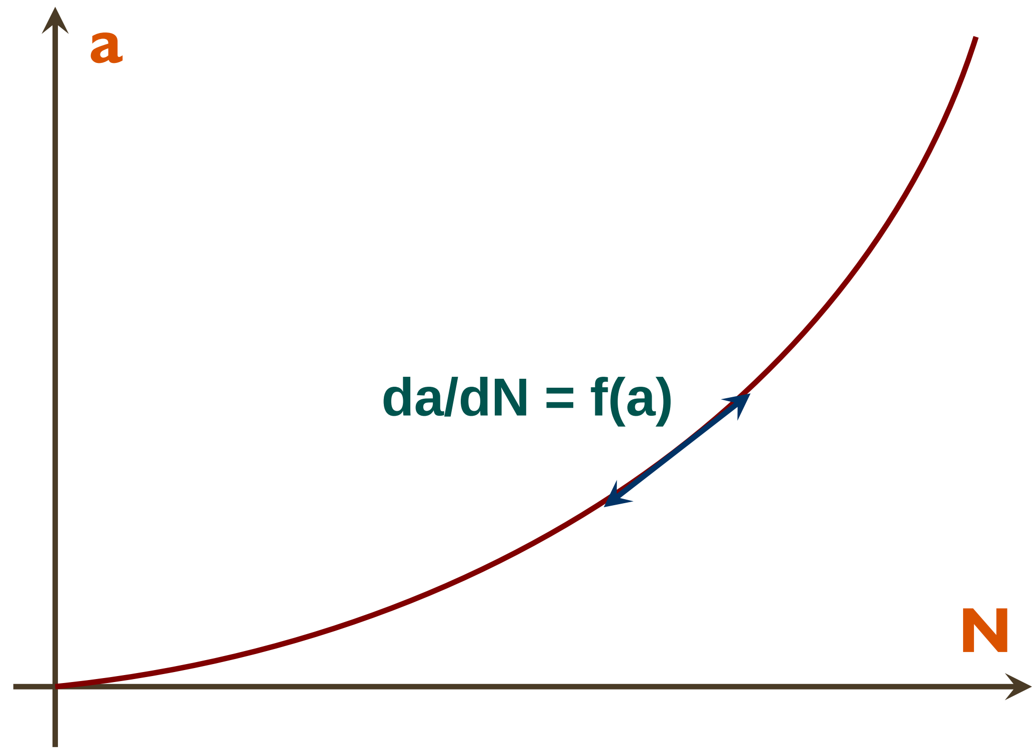
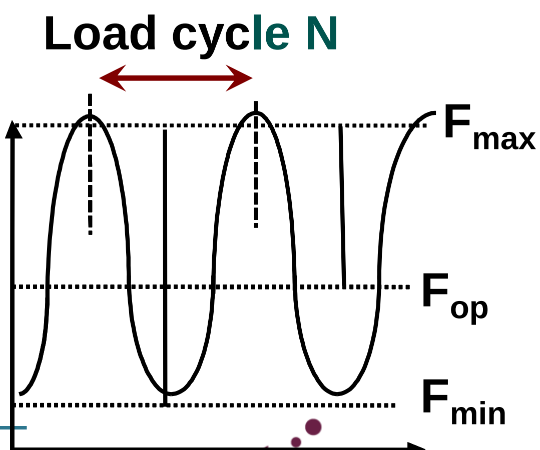




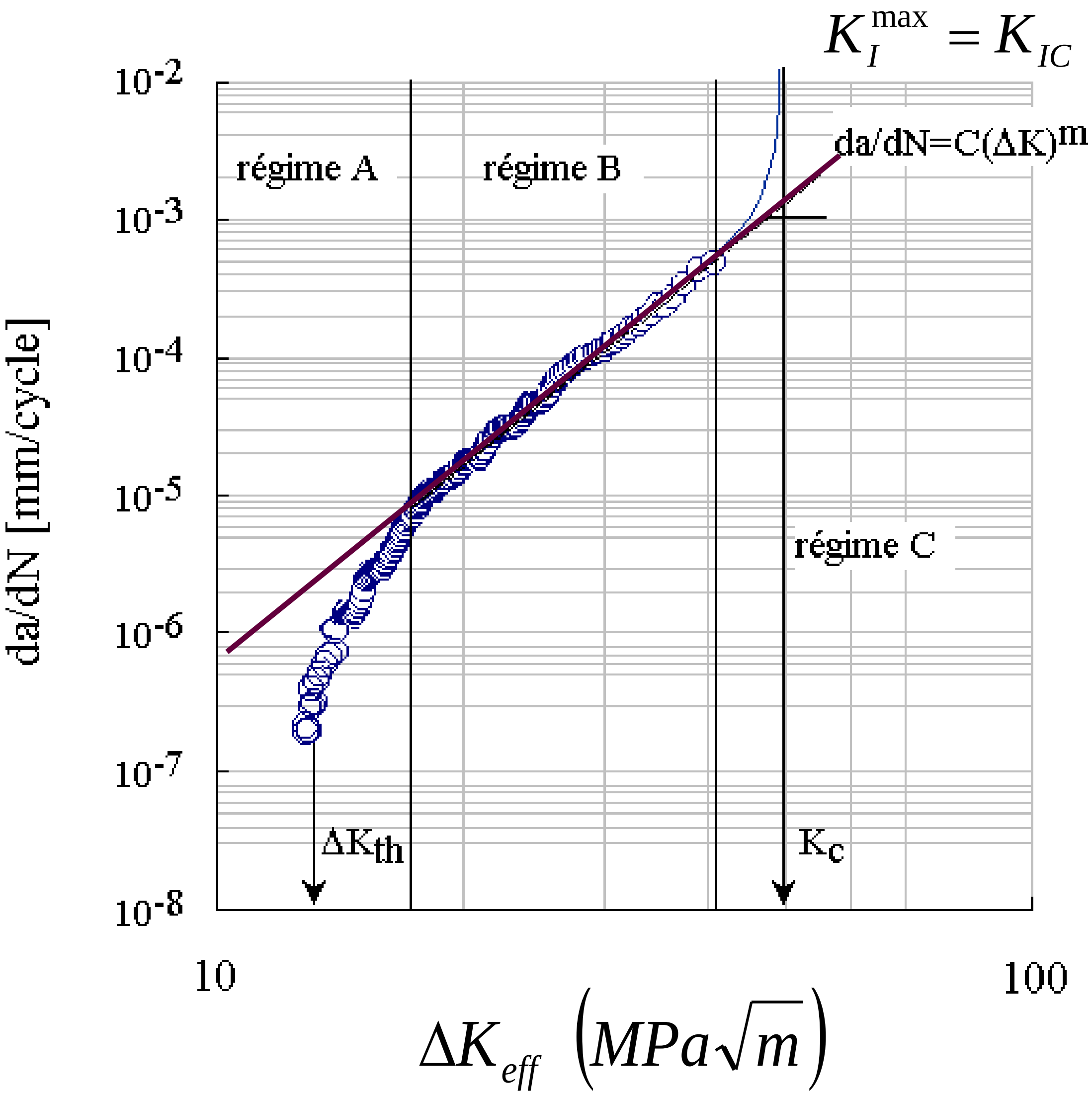


$R=F_{min}/F_{max}$

ΔF

ΔF_{eff}

Paris' law

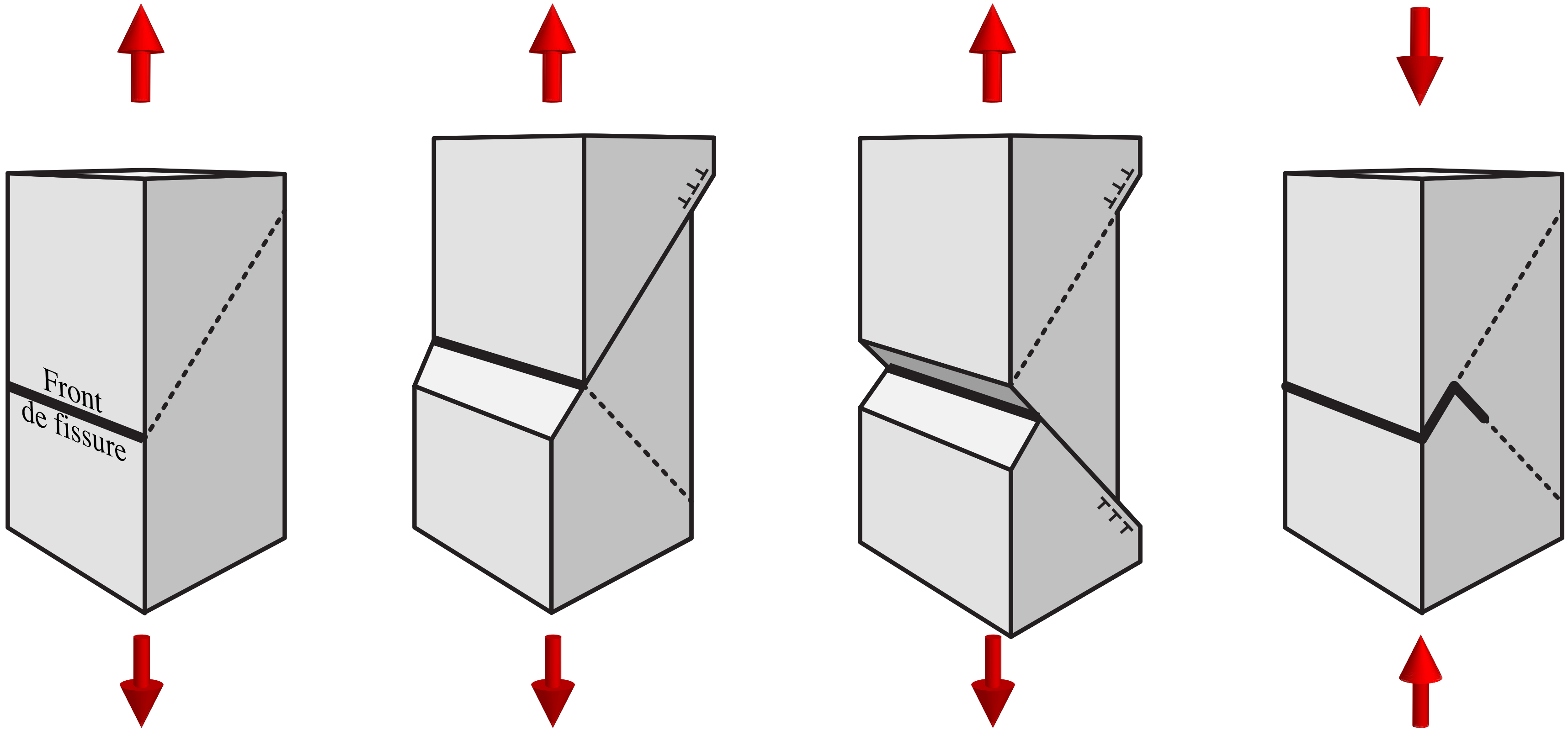


A - threshold regime
B – Paris' regime
C - unstable fracture

$$\Delta K_{eff} > \Delta K_{th}$$

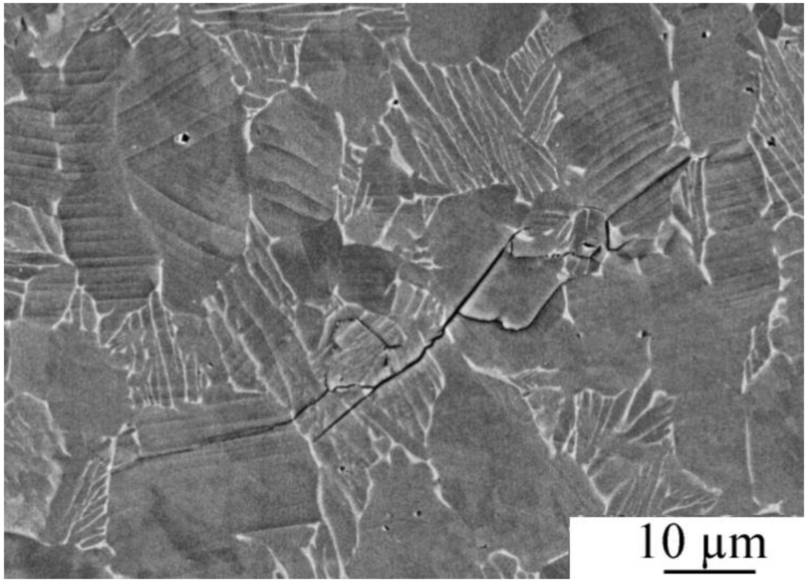
Subcritical crack growth if DK is over the non propagation threshold

Fatigue – Threshold regime

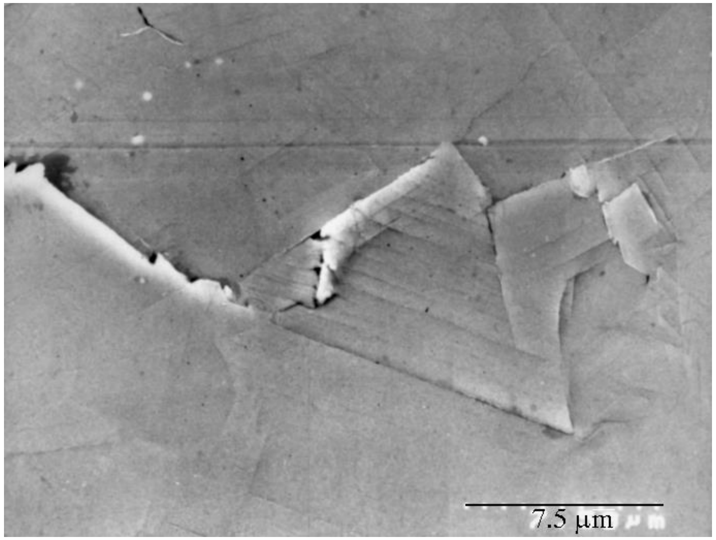


[Neumann,1969]

Fatigue – Threshold regime

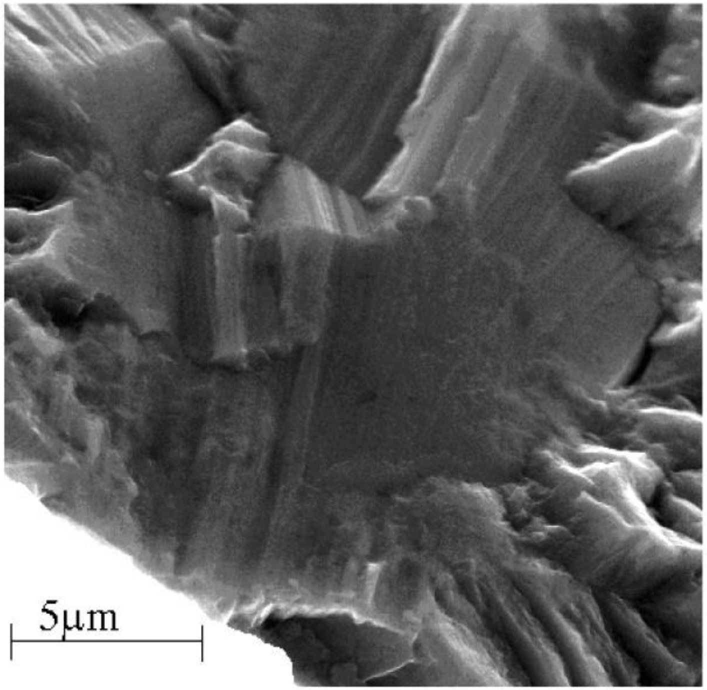


Titanium alloy TA6V [Le Biavant, 2000]. The fatigue crack grows along slip planes.

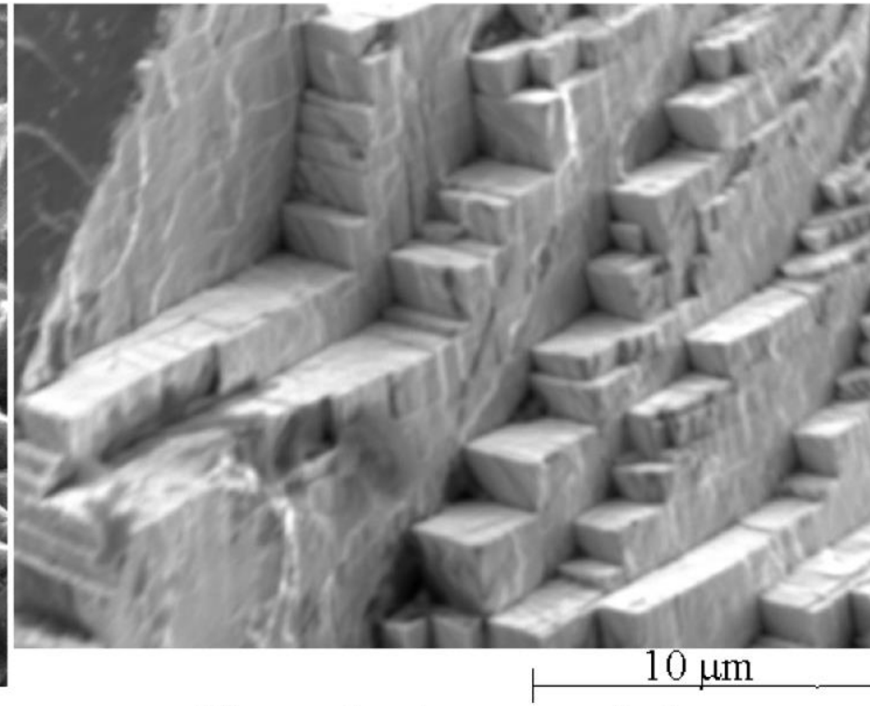


N18 nickel based superalloy at room temperature, [Pommier,1992]. The crack grows at the intersection between slip planes

Fatigue – Threshold regime – fracture surface



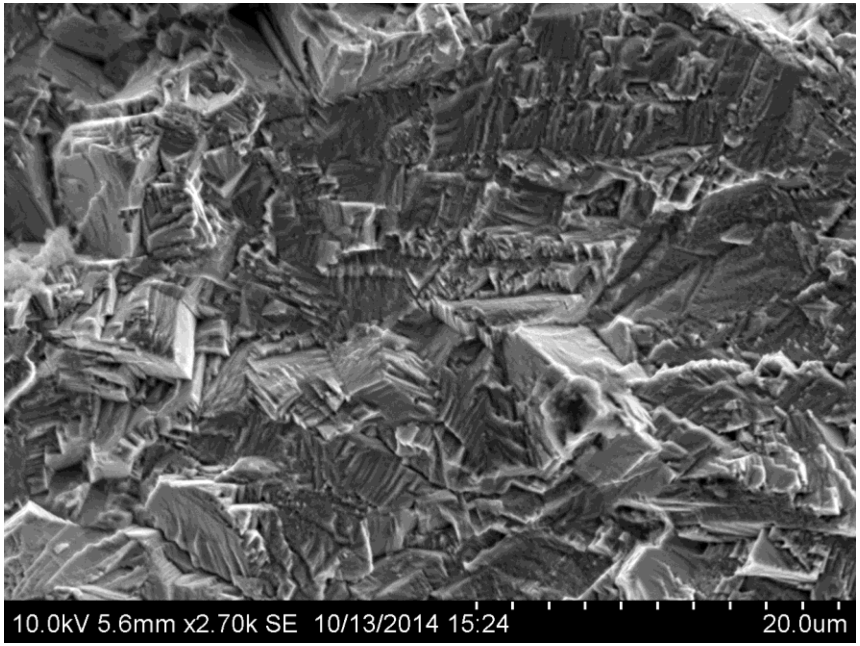
Titane TA6V (20°C)
(cliché M. Sampablo-Lauro)

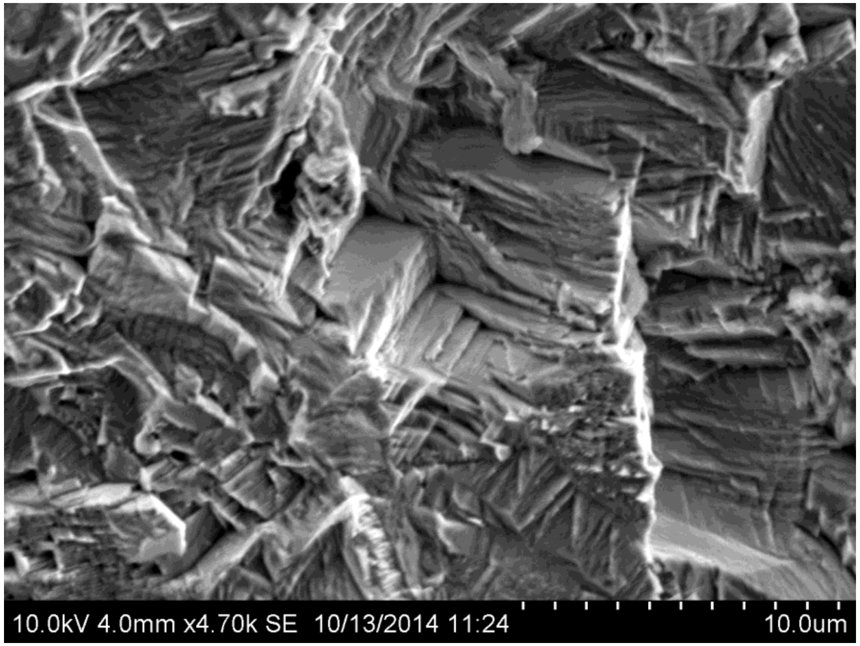


alliage de chrome-cobalt
(cliché M. Puget)

"pseudo-cleavage" facets at the initiation site

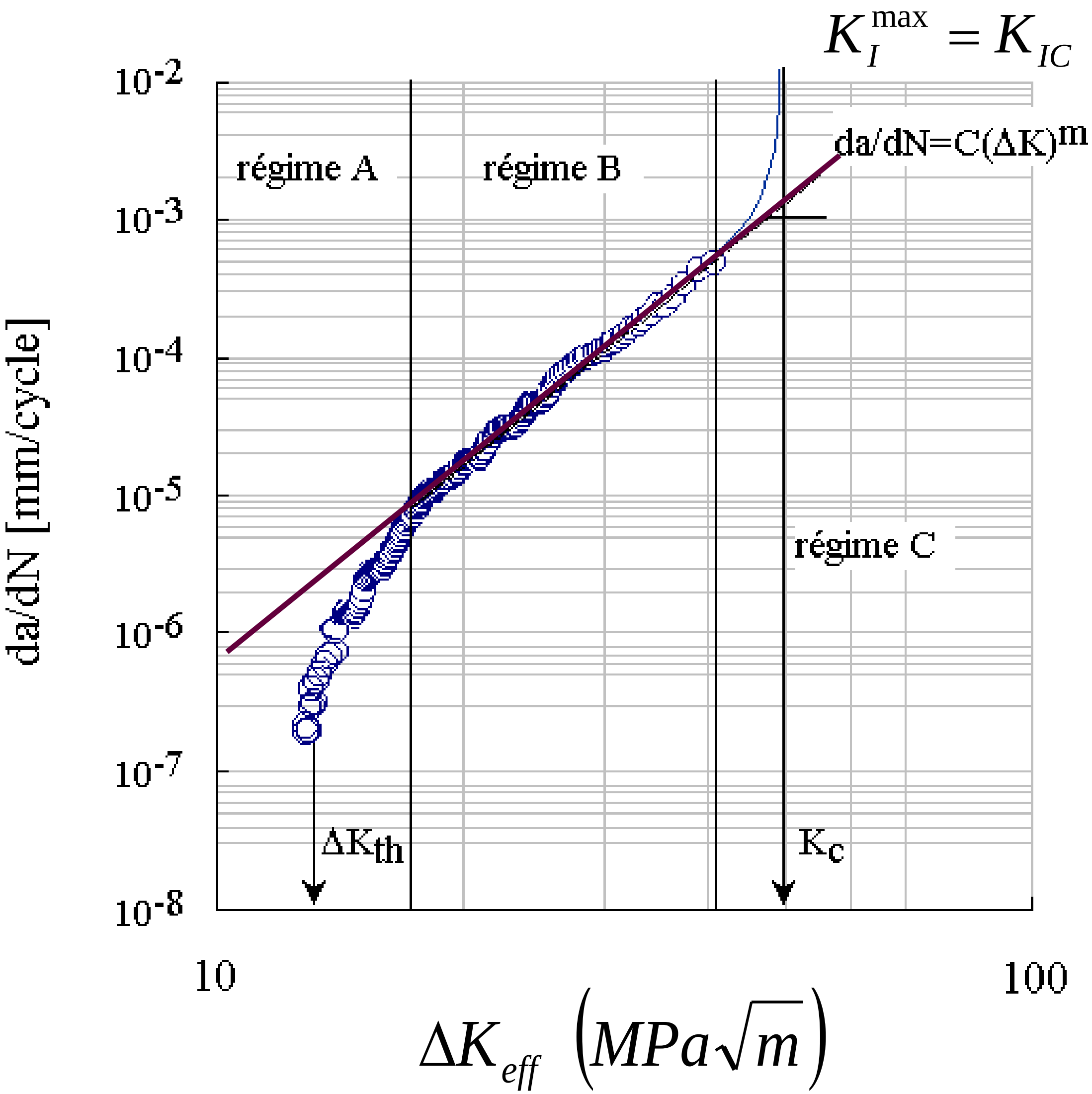
Fatigue – Threshold regime – fracture surface





INCO 718

Paris' law



A - threshold regime
B – Paris' regime
C - unstable fracture

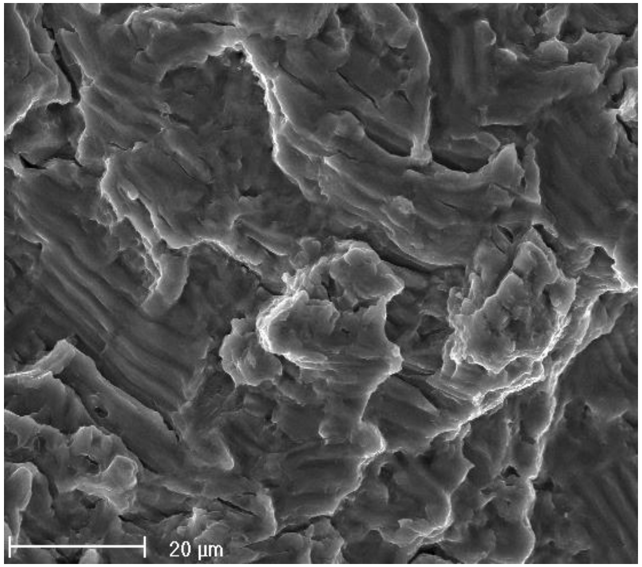
$$\Delta K_{eff} > \Delta K_{th}$$

Subcritical crack growth if DK is over the non propagation threshold

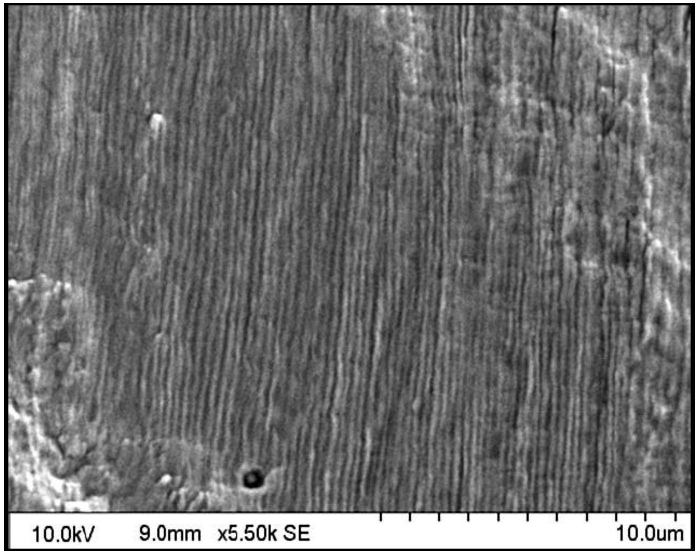
Paris' regime : crack growth by the striation process

[Laird,1967], [Pelloux, 1965]

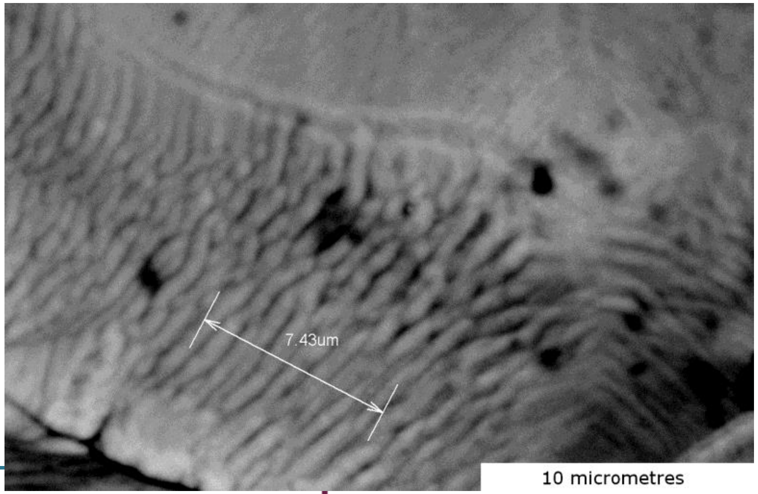
TA6V



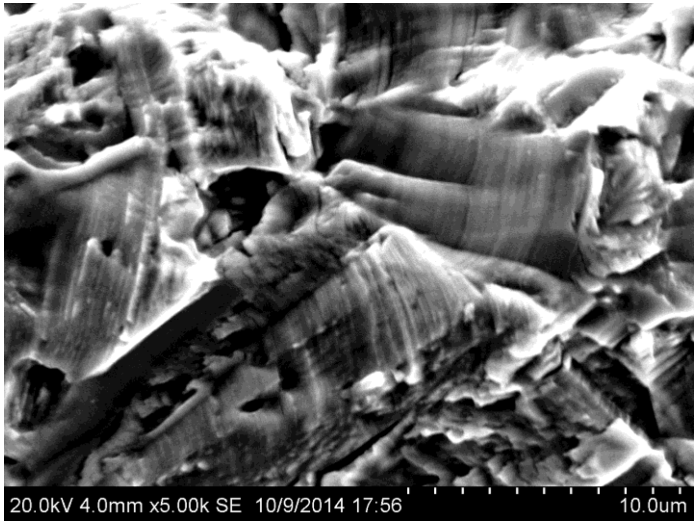
316L

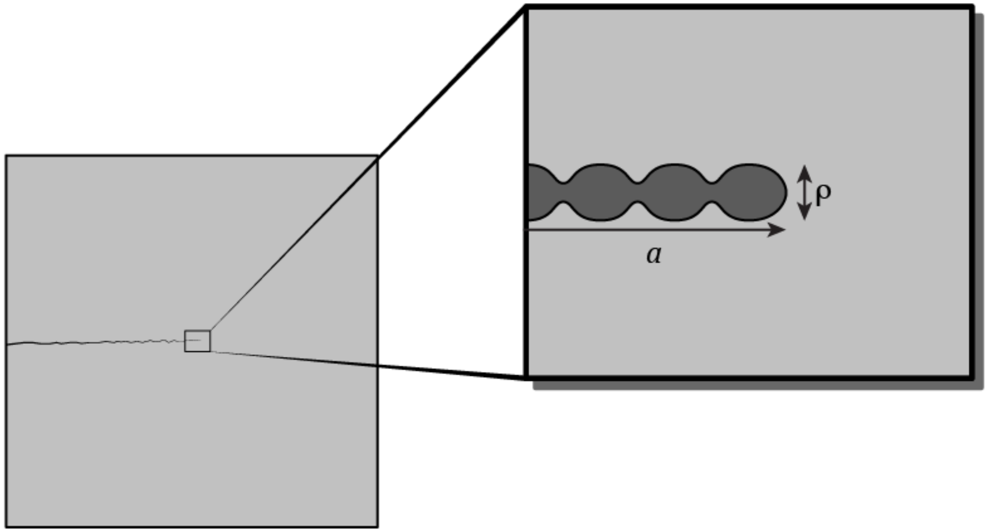


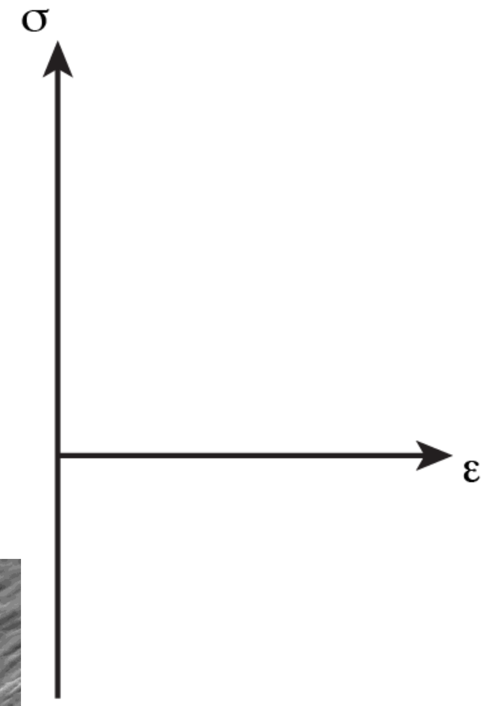
OFHC

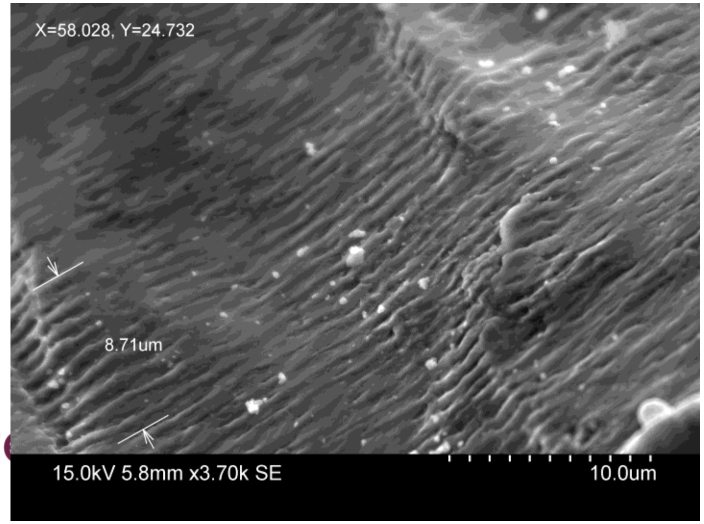


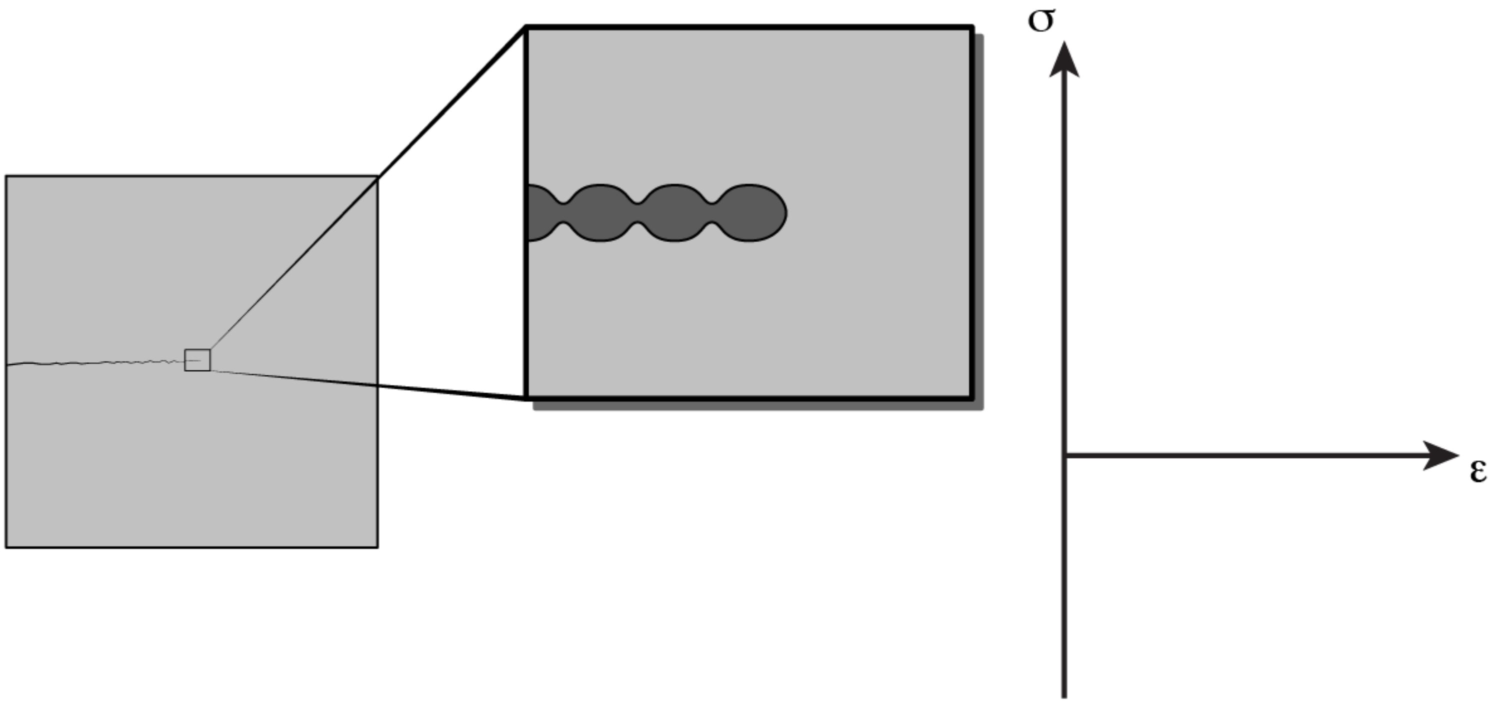
INCO 718

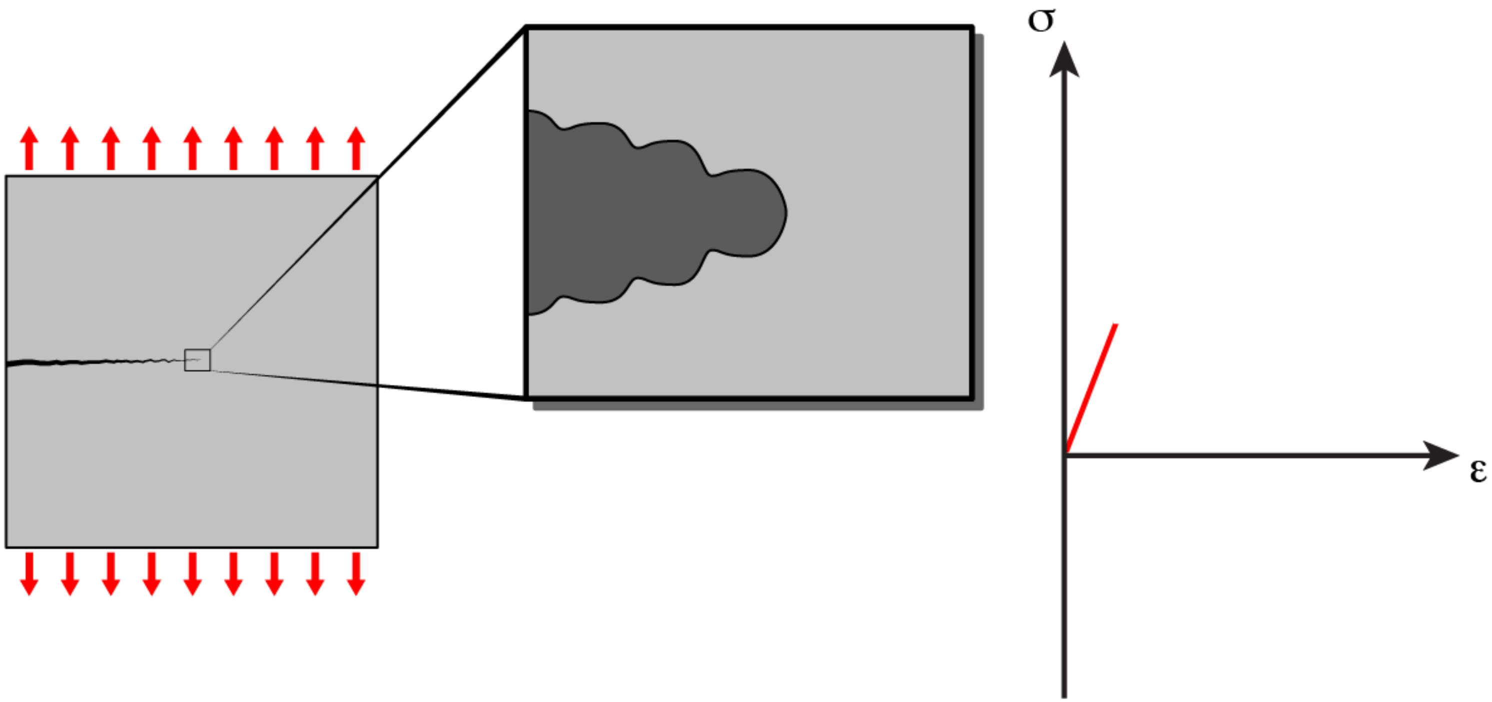


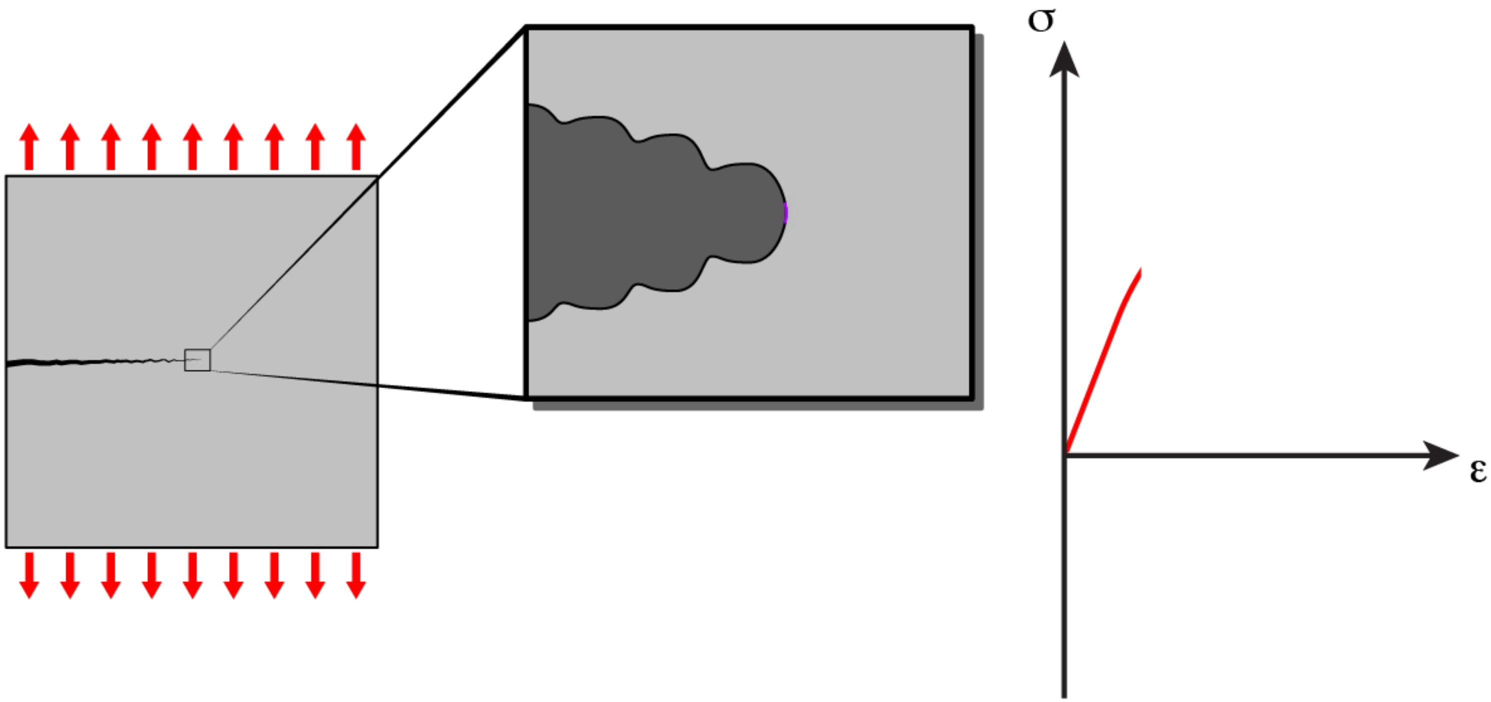

ρ
a

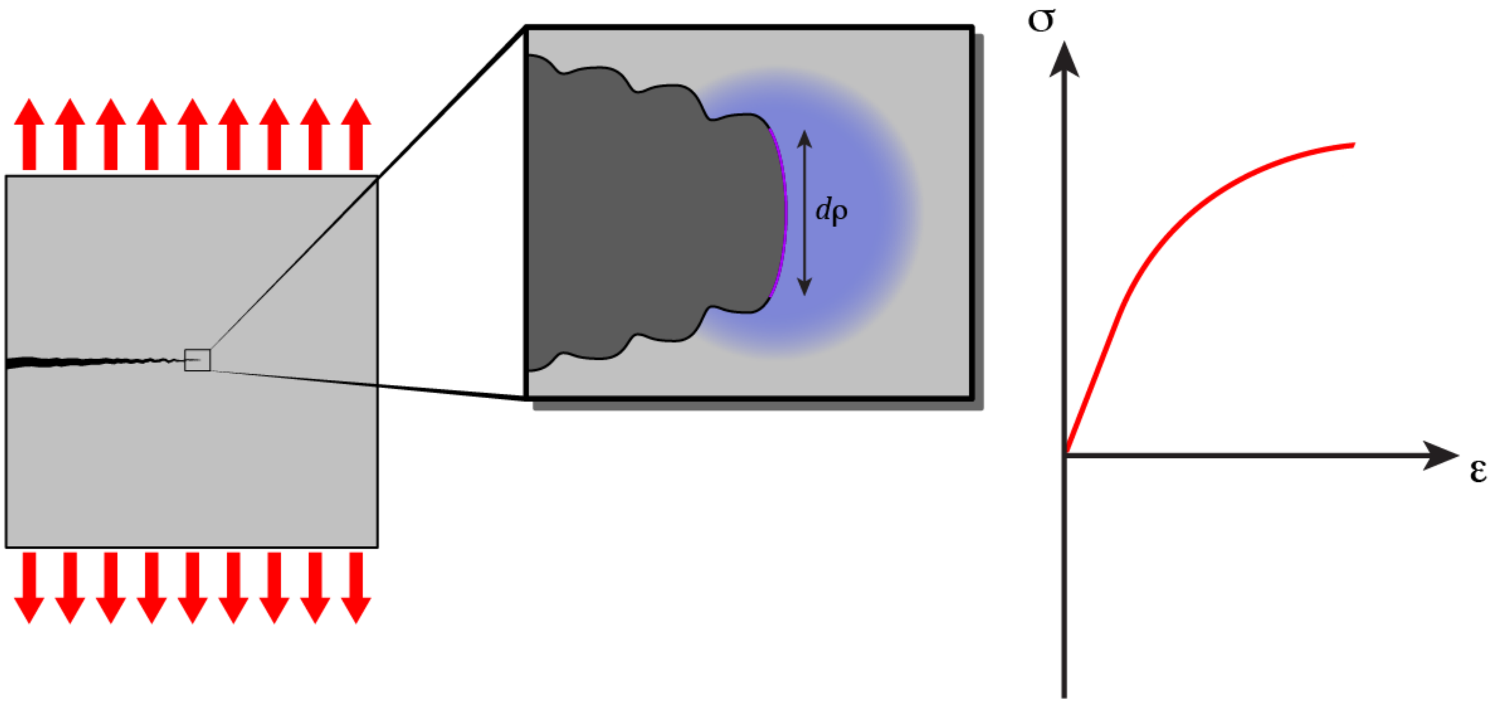

σ
ε


X=58.028, Y=24.732
8.71um
15.0kV 5.8mm x3.70k SE
10.0um

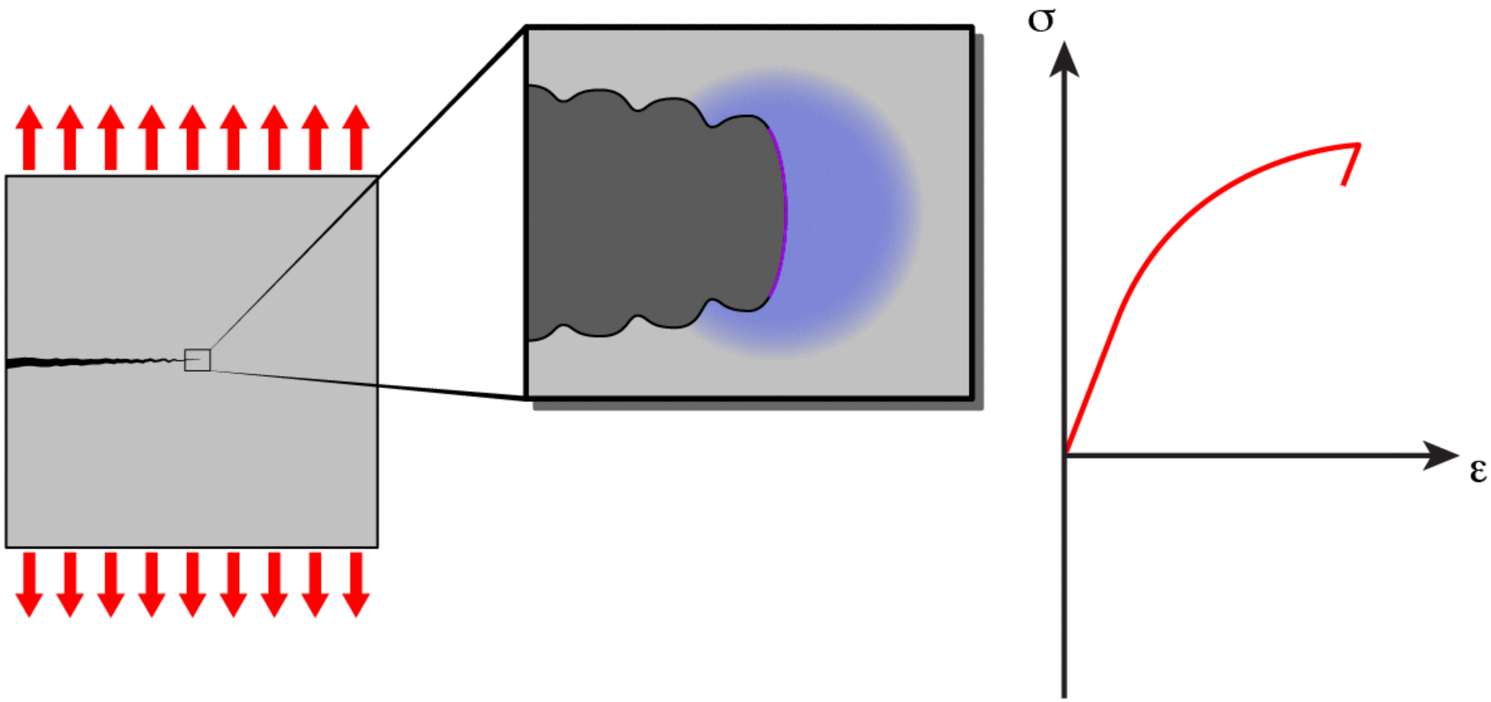

σ
ε

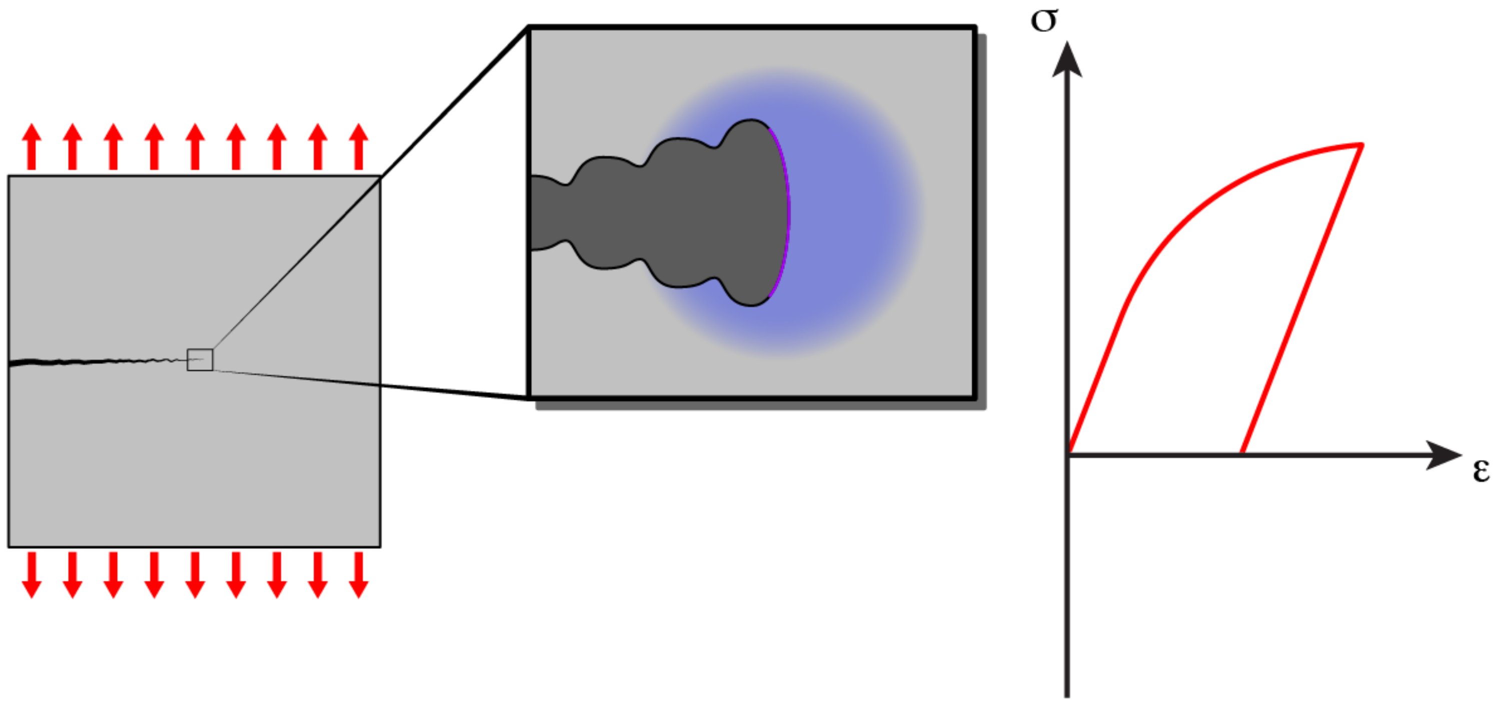

σ
ε


σ
ε



dρ
σ
ε


σ
ε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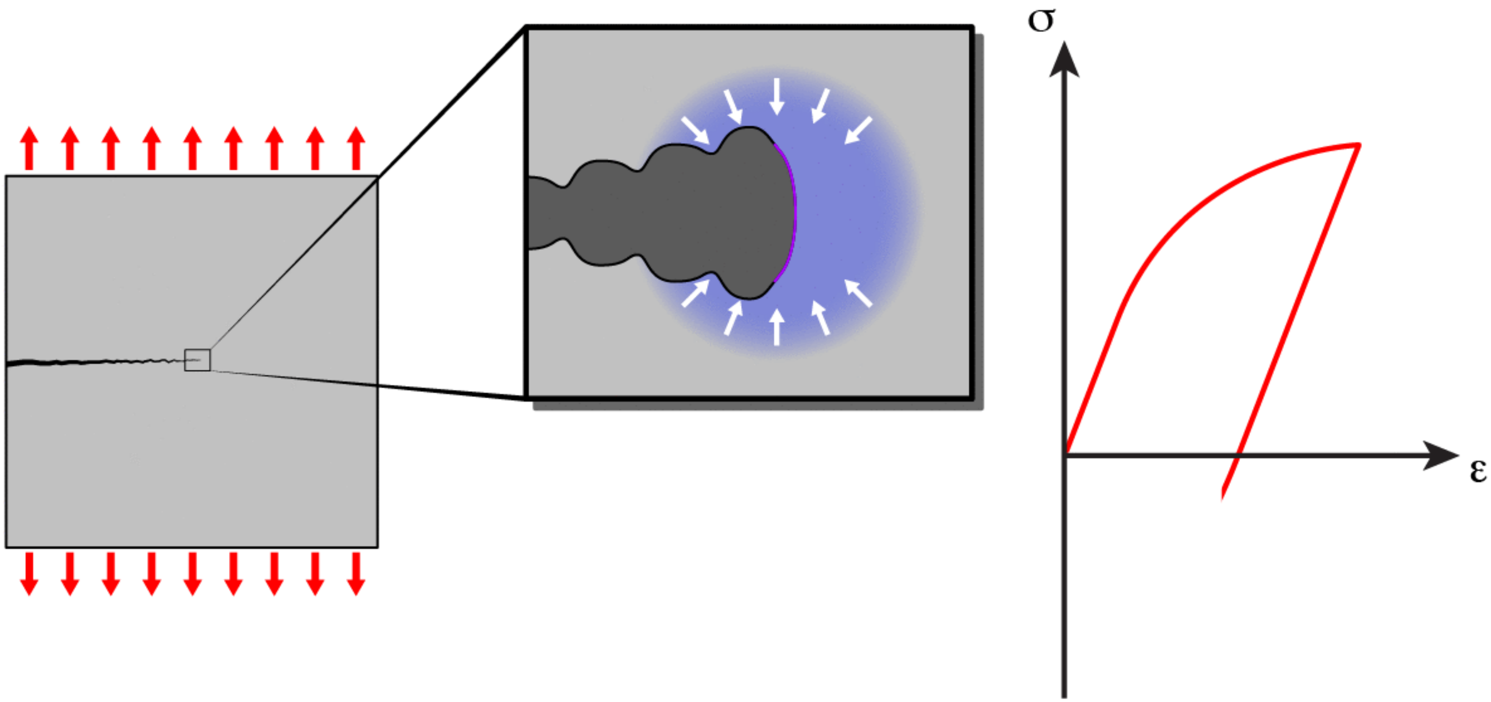
σ
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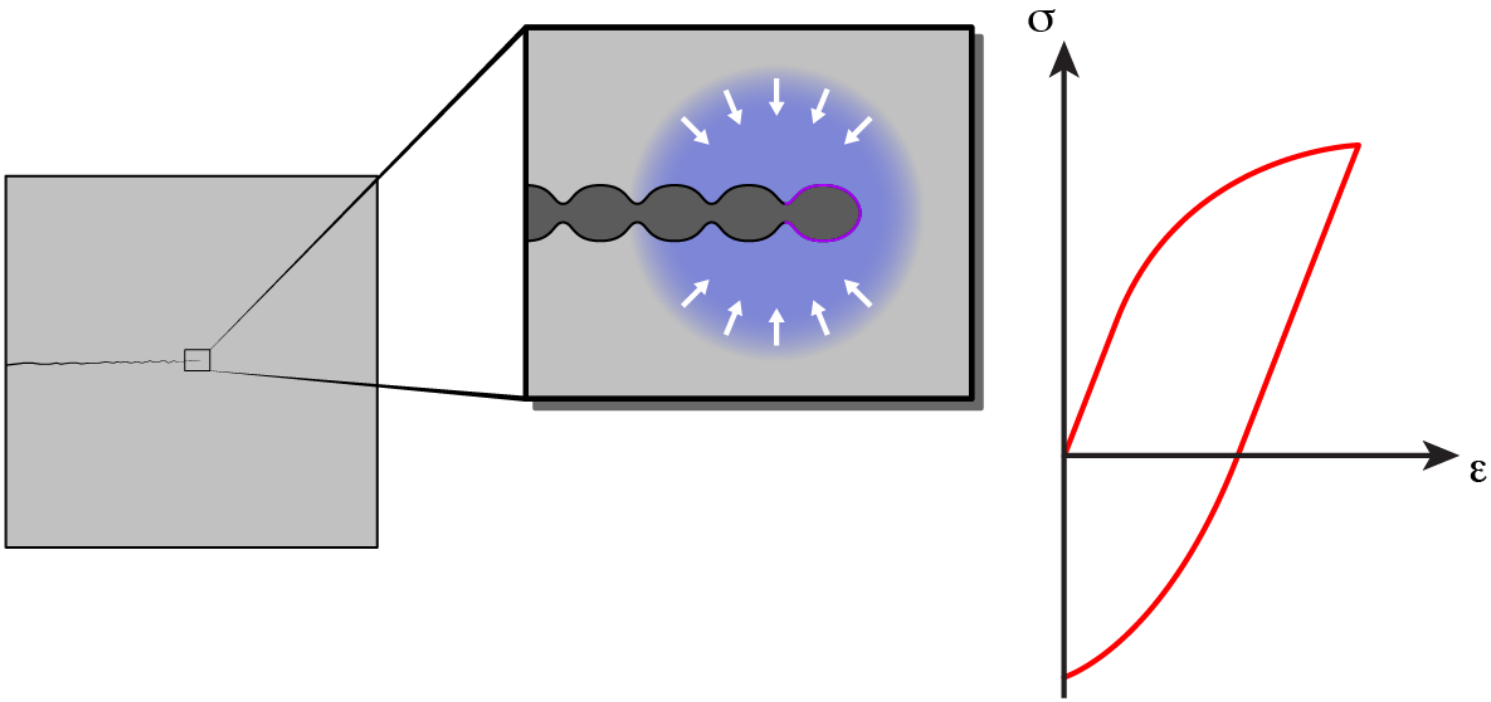
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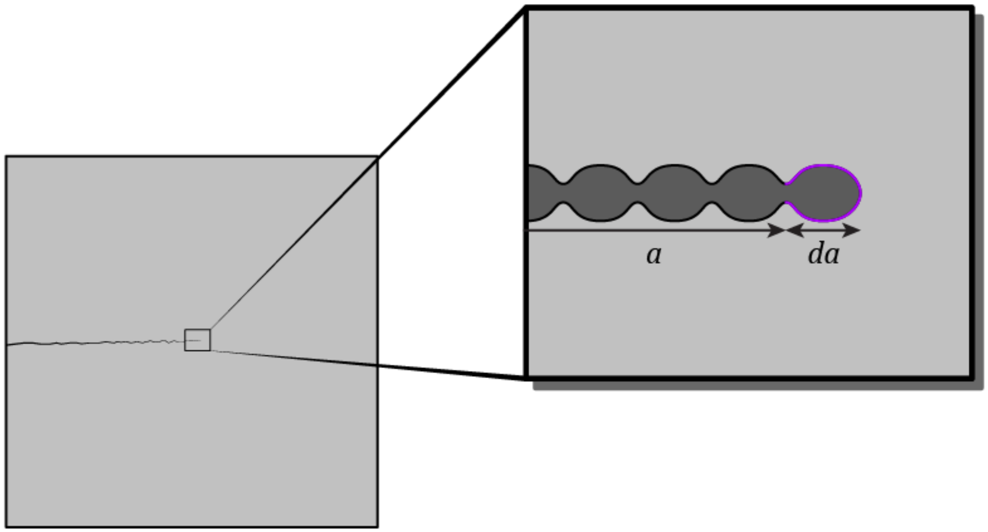
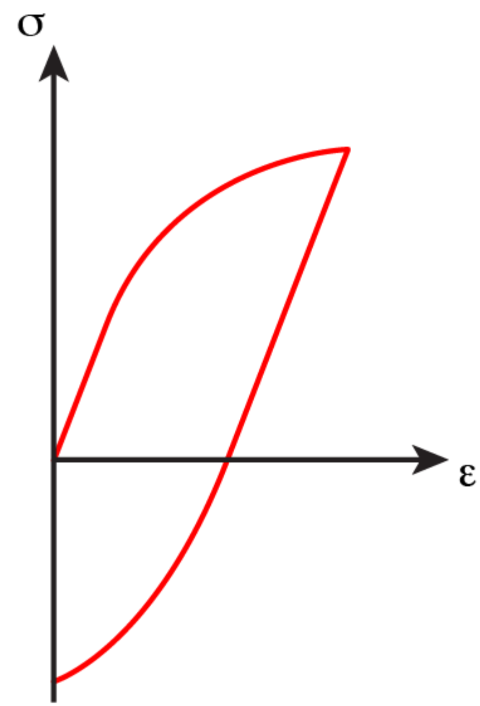
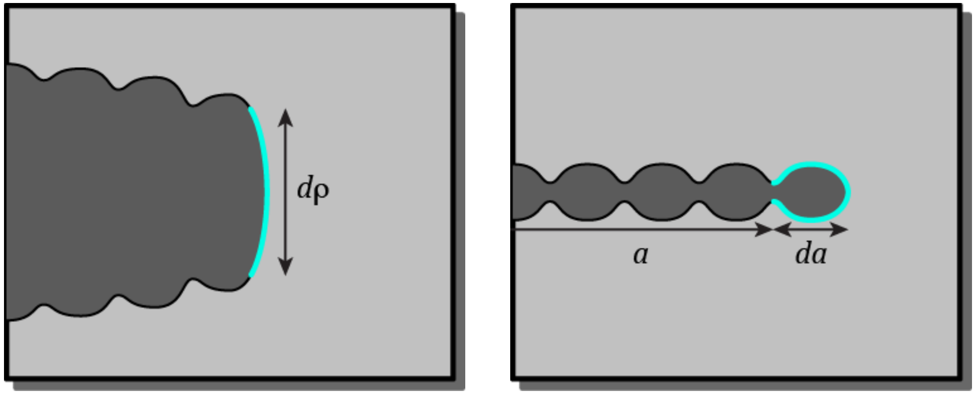

a
da


σ
ε



$$\boxed{\frac{da}{dN} = \alpha \frac{d\rho}{dN}}$$

Crack growth is governed by crack tip plasticity

Consequences

- the quantities of LEFM (KI, KII, KIII) control the behavior of the K-dominance area

- which controls the behavior of the plastic zone

- which controls crack growth by pure fatigue

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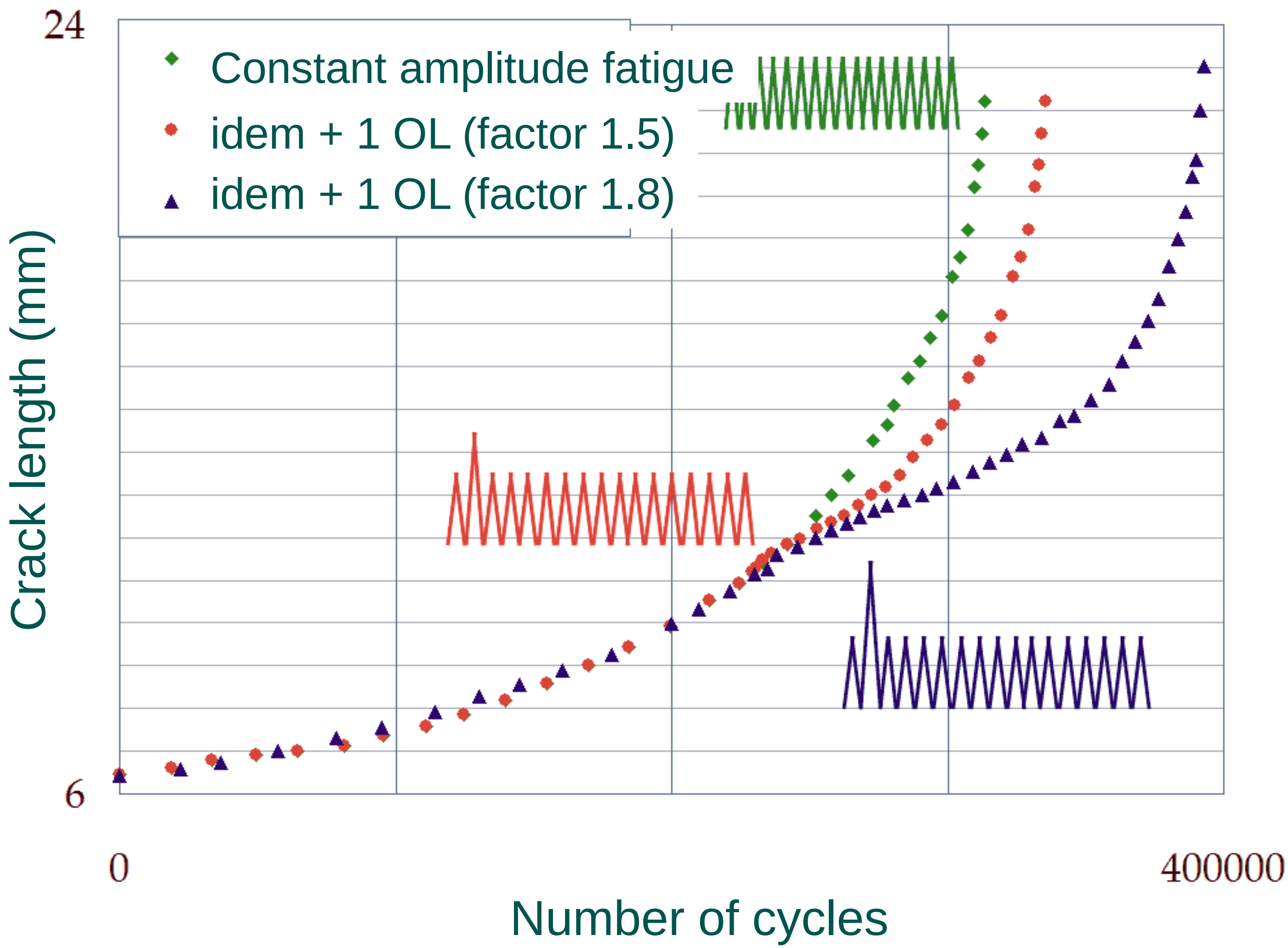
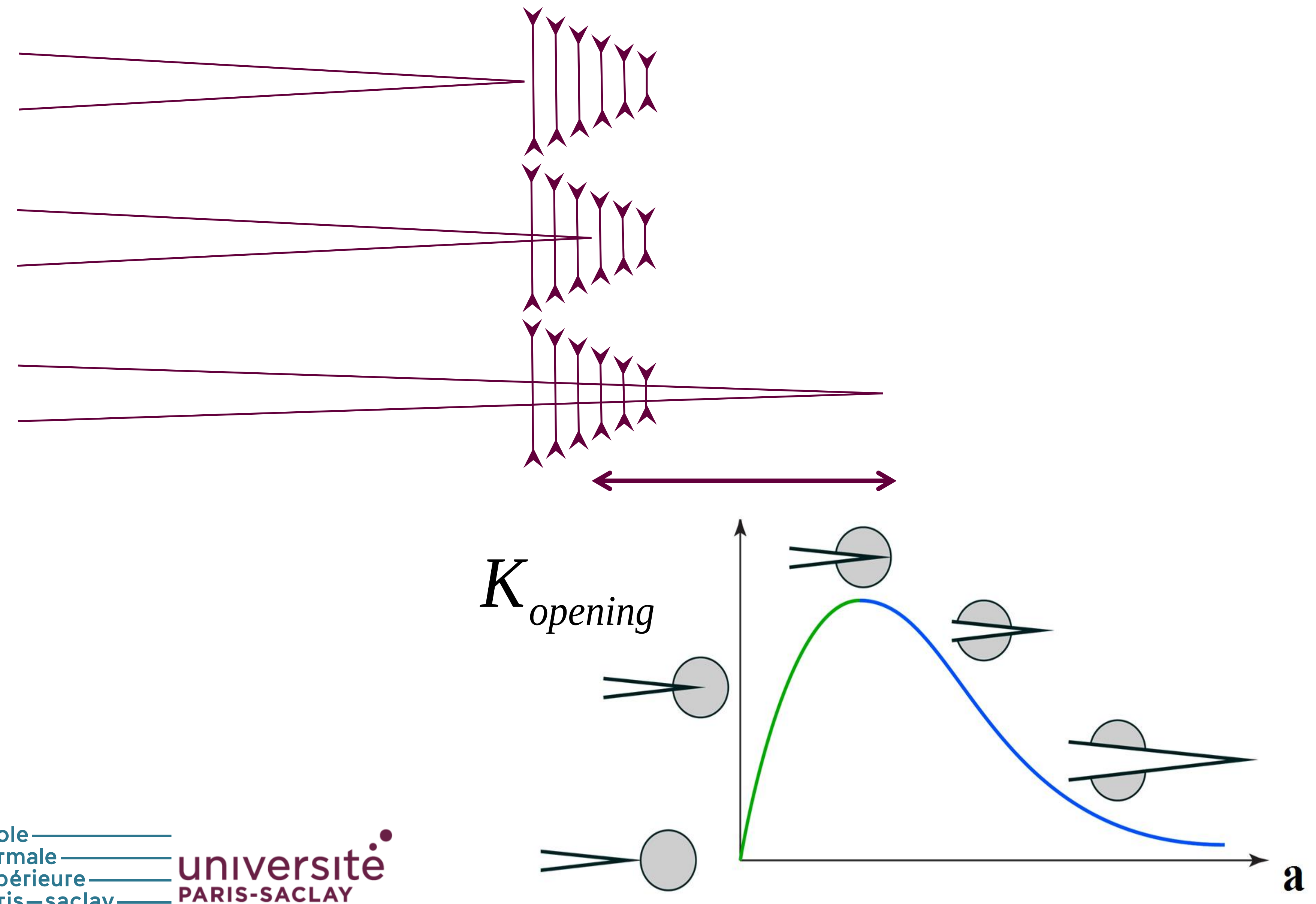
Long distance effect (overload)



CCT, 0.48% carbon steel, [Hamam et al. 2005]

Long distance effect (residual stresses)



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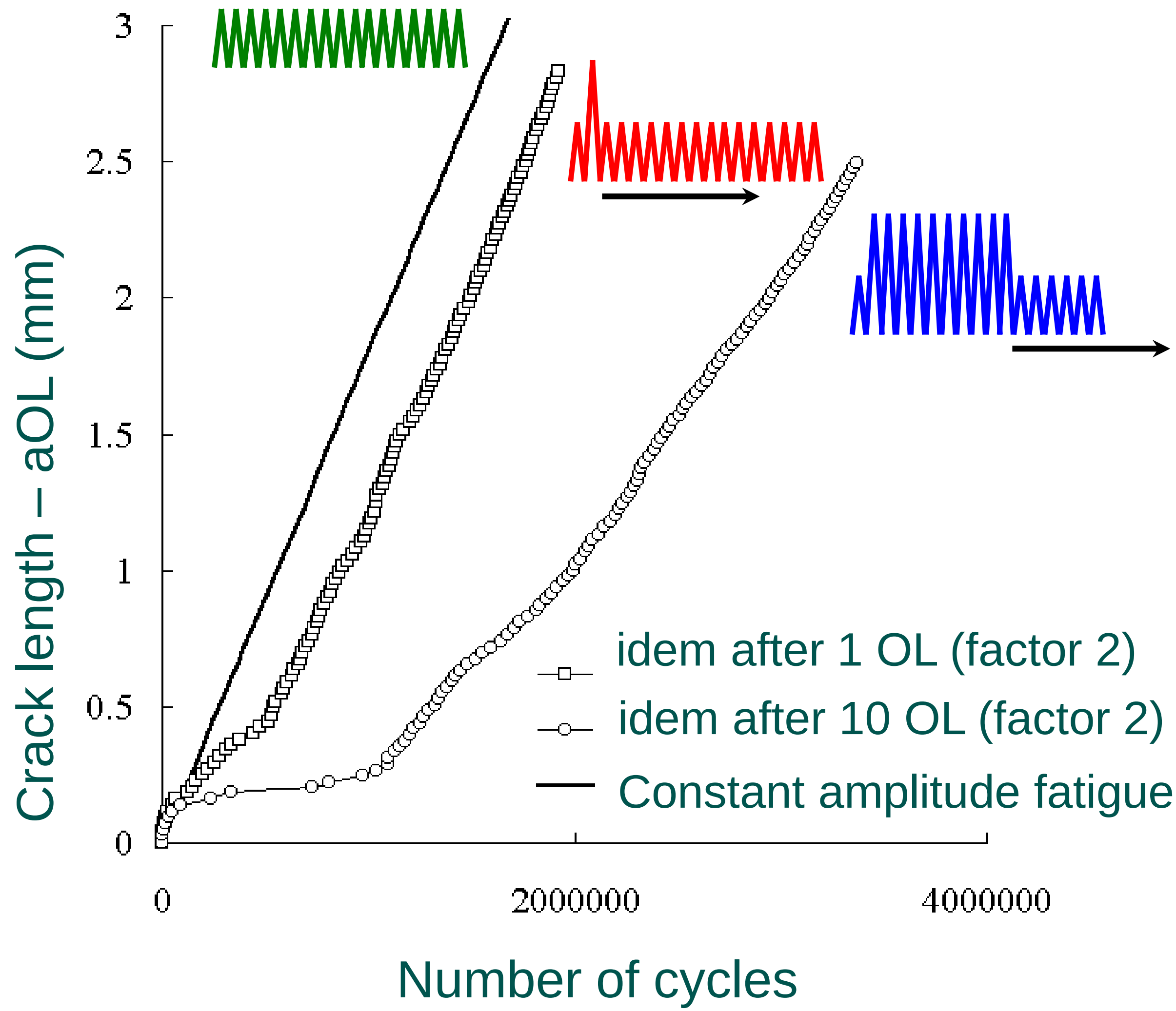
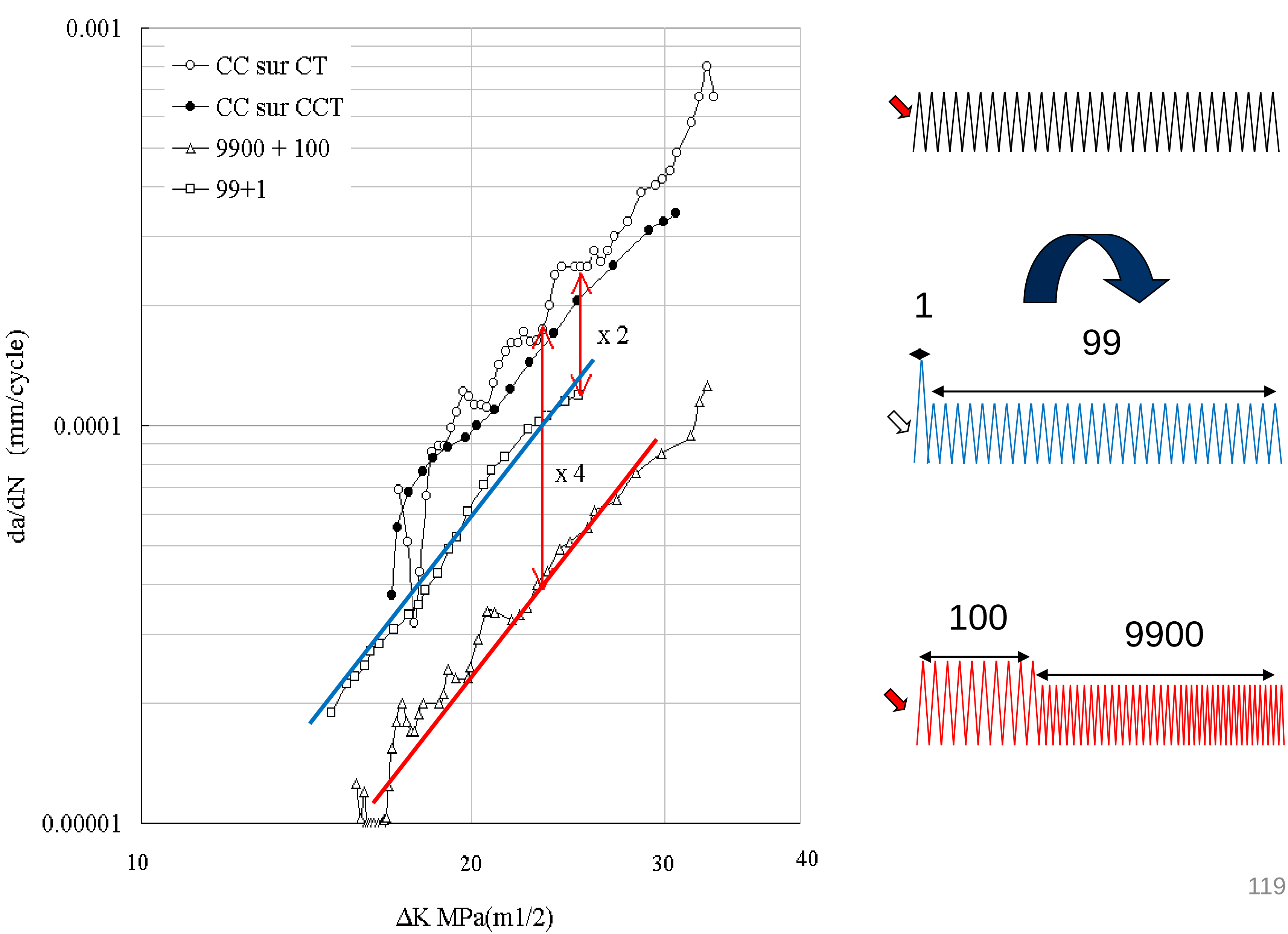
Short distance effect (repeated overloads)



CT, 316L austenitic stainless steel, [Pommier et al]

Short distance effect (block loadings)

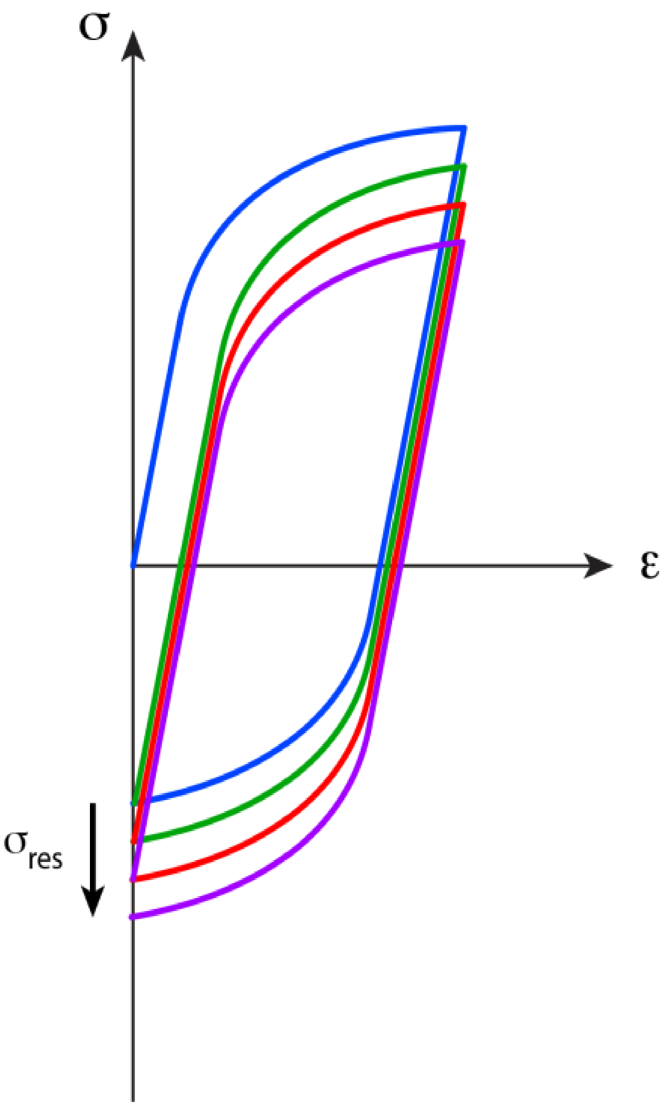


- If the plastic zone is well constrained inside the K-dominance area

- It is subjected to strain controlled conditions by the elastic bulk,

- Mean stress relaxation

- Material cyclic hardening



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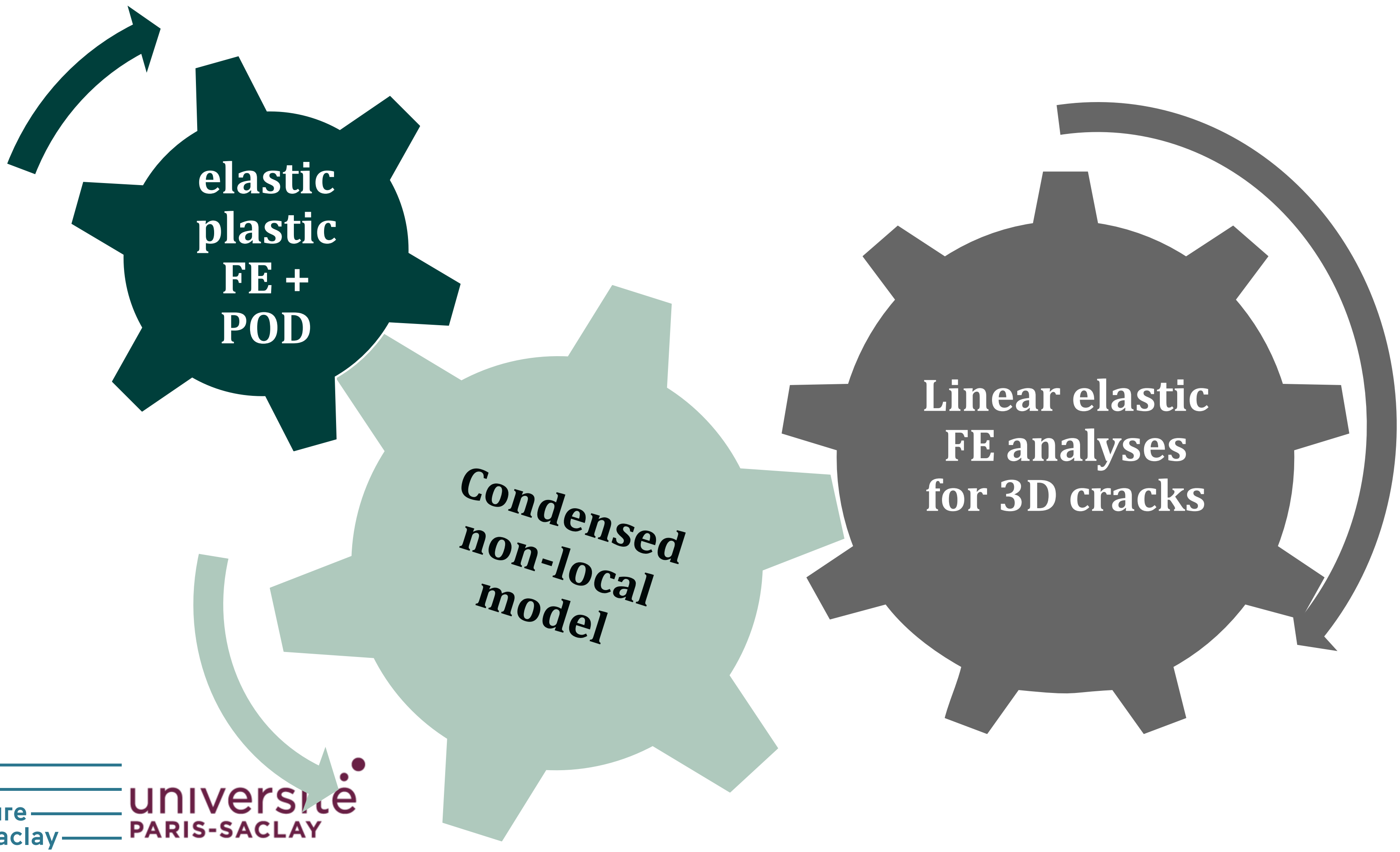
- **Issues**
  - **A very small plastic zone produces very large effects on the fatigue crack growth rate and direction**
  - **Finite element method : elastic plastic material, very fine mesh required, 3D cracks, huge number of cycles to be modelled, tricky post-treatment**
  - **Fastidious and time consuming**

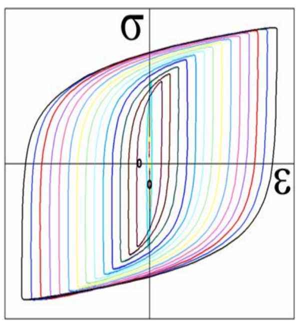
A simplified approach is needed: the elastic-plastic behaviour of the plastic zone is condensed a non-local elastic-plastic model tailored for cracks



Method

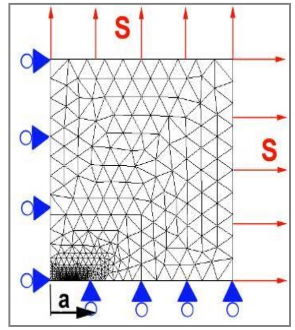
$$\frac{d\underline{\underline{\varepsilon}}}{dt} = f\left(\underline{\underline{\sigma}},\ldots\right)$$

Constitutive model LOCAL

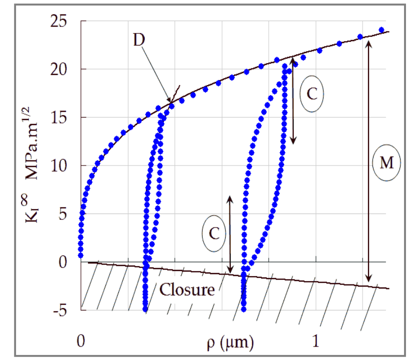


Tensile Push pull test

Expérimental input n°1

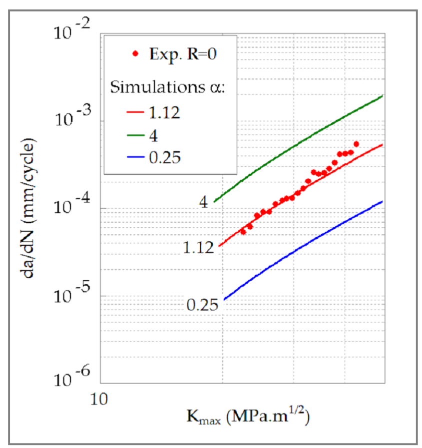


Scale transition



Generation of evolutions of ρ (CTOD) versus KI

Expérimental input n°2



Fatigue crack growth experiment

$$\frac{d\rho}{dt} = g\left(dK_I, K_I\ldots\right)$$

$$\frac{da}{dt} = \alpha\left|\frac{d\rho}{dt}\right|$$

Crack growth model, including history effects,

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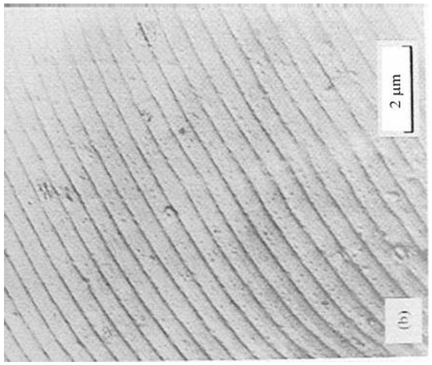
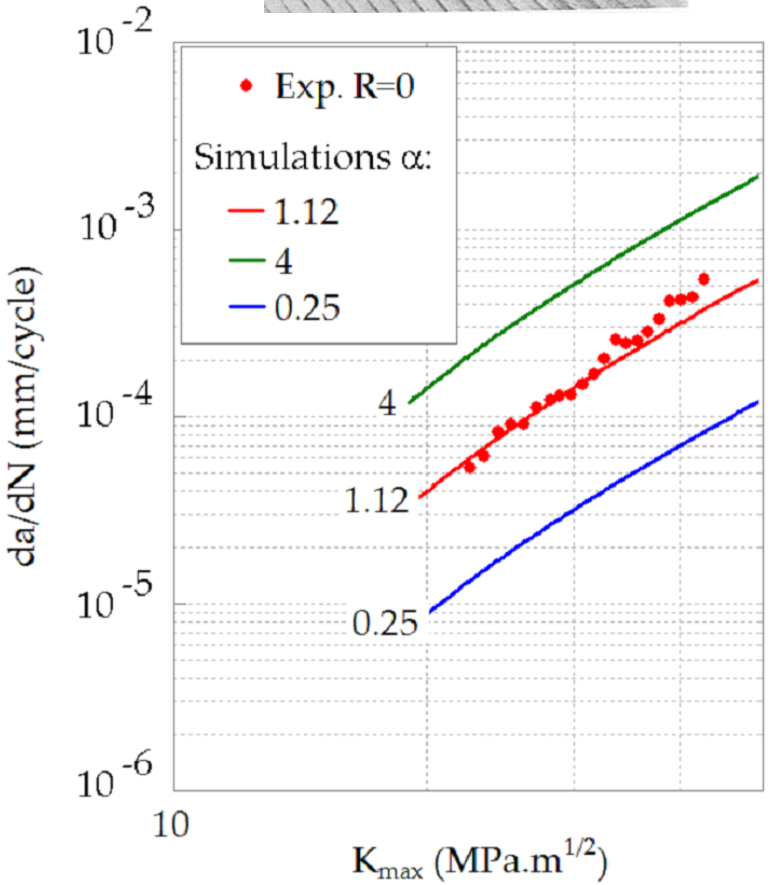
da/dt : rate of production of cracked area per unit length of the crack front

$$\frac{da}{dN} \propto \Delta CTOD$$



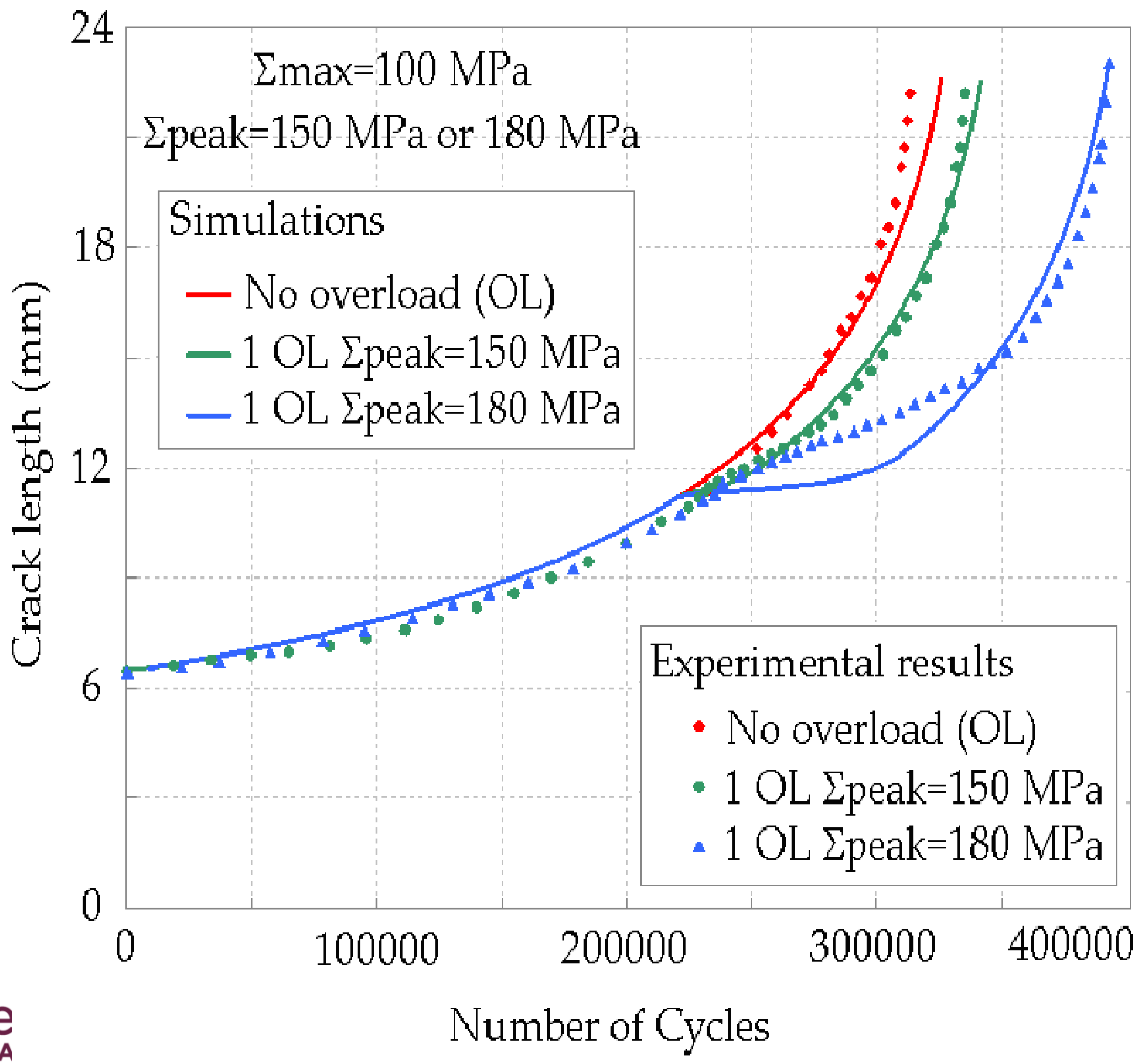
$$\approx$$

$$\frac{da}{dt} = \alpha \left| \frac{d\rho}{dt} \right|$$

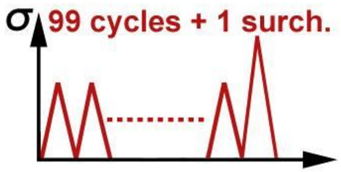


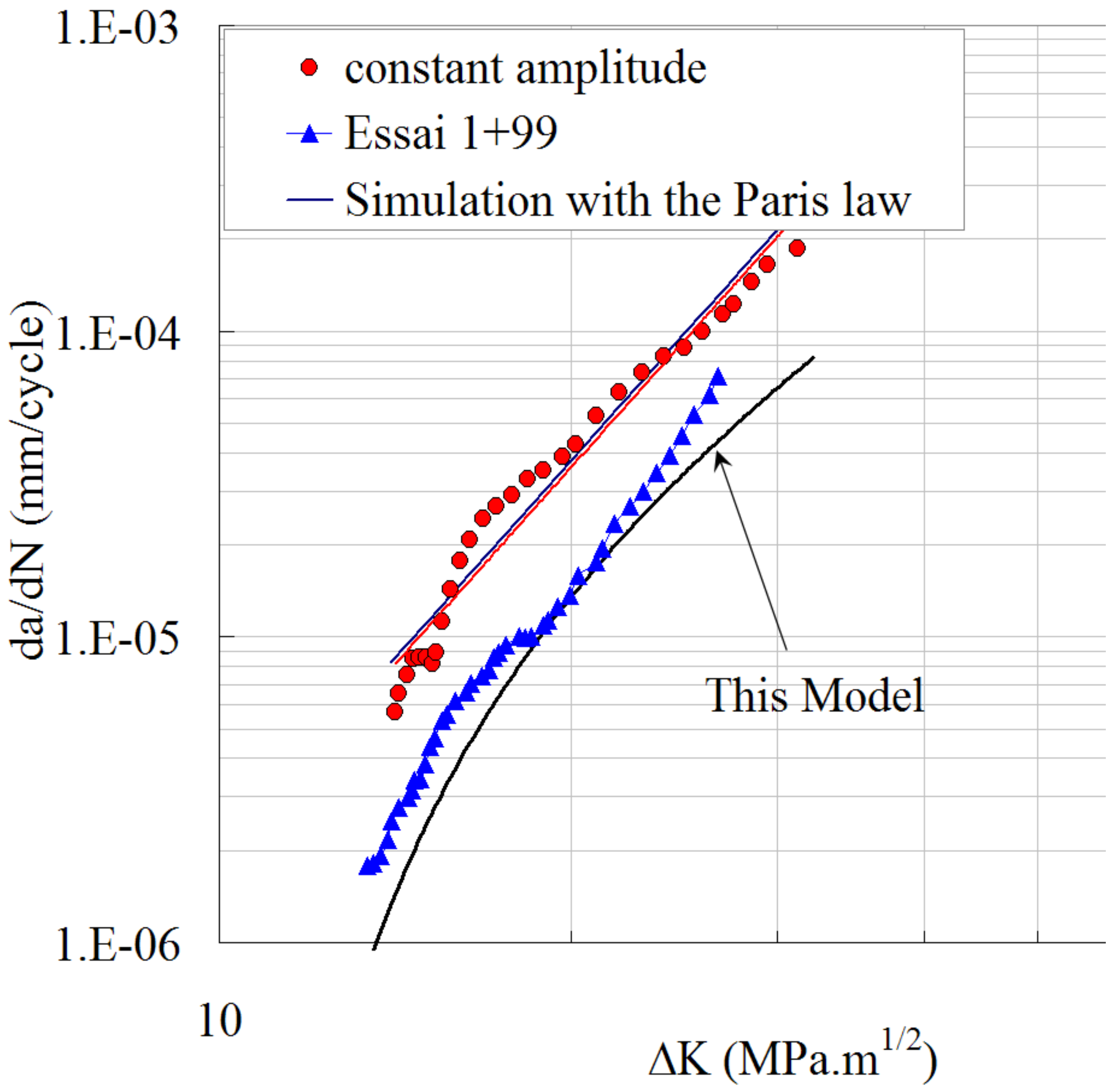
Adjust the coefficient a using one constant amplitude fatigue crack growth experiment

Single overload : long range retardation

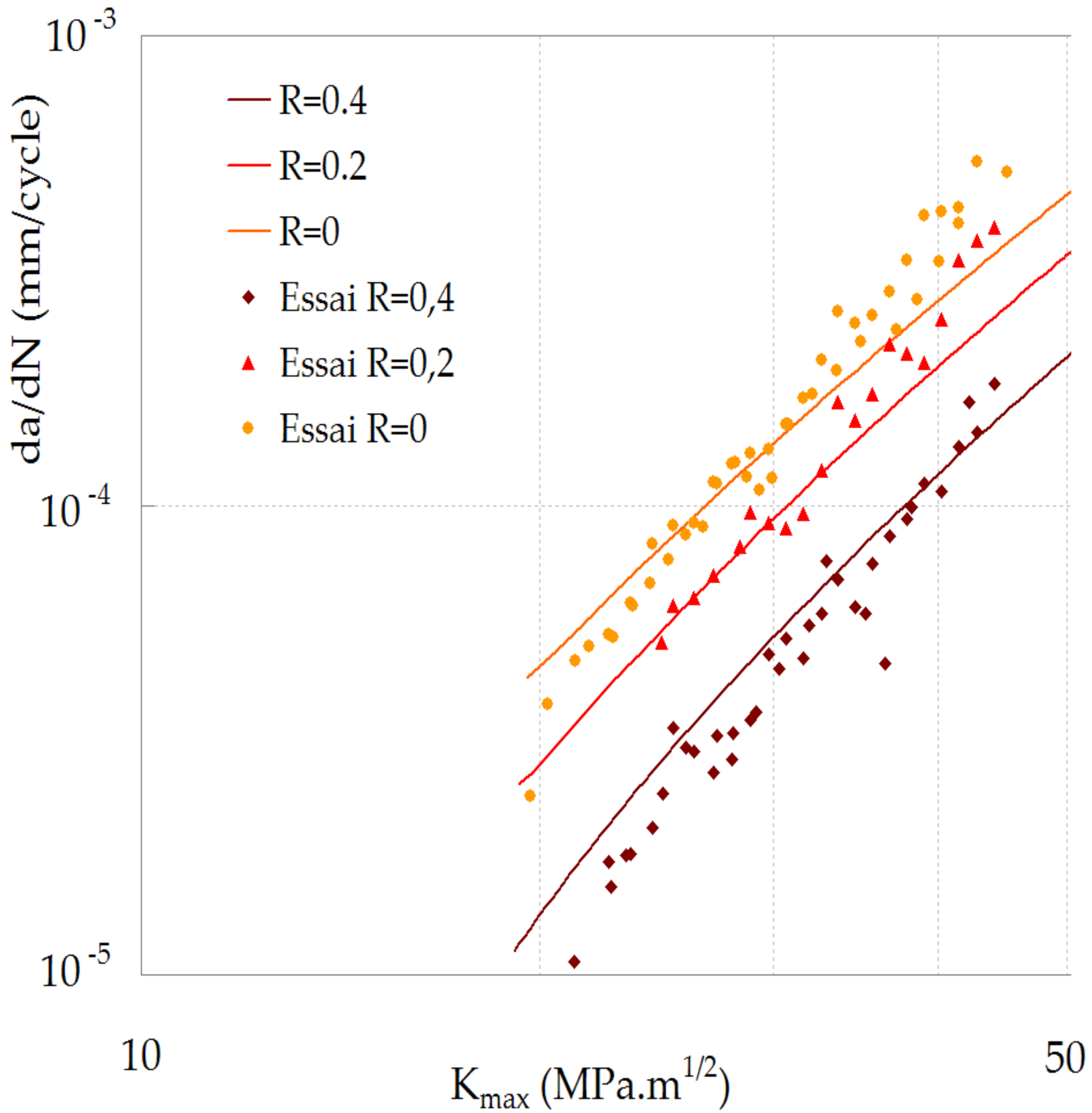


Block loading : short range retardation

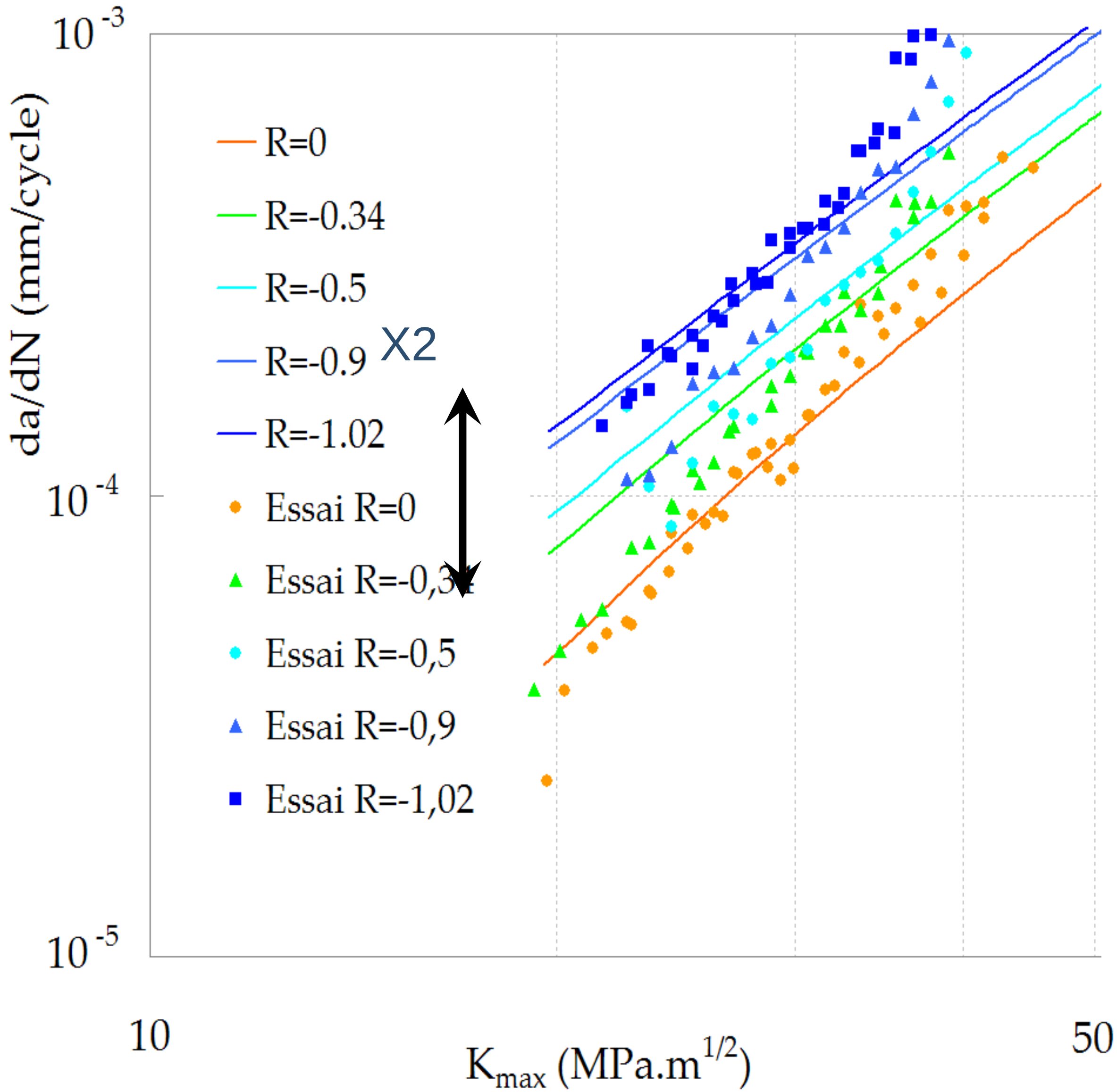




Stress ratio (mean stress) effect (R>0)



Stress ratio (mean stress) effect (R<0)



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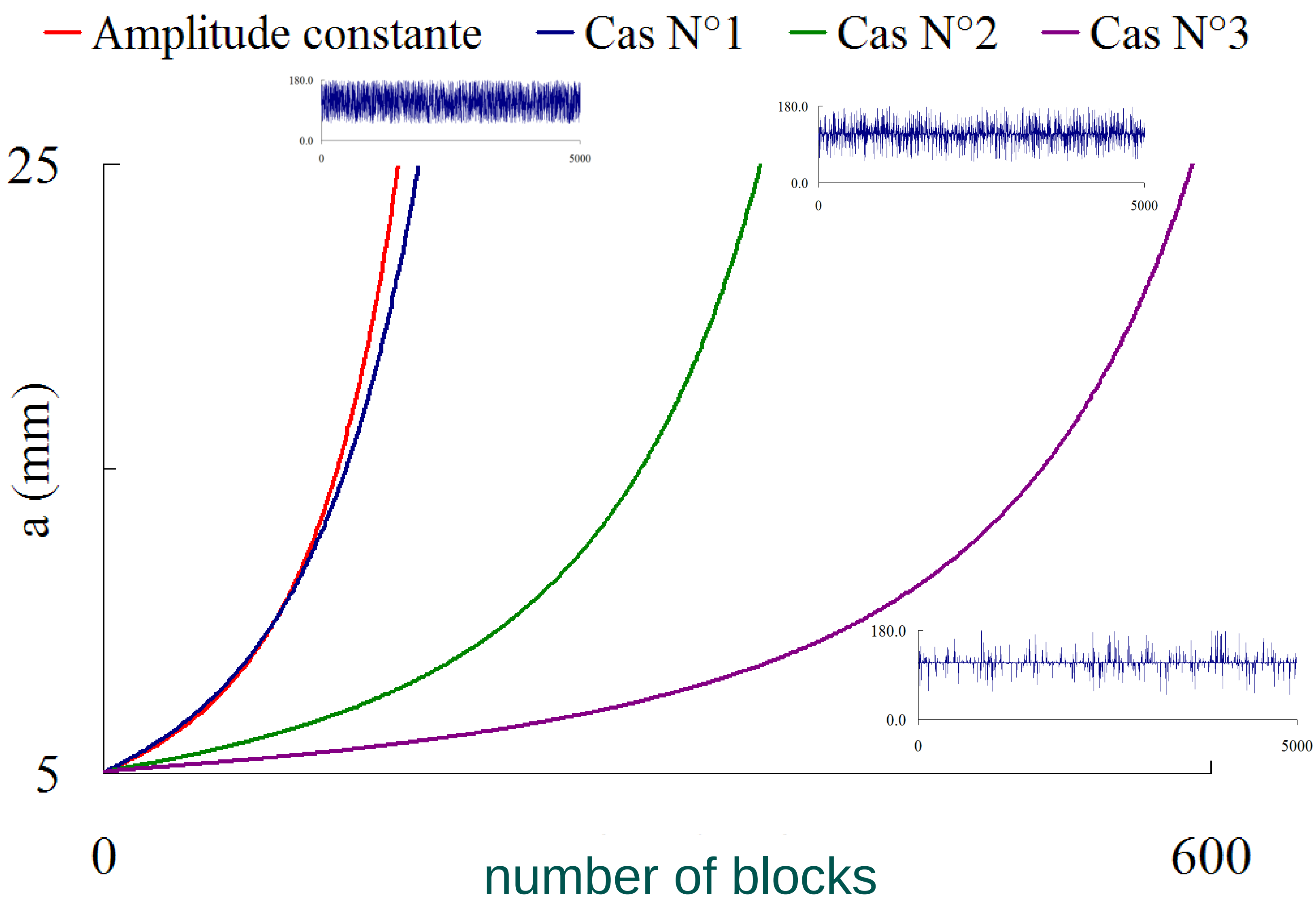
Random loading simulations



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Growth criteria in mixed mode conditions ?

$$\frac{da}{dN} = C\Delta K_{eq}^{m}$$

$$\Delta K_{eq} = \left(\Delta K_{I}{}^{n} + \beta \Delta K_{II}{}^{n} + \gamma \Delta K_{III}{}^{n}\right)^{1/n}$$

Same values of K_{max}, K_{min}, ΔK for each mode

Fatigue crack growth experiments

Crack growth rate

Crack path

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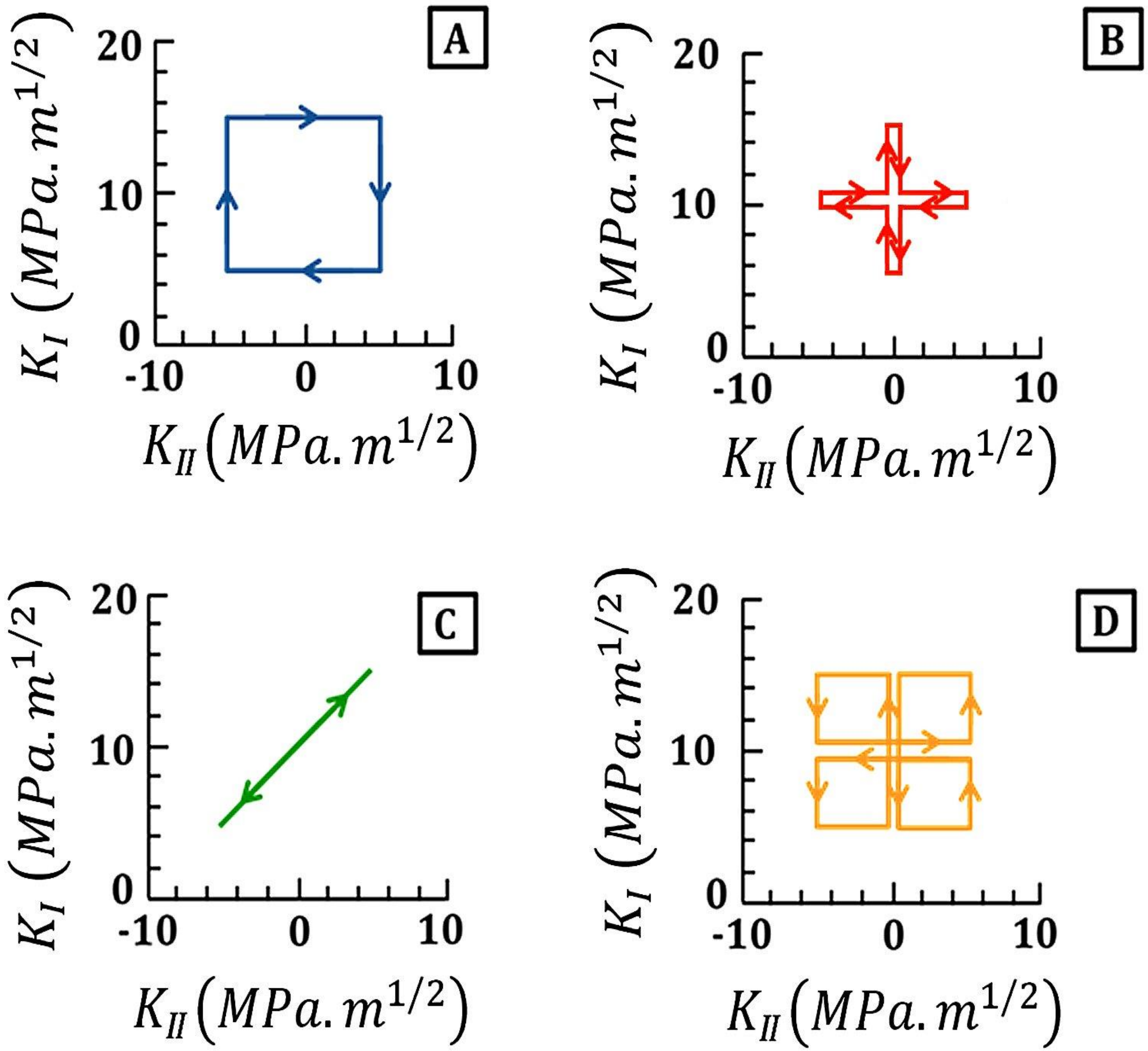
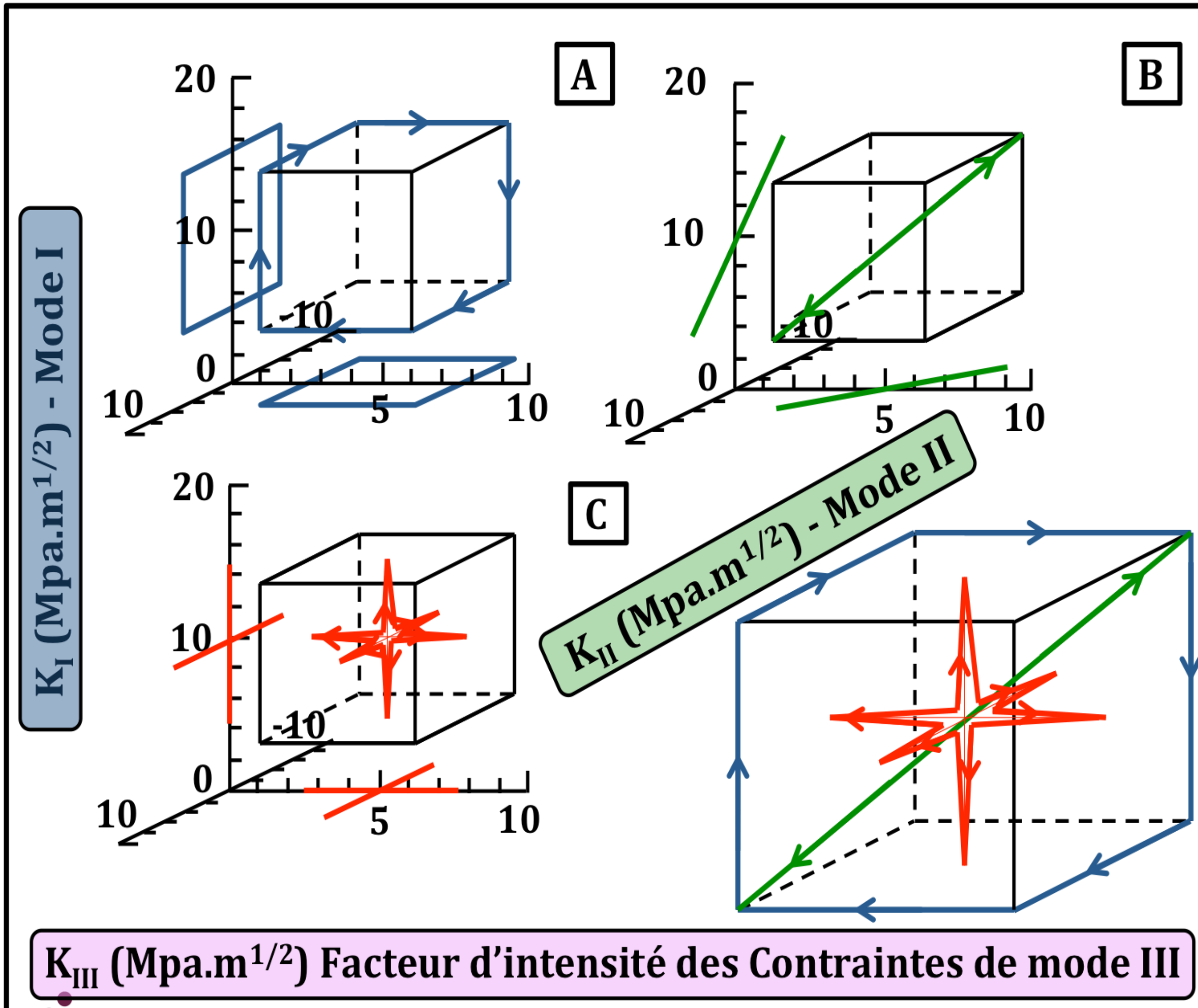
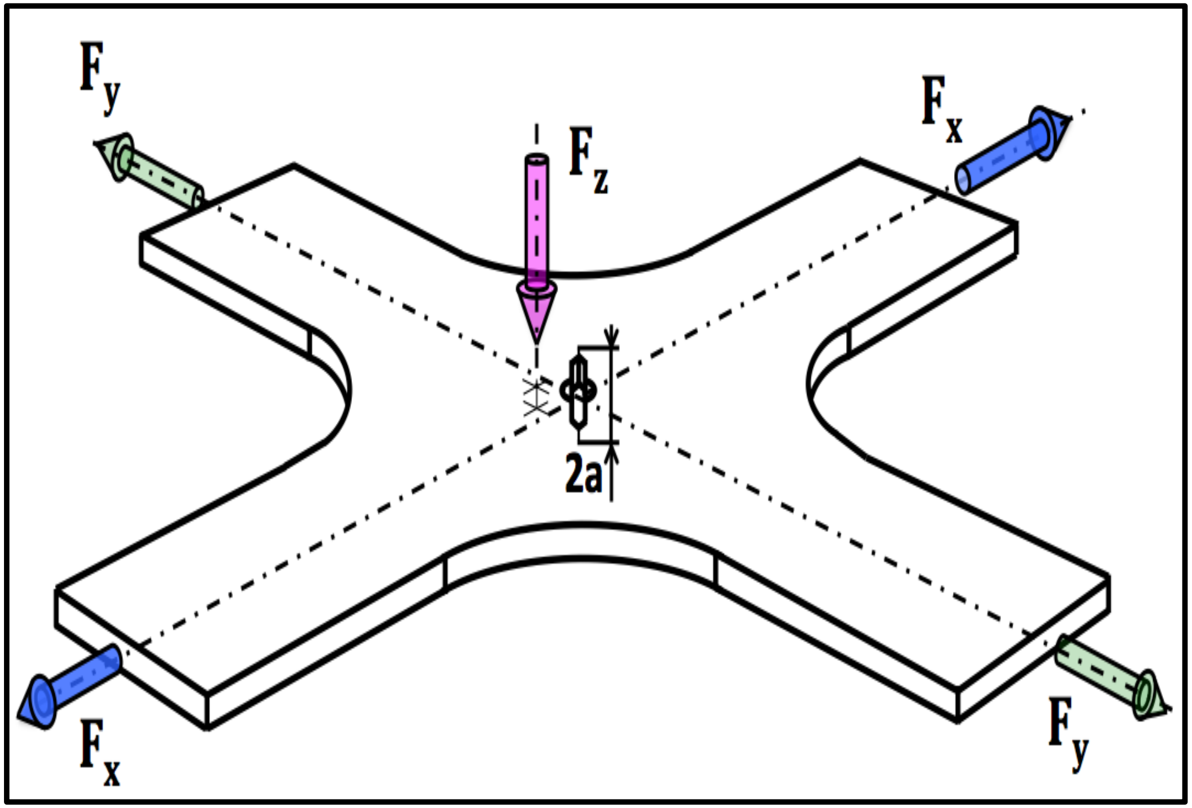
Load paths in mixed mode I+II



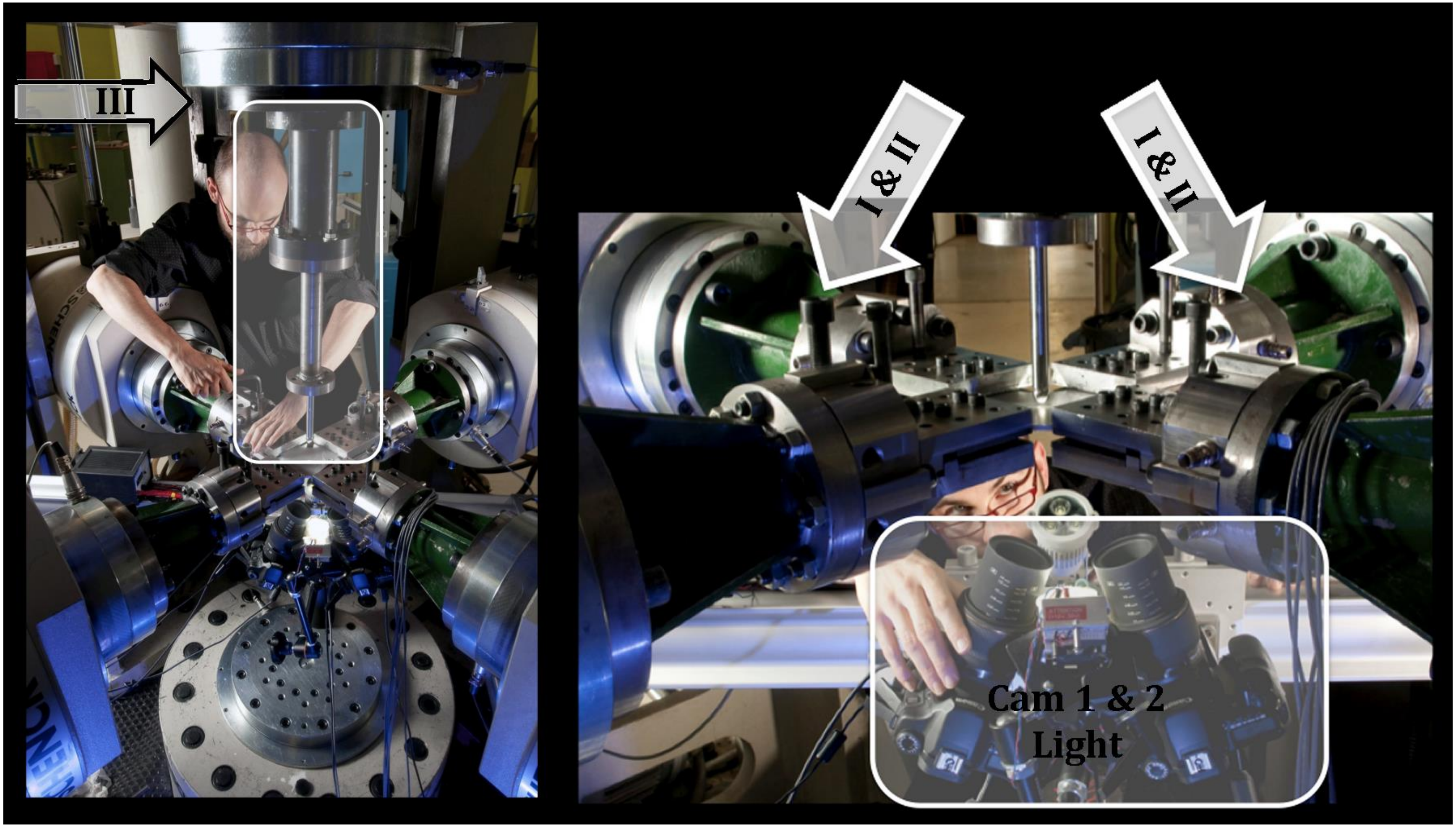
Load paths in mixed mode I+II+III





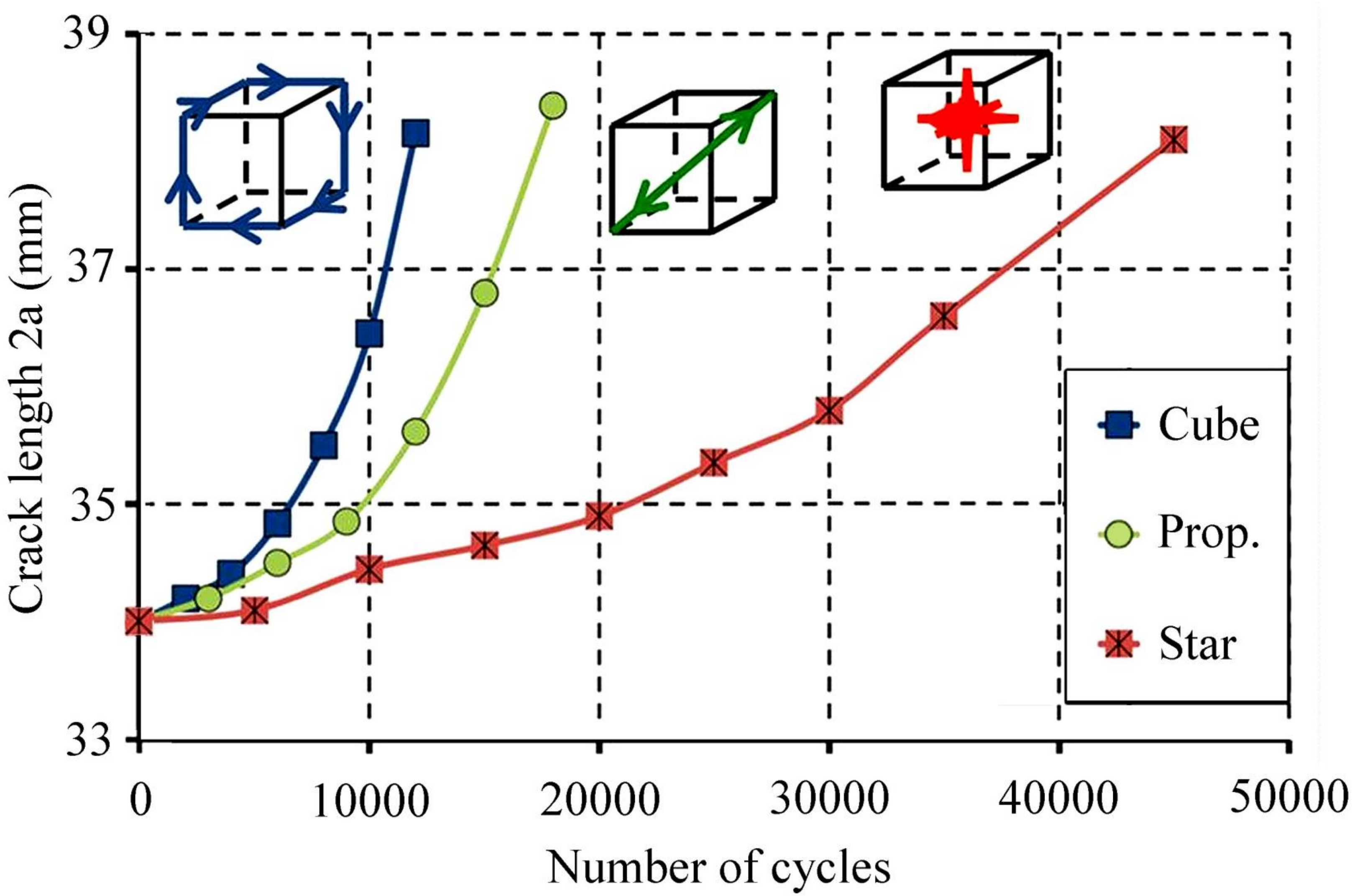
$$
\begin{pmatrix} K_I^\infty \\ K_{II}^\infty \\ K_{III}^\infty \end{pmatrix} = \begin{pmatrix} f_I(2a) & f_I(2a) & 0 \\ f_{II}(2a) & -f_{II}(2a) & 0 \\ 0 & 0 & f_{III}(2a) \end{pmatrix} \begin{pmatrix} F_X \\ F_Y \\ F_Z \end{pmatrix}
$$

Experimental protocol



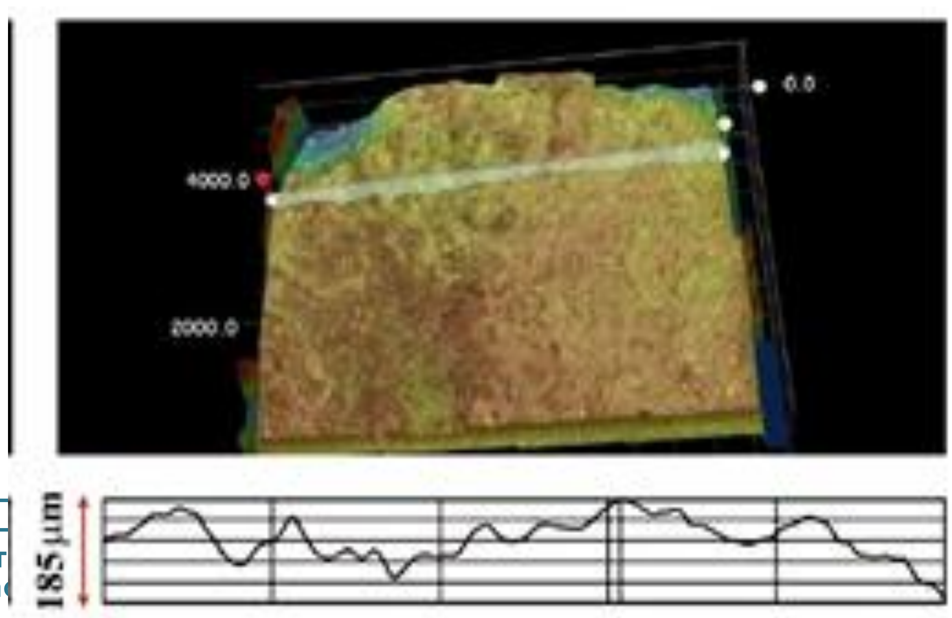
6 actuators hydraulic testing machine - ASTREE

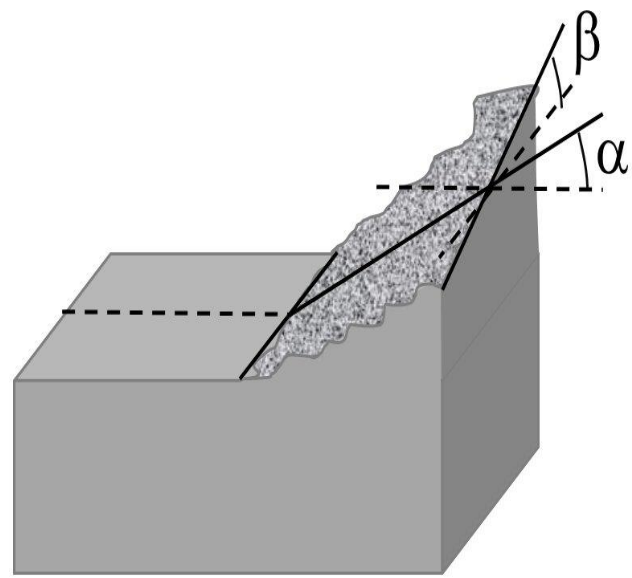
Fatigue crack growth in mixed mode I+II+III



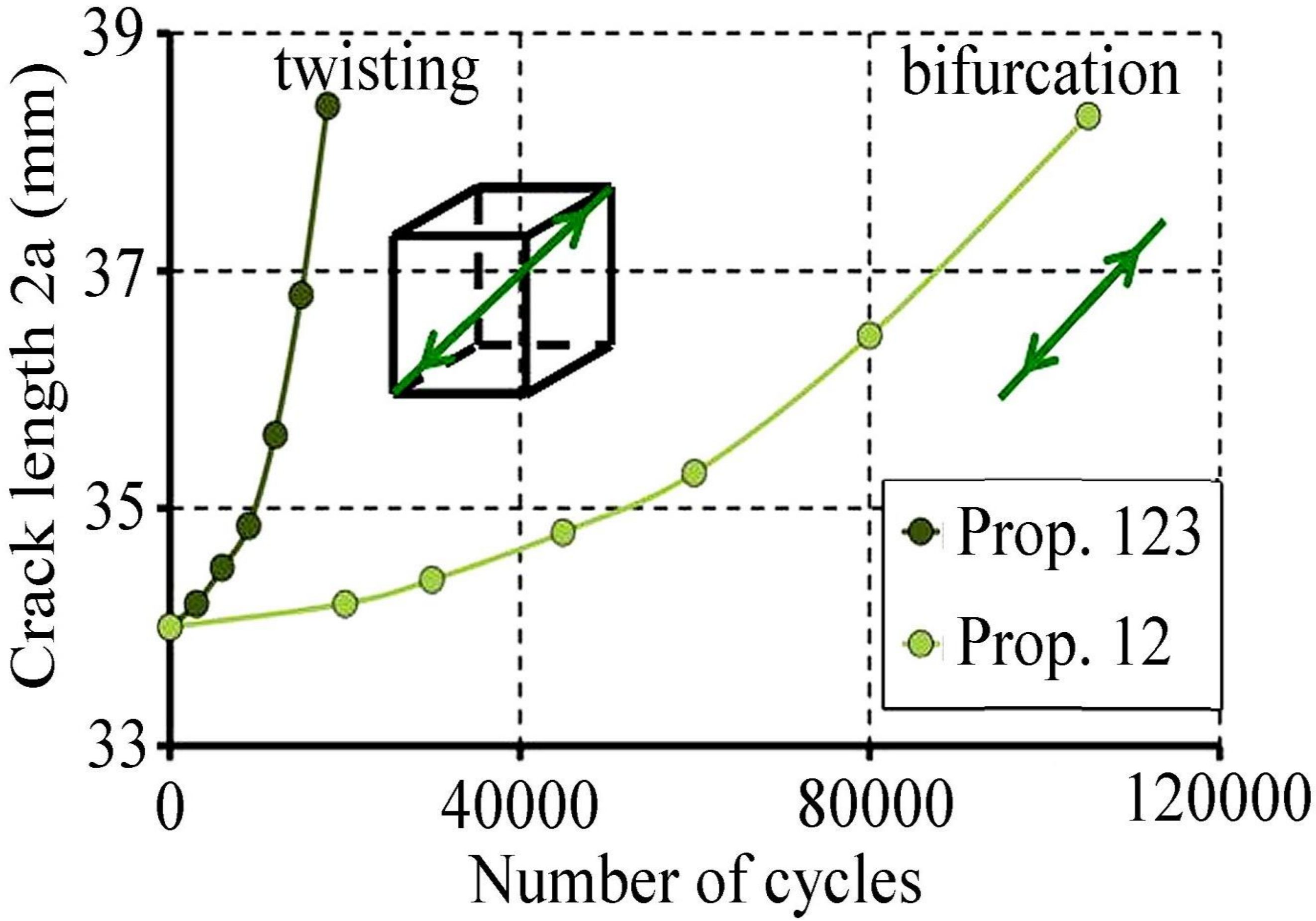
Crack path – mode I+II+III

Load path	"Prop."	" Cube "	" Star "
Bifurcation angle α	-10°	none	40°
Twist angle β	60°	15°	10°





Mode III contribution



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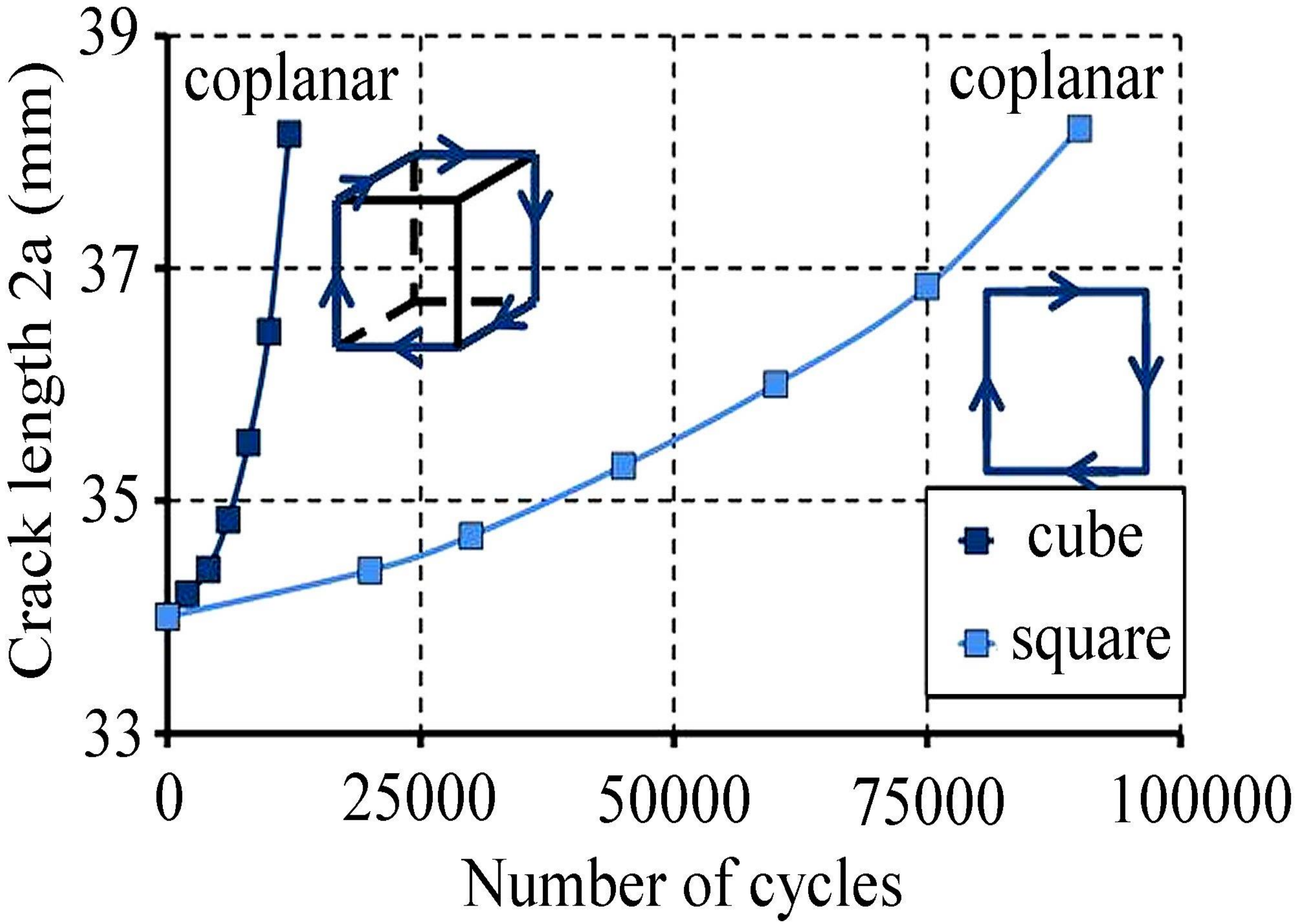
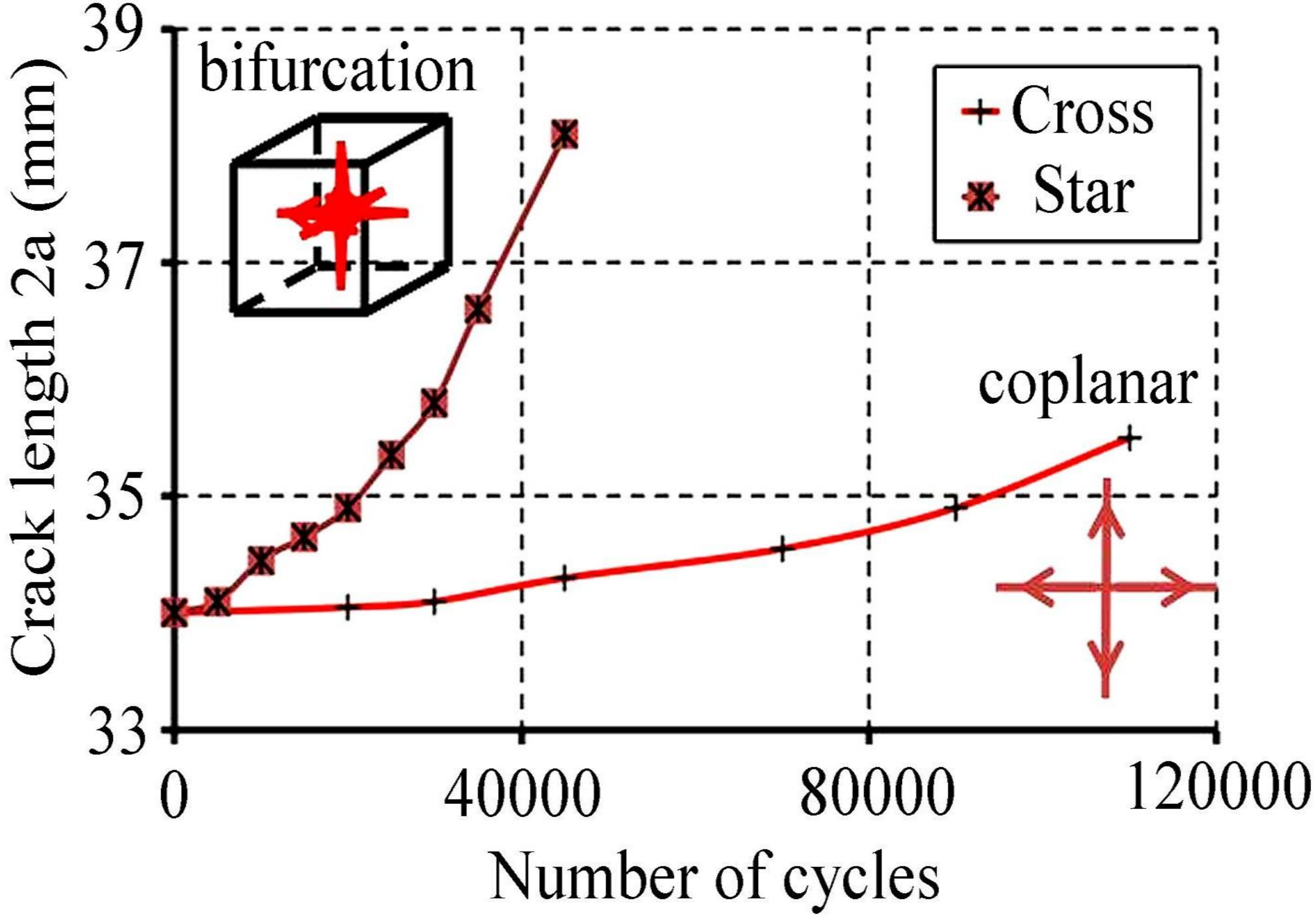
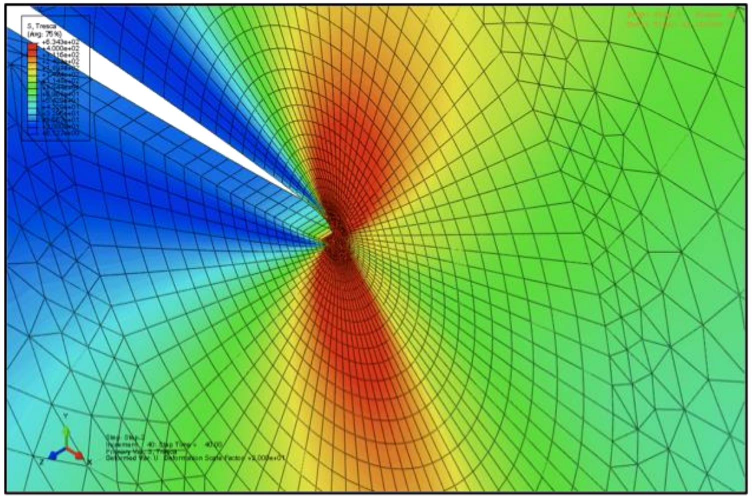
Mode III contribution

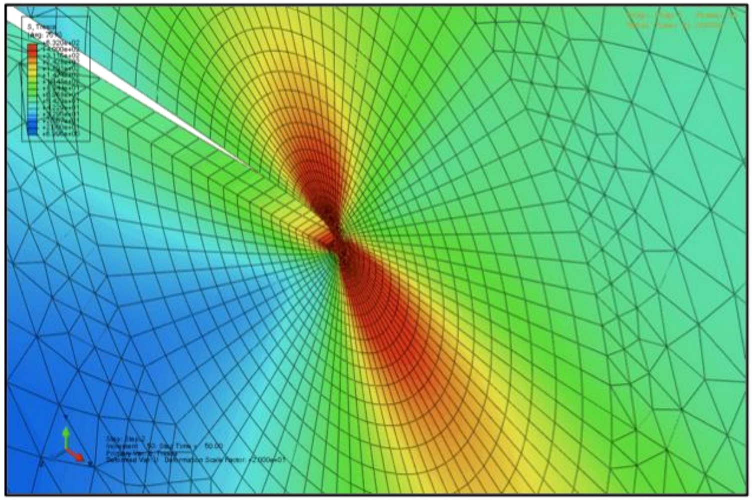


Mode III contribution



FE model and boundary conditions





Periodic BC along the two faces normal to the crack front

Prescribed displacements based on LEFM stress intensity factors

$K_I^{\infty} u_{bc\_nom}^{I}, K_{II}^{\infty} u_{bc\_nom}^{II}, K_{III}^{\infty} u_{bc\_nom}^{III}$

Elastic plastic material constitutive behaviour (kinematic and isotropic hardening identified experiments)

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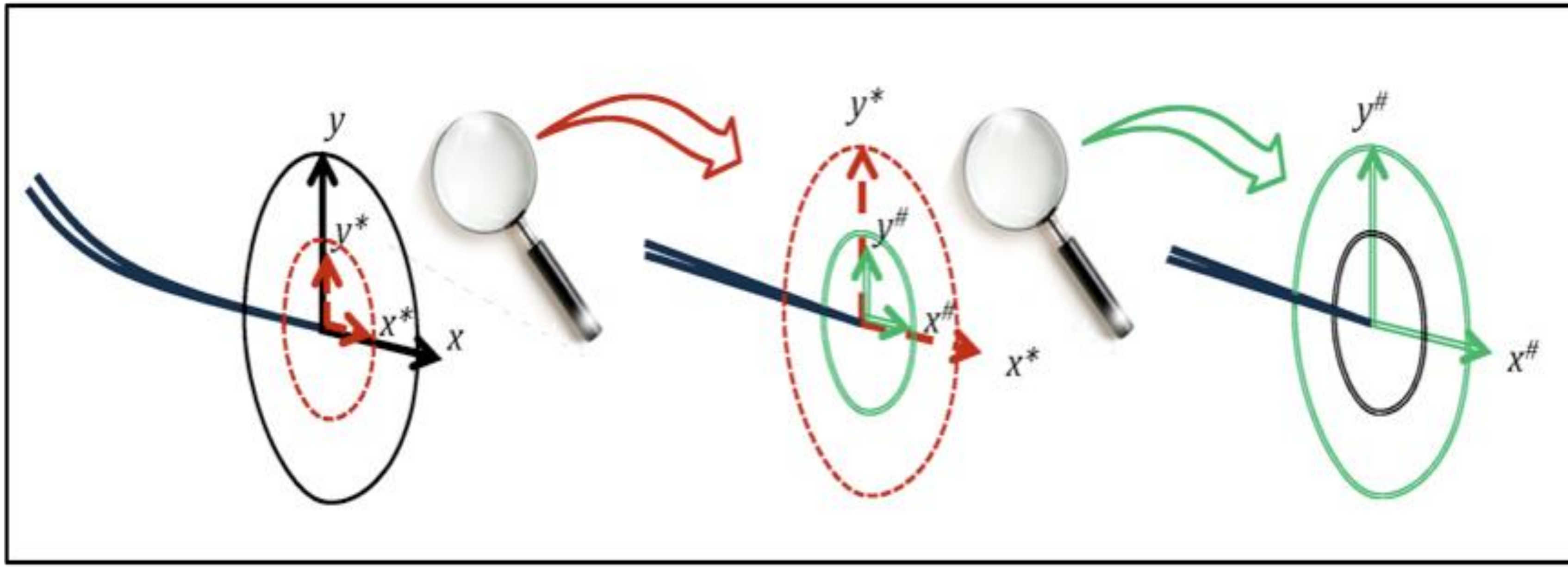

Crack : locally self similar geometry → locally self similar solution $f(\alpha r)=k(\alpha)f(r)$

Small scale yielding $f(r) \underset{r\to\infty}{\Longrightarrow} 0$

$$f_i(r) = f_i(0)e^{-\frac{r}{p}}$$

Velocity field : $f(r=0)$ finite

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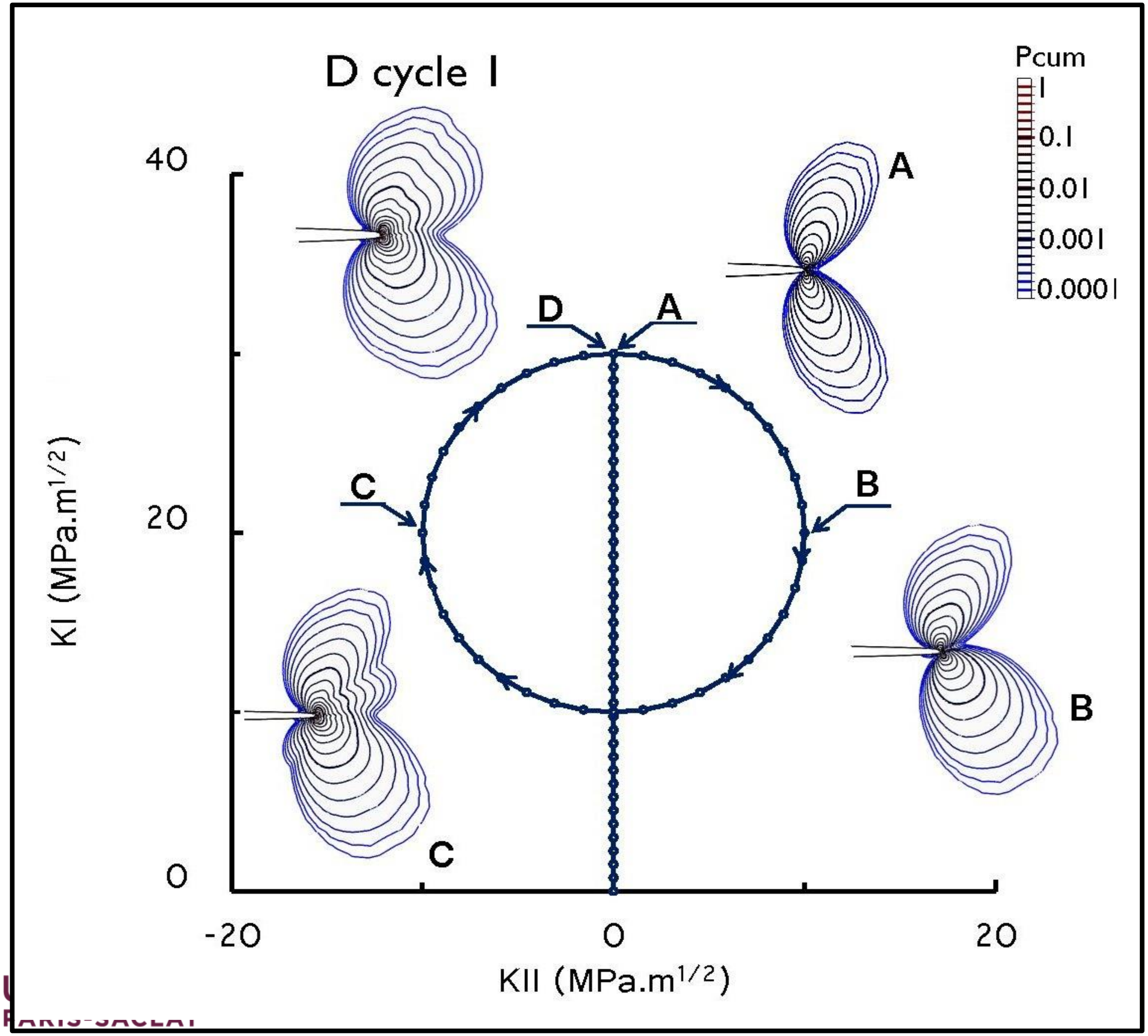
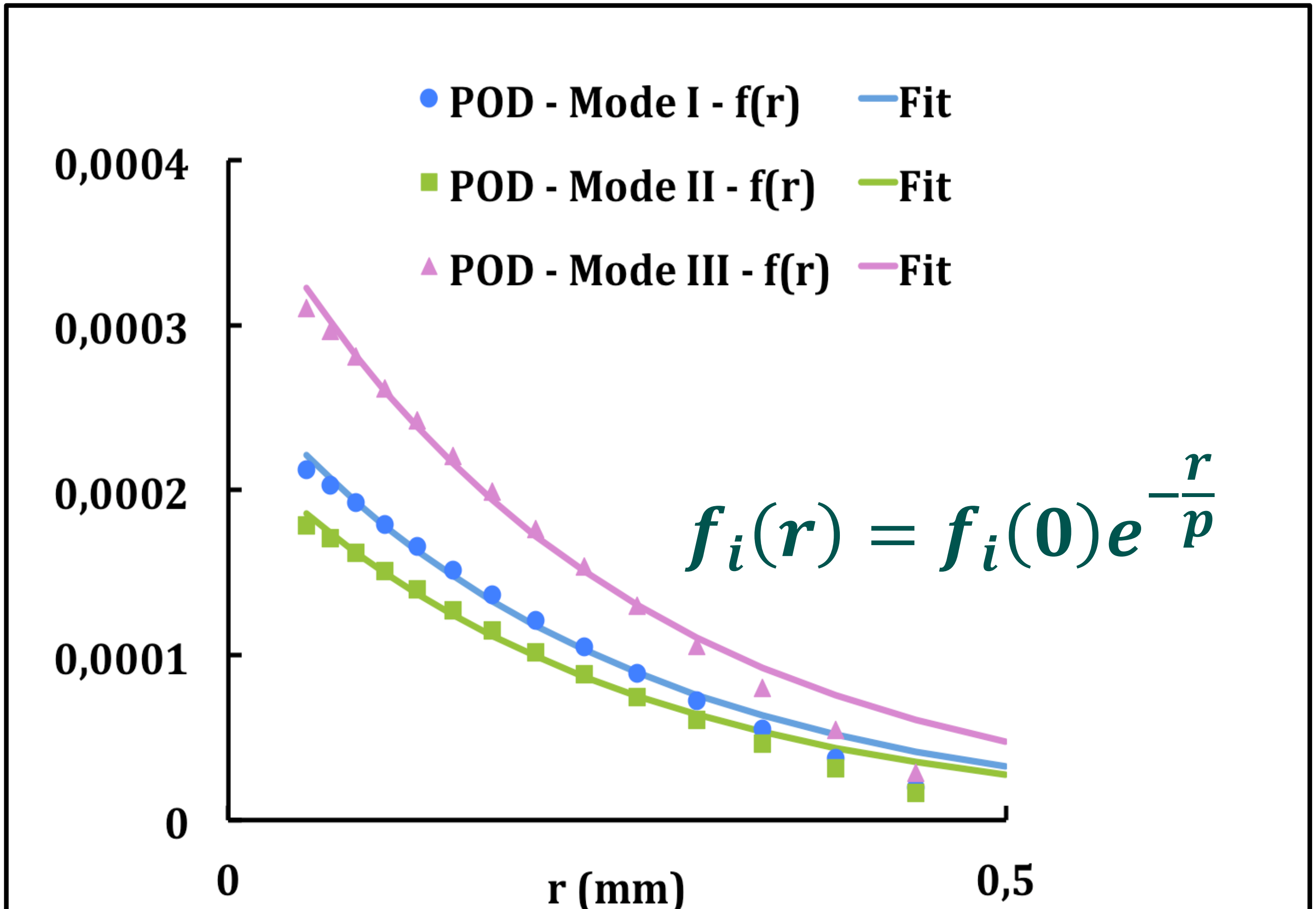
Cumulated equivalent plastic strain



radial distribution

$$POD2 \rightarrow \underline{u}_i^c(P) \approx \mathrm{f}(r)\ \underline{g}_i^c(\theta)$$



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POD based post treatment

$\underline{u}_i^e(P)$ Solution of an elastic FE analyses with $\boldsymbol{K_i^\infty}$=1MPa.m$^{1/2}$ for each mode

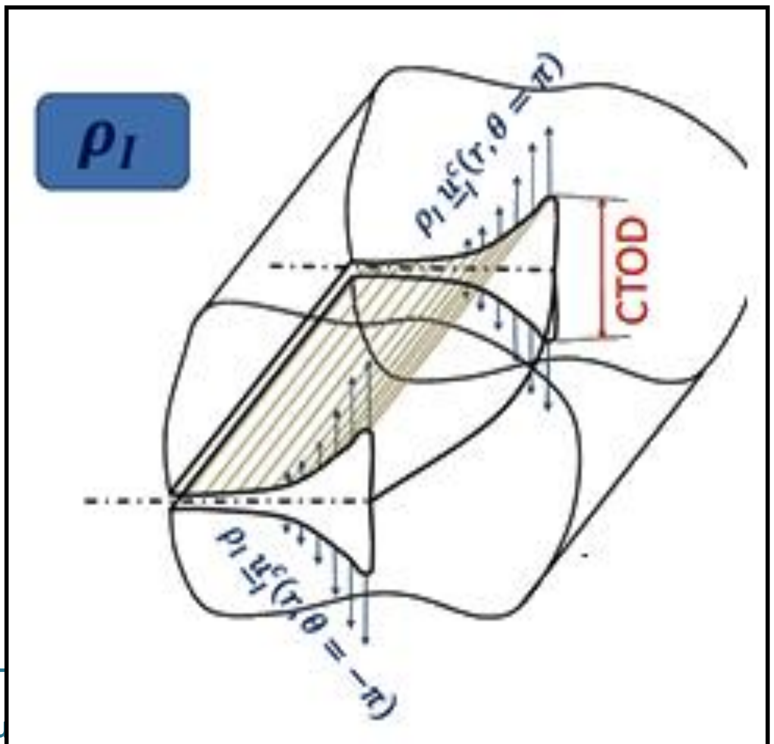
$$\underline{\boldsymbol{v}}_{\boldsymbol{i}}^{\boldsymbol{e}}(\boldsymbol{P},\boldsymbol{t}) = \dot{\tilde{\boldsymbol{K}}}_{\boldsymbol{i}}(\boldsymbol{t})\underline{\boldsymbol{u}}_{\boldsymbol{i}}^{\boldsymbol{e}}(\boldsymbol{P}) \qquad \dot{\tilde{\boldsymbol{K}}}_{\boldsymbol{i}}(\boldsymbol{t}) = \frac{\sum_{P\epsilon D}\underline{\boldsymbol{v}}^{\boldsymbol{EF\_i}}(\boldsymbol{P},\boldsymbol{t}).\underline{\boldsymbol{u}}_{\boldsymbol{i}}^{\boldsymbol{e}}(\boldsymbol{P})}{\sum_{P\epsilon D}\underline{\boldsymbol{u}}_{\boldsymbol{i}}^{\boldsymbol{e}}(\boldsymbol{P}).\underline{\boldsymbol{u}}_{\boldsymbol{i}}^{\boldsymbol{e}}(\boldsymbol{P})}$$

$$\underline{\boldsymbol{v}}^{\boldsymbol{résidu\_i}}(\boldsymbol{P},\boldsymbol{t}) = \underline{\boldsymbol{v}}^{\boldsymbol{EF\_i}}(\boldsymbol{P},\boldsymbol{t}) - \underline{\boldsymbol{v}}_{\boldsymbol{i}}^{\boldsymbol{e}}(\boldsymbol{P},\boldsymbol{t})$$

POD based post treatment

$$\underline{v}^{\text{résidu\_i}}(P,t) = \underline{v}^{EF\_i}(P,t) - \underline{v}_i^{e}(P,t)$$

$$POD1 \rightarrow \underline{v}^{\text{résidu\_i}}(P,t) \approx \dot{\rho}_i(t).\underline{u}_i^{c}(P)$$



$$POD2 \rightarrow \underline{u}_i^{c}(P) \approx \mathrm{f}(r)\ \underline{g}_i^{c}(\theta)$$

$$\underline{g}_{Iy}^{c}(\theta = \pi) = -\underline{g}_{Iy}^{c}(\theta = -\pi) = \frac{1}{2}$$

$$\lim_{r \to 0} f(r) = 1$$

POD based post treatment

$$\underline{v}(P,t) = \sum_{i=1}^{3} \underbrace{\dot{\tilde{K}}_i(t).\underline{u}_i^e(P)}_{\underline{v}_i^e(P,t)} + \underbrace{\dot{\rho}_i(t).\underline{u}_i^c(P)}_{\underline{v}_i^c(P,t)}$$

$\dot{\tilde{K}}_i(t)$

$\dot{\rho}_i(t)$

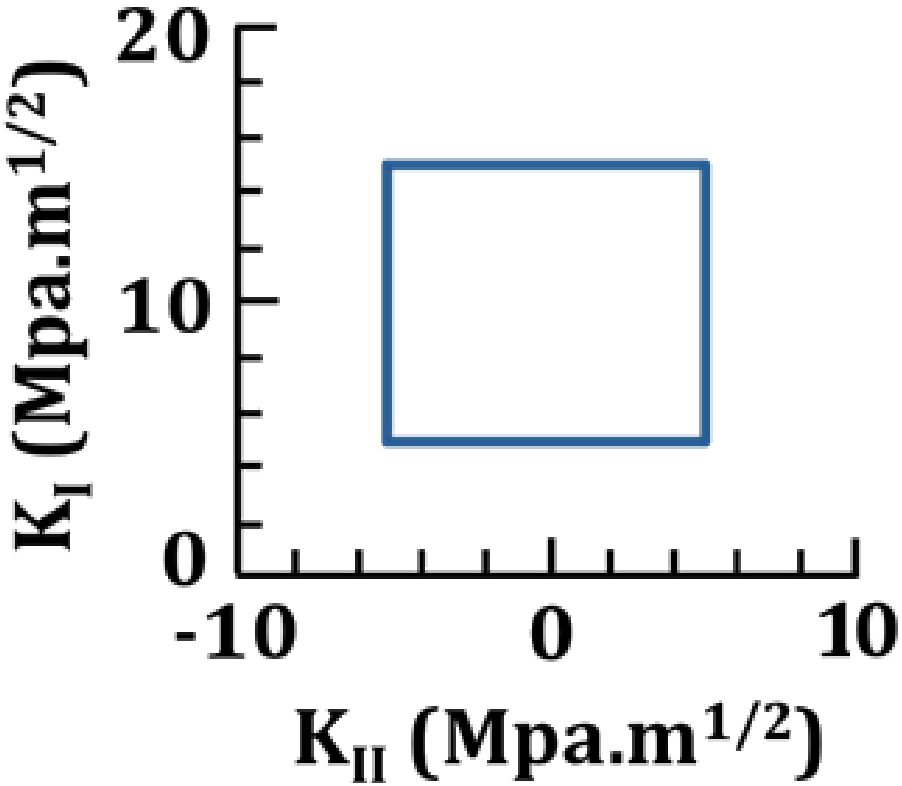
Intensity factors, **non-local variables**

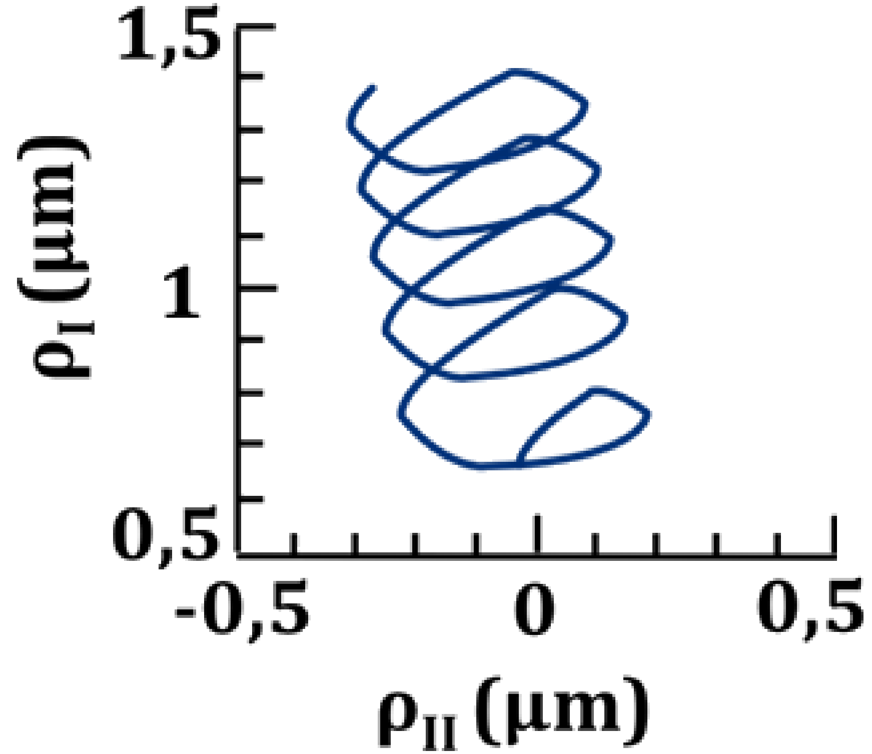
$\underline{u}_i^e(P)$

$\underline{u}_i^c(P)$

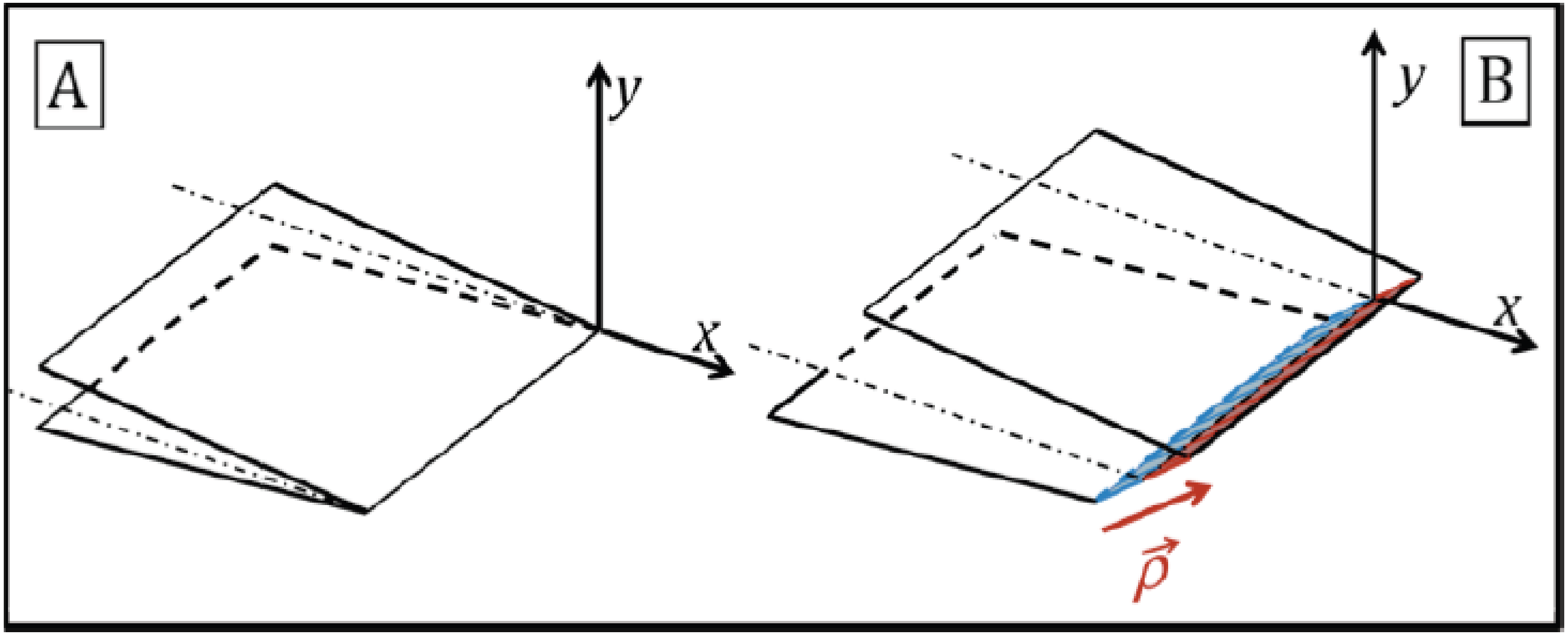
Field basis / weigthing functions tailored for cracks in elastic plastic materials

FE Simulations and results

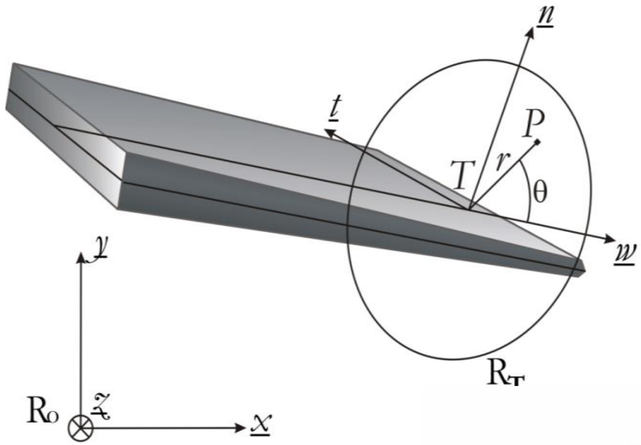




$$\dot{a}\underline{n}^{*} = \alpha\left(\underline{t} \wedge \underline{\dot{\rho}}\right) \Rightarrow \frac{\dot{a}}{\alpha} = \dot{\rho} = \sqrt{\dot{\rho}_I^2 + \dot{\rho}_{II}^2}$$

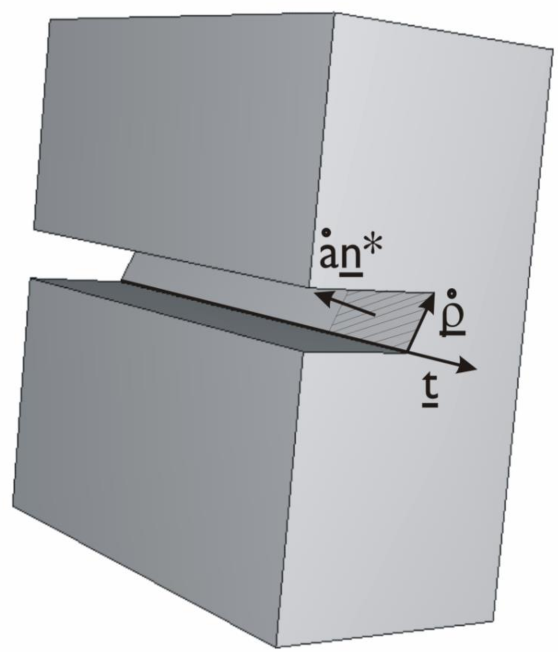


Crack propagation law

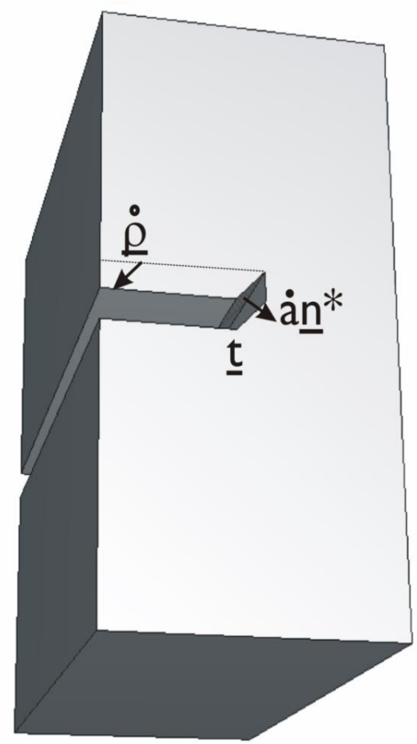


$$\dot{a}\underline{n}^{*} = \alpha\left(\underline{t} \wedge \underline{\dot{\rho}}\right)$$

In mode I, this law derives from the CTOD equation



In mode I+II+III, it derives from the Li's model



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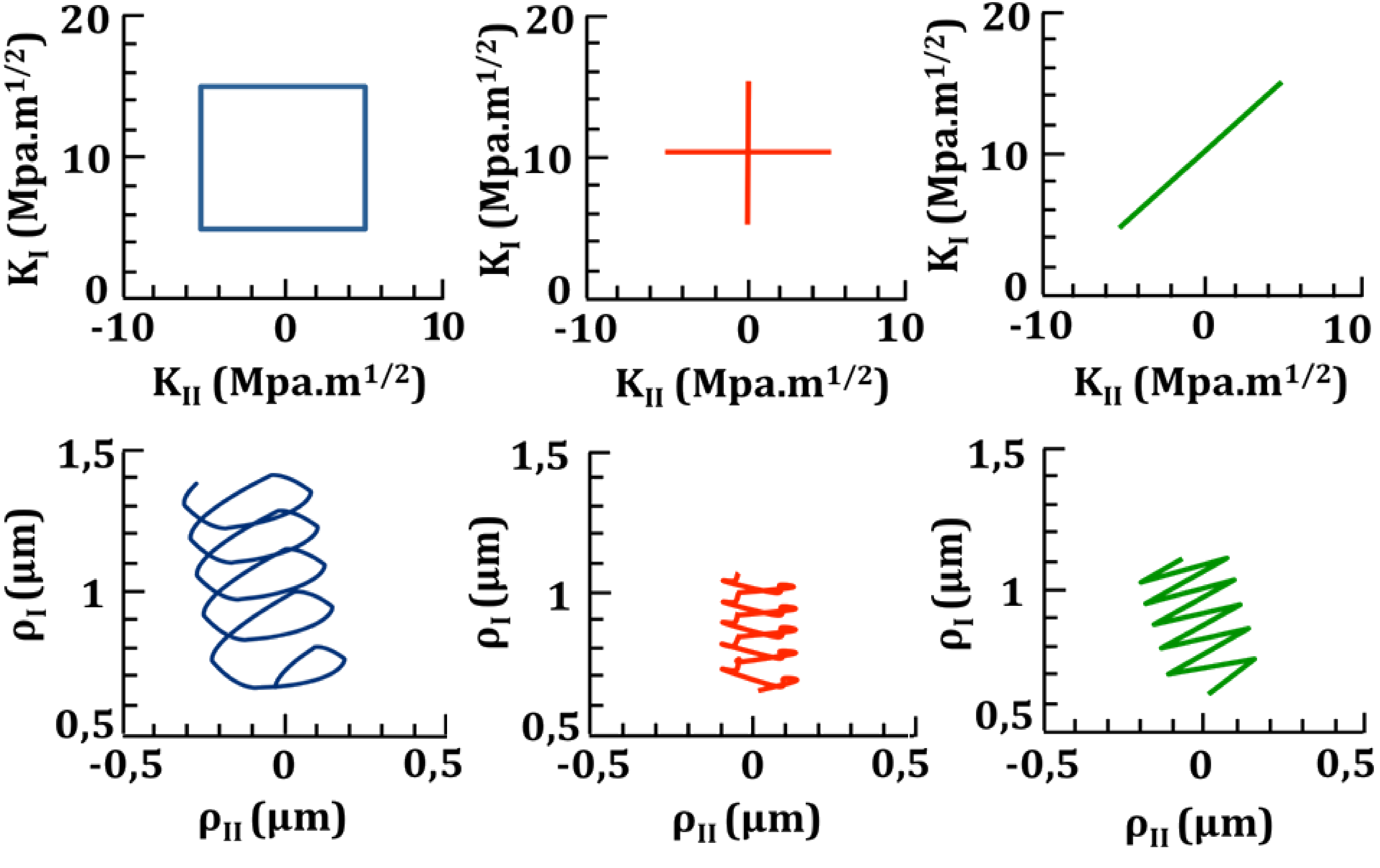
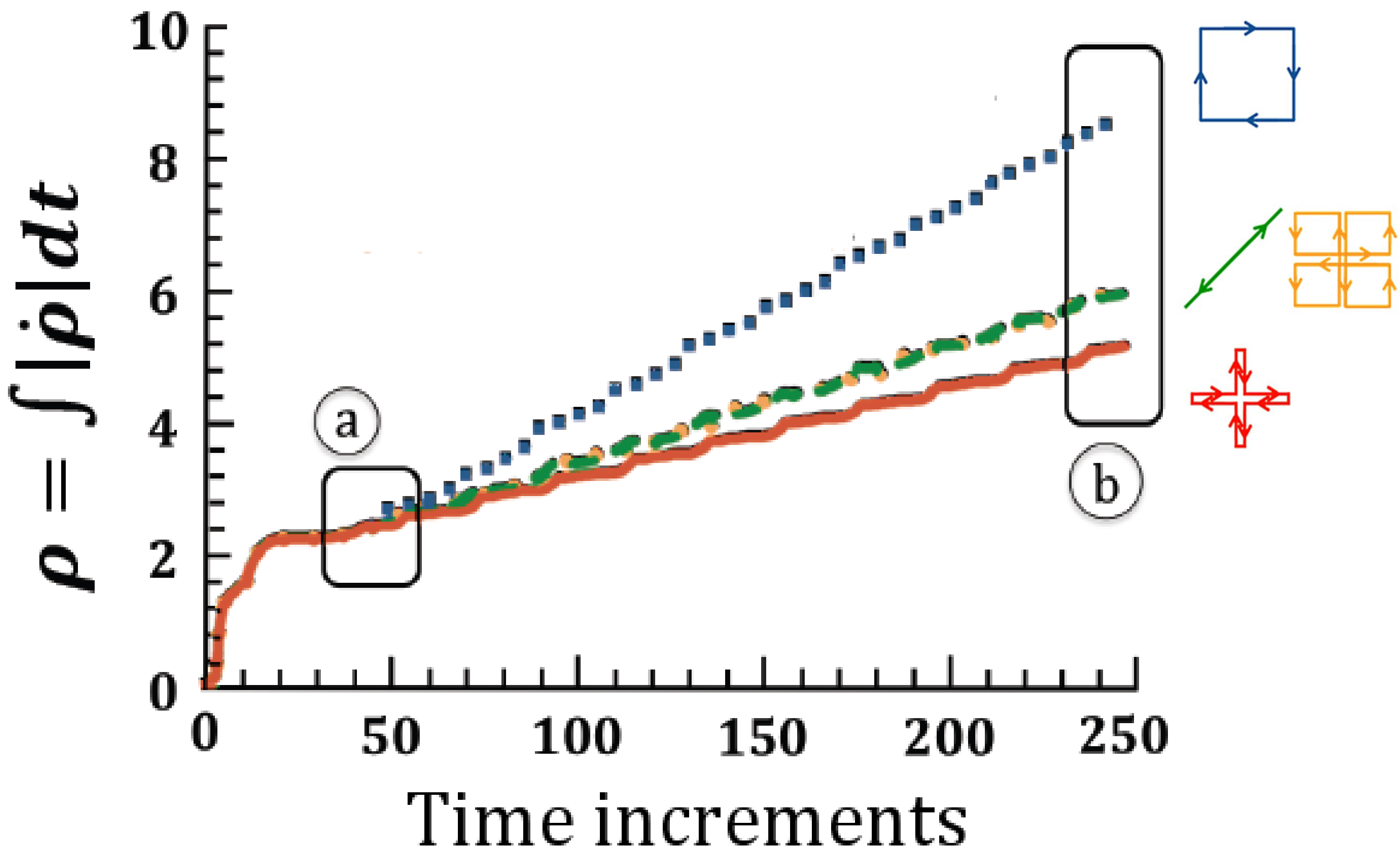
FE Simulations and results





$$\dot{a}\underline{n}^* = \alpha\left(\underline{t} \wedge \underline{\dot{\rho}}\right) \Rightarrow \frac{\dot{a}}{\alpha} = \dot{\rho} = \sqrt{\dot{\rho}_I^2 + \dot{\rho}_{II}^2}$$

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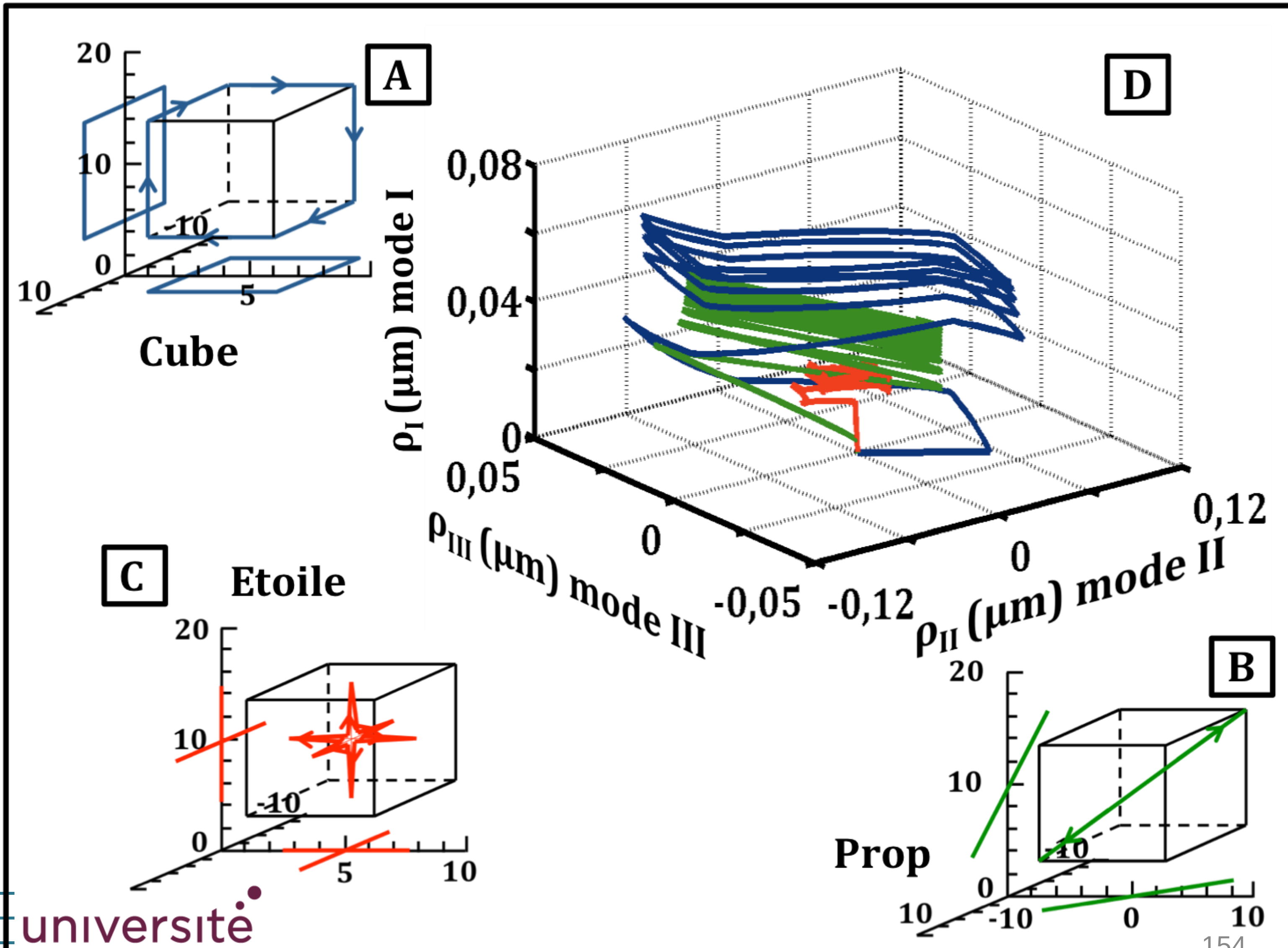
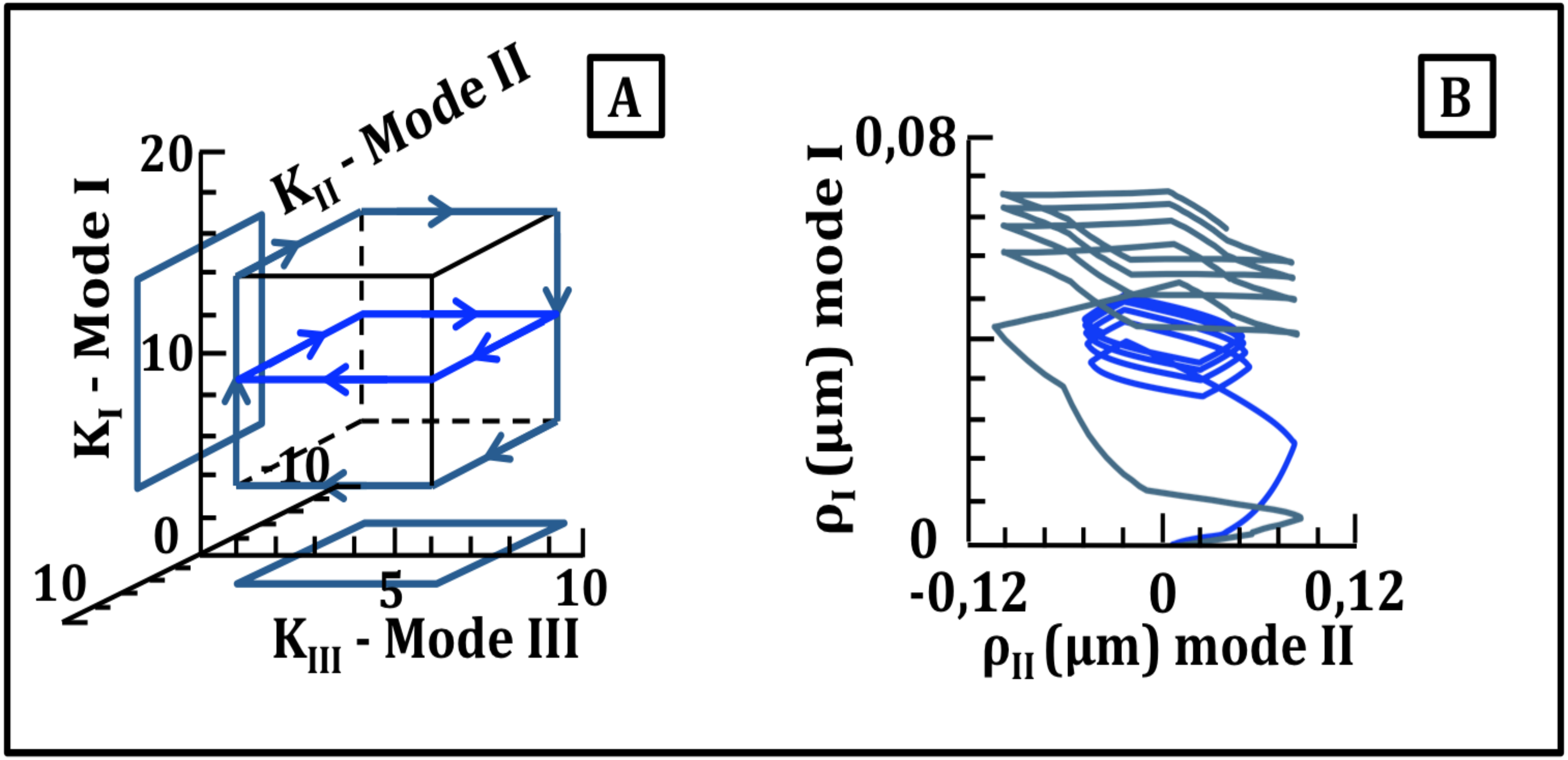
UPA

Intensity factor evolutions



Mode III contribution ?



A Mode III load step increases the amplitude of Mode I and of Mode II plastic flow

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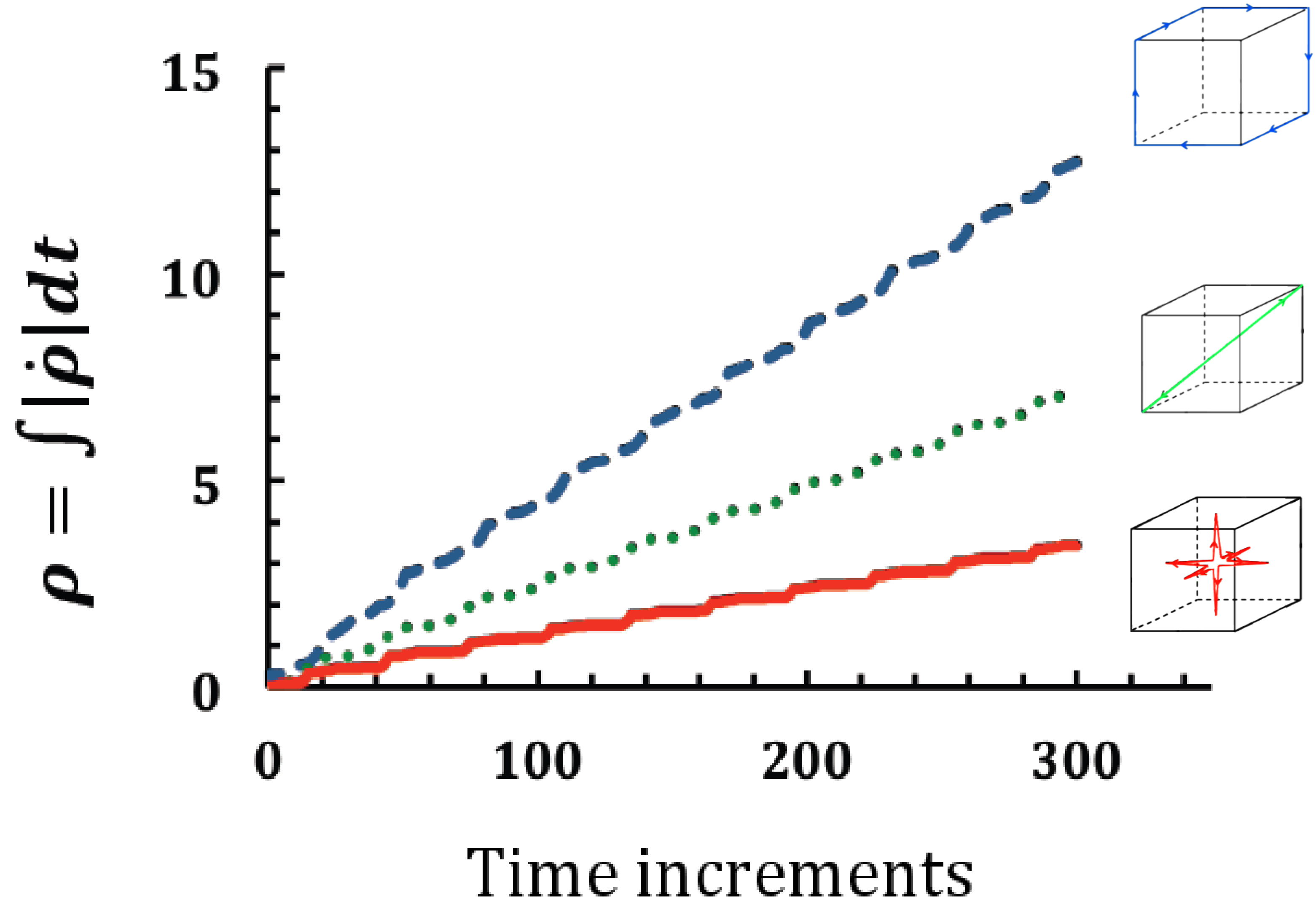


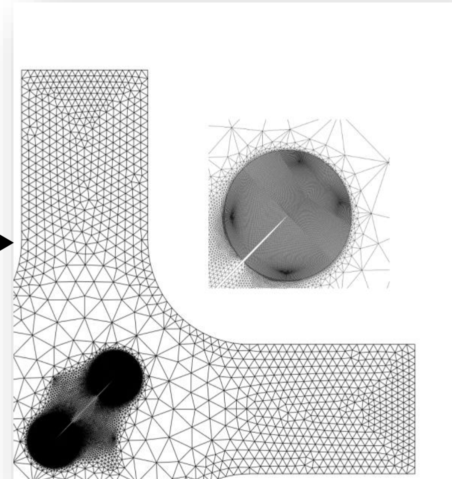
$$\dot{a}\underline{n}^{*} = \alpha\left(\underline{t} \wedge \underline{\dot{\rho}}\right) \Rightarrow \frac{\dot{a}}{\alpha} = \dot{\rho} = \sqrt{\dot{\rho}_{I}^{2} + \dot{\rho}_{II}^{2}}$$

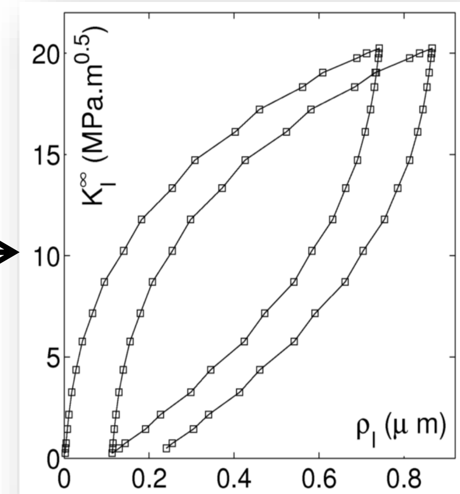
Approach

Material constitutive law, local and tensorial

$$\underline{\underline{\dot{\varepsilon}}} = f\left(\underline{\underline{\dot{\sigma}}}, etc.\right)$$

FE model $\underline{\tilde{v}}(P,t)$





$$\underline{\dot{\rho}} = (\dot{\rho}_I, \dot{\rho}_{II})$$

$$\underline{\dot{K}}^{\infty} = \left(\dot{K}_I^{\infty}, \dot{K}_{II}^{\infty}\right)$$

Crack tip region constitutive law, non-local and vectorial

$$\underline{\dot{\rho}} = g\left(\underline{\dot{K}}^{\infty}, etc.\right)$$

- Elastic domain (internal variables)
- Normal plastic flow rule
- Evolution equations

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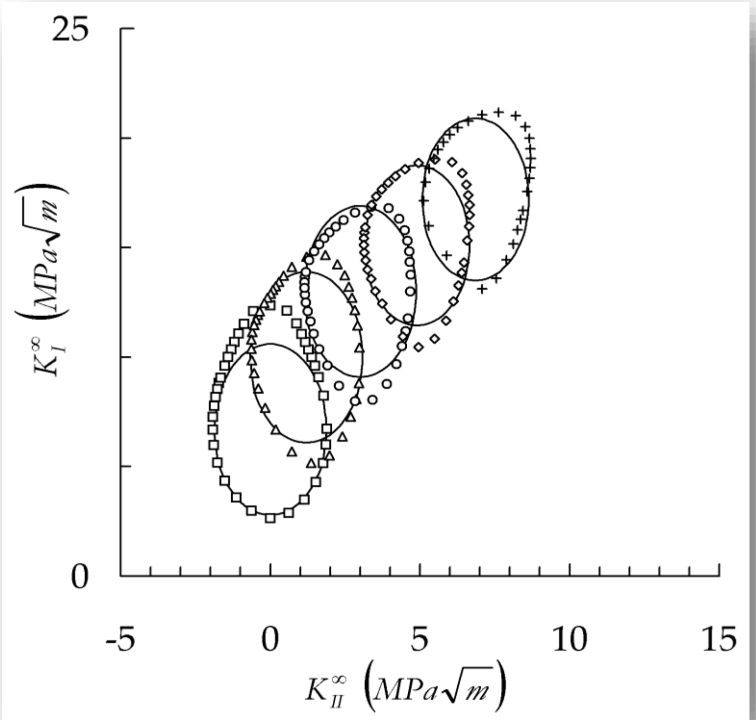
Elastic domain : generalized Von Mises Criterion

$$f_Y = \left(\frac{K_I^\infty - K_I^X}{K_I^Y}\right)^2 + \left(\frac{K_{II}^\infty - K_{II}^X}{K_{II}^Y}\right)^2 - 1$$



$$f_Y = \frac{|G_I|}{G_I^Y} + \frac{|G_{II}|}{G_{II}^Y} - 1$$

$$|G_i| = \frac{sign\left(K_i^\infty - K_i^X\right)\left(K_i^\infty - K_i^X\right)^2}{E^*}$$

Model

Yield criterion

$$f = \frac{\left(K_I^\infty - K_I^X\right)^2}{\left(K_I^Y\right)^2} + \frac{\left(K_{II}^\infty - K_{II}^X\right)^2}{\left(K_{II}^Y\right)^2} + \frac{\left(K_{III}^\infty - K_{III}^X\right)^2}{\left(K_{III}^Y\right)^2} - 1$$

$$f(G_I, G_{II}, G_{III}) = \frac{|G_I|}{G_I^Y} + \frac{|G_{II}|}{G_{II}^Y} + \frac{|G_{III}|}{G_{III}^Y} - 1$$

Flow rule

$$\dot{\rho}_i = \dot{\lambda}\frac{signe(G_i)}{G_i^Y}$$

Evolution equation

$$\underline{\dot{K}^X} = \underline{\underline{C}}\left[\underline{\dot{\rho}} - \frac{\Gamma\, K_{Xeq}^{M-1}}{1 + \Gamma\, K_{Xeq}^{M-1}}\left\langle \underline{d} \middle| \underline{\dot{\rho}} \right\rangle \underline{d}\right] \quad where \quad \underline{d} = \frac{\underline{K^X}}{K_{eq}^X}$$

Conclusions

- Fatigue crack growth experiments in Mixed mode I+II+III non proportionnal loading conditions

- Result : A load path effect is observed on fatigue crack growth and on the crack path

- Adding a mode III step to mixed mode I+II fatigue cycles increases the fatigue crack growth rate

- Elastic-plastic FE analyses show that accounting for plasticity allows predicting the load path effect and the effect of mode III Plasticity

- A simplified model has been developped to replace non-linear FE analyses