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ELEMENTS OF SPECTRAL THEORY

N. RAYMOND

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PROLEGOMENA

These notes are freely inspired by many books [LB03, Dav07, Hel13, Bre83, Rud80, RS, Kat95, Ray17, Zwo12]. They also owe very much to some lecture notes by Z. Ammari, C. Gérard and S. Vũ Ngọc. The Reader is warned that these notes are devoted to teaching. They only exist in order to refresh the memory of the Author and, consequently, some important comments do not appear in this electronic document. Hopefully, the students will also help correcting my mistakes, won't they?

1. PRELIMINARY CONSIDERATIONS

This section is here to help the reader revising some notions that he encountered in the past.

1.1. A question. We endow the space $L^2(I)$ with the usual scalar product

$$\langle u, v \rangle_{L^2(I)} = \int_I u \bar{v} \, dx.$$

We define

$$H^1(I) = \{\psi \in L^2(I) : \psi' \in L^2(I)\},$$

and we endow it with the following Hermitian form

$$\langle u, v \rangle_{H^1(I)} = \langle u, v \rangle_{L^2(I)} + \langle u', v' \rangle_{L^2(I)}.$$

Lemma 1.1. $(H^1(I), \langle \cdot, \cdot \rangle_{H^1(I)})$ is a Hilbert space.

We define

$$H_0^1(I) = \overline{\mathcal{C}_0^\infty(I)}^{H^1}.$$

Lemma 1.2. $(H_0^1(I), \langle \cdot, \cdot \rangle_{H^1(I)})$ is a Hilbert space.

For $J = (a, b)$, we let

$$(1.1) \quad \lambda_1 = \inf_{\substack{\psi \in H_0^1(J) \\ \psi \neq 0}} \frac{\int_J |\psi'|^2 \, dx}{\int_J |\psi|^2 \, dx},$$

and we would like to give an explicit value of λ_1 .

1.2. An answer.

Lemma 1.3 (Sobolev embedding). *The following assertions hold.*

(i) We have $H^1(\mathbb{R}) \subset \mathcal{C}^0(\mathbb{R})$ and, for all $\psi \in H^1(\mathbb{R})$,

$$\forall x \in \mathbb{R}, \quad |\psi(x)| \leq \frac{1}{\sqrt{2}} \|\psi\|_{H^1(\mathbb{R})}.$$

(ii) We have $H_0^1(J) \subset \mathcal{C}^0(\bar{J})$ and, for all $\psi \in H_0^1(J)$, $\psi(a) = \psi(b) = 0$ and

$$\forall x \in J, \quad |\psi(x)| \leq |J|^{\frac{1}{2}} \|\psi'\|_{L^2(J)}.$$

(iii) For all $\psi \in H_0^1(J)$, we have, for all $x, y \in J$,

$$|\psi(x) - \psi(y)| \leq \sqrt{|x - y|} \|\psi'\|_{L^2(J)}.$$

Proof. Let us deal with (i). We use the (unitary) Fourier transform to get, for all $\psi \in H^1(\mathbb{R})$,

$$\|\psi\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + \xi^2) |\widehat{\psi}(\xi)|^2 \, d\xi.$$

In particular, we deduce, by Cauchy-Schwarz, that $\widehat{\psi} \in L^1(\mathbb{R})$. By using the inverse Fourier transform, we get

$$\forall x \in \mathbb{R}, \quad \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\psi}(\xi) e^{ix\xi} \, d\xi.$$

By dominated convergence, we see that ψ is continuous. Moreover, it goes to 0 by the Riemann-Lebesgue lemma. In addition,

$$\forall x \in [0, 1], \quad |\psi(x)| \leq (2\pi)^{-\frac{1}{2}} \|\widehat{\psi}\|_{L^1(\mathbb{R})} \leq (2\pi)^{-\frac{1}{2}} \|\langle \xi \rangle^{-1}\|_{L^2(\mathbb{R})} \|\langle \xi \rangle \widehat{\psi}\|_{L^2(\mathbb{R})}.$$

Let us now consider (ii). Consider $\psi \in H_0^1(J)$. Let us extend ψ by zero outside J and denote by $\underline{\psi}$ this extension. We have $\underline{\psi} \in L^2(\mathbb{R})$. Since $\mathcal{C}_0^\infty(J)$ is dense in $H_0^1(J)$, we can consider a sequence $(\psi_n) \subset \mathcal{C}_0^\infty(J)$ converging to ψ in H^1 -norm. Note that, for all $n, p \in \mathbb{N}$,

$$(1.2) \quad \|\underline{\psi}_n - \underline{\psi}_m\|_{H^1(\mathbb{R})} = \|\psi_n - \psi_m\|_{H^1(J)}.$$

Thus, $(\underline{\psi}_n)$ converges in $H^1(\mathbb{R})$ to some v . Since $(\underline{\psi}_n)$ converges in $L^2(\mathbb{R})$ to $\underline{\psi}$, we get $v = \underline{\psi} \in H^1(\mathbb{R})$. By (i), we deduce that ψ is continuous on J . Coming back to (1.2) and using again (i), we get that $(\underline{\psi}_n)$ uniformly converges to $\underline{\psi}$. In particular, $\psi(a) = \psi(b) = 0$. Then, we write, for all $x \in J$,

$$\psi_n(x) = \int_a^x \psi_n'(t) dt,$$

so that

$$|\psi_n(x)| \leq |J|^{\frac{1}{2}} \|\psi_n'\|_{L^2(J)}.$$

Let us prove (iii). We use again the sequence (ψ_n) and we write, for all $x, y \in J$,

$$\psi_n(x) - \psi_n(y) = \int_x^y \psi_n'(t) dt,$$

and we use the Cauchy-Schwarz inequality. □

Lemma 1.4. *The infimum (1.1) is a minimum.*

Proof. Let (ψ_n) be a minimizing sequence such that $\|\psi_n\|_{L^2(J)} = 1$. In particular (ψ_n') is bounded in $L^2(J)$. Thus, (ψ_n) is equicontinuous on $[a, b]$ and pointwise bounded. We can apply the Ascoli theorem and we may assume that (ψ_n) uniformly converges to ψ on $[a, b]$ and thus in $L^2(J)$. We get $\|\psi\|_{L^2(J)} = 1$. Since (ψ_n) is bounded in $H_0^1(J)$, we can assume that it is weakly convergent (to ϕ) in $H_0^1(J)$, and thus (why?) in $\mathcal{D}'(J)$. We must have $\phi = \psi'$. We deduce

$$\liminf_{n \rightarrow +\infty} \|\psi_n'\|_{L^2(J)} \geq \|\psi'\|_{L^2(J)}$$

since (ψ_n') weakly converges in $L^2(J)$ to ψ' (why?). We get

$$\lambda_1 \geq \|\psi'\|_{L^2(J)}^2,$$

where $\psi \in H_0^1(J)$ and $\|\psi\|_{L^2(J)} = 1$. It follows that

$$\|\psi'\|_{L^2(J)}^2 = \lambda_1.$$

□

Lemma 1.5. *Let ψ be a minimum. Then, ψ satisfies, in the sense of distributions on J ,*

$$(1.3) \quad -\psi'' = \lambda_1 \psi.$$

Proof. Let $\varphi \in \mathcal{C}_0^\infty(J)$ and $\epsilon > 0$. We define

$$f(\epsilon) = \frac{\int_J |(\psi + \epsilon\varphi)'|^2 dx}{\int_J |\psi + \epsilon\varphi|^2 dx}.$$

We get $f'(0) = 0$ and then

$$\int_J \psi' \overline{\varphi}' dx = \lambda_1 \int_J \psi \overline{\varphi} dx.$$

□

Lemma 1.6. *The λ_1 for which there are non trivial solutions to (1.3) being 0 at a and at b are exactly the numbers $(b-a)^{-2} n^2 \pi^2$ where $n \in \mathbb{N}^*$. The corresponding solutions are proportional to $\sin(n\pi(b-a)^{-1} \cdot)$.*

1.3. Some density results. The Reader is invited to read [Bre83, Section IV. 4]. Let us consider a sequence of smooth non-negative functions $(\rho_n)_{n \in \mathbb{N}^*}$ such that $\int_{\mathbb{R}^d} \rho_n(x) dx = 1$ with $\text{supp} \rho_n = B(0, \frac{1}{n})$. Consider a smooth function with compact support $0 \leq \chi \leq 1$ equal to 1 in a neighborhood of 0, and define $\chi_n(\cdot) = \chi(n^{-1}\cdot)$.

Lemma 1.7. *Let $p \in [1, +\infty)$. Let $f \in L^p(\mathbb{R}^d)$. Then, $\rho_n \star f$ and $\chi_n(\rho_n \star f)$ converges to f in $L^p(\mathbb{R}^d)$. In particular, $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $(L^p(\mathbb{R}), \|\cdot\|_{L^p(\mathbb{R}^d)})$.*

Proof. Let $\varepsilon > 0$ and $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\|f - f_0\|_{L^p(\mathbb{R}^d)} \leq \varepsilon$.

We have

$$\rho_n \star f_0(x) - f_0(x) = \int_{\mathbb{R}^d} \rho_n(y)(f_0(x-y) - f_0(x)) dy,$$

and, by the Hölder inequality (with measure $\rho_n dy$),

$$\|\rho_n \star f_0 - f_0\|_{L^p(\mathbb{R}^d)}^p \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_n(y) |f_0(x-y) - f_0(x)|^p dy dx.$$

By using the uniform continuity of f_0 and the support of ρ_n , we see that $\rho_n \star f_0$ converges to f_0 in $L^p(\mathbb{R}^d)$. It remains to notice that

$$\|\rho_n \star (f - f_0)\|_{L^p(\mathbb{R}^d)} \leq \|f - f_0\|_{L^p(\mathbb{R}^d)},$$

to see that $\rho_n \star f$ converges to f in $L^p(\mathbb{R}^d)$.

Then, we consider

$$\|(1 - \chi_n)\rho_n \star f\|_{L^p(\mathbb{R}^d)}^p \leq \int_{\mathbb{R}^d} (1 - \chi_n(x))^p \int_{\mathbb{R}^d} \rho_n(y) |f(x-y)|^p dy dx,$$

and we get

$$\begin{aligned} \|(1 - \chi_n)\rho_n \star f\|_{L^p(\mathbb{R}^d)}^p &\leq \int_{|x| \geq n-1} \int_{\mathbb{R}^d} (1 - \chi_n(x+y))^p \rho_n(y) |f(x)|^p dy dx \\ &\leq \int_{|x| \geq n-1} |f(x)|^p dx, \end{aligned}$$

and the conclusion follows since $f \in L^p(\mathbb{R}^d)$. □

Lemma 1.8. *Let $k \in \mathbb{N}$. $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $(H^k(\mathbb{R}^d), \|\cdot\|_{H^k(\mathbb{R}^d)})$.*

Proof. Let us only deal with the case $k = 1$. Let $f \in H^1(\mathbb{R}^d)$. We let $f_n = \chi_n(\rho_n \star f)$.

First, notice that (f_n) converges to f in $L^2(\mathbb{R}^d)$. Then, we have

$$f'_n = \chi'_n \rho_n \star f + \chi_n \rho_n \star f'.$$

The first term converges to 0 in $L^2(\mathbb{R}^d)$ and the second one goes to f' in $L^2(\mathbb{R}^d)$. □

Consider

$$B^1(\mathbb{R}) = \{\psi \in H^1(\mathbb{R}) : x\psi \in L^2(\mathbb{R})\} \subset L^2(\mathbb{R}).$$

We let, for all $\varphi, \psi \in B^1(\mathbb{R})$,

$$Q(\varphi, \psi) = \langle \varphi, \psi \rangle_{H^1(\mathbb{R})} + \langle x\varphi, x\psi \rangle_{L^2(\mathbb{R})}.$$

Lemma 1.9. *$(B^1(\mathbb{R}), Q)$ is a Hilbert space.*

The following lemma will be convenient.

Lemma 1.10. *$\mathcal{C}_0^\infty(\mathbb{R})$ is dense in $(B^1(\mathbb{R}), \|\cdot\|_{B^1(\mathbb{R})})$.*

Proof. Let us recall Lemma 1.8. Let $f \in B^1(\mathbb{R})$. As in Lemma 1.8, we introduce the sequence $f_n = \chi_n(\rho_n \star f)$. We have seen that f_n goes to f in $H^1(\mathbb{R})$. Let us prove that xf_n goes to xf in $L^2(\mathbb{R})$. Since $xf \in L^2(\mathbb{R})$, $\chi_n(\rho_n \star (xf))$ goes to $xf \in L^2(\mathbb{R})$. We write

$$xf_n(x) - xf(x) = x\chi_n\rho_n \star f(x) - xf(x) = n^{-1}\chi_n\tilde{\rho}_n \star f(x) + \chi_n\rho_n \star (xf) - xf(x),$$

with $\tilde{\rho}_n(y) = n^2y\rho(ny)$. Then, we get

$$\|\chi_n\tilde{\rho}_n \star f\|_{L^2(\mathbb{R})} \leq \|\tilde{\rho}_n\|_{L^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R})} = \|(\cdot)\rho(\cdot)\|_{L^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$

The conclusion follows. \square

Exercise 1.11. Consider

$$\mathcal{V}_\pm = \{\psi \in L^2(\mathbb{R}) : (\pm\partial_x + x)\psi \in L^2(\mathbb{R})\} \subset L^2(\mathbb{R}).$$

We let, for all $\varphi, \psi \in \mathcal{V}_\pm$,

$$Q_\pm(\varphi, \psi) = \langle \varphi, \psi \rangle_{L^2(\mathbb{R})} + \langle (\pm\partial_x + x)\varphi, (\pm\partial_x + x)\psi \rangle_{L^2(\mathbb{R})}.$$

- i. Show that (\mathcal{V}_\pm, Q_\pm) is a Hilbert space.
- ii. Let $f \in \mathcal{V}$. Show that the sequence $f_n = \chi_n(\rho_n \star f)$ converges in \mathcal{V}_\pm .

Proposition 1.12. We have $H^1(\mathbb{R}_+) \subset \mathcal{C}^0(\overline{\mathbb{R}_+})$. Moreover, $\mathcal{C}_0^\infty(\overline{\mathbb{R}_+})$ is dense in $H^1(\mathbb{R}_+)$.

Proof. The main point is to prove the existence of an extension operator. The following lines are inspired by [Bre83, Lemme IX.2].

Let $\psi \in H^1(\mathbb{R}_+)$. We define $\underline{\psi}$ the function defined by $\underline{\psi}(x) = \psi(x)\mathbb{1}_{\mathbb{R}_+}(x) + \psi(-x)\mathbb{1}_{\mathbb{R}_-}(x)$. Let us prove that $\underline{\psi} \in H^1(\mathbb{R})$ and $\|\underline{\psi}\|_{H^1(\mathbb{R})}^2 = 2\|\psi\|_{H^1(\mathbb{R}_+)}^2$. Obviously, we have $\underline{\psi} \in L^2(\mathbb{R})$. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and consider

$$\begin{aligned} \langle \underline{\psi}, \varphi' \rangle_{L^2(\mathbb{R})} &= \int_0^{+\infty} \psi(t)\varphi'(t) dt + \int_{-\infty}^0 \psi(-t)\varphi'(t) dt \\ &= \int_0^{+\infty} \psi(t)(\varphi'(t) + \varphi'(-t)) dt \\ &= \int_0^{+\infty} \psi(t)\Phi'(t) dt, \end{aligned}$$

with $\Phi(t) = \varphi(t) - \varphi(-t)$, for all $t \in \mathbb{R}$. Consider a smooth even function χ being 0 on $(-\frac{1}{2}, \frac{1}{2})$ and 1 away from $(-1, 1)$. We let $\chi_n(t) = \chi(nt)$. We have $(\chi_n\Phi)|_{[0, +\infty)} \in \mathcal{C}_0^\infty(\mathbb{R}_+)$. Since $\psi \in H^1(\mathbb{R}_+)$, there exists un function $f \in L^2(\mathbb{R}_+)$ such that, for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_+)$,

$$\langle \psi, \phi' \rangle_{L^2(\mathbb{R}_+)} = -\langle f, \phi \rangle_{L^2(\mathbb{R}_+)}.$$

By changing ψ into $\chi_n\psi$, we have

$$\begin{aligned} \langle \underline{\psi}, \chi_n\varphi' \rangle_{L^2(\mathbb{R})} &= \int_0^{+\infty} \psi(t)\chi_n(t)\Phi'(t) dt \\ &= \int_0^{+\infty} \psi(t)(\chi_n\Phi)'(t) dt - \int_0^{+\infty} \psi(t)\chi_n'(t)\Phi(t) dt \\ &= -\langle f, \chi_n\Phi \rangle_{L^2(\mathbb{R}_+)} - \int_0^{+\infty} \psi(t)\chi_n'(t)\Phi(t) dt \end{aligned}$$

By using the behavior of Φ at 0 and a support consideration, we get

$$\lim_{n \rightarrow +\infty} \int_0^{+\infty} \psi(t)\chi_n'(t)\Phi(t) dt = 0,$$

and we deduce

$$\langle \underline{\psi}, \varphi' \rangle_{L^2(\mathbb{R})} = -\langle f, \Phi \rangle_{L^2(\mathbb{R}_+)},$$

and thus

$$|\langle \underline{\psi}, \varphi' \rangle_{L^2(\mathbb{R})}|^2 \leq 2\|f\|_{L^2(\mathbb{R}_+)}^2 \|\varphi\|_{L^2(\mathbb{R})}^2.$$

This proves that $\underline{\psi} \in H^1(\mathbb{R})$ and the relation between the H^1 -norms follow. Thus $\underline{\psi} \in \mathcal{C}^0(\mathbb{R})$. The conclusion about the density follows from Lemma 1.8. \square

2. UNBOUNDED OPERATORS

In this section, E and F are Banach spaces. For a slightly different presentation of the concepts in this section, one can consult [LB03, Chapitre 10].

2.1. Definitions.

Definition 2.1 (Unbounded operator). An operator is a pair $(\text{Dom}(T), T)$ where $\text{Dom}(T)$ is a linear subspace of E and T is a linear map from $\text{Dom}(T)$ to F . $\text{Dom}(T)$ is called the domain of T .

Definition 2.2 (Graph). The graph of $(\text{Dom}(T), T)$ is

$$\Gamma(T) = \{(x, Tx), x \in \text{Dom}(T)\} \subset E \times F.$$

Definition 2.3 (Graph norm). Let $(\text{Dom}(T), T)$ be an operator. For all $x \in \text{Dom}(T)$, we let

$$\|x\|_T = \|x\|_E + \|Tx\|_F.$$

Definition 2.4 (Extension). Let $(\text{Dom}(T), T)$ and $(\text{Dom}(S), S)$ two operators. We say that S is an extension of T and we write $T \subset S$ when $\Gamma(T) \subset \Gamma(S)$.

Definition 2.5 (Closed operator). $(\text{Dom}(T), T)$ is said closed when $\Gamma(T)$ is closed in $E \times F$.

Proposition 2.6. *The following assertions are equivalent.*

- i. $(\text{Dom}(T), T)$ is closed.
- ii. For all $(u_n) \in \text{Dom}(T)^\mathbb{N}$ such that $u_n \rightarrow u$ and $Tu_n \rightarrow v$, we have $u \in \text{Dom}(T)$ and $v = Tu$.
- iii. $(\text{Dom}(T), \langle \cdot, \cdot \rangle_T)$ is a Banach space.

Exercise 2.7. Take $E = F = L^2(\mathbb{R}^d)$. Prove that the operator $(H^2(\mathbb{R}^d), -\Delta)$ is closed.

Proposition 2.8. *Let $(\text{Dom}(T), T)$ be a closed operator. There exists $c > 0$ such that*

$$\forall u \in \text{Dom}(T), \quad \|Tu\| \geq c\|u\|,$$

if and only if T is injective with closed range.

Proof. Let us assume that the inequality holds. The injectivity is obvious. Let us consider (v_n) in the range of T such that (v_n) converges to $v \in H$. For all $n \in \mathbb{N}$, there exists $u_n \in \text{Dom}(T)$ such that $v_n = Tu_n$. We deduce that (u_n) is a Cauchy sequence so that it converges to some $u \in H$. Since T is closed, we find that $u \in \text{Dom}(T)$ and $v = Tu$.

Let us assume that T is injective with closed range. $(\text{ran } T, \|\cdot\|)$ is a Banach space. T induces a continuous bijection from $(\text{Dom}(T), \|\cdot\|_T)$ to $(\text{ran } T, \|\cdot\|)$. The inverse is continuous by the Banach theorem. \square

Exercise 2.9. Prove that there exists a constant $c > 0$ such that

$$\forall \varphi \in H^2(\mathbb{R}^d), \quad \|(-\Delta + 1)\varphi\|_{L^2(\mathbb{R}^d)} \geq c\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

Show that this holds for $c = 1$. What is the optimal c ?

Closed operators are generalizations of bounded operators.

Proposition 2.10. *Let $(\text{Dom}(T), T)$ be an operator. Assume that $\text{Dom}(T) = E$. Then, the operator $(\text{Dom}(T), T)$ is closed if and only if T is bounded.*

Example 2.11. Let $\Omega \subset \mathbb{R}^d$ and $K \in L^2(\Omega \times \Omega)$. For all $\psi \in L^2(\Omega)$, we let

$$T_K \psi(x) = \int_{\Omega} K(x, y) \psi(y) \, dy.$$

$T_K : L^2(\Omega) \rightarrow L^2(\Omega)$ is well-defined and bounded. Moreover, $\|T_K\| \leq \|K\|_{L^2(\Omega \times \Omega)}$.

Definition 2.12 (Closable operator). $(\text{Dom}(T), T)$ is said closable when it has a closed extension. In this case, the smallest (in the sense of graph inclusion) closed extension is called the closure of T and it is denoted by \overline{T} .

Proposition 2.13. *The following assertions are equivalent.*

- i. $(\text{Dom}(T), T)$ is closable.
- ii. $\overline{\Gamma(T)}$ is the graph of an operator.
- iii. For $(u_n) \in \text{Dom}(T)^{\mathbb{N}}$ such that $u_n \rightarrow 0$ and $Tu_n \rightarrow v$, we have $v = 0$.

In this case, we have $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.

Exercise 2.14. The closure of $(\mathcal{C}_0^\infty(\mathbb{R}^d), -\Delta)$ is $(H^2(\mathbb{R}^d), -\Delta)$.

Exercise 2.15. Take $E = L^2(\mathbb{R}^d)$ and $F = \mathbb{C}$. Consider the operator T defined on $\text{Dom}(T) = \mathcal{C}_0^\infty(\mathbb{R}^d)$ by $T\varphi = \varphi(0)$. T is not closable.

2.2. Adjoint and closedness.

2.2.1. *About the adjoint of bounded operators.* In this section, E and F are vector spaces. Let us recall what the adjoint of an operator is, in the case of bounded operators.

Definition 2.16. Let $T \in \mathcal{L}(E, F)$. For all $\varphi \in F' = \mathcal{L}(F, \mathbb{C})$, we let $T'(\varphi) = \varphi \circ T \in E'$.

Proposition 2.17. *Let $T \in \mathcal{L}(E, F)$. Then $T' \in \mathcal{L}(F', E')$ and $\|T\|_{\mathcal{L}(E, F)} = \|T'\|_{\mathcal{L}(F', E')}$.*

Proof. T' is clearly linear. Let us show that it is continuous. We have

$$\|T'\|_{\mathcal{L}(F', E')} = \sup_{\varphi \in F' \setminus \{0\}} \frac{\|T'\varphi\|_{E'}}{\|\varphi\|_{F'}} = \sup_{\varphi \in F' \setminus \{0\}} \sup_{x \in E \setminus \{0\}} \frac{\|T'\varphi(x)\|_F}{\|x\|_E \|\varphi\|_{F'}} \leq \|T\|_{\mathcal{L}(E, F)}.$$

For the converse inequality, we write, with a corollary of the Hahn-Banach theorem,

$$\|T\|_{\mathcal{L}(E, F)} = \sup_{x \in E \setminus \{0\}} \frac{\|Tx\|_F}{\|x\|_E} = \sup_{x \in E \setminus \{0\}} \sup_{\varphi \in F' \setminus \{0\}} \frac{\|\varphi(Tx)\|}{\|\varphi\|_{F'} \|x\|_E} \leq \|T'\|_{\mathcal{L}(F', E')}.$$

□

Definition 2.18. If $A \subset E$, we let

$$A^\perp = \{\varphi \in E' : \varphi|_A = 0\},$$

and, for all $B \subset E'$, we let

$$B^\circ = \{x \in E : \forall \varphi \in B, \varphi(x) = 0\}.$$

There is a deep connection between this notion of orthogonality and the adjoint.

Proposition 2.19. *Let $T \in \mathcal{L}(E, F)$. We have*

$$\ker T' = (\text{ran } T)^\perp, \quad \ker T = (\text{ran } T')^\circ.$$

Proof. The first equality is a reformulation of the definitions. We have

$$(\text{ran } T')^\circ = \{x \in E : \forall \varphi \in F' : T'\varphi(x) = 0\} = \ker T,$$

by the Hahn-Banach theorem. \square

Lemma 2.20. *Assume that $(E, \|\cdot\|)$ is a Banach space. Let us write $E = E_1 \oplus E_2$ with E_1 and E_2 closed. Then, the projections Π_{E_1} and Π_{E_2} are bounded.*

Proof. For all $x \in E$, there exists a unique $(x_1, x_2) \in E_1 \times E_2$ such that $x = x_1 + x_2$. We introduce the norm defined for all $x \in E$ by

$$\|x\|' = \|x_1\| + \|x_2\|.$$

$(E, \|\cdot\|')$ is a Banach space. We have

$$\forall x \in E, \quad \|x\| \leq \|x\|'.$$

By the Banach theorem, $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, and thus there exists $C > 0$ such that

$$\forall x \in E, \quad \|x\|' \leq C\|x\|.$$

\square

Let us recall the notion of codimension.

Definition 2.21. Let E be a vector space and E_1 and E_2 two subspaces such that $E = E_1 \oplus E_2$. Assume that $\dim E_2 < +\infty$. Then, all the supplements of E_1 are finite dimensional and have the same dimension. This dimension is called codimension of E_1 and denoted by $\text{codim } E_1$.

The notion of orthogonality is convenient to estimate the codimension.

Proposition 2.22. *Assume that E is a Banach space. Let us write $E = E_1 \oplus E_2$ with E_1 closed and E_2 finite dimensional. Then, we have $\dim E_1^\perp = \dim E_2 = \text{codim } E_1$.*

Proof. Consider $N \in \mathbb{N} \setminus \{0\}$. Let $(e_n)_{1 \leq n \leq N}$ be a basis of E_2 . We can consider $(e_n^*)_{1 \leq n \leq N}$ the dual basis. We consider $(e_n^* \Pi_{E_2})_{1 \leq n \leq N}$. By Lemma 2.20, this is a free family in E' being 0 on E_1 . Thus $\dim E_1^\perp \geq N$.

If $\varphi \in E_1^\perp$ and $x \in E$, we can write $x = x_1 + x_2$, with $(x_1, x_2) \in E_1 \times E_2$, and thus

$$\varphi(x) = \varphi(x_2) = \sum_{n=1}^N e_n^*(x_2) \varphi(e_n) = \sum_{n=1}^N e_n^*(\Pi_{E_2} x) \varphi(e_n),$$

so that $\dim E_1^\perp \leq N$. \square

2.2.2. The case of Hilbert spaces. Let us now assume that $E = F = \mathbb{H}$ is separable Hilbert space. In this case, we define the adjoint of an unbounded operator. But, first, let us discuss the bounded case.

Proposition 2.23 (Adjoint of a bounded operator). *Let $T \in \mathcal{L}(\mathbb{H})$ be a bounded operator. For all $x \in \mathbb{H}$, there exists a unique $T^*x \in \mathbb{H}$ such that*

$$\forall y \in \mathbb{H}, \quad \langle Ty, x \rangle = \langle y, T^*x \rangle.$$

T^* is a bounded operator (called the adjoint of T).

Proof. This is an application of the Riesz representation theorem. \square

There is, of course, a relation between T^* and T' .

Definition 2.24. Let us denote by $\mathcal{J} : \mathbb{H} \rightarrow \mathbb{H}'$ the application defined by

$$\forall u \in \mathbb{H}, \quad \forall \varphi \in \mathbb{H}, \quad \mathcal{J}(u)(\varphi) = \langle \varphi, u \rangle.$$

We recall that J is a bijective isometry by the Riesz representation theorem.

Proposition 2.25. Let $T \in \mathcal{L}(\mathbb{H})$. We have $T^* = \mathcal{J}^{-1}T'\mathcal{J}$.

Proof. Consider $(x, y) \in \mathbb{H}^2$ and

$$\langle x, \mathcal{J}^{-1}T'\mathcal{J}y \rangle = T'\mathcal{J}y(x) = (\mathcal{J}y)(Tx) = \langle Tx, y \rangle = \langle x, T^*y \rangle.$$

□

Exercise 2.26. We let $\mathbb{H} = \ell^2(\mathbb{Z}, \mathbb{C})$, equipped with the usual Hermitian scalar product. For all $u \in \mathbb{H}$, we let, for all $n \in \mathbb{Z}$, $(S_-u)_n = u_{n-1}$ and $(S_+u)_n = u_{n+1}$.

- i. Show that S_- and S_+ are bijective isometries.
- ii. Prove that $S_{\pm}^* = S_{\mp}$.

Proposition 2.27 (Adjoint of an unbounded operator). Let $(\text{Dom}(T), T)$ be an operator with dense domain. We let

$$\text{Dom}(T^*) = \{x \in \mathbb{H} : \text{Dom}(T) \ni y \mapsto \langle Ty, x \rangle \text{ is continuous for the topology of } \mathbb{H}\}.$$

For all $x \in \text{Dom}(T^*)$, there exists a unique $T^*x \in \mathbb{H}$ such that

$$\forall y \in \text{Dom}(T), \quad \langle Ty, x \rangle = \langle y, T^*x \rangle.$$

$(\text{Dom}(T^*), T^*)$ is an operator. It is called adjoint of T .

Proof. It is a consequence of the Riesz representation theorem. □

Definition 2.28. We say that $(\text{Dom}(T), T)$ is self-adjoint when $T = T^*$.

Example 2.29. Consider Exercise 2.26. Then $S_+ + S_-$ is self-adjoint.

Example 2.30. Let us consider (X, \mathcal{A}, μ) a measure space, with a σ -finite measure μ . We let $\mathbb{H} = L^2(X, \mathcal{A}, \mu)$ and consider a \mathbb{C} -valued measurable function f . We define

$$\text{Dom}(T_f) = \{\psi \in \mathbb{H} : f\psi \in \mathbb{H}\},$$

and, for all $\psi \in \text{Dom}(T_f)$, $T_f\psi = f\psi$.

- i. If $f \in L^\infty(X, \mathcal{A}, \mu)$, we have $\text{Dom}(T_f) = \mathbb{H}$ and T_f is bounded.
- ii. The domain $\text{Dom}(T_f)$ is dense in \mathbb{H} .
- iii. The domain of the adjoint of T_f is given by $\text{Dom}(T_f)$ and $T_f^* = T_{\bar{f}}$. In particular, when f is real-valued, T_f is self-adjoint.

Exercise 2.31. Take $\mathbb{H} = L^2(\mathbb{R})$. Consider $\text{Dom}(T) = H^1(\mathbb{R})$ and $T = -i\partial_x$. What is $(\text{Dom}(T^*), T^*)$? And if we choose $\text{Dom}(T) = \mathcal{C}_0^\infty(\mathbb{R})$?

Proposition 2.32. Let us define $J : \mathbb{H} \times \mathbb{H} \ni (x, y) \mapsto (-y, x) \in \mathbb{H} \times \mathbb{H}$. We equip $\mathbb{H} \times \mathbb{H}$ with the natural scalar product. If T is an operator with dense domain, we have

$$\Gamma(T^*) = J(\Gamma(T))^\perp, \quad \overline{\Gamma(T)} = J(\Gamma(T^*))^\perp.$$

In particular, T^* is closed.

Proposition 2.33. Let $(\text{Dom}(T), T)$ and $(\text{Dom}(S), S)$ two operators. If $T \subset S$, we have $S^* \subset T^*$.

Proposition 2.34. Let us consider a densely defined operator T . Then, T is closable if and only if $\text{Dom}(T^*)$ is dense. In this case, $(T^*)^* = \bar{T}$.

Proof. Assume that $\text{Dom}(T^*)$ is dense. Then, we have $\Gamma((T^*)^*) = \overline{\Gamma(T)}$. Thus, $\overline{\Gamma(T)}$ is a graph and $(T^*)^* = \overline{T}$.

Assume that T is closable. Let $v \in \text{Dom}(T^*)^\perp$. We have $(0, v) \in J(\Gamma(T^*))^\perp = \overline{\Gamma(T)} = \Gamma(\overline{T})$. Thus, $v = 0$. \square

Proposition 2.35. *If T is closable with dense domain, we have $\overline{T}^* = T^*$.*

Proof. We have $\overline{T}^* = (T^*)^{**} = \overline{T^*} = T^*$. \square

Proposition 2.36. *Let us consider a densely defined operator T . We have*

$$\ker(T^*) = \text{ran}(T)^\perp, \quad \ker(T^*)^\perp = \overline{\text{ran}(T)}.$$

In particular, T^ is injective if and only if T has a dense range.*

Proof. Let $x \in \ker T^*$ and $y \in \text{ran}(T)$. We write $y = Tz$ with $z \in \text{Dom}(T)$. We have $\langle x, Tz \rangle = 0$.

Let $y \in \text{H}$ such that $\langle y, Tx \rangle = 0$ for all $x \in \text{Dom}(T)$. We deduce that $y \in \text{Dom}(T^*)$ and that $\langle T^*y, x \rangle = 0$ for all $x \in \text{Dom}(T)$. Thus, $T^*y = 0$. \square

2.2.3. Creation and annihilation operators. Let us discuss two important examples to illustrate the above abstract propositions. Let us introduce the following differential operators, acting on $\text{Dom}(a) = \text{Dom}(c) = \mathcal{S}(\mathbb{R})$,

$$a = \frac{1}{\sqrt{2}}(\partial_x + x), \quad c = \frac{1}{\sqrt{2}}(-\partial_x + x).$$

The domains of their adjoints are

$$\begin{aligned} \text{Dom}(a^*) &= \{\psi \in \text{L}^2(\mathbb{R}) : (-\partial_x + x)\psi \in \text{L}^2(\mathbb{R})\}, \\ \text{Dom}(c^*) &= \{\psi \in \text{L}^2(\mathbb{R}) : (\partial_x + x)\psi \in \text{L}^2(\mathbb{R})\}, \end{aligned}$$

and,

$$\begin{aligned} \forall \psi \in \text{Dom}(a^*), \quad a^*\psi &= \frac{1}{\sqrt{2}}(-\partial_x + x)\psi, \\ \forall \psi \in \text{Dom}(c^*), \quad c^*\psi &= \frac{1}{\sqrt{2}}(\partial_x + x)\psi. \end{aligned}$$

In particular, we see that $a \subset c^*$ and $c \subset a^*$. Thus, a and c are closable and their closures satisfy $\overline{a} \subset c^*$ and $\overline{c} \subset a^*$.

Lemma 2.37. *We have*

$$\text{Dom}(\overline{a}) = \text{Dom}(\overline{c}) = \text{B}^1(\mathbb{R}).$$

Proof. For all $u \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned} 2\|au\|^2 &= \|u'\|^2 + \|xu\|^2 - \|u\|^2 \\ 2\|cu\|^2 &= \|u'\|^2 + \|xu\|^2 + \|u\|^2. \end{aligned}$$

Now, take $u \in \text{Dom}(\overline{a})$. By definition, we have $(u, \overline{a}u) \in \overline{\Gamma(a)}$. There exists $(u_n) \in \text{Dom}(a)^{\mathbb{N}}$ such that (u_n) converges to u and (au_n) converges to $\overline{a}u$. We deduce that (u'_n) and (xu_n) are Cauchy sequences in $\text{L}^2(\mathbb{R})$. We get that $u' \in \text{L}^2(\mathbb{R})$ and $xu \in \text{L}^2(\mathbb{R})$. We get $\text{Dom}(\overline{a}) \subset \text{B}^1(\mathbb{R})$. We proceed in the same way for \overline{c} .

Let us now deal with the reversed inclusion. Take $u \in \text{B}^1(\mathbb{R})$. By Lemma 1.10, there exists a sequence (u_n) of smooth functions with compact support such that u_n converges to u in $\text{B}^1(\mathbb{R})$.

In particular, (au_n) and (cu_n) are convergent in $L^2(\mathbb{R})$. We deduce that $u \in \text{Dom}(\bar{a})$ and $u \in \text{Dom}(\bar{c})$. □

Now, we use the results of Exercise 1.11. For example, if $\psi \in \text{Dom}(c^*)$, we have $\psi \in L^2(\mathbb{R})$ and $(\partial_x + x)\psi \in L^2(\mathbb{R})$. There exists $(\psi_n) \in \mathcal{S}^{\mathbb{N}}$ such that ψ_n converges to ψ and $(\partial_x + x)\psi_n$ converges to $(\partial_x + x)\psi \in L^2(\mathbb{R})$. We get that (ψ'_n) and $(x\psi_n)$ are Cauchy sequences. Thus $\psi \in B^1(\mathbb{R})$. We deal with a^* in the same way. We get that

$$\text{Dom}(c^*) = \text{Dom}(a^*) = B^1(\mathbb{R}).$$

We deduce that

$$\bar{a} = c^*, \quad \bar{c} = a^*.$$

By using Propositions 2.34 and 2.35, we get

$$a^* = \bar{a}^* = \bar{c}, \quad c^* = \bar{c}^* = \bar{a}.$$

In other words, the closures of a and c are adjoint of each other and they share the same domain $B^1(\mathbb{R})$.

2.3. Self-adjoint operators and essentially self-adjoint operators.

2.3.1. Symmetric and self-adjoint operators.

Definition 2.38. A densely defined operator T is said symmetric if $T \subset T^*$. It is said self-adjoint if $T = T^*$.

Note that a symmetric and densely defined operator is closable.

Proposition 2.39. *The operator T is symmetric if and only if*

$$\forall u, v \in \text{Dom}(T), \quad \langle Tu, v \rangle = \langle u, Tv \rangle.$$

In particular, for all $u \in \text{Dom}(T)$, $\langle Tu, u \rangle \in \mathbb{R}$.

Exercise 2.40. Take $H = L^2(\mathbb{R})$. Show that $(\mathcal{C}_0^\infty(\mathbb{R}), -i\partial_x)$ is symmetric.

Exercise 2.41. Let $P \in \mathbb{R}[X]$ be a polynomial of degree n . Show that the differential operator $(H^n(\mathbb{R}^d), P(D))$ is symmetric. Here $D = -i\partial_x$. Use the Fourier transform and Example 2.30.

Exercise 2.42. Give an example of non-symmetric operator.

Proposition 2.43. *Let us consider a symmetric operator T . Let $z = \alpha + i\beta$ with $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^*$. Then,*

$$\forall u \in \text{Dom}(T), \quad \|(T - z)u\| \geq |\beta|\|u\|.$$

If, moreover, T is closed, $T - z$ is injective with closed range.

Proof. Let $u \in \text{Dom}(T)$. We have

$$\|(T - z)u\|^2 = \|(T - \alpha)u - i\beta u\|^2 = \|(T - \alpha)u\|^2 + \beta^2\|u\|^2 + 2\text{Re} \langle (T - \alpha)u, (-i\beta)u \rangle,$$

and thus

$$\|(T - z)u\|^2 = \|(T - \alpha)u\|^2 + \beta^2\|u\|^2 \geq \beta^2\|u\|^2.$$

□

Proposition 2.44. *Let us consider a symmetric operator T . Then, T is closable and*

$$T \subset \bar{T} \subset T^*.$$

T is self-adjoint if and only if $T = \bar{T} = T^$.*

Proposition 2.45. *Let T be a closed and symmetric operator. Then, T is self-adjoint if and only if T^* is symmetric.*

Proof. Assume that T^* is symmetric. We have $T \subset \bar{T} \subset T^* \subset (T^*)^* = \bar{T} = T$. □

Proposition 2.46. *Let T be a symmetric operator. The following assertions are equivalent.*

- i. T is self-adjoint.
- ii. T is closed and $\ker(T^* \pm i) = \{0\}$.
- iii. $\text{ran}(T \pm i) = H$.

Proof. If T is self-adjoint, T is closed and $\ker(T^* \pm i) = \ker(T \pm i) = \{0\}$.

Then, we assume that T is closed and $\ker(T^* \pm i) = \{0\}$. Thus, $T \mp i$ has a dense range and the range is closed.

Finally, we assume that $\text{ran}(T \pm i) = H$. Let us prove that $\text{Dom}(T^*) \subset \text{Dom}(T)$. Take $u \in \text{Dom}(T^*)$ and consider $(T^* - i)u$. There exists $v \in \text{Dom}(T)$ such that $(T^* - i)u = (T - i)v$. Then, $(T^* - i)u = (T^* - i)v$ and $\ker(T^* - i) = \{0\}$. We deduce that $u - v = 0$. □

Exercise 2.47. Take $H = L^2(\mathbb{R}^d)$. Consider $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ and the operator $-\Delta + V$ with domain $H^2(\mathbb{R}^d)$. Is it self-adjoint?

Exercise 2.48. Take $H = L^2(\mathbb{R}_+)$.

- i. Is the operator $(H^1(\mathbb{R}_+), -i\partial_x)$ symmetric?
- ii. Is the operator $(H_0^1(\mathbb{R}_+), -i\partial_x)$ symmetric?
- iii. Show that the domain of the adjoint of $(H_0^1(\mathbb{R}_+), -i\partial_x)$ is $H^1(\mathbb{R}_+)$.
- iv. By using Proposition 2.46, prove that $(H_0^1(\mathbb{R}_+), -i\partial_x)$ is not self-adjoint.

Exercise 2.49. Take $H = L^2(\mathbb{R}_+)$. We let $\text{Dom}(T) = \{\psi \in H^2(\mathbb{R}_+) : u'(0) = -u(0)\}$ and $T = -\partial_x^2$. Is this operator self-adjoint? We recall that $H^2(\mathbb{R}_+)$ is continuously embedded in $\mathcal{C}^1(\overline{\mathbb{R}_+})$ (see Proposition 1.12).

2.3.2. Essentially self-adjoint operators.

Definition 2.50. A symmetric operator is essentially self-adjoint if its closure is self-adjoint.

Proposition 2.51. *Let T be a symmetric operator. Then, T is essentially self-adjoint if and only if $\bar{T} = T^*$.*

Proof. If T is essentially self-adjoint, we have $\bar{T}^* = \bar{T}$. Since $T^* = \bar{T}$, the conclusion follows. Conversely, let us assume that $\bar{T} = T^*$. Then $\bar{T}^* = T^{**} = \bar{T}$. □

Exercise 2.52. Take $H = L^2(\mathbb{R}^d)$. We take $(\mathcal{C}_0^\infty(\mathbb{R}^d), -\Delta)$. Is this operator essentially self-adjoint? What is the adjoint?

Proposition 2.53. *If T is essentially self-adjoint, it has a unique self-adjoint extension.*

Proof. Let us consider S a self-adjoint extension of T . We have $T \subset S$ so that $\bar{T} \subset S = S^* \subset T^* = \bar{T}$. □

Proposition 2.54. *The following assertions are equivalent.*

- i. T is essentially self-adjoint.
- ii. $\overline{\ker(T^* \pm i)} = \{0\}$.
- iii. $\text{ran}(T \pm i) = H$.

Proof. If T is essentially self-adjoint, \bar{T} is self-adjoint and thus $\ker(\bar{T}^* \pm i) = \ker(T^* \pm i) = \{0\}$. If $\overline{\ker(T^* \pm i)} = \{0\}$, then $\text{ran}(T \pm i) = H$. Finally, we assume that $\text{ran}(T \pm i) = H$. We get $\ker(T^* \pm i) = \{0\}$ and $\text{ran}(\bar{T} \pm i) = H$. Therefore \bar{T} is self-adjoint. □

Exercise 2.55. Take $H = L^2(I)$ with $I = (0, 1)$. Consider $(\mathcal{C}_0^\infty(I), -\partial_x^2)$. Is it essentially self-adjoint?

2.3.3. *A criterion for essential self-adjointness for Schrödinger operators.*

Lemma 2.56. Let $f \in L_{loc}^2(\mathbb{R}^d)$ such that $\Delta f \in L_{loc}^2(\mathbb{R}^d)$. Then, there exists a sequence $(f_n) \in \mathcal{C}_0^\infty(\mathbb{R}^d)^\mathbb{N}$ such that (f_n) tends to f and (Δf_n) tends to Δf in $L_{loc}^2(\mathbb{R}^d)$.

Proof. It is sufficient to adapt the proof of Lemma 1.8. \square

Lemma 2.57. Let φ and χ two smooth functions with compact supports, with χ real-valued. We have

$$\int_{\mathbb{R}^d} \chi^2 |\nabla \varphi|^2 dx \leq 2 \|\chi \Delta \varphi\| \|\chi \varphi\| + 4 \|(\nabla \chi) \varphi\|^2.$$

Proof. We write

$$\langle \Delta \varphi, \chi^2 \varphi \rangle = \langle \nabla \varphi, \nabla(\chi^2 \varphi) \rangle = \|\chi \nabla \varphi\|^2 + 2 \langle \chi \nabla \varphi, (\nabla \chi) \varphi \rangle.$$

By using the Cauchy-Schwarz inequality, we have

$$2 |\langle \chi \nabla \varphi, (\nabla \chi) \varphi \rangle| \leq \frac{1}{2} \|\chi \nabla \varphi\|^2 + 2 \|(\nabla \chi) \varphi\|^2.$$

We deduce the desired estimate. \square

Lemma 2.58. Let $f \in L_{loc}^2(\mathbb{R}^d)$ such that $\Delta f \in L_{loc}^2(\mathbb{R}^d)$. Then $f \in H_{loc}^1(\mathbb{R}^d)$.

Proof. We consider the sequence (f_n) given in Lemma 2.56 and we use Lemma 2.57 with $\varphi = f_n - f_p$. We easily deduce that (∇f_n) is convergent in $L_{loc}^2(\mathbb{R}^d)$ and that the limit is ∇f in the sense of distributions. \square

Lemma 2.59. Let $f \in L_{loc}^2(\mathbb{R}^d)$ such that $\Delta f \in L_{loc}^2(\mathbb{R}^d)$. Then $f \in H_{loc}^2(\mathbb{R}^d)$.

Proof. Let χ be a smooth function with compact support. We have just to show that $\chi f \in H^2(\mathbb{R}^d)$. We have $\Delta(\chi f) = \chi \Delta f + 2 \nabla \chi \cdot \nabla f + f \Delta \chi \in L^2(\mathbb{R}^d)$ by Lemma 2.58. Thus, by considering the Fourier transform of χf , we easily find that $\langle \xi \rangle^2 \widehat{\chi f} \in L^2(\mathbb{R}^d)$ and we deduce that $\chi f \in H^2(\mathbb{R}^d)$. \square

Proposition 2.60. Let us consider $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ and the operator T with domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$ acting as $-\Delta + V$. We assume that T is semi-bounded from below, i.e., there exists $C \in \mathbb{R}$ such that

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad \langle Tu, u \rangle \geq C \|u\|^2.$$

Then, T is essentially self-adjoint.

Proof. We follow the presentation in [Hel13, Theorem 9.15]. Up to a translation of V , we can assume that $C = 1$. Let us prove that the range of $T \pm i$ is dense. Let us consider $f \in L^2(\mathbb{R}^d)$ such that, for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\langle f, (T \pm i)u \rangle = 0.$$

We get, in the sense of distributions, that

$$(-\Delta + V \mp i)f = 0.$$

With Lemma 2.59, we get that $f \in H_{loc}^2(\mathbb{R}^d)$. By induction, we get that $f \in H_{loc}^\infty(\mathbb{R}^d)$. From this and the Sobolev embedding $H^s(\mathbb{R}^d) \rightarrow \mathcal{C}^0(\mathbb{R}^d)$ when $s > \frac{d}{2}$, we deduce that $f \in \mathcal{C}^\infty(\mathbb{R}^d)$.

Now, take $u \in \mathcal{C}^\infty(\mathbb{R}^d)$ and consider $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ supported in $B(0, 2)$ and equal to 1 on $B(0, 1)$. For all $n \geq 1$, we let, for all $x \in \mathbb{R}^d$,

$$\chi_n(x) = \chi(n^{-1}x).$$

We write

$$\langle f, (T \pm i)(\chi_n^2 u) \rangle = 0,$$

and we have

$$\langle f, (T \pm i)(\chi_n^2 u) \rangle = \int_{\mathbb{R}^d} (\nabla f \nabla(\chi_n^2 \bar{u}) + (V \mp i)\chi_n^2 f \bar{u}) dx.$$

We get

$$\int_{\mathbb{R}^d} \nabla f \nabla(\chi_n^2 \bar{u}) dx = \int_{\mathbb{R}^d} \chi_n \nabla f \nabla(\chi_n \bar{u}) dx + \int_{\mathbb{R}^d} \nabla f \cdot (\nabla \chi_n) \chi_n \bar{u} dx.$$

Thus,

$$\int_{\mathbb{R}^d} \nabla f \nabla(\chi_n^2 \bar{u}) dx = \int_{\mathbb{R}^d} \nabla(\chi_n f) \nabla(\chi_n \bar{u}) dx - \int_{\mathbb{R}^d} f \nabla \chi_n \cdot \nabla(\chi_n \bar{u}) dx + \int_{\mathbb{R}^d} \nabla f \cdot (\nabla \chi_n) \chi_n \bar{u} dx,$$

and

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla f \nabla(\chi_n^2 \bar{u}) dx &= \int_{\mathbb{R}^d} \nabla(\chi_n f) \nabla(\chi_n \bar{u}) dx - \int_{\mathbb{R}^d} f |\nabla \chi_n|^2 \bar{u} dx \\ &\quad + \int_{\mathbb{R}^d} \nabla f \cdot (\nabla \chi_n) \chi_n \bar{u} dx - \int_{\mathbb{R}^d} f \chi_n \nabla \chi_n \cdot \nabla \bar{u} dx. \end{aligned}$$

We can choose $u = f$, take the real part to get

$$\int_{\mathbb{R}^d} |\nabla(\chi_n f)|^2 + V |\chi_n f|^2 dx = \int_{\mathbb{R}^d} |f \nabla \chi_n|^2 dx.$$

The r.h.s. goes to zero when n goes to $+\infty$. By assumption, this implies that

$$\liminf_{n \rightarrow +\infty} \|\chi_n f\|^2 = 0.$$

The conclusion follows from the Fatou lemma. \square

Example 2.61. The operator with domain $\mathcal{C}_0^\infty(\mathbb{R})$ acting as $-\partial_x^2 + x^2$ is essentially self-adjoint. Show that, in fact, this operator is bounded from below by 1.

Exercise 2.62. Take $H = L^2(\mathbb{R}^2)$. We take $\text{Dom}(T) = \mathcal{C}_0^\infty(\mathbb{R}^2)$. For $\psi \in \text{Dom}(T)$, we let $T\psi = (-\partial_{x_1}^2 + (-i\partial_{x_2} - x_1)^2)\psi$. Is this operator essentially self-adjoint?

2.4. Lax-Milgram theorems. In this section, we adopt the presentation in [Hel13, Chapter 3].

Theorem 2.63. Let \mathcal{V} be a Hilbert space. Let Q be a continuous sesquilinear form on $\mathcal{V} \times \mathcal{V}$. Assume that there exists $\alpha > 0$ such that, for all $u \in \mathcal{V}$, we have

$$|Q(u, u)| \geq \alpha \|u\|_{\mathcal{V}}^2,$$

The operator \mathcal{A} defined by

$$\forall u, v \in \mathcal{V}, \quad Q(u, v) = \langle \mathcal{A}u, v \rangle_{\mathcal{V}}$$

is a continuous isomorphism of \mathcal{V} onto \mathcal{V} with bounded inverse.

Proof. The operator \mathcal{A} is well defined by the Riesz representation theorem. It is bounded since Q is continuous. It is clearly injective. Let us show that the range of \mathcal{A} is closed and dense. We easily get that $\text{ran } \mathcal{A}^\perp = \{0\}$ so that the range is dense. Moreover, we have, by Cauchy-Schwarz, for all $u \in \mathcal{V}$,

$$\alpha \|u\|_{\mathcal{V}} \leq \|\mathcal{A}u\|_{\mathcal{V}}.$$

Thus, the range is closed. \square

Theorem 2.64. *In addition to the hypotheses of Theorem 2.63, assume that H is a Hilbert space such that \mathcal{V} is continuously embedded and dense in H . Then the operator \mathcal{L} defined by*

$$\forall u \in \text{Dom}(\mathcal{L}), \quad \forall v \in \mathcal{V}, \quad Q(u, v) = \langle \mathcal{L}u, v \rangle_H$$

where

$$\text{Dom}(\mathcal{L}) = \{u \in \mathcal{V} : \text{the map } v \mapsto Q(u, v) \text{ is continuous on } \mathcal{V} \text{ for the norm of } H\},$$

satisfies the following properties:

- (i) \mathcal{L} is bijective from $\text{Dom}(\mathcal{L})$ onto H ,
- (ii) \mathcal{L} is closed.
- (iii) $\text{Dom}(\mathcal{L})$ is dense in \mathcal{V} and in H ,
- (iv) If \tilde{Q} is the adjoint sesquilinear form defined by

$$\forall u, v \in \mathcal{V}, \quad \tilde{Q}(u, v) = \overline{Q(v, u)},$$

then $\tilde{\mathcal{L}} = \mathcal{L}^*$.

Proof. By density and the Riesz theorem, \mathcal{L} is well defined.

Let us deal with (i). For all $u \in \text{Dom}(\mathcal{L})$, we have

$$\|\mathcal{L}u\| \|u\| \geq \alpha \|u\|_{\mathcal{V}}^2 \geq \alpha c \|u\|^2,$$

where $c > 0$ is such that

$$\forall u \in \mathcal{V}, \quad c \|u\| \leq \|u\|_{\mathcal{V}}.$$

We deduce that \mathcal{L} is injective.

Let us prove the surjectivity. Let $w \in H$. We look for $u \in \text{Dom}(\mathcal{L})$ such that $\mathcal{L}u = w$. This is equivalent to

$$\forall \varphi \in H, \quad \langle \mathcal{L}u, \varphi \rangle = \langle w, \varphi \rangle.$$

We notice that $\mathcal{V} \ni \varphi \mapsto \langle w, \varphi \rangle$ is continuous. Thus, there exists $v \in \mathcal{V}$ such that, for all $\varphi \in \mathcal{V}$,

$$\langle w, \varphi \rangle = \langle v, \varphi \rangle_{\mathcal{V}}.$$

We let $u = \mathcal{A}^{-1}v \in \mathcal{V}$ so that

$$\forall \varphi \in \mathcal{V}, \quad \langle w, \varphi \rangle = Q(u, \varphi).$$

We deduce that $u \in \text{Dom}(\mathcal{L})$ and

$$\forall \varphi \in \mathcal{V}, \quad \langle w, \varphi \rangle = \langle \mathcal{L}u, \varphi \rangle.$$

By density, we get $\mathcal{L}u = w$.

Therefore \mathcal{L} is bijective. We get that \mathcal{L}^{-1} is continuous and $\|\mathcal{L}^{-1}\| \leq (\alpha c)^{-1}$. Therefore \mathcal{L} is closed. This proves (ii).

Now, we prove (iii). Let $u \in \text{Dom}(\mathcal{L})^{\perp \mathcal{V}}$. We have

$$\forall v \in \text{Dom}(\mathcal{L}), \quad \langle u, v \rangle_{\mathcal{V}} = 0.$$

The operator $\mathcal{A}^* \in \mathcal{L}(\mathcal{V})$ is bijective. There exists a unique $w \in \mathcal{V}$ such that $u = \mathcal{A}^*w$. Thus,

$$\forall v \in \text{Dom}(\mathcal{L}), \quad \langle w, \mathcal{A}v \rangle_{\mathcal{V}} = 0,$$

so that

$$\forall v \in \text{Dom}(\mathcal{L}), \quad Q(v, w) = 0,$$

and therefore

$$\forall v \in \text{Dom}(\mathcal{L}), \quad \langle \mathcal{L}v, w \rangle = 0.$$

By surjectivity of \mathcal{L} , we get $w = 0$ and then $u = 0$.

Let us now deal with (iv). Let us prove that $\mathcal{L}^* \subset \widetilde{\mathcal{L}}$. Let $u \in \text{Dom}(\mathcal{L}^*)$. For all $\varphi \in \text{Dom}(\mathcal{L})$, $\langle \mathcal{L}\varphi, u \rangle = \langle \varphi, \mathcal{L}^*u \rangle$. We notice that $\mathcal{V} \ni \varphi \mapsto \langle \varphi, \mathcal{L}^*u \rangle$ is continuous for $\|\cdot\|_{\mathcal{V}}$. Thus, there exists $v \in \mathcal{V}$ such that

$$\forall \varphi \in \mathcal{V}, \quad \langle \varphi, \mathcal{L}^*u \rangle = \langle \varphi, v \rangle_{\mathcal{V}}.$$

In particular, we have

$$\forall \varphi \in \text{Dom}(\mathcal{L}), \quad \langle \mathcal{L}\varphi, u \rangle = \langle \varphi, \mathcal{L}^*u \rangle = \langle \varphi, v \rangle_{\mathcal{V}}.$$

There exists $w \in \mathcal{V}$ such that $v = \mathcal{A}^*w$ and thus

$$\forall \varphi \in \text{Dom}(\mathcal{L}), \quad \langle \mathcal{L}\varphi, u \rangle = \langle \varphi, \mathcal{L}^*u \rangle = \langle \varphi, v \rangle_{\mathcal{V}} = Q(\varphi, w) = \langle \mathcal{L}\varphi, w \rangle.$$

By surjectivity of \mathcal{L} , we get $u = w \in \mathcal{V}$. Then

$$\forall \varphi \in \text{Dom}(\mathcal{L}), \quad \overline{Q(u, \varphi)} = Q(\varphi, u) = \langle \varphi, \mathcal{L}^*u \rangle.$$

We get $u \in \text{Dom}(\widetilde{\mathcal{L}})$ and

$$\forall \varphi \in \text{Dom}(\mathcal{L}), \quad \langle \widetilde{\mathcal{L}}u, \varphi \rangle = \langle \varphi, \mathcal{L}^*u \rangle.$$

By density of $\text{Dom}(\mathcal{L})$, we deduce that $\mathcal{L}^* \subset \widetilde{\mathcal{L}}$. Let us now prove the converse inclusion. Let $u \in \text{Dom}(\widetilde{\mathcal{L}})$. We have

$$\forall \varphi \in \text{Dom}(\mathcal{L}), \quad \langle \mathcal{L}\varphi, u \rangle = Q(\varphi, u) = \overline{Q(u, \varphi)} = \overline{\langle \widetilde{\mathcal{L}}u, \varphi \rangle} = \langle \varphi, \widetilde{\mathcal{L}}u \rangle.$$

It follows that $u \in \text{Dom}(\mathcal{L}^*)$ and that $\mathcal{L}^*u = \widetilde{\mathcal{L}}u$. □

2.5. Examples.

2.5.1. Dirichlet Laplacian. Let $\Omega \subset \mathbb{R}^d$ be an open set. Here, we consider $\mathcal{V} = H_0^1(\Omega)$ and we define the sesquilinear form

$$Q^{\text{Dir}}(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + u\overline{v} \, dx.$$

The form Q^{Dir} is Hermitian, continuous, and coercive on \mathcal{V} . In Theorem 2.64, we have $\mathcal{A} = \text{Id}_{\mathcal{V}}$.

The self-adjoint operator $\mathcal{L}^{\text{Dir}} - \text{Id}$ given by Theorem 2.64 is called Dirichlet Laplacian on Ω . The domain of \mathcal{L}^{Dir} is

$$\text{Dom}(\mathcal{L}^{\text{Dir}}) = \{\psi \in H_0^1(\Omega) : -\Delta\psi \in L^2(\Omega)\}.$$

If the boundary of Ω is smooth, we have

$$\text{Dom}(\mathcal{L}) = H_0^1(\Omega) \cap H^2(\Omega).$$

This characterization of the domain is not true if the boundary is not smooth.

2.5.2. Neumann Laplacian. Let $\Omega \subset \mathbb{R}^d$ be an open set. Here, we consider $\mathcal{V} = H^1(\Omega)$ and we define the sesquilinear form

$$Q^{\text{Neu}}(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} + u\overline{v} \, dx.$$

The form Q is Hermitian, continuous, and coercive on \mathcal{V} . In Theorem 2.64, we have $\mathcal{A} = \text{Id}_{\mathcal{V}}$.

The self-adjoint operator $\mathcal{L}^{\text{Neu}} - \text{Id}$ given by Theorem 2.64 is called Neumann Laplacian on Ω . If the boundary of Ω is smooth, the domain of \mathcal{L}^{Neu} is

$$\text{Dom}(\mathcal{L}^{\text{Neu}}) = \{\psi \in H^1(\Omega) : -\Delta\psi \in L^2(\Omega), \quad \nabla\psi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

We have

$$\text{Dom}(\mathcal{L}^{\text{Neu}}) = \{\psi \in H^1(\Omega) \cap H^2(\Omega) : \nabla\psi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

This characterization of the domain is not true if the boundary is not smooth.

2.5.3. *Harmonic oscillator.* Let us consider the operator

$$\mathcal{H}_0 = (\mathcal{C}_0^\infty(\mathbb{R}), -\partial_x^2 + x^2).$$

This operator is essentially self-adjoint as we have seen in Example 2.61. Let us denote by \mathcal{H} its closure. The operator \mathcal{H} is called the harmonic oscillator. We have

$$\text{Dom}(\mathcal{H}) = \text{Dom}(\mathcal{H}_0^*) = \{\psi \in L^2(\mathbb{R}) : (-\partial_x^2 + x^2)\psi \in L^2(\mathbb{R})\}.$$

We recall Lemma 1.9.

Theorem 2.63 can be applied and $\mathcal{A} = \text{Id}$. Let us now consider Theorem 2.64 with $H = L^2(\mathbb{R})$. The assumptions are satisfied since \mathcal{V} is continuously embedded and dense in $L^2(\mathbb{R})$. The operator \mathcal{L} associated with Q is self-adjoint, its domain is

$$\text{Dom}(\mathcal{L}) = \{\psi \in B^1(\mathbb{R}) : (-\partial_x^2 + x^2)\psi \in L^2(\mathbb{R})\}.$$

The operator \mathcal{L} satisfies in particular

$$\langle (-\partial_x^2 + x^2)u, v \rangle = Q(u, v) = \langle \mathcal{L}u, v \rangle,$$

for all $u, v \in \mathcal{C}_0^\infty(\mathbb{R})$. This shows that \mathcal{L} is a self-adjoint extension of \mathcal{H}_0 . Thus, $\mathcal{L} = \mathcal{H}$.

3. SPECTRUM

3.1. Definitions and basic properties.

3.1.1. *Holomorphic functions valued in a Banach space.* Let E be a Banach space.

Definition 3.1. Let Ω be a non-empty open set in \mathbb{C} . We say that $f : \Omega \rightarrow E$ is holomorphic when, for all $z_0 \in \Omega$, the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. It is denoted by $f'(z_0)$.

Lemma 3.2. Let $A \subset E$ such that $\ell(A)$ is bounded for all $\ell \in E'$. Then A is bounded.

Proposition 3.3. Let $f : \Omega \rightarrow E$. f is holomorphic if and only if it is weakly holomorphic, i.e., $\ell \circ f$ is holomorphic on Ω for all $\ell \in E'$.

Proof. Let us assume that $\ell \circ f$ is holomorphic on Ω for all $\ell \in E'$. Let us first prove that f is continuous. Take $z_0 \in \Omega$ and define for $r > 0$ such that $D(z_0, r) \subset \Omega$,

$$A = \left\{ \frac{f(z) - f(z_0)}{z - z_0}, z \in D(z_0, r) \setminus \{z_0\} \right\} \subset E.$$

We observe that $\ell(A)$ is bounded for all $\ell \in E'$. We deduce that A is bounded. This proves the continuity of f at z_0 .

Take $z_0 \in \Omega$ and Γ a circle with center z_0 and radius r such that $\overline{D(z_0, r)} \subset \Omega$. Since f is continuous, we can define, for $z \in D(z_0, r)$,

$$F(z) = \frac{1}{2i\pi} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

By the Cauchy formula, we get, for all $\ell \in E'$ and $z \in D(z_0, r)$,

$$\ell \circ f(z) = \frac{1}{2i\pi} \int_{\Gamma} \frac{\ell \circ f(\zeta)}{\zeta - z} d\zeta.$$

Using the Riemannian sums, we find

$$\ell(f(z) - F(z)) = 0.$$

By the Hahn-Banach theorem, we deduce that $F(z) = f(z)$. From this, it is easy to show that f has a power series expansion and thus it is holomorphic. \square

By using the classical Liouville theorem, we get the following.

Corollary 3.4. *Let $f : \mathbb{C} \rightarrow E$ be holomorphic. If f is bounded, then it is constant.*

3.1.2. *Basic definitions and properties.* Let T be a closed operator on H or a bounded operator on a Banach space E .

Definition 3.5. We let

$$\rho(T) = \{z \in \mathbb{C} : T - z \text{ is bijective}\}, \quad \text{sp}(T) = \mathbb{C} \setminus \rho(T).$$

$\rho(T)$ is called the resolvent set of T . $\text{sp}(T)$ is called the spectrum of T .

Definition 3.6. An eigenvalue of T is a number $\lambda \in \mathbb{C}$ such that $\ker(T - \lambda) \neq \{0\}$. The set formed by the eigenvalues is called point spectrum.

Proposition 3.7. *In finite dimension, the spectrum coincides with the point spectrum.*

Exercise 3.8. Here $H = \mathbb{C}^n$. Let $\varepsilon \geq 0$ and define the matrix $M_n(\varepsilon) = (m_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ with $m_{n,1} = \varepsilon$, $m_{i,i+1} = 1$ for all $i \in \{1, \dots, n-1\}$, and 0 otherwise.

- i. What is the spectrum of $M_n(\varepsilon)$?
- ii. What is the behavior of the spectrum when n goes to $+\infty$?

Exercise 3.9. What are the spectra of \bar{a} and \bar{c} defined in Section 2.2.3?

Proposition 3.10. *For all $z \in \rho(T)$, $T - z$ is bijective with bounded inverse. In this case, we let $R_T(z) = (T - z)^{-1}$.*

Proposition 3.11. *$\rho(T)$ is an open set and $\rho(T) \ni z \mapsto R_T(z)$ is holomorphic.*

Lemma 3.12 (Weyl sequences). *Let us consider an unbounded closed operator $(T, \text{Dom}(T))$. Assume that there exists a sequence $(u_n) \in \text{Dom}(T)$ such that $\|u_n\|_H = 1$, (u_n) and*

$$\lim_{n \rightarrow +\infty} (T - \lambda)u_n = 0$$

in H . Then $\lambda \in \text{sp}(T)$.

A sequence (u_n) as in Lemma 3.12 is called a Weyl sequence.

Example 3.13. We let $H = L^2(I)$, with $I = (0, 1)$. Take $f \in \mathcal{C}^0([0, 1], \mathbb{C})$. We consider the operator $T : L^2(I) \ni \psi \mapsto f\psi \in L^2(I)$. T is bounded and $\|T\| \leq \|f\|_\infty$.

- i. If $\lambda \notin \text{ran}(f)$, then, the multiplication operator by $(f - \lambda)^{-1}$ is bounded and it is the inverse of T . In particular, this shows that $\text{sp}(T) \subset \text{ran}(f)$.
- ii. Let us now take $x_0 \in (0, 1)$ and let $\lambda = f(x_0)$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\|\chi\|_{L^2(\mathbb{R})} = 1$. For n large enough, we consider the sequence

$$u_n(x) = \sqrt{n}\chi(n(x - x_0)),$$

and we notice that the support of u_n is included in $[0, 1]$. Moreover, we have $\|u_n\|_H = 1$. By dominated convergence, we have

$$\lim_{n \rightarrow +\infty} (T - \lambda)u_n = 0.$$

This shows that $\lambda \in \text{sp}(T)$. We get $f(I) \subset \text{sp}(T)$. Since the spectrum is closed and f continuous, we get $f([0, 1]) \subset \text{sp}(T)$.

- iii. If λ is an eigenvalue of T , there exists $\psi \in L^2(I)$ such that $\|\psi\|_H = 1$ and $(f - \lambda)\psi = 0$. Thus the measure of $\{f = \lambda\}$ is positive. Conversely, if $A = \{f = \lambda\}$ has a non zero measure, $\mathbb{1}_A$ is not zero and satisfies $T\mathbb{1}_A = \lambda\mathbb{1}_A$.

Actually, we can generalize this last example.

Exercise 3.14. We use the notations of Example 2.30. We define the essential range of f as

$$\text{ran}_{\text{ess}}(f) = \{\lambda \in \mathbb{C} : \forall \varepsilon > 0, \mu(\{|f - \lambda| > \varepsilon\}) > 0\}.$$

- i. Prove that, if $\lambda \notin \text{ran}_{\text{ess}}(f)$, then $\lambda \in \rho(T_f)$.
- ii. Let $\lambda \in \text{ran}_{\text{ess}}(f)$ and $\varepsilon > 0$. By using $A_\varepsilon = \{|f - \lambda| > \varepsilon\}$, find a function $\psi_\varepsilon \in \text{Dom}(T_f)$ such that $\|(T_f - \lambda)\psi_\varepsilon\|_H \leq \varepsilon\|\psi_\varepsilon\|_H$.
- iii. Conclude that $\text{ran}_{\text{ess}}(f) = \text{sp}(T_f)$.

Exercise 3.15. Here $H = \ell^2(\mathbb{Z})$. We recall that $L^2(\mathbb{S}^1, \mathbb{C})$ is isometric to $\ell^2(\mathbb{Z})$ via the Fourier series and the Parseval formula.

- i. For all $u \in H$, we let, for all $n \in \mathbb{Z}$, $(S_-u)_n = u_{n-1}$. By using the result of Exercise 3.14 (or Exercise 3.13) and the Fourier series, find the spectrum of S_- . What is the point spectrum of S_- ?
- ii. For all $u \in H$, we let, for all $n \in \mathbb{Z}$, $(Tu)_n = u_{n-1} + u_{n-1}$. Find the spectrum of T .

Proposition 3.16 (Resolvent formula). *For all $z_1, z_2 \in \rho(T)$, we have*

$$R_T(z_1)R_T(z_2) = R_T(z_2)R_T(z_1),$$

and

$$(z_1 - z_2)R_T(z_1)R_T(z_2) = R_T(z_1) - R_T(z_2).$$

3.1.3. *About the bounded case.*

Definition 3.17 (Spectral radius). Let $T \in \mathcal{L}(E)$. We let

$$r(T) = \sup_{\lambda \in \text{sp}(T)} |\lambda|.$$

Proposition 3.18. *Let $T \in \mathcal{L}(E)$. Then,*

$$r(T) \leq \|T\|.$$

Moreover,

$$r(T) = \limsup_{n \rightarrow +\infty} \|T^n\|^{\frac{1}{n}}.$$

Proof. For all $z \in \mathbb{C}$ with $|z| > \|T\|$, we have

$$T - z = z(z^{-1}T - \text{Id}),$$

so that

$$(3.1) \quad R_T(z) = (T - z)^{-1} = z^{-1}(z^{-1}T - \text{Id})^{-1} = -z^{-1} \sum_{n=0}^{+\infty} T^n z^{-n},$$

If $\lambda \in \text{sp}(T)$, we have $\lambda^n \in \text{sp}(T^n)$ and thus, for all $n \in \mathbb{N}^*$,

$$r(T) \leq \|T^n\|^{\frac{1}{n}}.$$

Moreover, R_T is holomorphic on $\{z \in \mathbb{C} : |z| > r(T)\}$ (and not on the exterior of a smaller disk) so that, by the Hadamard formula,

$$r(T) \geq \limsup_{n \rightarrow +\infty} \|T^n\|^{\frac{1}{n}}.$$

□

Lemma 3.19. Let $T \in \mathcal{L}(E)$. The sequence is $(\|T^n\|^{\frac{1}{n}})_{n \in \mathbb{N}^*}$ convergent to $\inf_{n \in \mathbb{N}^*} \|T^n\|^{\frac{1}{n}}$.

Proof. We can assume that $T^n \neq 0$ for all $n \in \mathbb{N}^*$. We let $u_n = \|T^n\|$. We have

$$\forall n, p \in \mathbb{N}^*, \quad u_{n+p} \leq u_n + u_p.$$

Let $p \in \mathbb{N}^*$. We write $n = qp + r$ with $r \in [0, p)$. We have

$$u_n \leq qu_p + u_r.$$

Thus,

$$\frac{u_n}{n} \leq \frac{u_p}{p} + \frac{u_r}{n}.$$

We have, for all $p \in \mathbb{N}^*$,

$$\limsup_{n \rightarrow +\infty} \frac{u_n}{n} \leq \frac{u_p}{p}.$$

□

Proposition 3.20. If $T \in \mathcal{L}(E)$, then $\text{sp}(T) \neq \emptyset$.

Proof. We use Proposition 3.11 and (3.1) to see that, if $\rho(T) = \mathbb{C}$, R_T is bounded on \mathbb{C} . Then, we apply Corollary 3.4 to see that R_T is constant. We again use (3.1) to notice that R_T goes to 0 at infinity. So $R_T = 0$ and this is a contradiction. □

3.1.4. Spectrum of the adjoint.

Proposition 3.21. Let $(\text{Dom}(T), T)$ be a closed and densely defined operator. Then, $T : \text{Dom}(T) \rightarrow H$ is bijective if and only if $T^* : \text{Dom}(T^*) \rightarrow H$ is bijective. In this case, $(T^*)^{-1} = (T^{-1})^*$.

Proof. Assume that T is bijective. Since $\text{ran}(T) = H$, we get that $\ker(T^*) = \{0\}$. Moreover, we also get that $\text{ran}(T^*)$ is dense in H since $\ker(T) = \{0\}$. Consider the bounded operator $T^{-1} = H \rightarrow H$ (by the Banach theorem). Its adjoint $(T^{-1})^* : H \rightarrow H$ is also bounded:

$$\forall y \in H, \quad \|(T^{-1})^*y\| \leq C\|y\|.$$

If $x \in \text{Dom}(T^*)$ and $v \in H$, we have

$$\langle (T^{-1})^*T^*x, v \rangle = \langle T^*x, T^{-1}v \rangle = \langle x, TT^{-1}v \rangle = \langle x, v \rangle,$$

so that

$$(T^{-1})^*T^* = \text{Id}_{\text{Dom}(T^*)}.$$

We deduce that $\text{ran}(T^*)$ is closed. Thus T^* is bijective.

If T^* is bijective, we use Proposition 2.34 to get $T^{**} = T$. Thus, T is bijective. Note also that, for all $u \in H$ and $v \in \text{Dom}(T)$,

$$\langle (T^{-1})^*u, Tv \rangle = \langle u, v \rangle,$$

so that $(T^{-1})^*u \in \text{Dom}(T^*)$ and $T^*(T^{-1})^* = \text{Id}_H$.

□

Corollary 3.22. Let $(\text{Dom}(T), T)$ be a closed and densely defined operator. Then, $\overline{\text{sp}(T)} = \text{sp}(T^*)$. Here the bar denotes the complex conjugation.

Proof. Let $z \in \mathbb{C}$ and apply Proposition 3.21 to $T - z$. □

3.2. Spectral radius and resolvent bound in the self-adjoint case.

Definition 3.23 (Normal operator). Let $T \in \mathcal{L}(H)$. T is normal when $TT^* = T^*T$.

Proposition 3.24. Let $T \in \mathcal{L}(H)$ be a normal operator. Then,

$$r(T) = \|T\|.$$

Proof. Let us start to deal with the case when $T = T^*$. We have $\|T^2\| = \|T\|^2$. Indeed, we have, for all $S \in \mathcal{L}(H)$,

$$\|S\| = \sup_{u \neq 0, v \neq 0} \frac{|\langle Su, v \rangle|}{\|u\| \|v\|}.$$

For $S = T^2 = T^*T$, we find

$$\|T^2\| = \sup_{u \neq 0, v \neq 0} \frac{|\langle Tu, Tv \rangle|}{\|u\| \|v\|} \geq \sup_{u \neq 0} \frac{\|Tu\|^2}{\|u\|^2} = \|T\|^2.$$

But, we obviously have $\|T^2\| \leq \|T\|^2$. By Lemma 3.19, we have

$$\lim_{n \rightarrow +\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow +\infty} \|T^{2^n}\|^{\frac{1}{2^n}} = \|T\|.$$

Let us now assume that T is normal. Note that T^*T is self-adjoint so that $r(T^*T) = \|T^*T\| = \|T\|^2$. We have used that the general facts that $\|S^*\| = \|S\|$, and $\|S^*S\| = \|S\|^2$. Since T is normal, we have

$$r(T^*T) = \lim_{n \rightarrow +\infty} \|(T^*T)^n\|^{\frac{1}{n}} = \lim_{n \rightarrow +\infty} \|(T^n)^*(T^n)\|^{\frac{1}{n}} = \left(\lim_{n \rightarrow +\infty} \|(T^n)\|^{\frac{1}{n}} \right)^2 = r(T)^2.$$

□

Corollary 3.25. Let $T \in \mathcal{L}(H)$ be a normal operator. If $\text{sp}(T) = \{0\}$, then $T = 0$.

3.2.1. Resolvent bounds.

Proposition 3.26. Let $T \in \mathcal{L}(H)$ be a normal operator. For all $z \notin \text{sp}(T)$, we have

$$\|(T - z)^{-1}\| = \frac{1}{\text{dist}(z, \text{sp}(T))}.$$

Exercise 3.27. Consider that $H = \mathbb{C}^d$, with $d \geq 2$ and equipped with the canonical scalar product.

- i. Let $T \in \mathcal{L}(H)$. We assume that $d \geq 3$ and that, for all strict subspace F of H such that $T(F) \subset F$, $T|_F$ is normal.
 - a. Assume that T has at least two distinct eigenvalues. By using the decomposition in characteristic subspaces, show that T is diagonalizable. Prove then that the characteristic subspaces are orthogonal.
 - b. Assume that T has only one eigenvalue λ and let $N = T - \lambda \text{Id}$. Prove that $N = 0$.
 - c. Conclude that T is normal.
- ii. Let $T \in \mathcal{L}(H)$ be a non-normal operator.
 - a. Show that there exists $F \subset H$ of dimension two and invariant by T such that $S := T|_F$ is non-normal.
 - b. Prove that there exists a (z_n) sequence (in the resolvent set of S) converging to an element λ in the spectrum of S and such that

$$\|(S - z_n)^{-1}\| > \frac{1}{\text{dist}(z_n, \text{sp}(S))}.$$

c. Deduce that there exists z in the resolvent set of T such that

$$\|(T - z)^{-1}\| > \frac{1}{\text{dist}(z, \text{sp}(T))}.$$

Proposition 3.28. *Let $(T, \text{Dom}(T))$ be a self-adjoint operator. For all $z \notin \text{sp}(T)$, we have*

$$\|(T - z)^{-1}\| = \frac{1}{\text{dist}(z, \text{sp}(T))}.$$

Proof. Let $z \notin \text{sp}(T)$. We have $((T - z)^{-1})^* = (T - \bar{z})^{-1}$. The operators $(T - \bar{z})^{-1}$ and $(T - z)^{-1}$ commute. Thus $(T - z)^{-1}$ is normal and

$$\|(T - z)^{-1}\| = r((T - z)^{-1}) = \frac{1}{\text{dist}(z, \text{sp}(T))},$$

since

$$\text{sp}((T - z)^{-1}) = \{(\lambda - z)^{-1}, \lambda \in \text{sp}(T)\}.$$

□

3.3. About the Riesz projections.

3.3.1. Properties.

Proposition 3.29. *Let us consider an unbounded closed operator $(T, \text{Dom}(T))$ and λ an isolated element of $\text{sp}(T)$. Let Γ_λ be a contour that enlases only λ as element of the spectrum of T and define*

$$P_\lambda := \frac{1}{2i\pi} \int_{\Gamma_\lambda} (z - T)^{-1} dz.$$

The bounded operator $P_\lambda : \mathbb{H} \rightarrow \text{Dom}(T) \subset \mathbb{H}$ commutes with T and does not depend on the choice of Γ_λ . P_λ is a projection and

$$(3.2) \quad P_\lambda - \text{Id} = \frac{1}{2i\pi} \int_{\Gamma_\lambda} (\zeta - \lambda)^{-1} (T - \lambda) (\zeta - T)^{-1} d\zeta.$$

We say that λ has finite algebraic multiplicity when the rank of P_λ is finite.

Proof. We notice P_λ is well defined and bounded since the integral is understood in the Riemannian sense. Then P_λ commutes with T since T is closed and the integral can be approximated by the Riemannian sums. The fact that P_λ does not depend on the contour enlasing λ comes from the holomorphy of the resolvent.

There exist $0 < r < \tilde{r}$ such that

$$P_\lambda = \frac{1}{2i\pi} \int_{C(\lambda, r)} (z - T)^{-1} dz = \frac{1}{2i\pi} \int_{C(\lambda, \tilde{r})} (z - T)^{-1} dz.$$

Thus,

$$P_\lambda^2 = \frac{1}{(2i\pi)^2} \int_{z \in C(\lambda, r)} \int_{w \in C(\lambda, \tilde{r})} R_T(z) R_T(w) dw dz,$$

and, by the resolvent formula and the choice of r and \tilde{r} ,

$$P_\lambda^2 = \frac{1}{(2i\pi)^2} \int_{w \in C(\lambda, \tilde{r})} \int_{z \in C(\lambda, r)} \frac{R_T(z) - R_T(w)}{z - w} dz dw.$$

Thus,

$$P_\lambda^2 = \frac{1}{(2i\pi)^2} \int_{z \in C(\lambda, r)} \int_{w \in C(\lambda, \tilde{r})} \frac{R_T(z)}{z - w} dw dz = \frac{2i\pi}{(2i\pi)^2} \int_{w \in C(\lambda, r)} R_T(z) dz = P_\lambda.$$

□

Lemma 3.30. *Let us consider an unbounded closed operator $(T, \text{Dom}(T))$ and λ an isolated element of $\text{sp}(T)$. Then we have either $1 \in \text{sp}(P_\lambda^*)$ or $1 \in \text{sp}(P_\lambda)$. In any case, we have $P_\lambda \neq 0$.*

Proof. Before starting the proof, let us observe that $\lambda \in \text{sp}(T)$ iff $\bar{\lambda} \in \text{sp}(T^*)$.

We have just to consider the following cases:

- i. $T - \lambda$ is injective with closed range. We have $\ker(T^* - \bar{\lambda}) \neq \{0\}$ and we consider $0 \neq u \in \ker(T^* - \bar{\lambda})$. We have

$$P_\lambda^* = \frac{1}{2i\pi} \int_{\Gamma_\lambda} (\zeta - T^*)^{-1} d\zeta.$$

We apply Formula (3.2) to $\bar{\lambda}$, $\overline{\Gamma_\lambda}$ and T^* to get that $P_\lambda^* u = u$.

- ii. or there exists a Weyl sequence (u_n) associated with λ : the sequence $((T - \lambda)u_n)$ goes to zero and $\|u_n\| = 1$. Again with Formula (3.2), we have $(P_\lambda - \text{Id})u_n \rightarrow 0$, and thus $1 \in \text{sp}(P_\lambda)$ (by Lemma 3.12).

□

3.3.2. About the finite dimension. In this section, we discuss the case when H is finite dimensional and $T \in \mathcal{L}(H)$.

Proposition 3.31. *Let $\lambda \in \text{sp}(T)$. Then, λ is an eigenvalue. If Γ_λ is a contour enlacing only λ , then P_λ is the projection on the algebraic eigenspace associated with λ .*

Proof. It is well known that $H = \bigoplus_{j=1}^k H_j$ where the H_j are the algebraic eigenspaces associated with the distinct eigenvalues. They are stable under T . We can assume that H_1 is associated with λ . There exists a basis of H such that the matrix of T is block diagonal (T_1, \dots, T_k) where the T_j is the (upper triangular) matrix of $T|_{H_j}$. In this adapted basis, the matrix of P_λ is also block diagonal $(P_{\lambda,1}, \dots, P_{\lambda,k})$. By holomorphy, we have $P_{\lambda,j} = 0$ when $j \neq 1$. By considering the structure of the triangular matrix of $(T_1 - z)^{-1}$, we see that $P_{\lambda,1} = \text{Id}$. Therefore P_λ is the projection on H_1 and associated with the direct sum. □

Let us come back to the infinite dimensional situation.

Corollary 3.32. *If $\lambda \in \text{sp}(T)$ is isolated with finite algebraic multiplicity, then it is an eigenvalue.*

Proof. The projection $P = P_\lambda$ commutes with T . Thus we may write

$$T = T|_{\text{ran } P} \oplus T|_{\ker P}.$$

The spectrum of T is the union of the corresponding spectra and λ is still isolated in these spectra.

By definition, we have

$$\frac{1}{2i\pi} \int_{\Gamma} (\zeta - T|_{\ker P})^{-1} d\zeta = 0.$$

Thus, λ does not belong to $\text{sp}(T|_{\ker P})$. Therefore, λ belongs to the spectrum of the "matrix" $T|_{\text{ran } P}$ and it is an eigenvalue. □

3.3.3. Fredholm operators: definition and first properties.

Definition 3.33. Let E and F two Banach spaces. An application $T \in \mathcal{L}(E, F)$ is said to be Fredholm when $\dim \ker T < +\infty$, $\text{codim } \text{ran } T < +\infty$. By definition, we call index of T the following number

$$\text{ind } T = \dim \ker(T) - \text{codim } \text{ran}(T).$$

The set of the Fredholm operators from E to F is denoted by $\text{Fred}(E, F)$.

Example 3.34. Consider $H = \ell^2(\mathbb{N})$ and, for $u \in H$, define Tu by $(Tu)_n = u_{n+1}$ for all $n \in \mathbb{N}$. T is a Fredholm operator of index 1.

We will see in Lemma 5.1 that $\text{ran } T$ is closed, but we can already give an elementary proof.

Proposition 3.35. Let $T \in \text{Fred}(E, F)$. Then $\text{ran } T$ is closed.

Proof. Let us write $E = \ker T \oplus \tilde{E}$, with \tilde{E} closed. Then, $T : \tilde{E} \rightarrow F$ is injective. Let us also write $F = \text{ran } T \oplus \tilde{F}$, with \tilde{F} finite dimensional. Consider a basis $(f_j)_{1 \leq j \leq n}$ of \tilde{F} .

Let us consider the application

$$S : \tilde{E} \times \mathbb{C}^n \ni (x, v) \mapsto Tx + \sum_{j=1}^n v_j f_j \in F.$$

S is continuous and bijective between two Banach spaces. Thus, its inverse is continuous and there exists $C > 0$ such that, for all $f \in F$,

$$\|S^{-1}f\|_{\tilde{E} \times \mathbb{C}^n} \leq C\|f\|_F,$$

and, for all $(x, v) \in \tilde{E} \times \mathbb{C}^n$,

$$\|x\|_E + \|v\|_{\mathbb{C}^n} \leq C\|S(x, v)\|_F.$$

Taking $v = 0$, we easily deduce that $\text{ran } T$ is closed. □

In the case of an unbounded operator $T : \text{Dom}(T) \subset E \rightarrow F$, we say that T is Fredholm when T is closed and $T \in \mathcal{L}(\text{Dom}(T), \|\cdot\|_T, F)$ is Fredholm.

Proposition 3.36. In the case when E and F have finite dimension, we have $T \in \text{Fred}(E, F)$ and $\text{ind } T = \dim E - \dim F$.

Proposition 3.37. Let $T \in \mathcal{L}(E, F)$. Then, T is Fredholm if and only if $\dim \ker T < +\infty$ and $\dim \ker T' < +\infty$, and $\text{ran}(T)$ is closed. In this case, we have

$$\text{ind } T = \dim \ker(T) - \dim \ker(T').$$

Proof. By Propositions 2.19 and 2.22, we have $\ker(T') = \text{ran}(T)^\perp$ and $\dim \text{ran}(T)^\perp = \text{codim } \text{ran}(T)$ since the range of T is closed (see Proposition 3.35). □

The following consequence can actually be proved directly.

Proposition 3.38. Let $(T, \text{Dom}(T))$ be a closed operator on H . T is a Fredholm operator when $\dim \ker(T) < +\infty$, $\dim \ker(T^*) < +\infty$, and $\text{ran}(T)$ is closed. The index of T is

$$\text{ind } T = \dim \ker(T) - \dim \ker(T^*).$$

A remarkable property is the following.

Proposition 3.39. Let $T \in \text{Fred}(E, F)$ with index 0. Then, T is injective if and only if T is surjective.

3.3.4. Spectrum and Fredholm operators.

Definition 3.40. We define

- i. essential spectrum: $\lambda \in \text{sp}_{\text{ess}}(T)$ if $T - \lambda$ is not Fredholm with index 0 from $\text{Dom}(T)$ into H ,
- ii. discrete spectrum: $\lambda \in \text{sp}_{\text{dis}}(T)$ if λ is isolated in the spectrum of T , with finite algebraic multiplicity and such that $\text{ran}(T - \lambda)$ is closed.

Note that $\text{sp}_{\text{ess}}(T) \subset \text{sp}(T)$. We will illustrate these general definitions in the next sections, especially in the case of self-adjoint operators when these sets are complementary.

4. COMPACT OPERATORS

4.1. Reminders. The proofs of the following reminded results can be found in [LB03, Chapitre 3] or in [Bre83, Chapter VI].

Definition 4.1. Let E and F be two Banach spaces. A linear map T is said to be compact when $T(B_E(0, 1))$ is relatively compact (or, equivalently, precompact) in F .

Proposition 4.2. *The following assertions are equivalent.*

- i. $T \in \mathcal{K}(E, F)$ is compact.
- ii. For all $B \subset E$ with B bounded, $T(B)$ is relatively compact in F .
- iii. For all bounded sequence $(u_n) \in E^{\mathbb{N}}$, (Tu_n) has a convergent subsequence.

Proposition 4.3. $\mathcal{K}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$.

Proposition 4.4. $\mathcal{K}(E, F)$ is a left and right ideal of $\mathcal{L}(E, F)$.

Proposition 4.5. If $T \in \mathcal{L}(E, F)$ has finite rank, it is compact.

Proposition 4.6. If $T \in \mathcal{K}(E, F)$ is compact, it transforms weakly convergent sequences into convergent sequences. The converse is true when E is reflexive.

Proposition 4.7. $T \in \mathcal{L}(E, F)$ is a compact operator if and only if $T' \in \mathcal{L}(F', E')$ is a compact operator.

Proposition 4.8. Let $T \in \mathcal{K}(E)$ be a compact operator. Then $\text{Id}_E + K$ is Fredholm.

Proof. By the Riesz theorem, we have that $\dim \ker(\text{Id}_E + K) < +\infty$. By Proposition 4.7, we have $T' \in \mathcal{K}(E')$ and thus $\dim \ker(\text{Id}_{E'} + K') < +\infty$. Then, let us show that $\text{ran}(\text{Id}_E + K)$ is closed. Let us consider a sequence (u_n) such that $(u_n + Tu_n)$ converges to f . We let

$$d_n = \text{dist}(u_n, \ker(\text{Id}_E + K)),$$

and we notice that there exists v_n such that $d_n = \|u_n - v_n\|$. Let us show that (d_n) is bounded. If it were not the case, up to a subsequence extraction, we could assume that (d_n) tends to $+\infty$. We write

$$u_n + Tu_n = u_n - v_n + T(u_n - v_n).$$

Letting

$$w_n = \frac{u_n - v_n}{d_n},$$

we would get $(w_n + Tw_n)$ converges to 0. By compactness of T , we can assume that (w_n) converges to some $w \in \ker(\text{Id}_E + K)$. But, we have

$$\text{dist}(w_n, \ker(\text{Id}_E + K)) = 1,$$

and we get a contradiction. Thus, (d_n) is bounded and we can assume that $(u_n - v_n)$ converges, and the closedness of the range follows.

We conclude with Proposition 3.37. □

4.2. A compactness criterion. In order to prove that an operator is compact, the following criterion of relative compactness in $L^p(\Omega)$ will be useful. The proof of this criterion is based on the Ascoli theorem (see [Bre83, Section IV.5]).

Theorem 4.9 (Riesz-Fréchet-Kolmogorov). *Let $\Omega \subset \mathbb{R}^N$ be an open set and \mathcal{F} a bounded subset of $L^p(\Omega)$, with $p \in [1, +\infty)$. We assume that*

$$\forall \varepsilon > 0, \exists \omega \subset\subset \Omega, \quad \forall f \in \mathcal{F}, \quad \|f\|_{L^p(\Omega \setminus \omega)} \leq \varepsilon$$

and that

$$\forall \varepsilon > 0, \forall \omega \subset\subset \Omega, \exists \delta > 0, \quad \delta < \text{dist}(\omega, \mathbb{C}\Omega), \quad \forall |h| \leq \delta, \quad \forall f \in \mathcal{F}, \quad \|\tau_h f\|_{L^p(\omega)} \leq \varepsilon,$$

where $\tau_h f(x) = f(x+h) - f(x)$. Then, \mathcal{F} is relatively compact in $L^p(\Omega)$.

How to get the control of the translations in practice? By using a density argument and the Taylor formula, we can get the following proposition (see [Bre83, Proposition IX.3]).

Proposition 4.10. *Let $p \in (1, +\infty]$ and $u \in L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$ if and only if there exists $C > 0$ such that, for all $\omega \subset\subset \Omega$ and $h \in (0, \text{dist}(\omega, \mathbb{C}\Omega))$, we have*

$$\|\tau_h u\|_{L^p(\omega)} \leq C|h|.$$

In this case, we can take $C = \|\nabla u\|_{L^p(\Omega)}$. If $p = 1$ and $u \in W^{1,1}(\Omega)$, we still have

$$\|\tau_h u\|_{L^1(\omega)} \leq \|\nabla u\|_{L^1(\Omega)}|h|.$$

Working slightly more, we can prove the following important theorem [Bre83, Theorem IX.16].

Theorem 4.11. *Let Ω be a bounded open subset of \mathbb{R}^d with \mathcal{C}^1 boundary. Let $p \in [1, +\infty]$. The injection $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact.*

Exercise 4.12. Consider the operator $\mathcal{L} = -\Delta$ with domain $H^2(\mathbb{R}^d)$ and take $\lambda \in \mathbb{R}_-$.

- i. Show that $\lambda \in \rho(\mathcal{L})$.
- ii. Consider then a function $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{C})$ such that ∇V is bounded and $\lim_{|x| \rightarrow +\infty} V(x) = 0$. Prove that $V(\mathcal{L} - \lambda)^{-1} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is compact.

Exercise 4.13. Consider

$$B^1(\mathbb{R}) = \{\psi \in H^1(\mathbb{R}) : x\psi \in L^2(\mathbb{R})\} \subset L^2(\mathbb{R}).$$

Prove that the injection of $B^1(\mathbb{R})$ in $L^2(\mathbb{R})$ is a compact operator.

Actually, there is a direct proof that the injection of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact (for any Ω).

Lemma 4.14. *Let Ω be an open set in \mathbb{R}^d . For all $u \in H_0^1(\Omega)$, consider its extension by zero outside Ω , denoted by \underline{u} . Then $\underline{u} \in H_0^1(\mathbb{R}^d)$ and $\|\underline{u}\|_{H^1(\mathbb{R}^d)} = \|u\|_{H^1(\Omega)}$.*

Proof. Clearly, $\underline{u} \in L^2(\mathbb{R}^d)$ and $\|\underline{u}\|_{L^2(\mathbb{R}^d)} = \|u\|_{L^2(\Omega)}$. We know that, by definition, $\mathcal{C}_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. Consider a sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\Omega)$ converging to u in H^1 -norm. For all $n \in \mathbb{N}$, we have $\underline{u}_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. For all $n, p \in \mathbb{N}$, we have

$$\|u_n - u_p\|_{H^1(\Omega)} = \|\underline{u}_n - \underline{u}_p\|_{H^1(\mathbb{R}^d)}.$$

Thus, (\underline{u}_n) is a Cauchy sequence in $H^1(\mathbb{R}^d)$. We deduce that (\underline{u}_n) converges in $H^1(\mathbb{R}^d)$ to some $v \in H^1(\mathbb{R}^d)$. We have $v = \underline{u}$ and the equality of the norms. \square

Theorem 4.15 (Kato-Rellich). *Let Ω be an open set in \mathbb{R}^d . The injection of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact.*

Proof. Let us prove that, if $(u_n)_{n \in \mathbb{N}}$ weakly converges to u in $H_0^1(\Omega)$, it strongly converges to u in $L^2(\Omega)$. The sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Let $\varepsilon > 0$.

For all $n \in \mathbb{N}$, we let $f_n = \widehat{u_n}$ and we define $f = \widehat{u}$. By the Parseval formula, it is sufficient to show that f_n converges to f in $L^2(\mathbb{R}^d)$.

We notice that, for all $\xi \in \mathbb{R}^d$,

$$f_n(\xi) = \int_{\Omega} u_n(x) e^{-ix \cdot \xi} dx,$$

so that

$$|f_n(\xi)| \leq |\Omega|^{\frac{1}{2}} \|u_n\|_{L^2(\Omega)} \leq C.$$

We recall that $(u_n)_{n \in \mathbb{N}}$ weakly converges to u in $H_0^1(\Omega)$ and, in particular, for all $\varphi \in L^2(\Omega)$,

$$\int_{\Omega} u_n \overline{\varphi} dx \rightarrow \int_{\Omega} u \overline{\varphi} dx.$$

We choose $\varphi(x) = e^{ix \cdot \xi}$ and thus, for all $\xi \in \mathbb{R}^d$, $f_n(\xi) \rightarrow f(\xi)$.

Moreover, we have

$$\|u_n\|_{H^1(\Omega)}^2 = \|\underline{u_n}\|_{H^1(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \langle \xi \rangle^2 |f_n(\xi)|^2 d\xi.$$

In particular, there exists $R > 0$ such that, for all $n \in \mathbb{N}$,

$$\int_{|\xi| > R} |f_n(\xi)|^2 d\xi \leq \varepsilon.$$

Up to changing R , we also have

$$\int_{|\xi| > R} |f(\xi)|^2 d\xi \leq \varepsilon.$$

Let us now write

$$\|f_n - f\|_{L^2(\mathbb{R}^d)}^2 = \int_{|x| \leq R} |f_n(\xi) - f(\xi)|^2 d\xi + \int_{|x| > R} |f_n(\xi) - f(\xi)|^2 d\xi.$$

We deal with the first integral by using the dominated convergence theorem (the sequence (f_n) is uniformly bounded). \square

4.3. Operators with compact resolvent. In practice, we meet unbounded and closed operators. To describe the spectrum of such operators, we can consider their resolvents (which are bounded) and prove, in good situations, that they are compact. The next propositions explain that it is sufficient to prove that the injection of the domain in the ambient Hilbert space is compact.

Proposition 4.16. *Let $(T, \text{Dom}(T))$ be a closed operator and $z_0 \in \rho(T)$. If $(T - z_0)^{-1}$ is compact, then, for all $z \in \rho(T)$, $(T - z)^{-1}$ is compact.*

Proof. It follows from the resolvent formula (Proposition 3.16) and from the fact that the algebra of compact operators is an ideal of $\mathcal{L}(\mathbb{H})$. \square

Let us provide a useful criterion for the compactness of a resolvent.

Proposition 4.17. *A closed operator $(T, \text{Dom}(T))$ has compact resolvent if and only if the injection $(\text{Dom}(T), \|\cdot\|_T) \hookrightarrow \mathbb{H}$ is compact.*

Proof. Assume that the injection is compact. Thanks to the closed graph theorem, for $z \notin \text{sp}(T)$, $(T - z)^{-1} : (\mathbb{H}, \|\cdot\|_{\mathbb{H}}) \rightarrow (\text{Dom}(T), \|\cdot\|_T)$ is bounded. The conclusion follows since the compact operators form an ideal in the algebra of bounded operators.

Conversely, assume that the resolvent is compact. Take $z_0 \in \rho(T)$ and consider

$$\{u \in \text{Dom}(T) : \|u\| + \|Tu\| \leq 1\} \subset \{u \in \text{Dom}(T) : \|u\| + \|(T - z_0)u\| \leq 1 + |z_0|\},$$

and then

$$\{u \in \text{Dom}(T) : \|u\| + \|(T - z_0)u\| \leq 1 + |z_0|\} \subset (T - z_0)^{-1}(B(0, 1 + |z_0|)).$$

□

Proposition 4.18. *Consider two Hilbert spaces \mathcal{V} and \mathbb{H} such that $\mathcal{V} \subset \mathbb{H}$ with continuous injection and with \mathcal{V} dense in \mathbb{H} . Assume that Q is a continuous, coercive and Hermitian sesquilinear form on \mathcal{V} and let T be the self-adjoint operator associated with Q . Let us denote by $\|\cdot\|_Q$ the norm induced by Q , i.e., $\|u\|_Q = \sqrt{Q(u, u)}$, and by $\|\cdot\|_T$ the graph norm on $\text{Dom}(T)$.*

If $(\text{Dom}(Q), \|\cdot\|_Q) \hookrightarrow \mathbb{H}$ is compact then T has compact resolvent.

Proof. It is sufficient to notice that, by the Cauchy-Schwarz inequality, $(\text{Dom}(T), \|\cdot\|_T) \rightarrow (\text{Dom}(Q), \|\cdot\|_Q)$ is bounded. □

Remark 4.19. The converse is true. See Exercise 6.5.

Exercise 4.20. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Prove that the Dirichlet Laplacian on Ω has compact resolvent.

Exercise 4.21. Prove that the harmonic oscillator defined in Section 2.5.3 has compact resolvent.

5. FREDHOLM THEORY

In this section, we follow [Zwo12, Appendix D].

5.1. Grushin formalism. In this section, we consider two Banach spaces X_1 and X_2 .

Lemma 5.1. *Let $T \in \mathcal{L}(X_1, X_2)$ be a Fredholm operator. We write $X_1 = \ker(T) \oplus \tilde{X}_1$ (with \tilde{X}_1 a closed subspace) and $X_2 = \text{ran}(T) \oplus \tilde{X}_2$.*

We let $n_+ = \dim \ker(T)$ and $n_- = \text{codim} \text{ran}(T) = \dim \tilde{X}_2$. We introduce $(k_j)_{1 \leq j \leq n_+}$ a basis of $\ker(T)$ and $(k'_j)_{1 \leq j \leq n_-}$ a basis of \tilde{X}_2 . We let

$$\mathcal{M} = \begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix},$$

where $R_- : \mathbb{C}^{n_-} \rightarrow X_2$ is defined by $R_- \alpha = \sum_{j=1}^{n_-} \alpha_j k'_j$ and $R_+ : X_1 \rightarrow \mathbb{C}^{n_+}$ is defined by $R_+(u) = (k_j^(u))_{1 \leq j \leq n_+}$. Then, $\mathcal{M} : X_1 \times \mathbb{C}^{n_-} \rightarrow X_2 \times \mathbb{C}^{n_+}$ is bijective (with bounded inverse) and the range of T is closed.*

Proof. Let us consider $(f, d) \in X_2 \times \mathbb{C}^{n_+}$ and look for $(e, c) \in X_1 \times \mathbb{C}^{n_-}$ such that $\mathcal{M} \begin{pmatrix} e \\ c \end{pmatrix} = \begin{pmatrix} f \\ d \end{pmatrix}$. This is equivalent to

$$Te = f - R_-c, \quad R_+e = d,$$

f can be uniquely written as $f = g + f_0$ with $g \in \text{ran} T$ and $f_0 \in \tilde{X}_2$. Thus, we must choose for c the coordinates of f_0 in the basis $(k'_j)_{1 \leq j \leq n_-}$. Now, we can write $e = k + e_0$ with $k \in \ker T$ and $e_0 \in \tilde{X}_1$. The constraint $R_+e = d$ means that the coordinates of k in the basis

$(k_j)_{1 \leq j \leq n_+}$ are d . Then, we are reduced to solve $Te_0 = f - R_-c$, but T induces a bijection from $\tilde{X}_1 \rightarrow \text{ran } T$. Therefore, e_0 is uniquely determined.

To see that the range of T is closed, we write $\text{ran } (T) = T\tilde{X}_1 = \begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} \tilde{X}_1 \\ 0 \end{pmatrix}$.

□

Lemma 5.2. *Let $T \in \mathcal{L}(X_1, X_2)$ and consider the operator matrix*

$$\begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix},$$

with $R_- : \mathbb{C}^{n_-} \rightarrow X_2$ and $R_+ : X_1 \rightarrow \mathbb{C}^{n_+}$ bounded. Assume that \mathcal{M} is bijective. We denote by \mathcal{E} its (bounded) inverse:

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_0 \end{pmatrix}.$$

Then, T is a Fredholm operator and we have $\text{ind } T = \text{ind } (E_0) = n_+ - n_-$ and T is bijective if and only if E_0 is bijective.

Proof. We write that \mathcal{E} is the inverse on the right:

$$\begin{aligned} TE + R_-E_- &= \text{Id} \\ R_+E_+ &= \text{Id} \\ TE_+ + R_-E_0 &= 0 \\ R_+E &= 0 \end{aligned}$$

and on the left:

$$\begin{aligned} ET + E_+R_+ &= \text{Id} \\ E_-R_- &= \text{Id} \\ E_-T + E_0R_+ &= 0 \\ ER_- &= 0. \end{aligned}$$

From this, we get that R_+ and E_- are surjective and that R_- and E_+ are injective. By elementary considerations, we see that if T is bijective, E_0 must be so. Conversely, suppose that E_0 is bijective. Then, consider

$$(5.1) \quad E - E_+E_0^{-1}E_-,$$

and check that it is the inverse of T .

Let us finally discuss the result about the Fredholm property and the index. We can check that the injective application E_+ sends $\ker E_0$ into $\ker(T)$ and that $E_+ : \ker E_0 \rightarrow \ker(T)$ is a bijection. Let us consider a subspace H such that $\mathbb{C}^{n_-} = \text{ran } E_0 \oplus H$. We recall that $E_- : \mathbb{C}^{n_-} \rightarrow \mathbb{C}^{n_-}$ is surjective and notice that $E_- : \text{ran } (T) \rightarrow \text{ran } E_0$. Then, consider the induced map, denoted by E^\sharp , $\tilde{X}_2 \ni x \mapsto \Pi_H E_-(x) \in H$. E^\sharp is surjective. E^\sharp is injective. Indeed, if $\Pi_H E_-v = 0$ with $v \in \tilde{X}_2$, we have $E_-v \in \text{ran } E_0$ so that we can write $E_-v = E_0w$ and we deduce that $v \in \text{ran } (T)$ and thus $v = 0$.

We deduce that T is Fredholm and that

$$\text{ind } T = \text{ind } E_0 = n_+ - n_-.$$

□

5.2. On the spectrum of compact operators. In the following theorem, we recall some fundamental facts about compact operators. In particular, we will notice that the non-zero spectrum of a compact operator is discrete.

Theorem 5.3 (Fredholm alternative). *Let $T \in \mathcal{L}(E)$ be a compact operator. Then, we have*

- (i) *If E is of infinite dimension, then $0 \in \text{sp}(T)$.*
- (ii) *For all $z \in U = \mathbb{C} \setminus \{0\}$, $T - z$ is a Fredholm operator of index 0.*
- (iii) *$\ker(T - \text{Id}) = \{0\}$ if and only if $\text{range}(T - \text{Id}) = E$.*
- (iv) *The elements of $\text{sp}(T) \setminus \{0\}$ are isolated with finite algebraic multiplicity and the only possible accumulation point of the spectrum is 0.*
- (v) *The non-zero spectrum of T is discrete.*

Proof. The point (i) is a consequence of the fact that the set of compact operators forms a ideal of bounded operators and from the Riesz theorem. Let us consider the point (ii). By Proposition 4.8, $T - z$ is a Fredholm operator.

Then, by Lemma 5.2 applied to the operator $T + P$ (with $P = (z_1 - z)\text{Id}$ and $T = T - z_1$, $z_1 \in U$), the application $U \ni z \mapsto \text{ind}(T - z\text{Id})$ is locally constant and thus constant since U is connected. For z large enough, we know that $T - z\text{Id}$ is bijective and thus of index 0. From this, we deduce the point (iii). Let us now prove the point (iv). Let us introduce

$$V := \{z \in U : \exists r > 0 : D(z, r) \subset \text{sp}(T)\}.$$

V is open by definition. Let us prove that is closed in U . Let us consider a sequence $V \ni z_n \rightarrow z_\infty \in U$. We apply again Lemma 5.2 (with $P = (z - z_\infty)\text{Id}$ and $T = T - z_\infty$). In a neighborhood of z_∞ , $T - z$ is not bijective if and only if $\det E_0(z) = 0$. But $\det E_0$ is holomorphic in a neighborhood of z_∞ . Therefore, its zeros are isolated unless $\det E_0 = 0$. By definition z_∞ , we must have $\det E_0 = 0$ in a neighborhood of z_∞ . Thus $z_\infty \in V$. We deduce that $V = U$ or $V = \emptyset$ and we get that $V = \emptyset$. Now let us consider $z_1 \in \text{sp}(T) \setminus \{0\}$. Then, in a neighborhood of z_1 , $T - z$ is not bijective if and only if $\det E_0(z) = 0$. Since $V = \emptyset$, $\det E_0$ is not zero near z_1 and thus (by holomorphy), its zeros are isolated. Finally, we recall (5.1) and thus we have, near each point of the spectrum in U ,

$$(T - z)^{-1} = E(z) - E_+(z)E_0^{-1}(z)E_-(z),$$

and we deduce that the resolvent is meromorphic in U . The operator coefficients of the poles are finite rank operators and we deduce the result about the multiplicity by using the Riesz projections. \square

Remark 5.4. The Reader is invited to compare our presentation to the one in [Bre83, Section VI.3].

Proposition 5.5. *Let $(\text{Dom}(T), T)$ be a closed operator. Assume that the resolvent set is not empty and that the resolvent is compact. Then, the spectrum of T is discrete.*

Remark 5.6. Even if a closed operator has compact resolvent (with a non empty resolvent set), the discrete spectrum might be finite (and even empty!).

5.3. On the index of Fredholm operators.

Proposition 5.7. *Let $T \in \mathcal{L}(E, F)$ be Fredholm, then so is $T' \in \mathcal{L}(F', E')$ and $\text{ind } T' = -\text{ind } T$.*

Proof. It is sufficient to use Lemmas 5.1 and 5.2. \square

Proposition 5.8. Let $T \in \mathcal{L}(E, F)$. Then T is Fredholm if and only if there exist $S \in \mathcal{L}(F, E)$, $K_1 \in \mathcal{K}(E)$, and $K_2 \in \mathcal{K}(F)$ such that

$$ST = \text{Id}_E + K_1, \quad TS = \text{Id}_F + K_2.$$

In this case, such an S is Fredholm.

Proof. If T is Fredholm, by using Lemma 5.1 and the proof of Lemma 5.2, we get the operators S , K_1 and K_2 as required.

Conversely, we use Proposition 4.8 to see that $\dim \ker T < +\infty$ and Propositions 5.7 and 4.7 to get $\dim \ker T' < +\infty$. \square

Corollary 5.9. Let $T \in \mathcal{L}(X_1, X_2)$ and $U \in \mathcal{L}(X_2, X_3)$ and be Fredholm operators. Then UT is a Fredholm operator and

$$\text{ind}(UT) = \text{ind} U + \text{ind} T.$$

Proof. The fact that UT is Fredholm comes from Proposition 5.8. For $t \in [0, \frac{\pi}{2}]$, consider the operator from $X_2 \times X_1$ to $X_3 \times X_2$,

$$L_t = \begin{pmatrix} U & 0 \\ 0 & \text{Id}_{X_2} \end{pmatrix} \begin{pmatrix} \cos t \text{Id}_{X_2} & -\sin t \text{Id}_{X_2} \\ \sin t \text{Id}_{X_2} & \cos t \text{Id}_{X_2} \end{pmatrix} \begin{pmatrix} \text{Id}_{X_2} & 0 \\ 0 & T \end{pmatrix}.$$

This is a product of three Fredholm operators. Thus $(L_t)_{t \in [0, \frac{\pi}{2}]}$ is a family of Fredholm operators. We have $\text{ind} L_0 = \text{ind} U + \text{ind} T$ and $\text{ind} L_{\frac{\pi}{2}} = \text{ind}(UT)$. Since $[0, \frac{\pi}{2}] \ni t \mapsto L_t$ is continuous and the index locally constant, the conclusion follows. \square

Exercise 5.10. With the notations of Exercise 4.12, prove that $\lambda \notin \text{sp}_{\text{ess}}(\mathcal{L} + V)$.

Corollary 5.11. Let $T \in \mathcal{L}(X_1, X_2)$ a Fredholm operator and $K \in \mathcal{K}(X_1, X_2)$. Then $T + K$ is Fredholm and $\text{ind}(T + K) = \text{ind} T$.

Proof. It follows from Proposition 5.8 and Corollary 5.9. \square

5.4. Toeplitz operators on the circle. The following presentation is inspired by a course given by G. Lebeau at the École Polytechnique.

In this section, we consider $H = L^2(\mathbb{S}^1, \mathbb{C})$. If $u \in H$, we denote by $(u_n)_{n \in \mathbb{Z}}$ the family of the Fourier coefficients of u :

$$\forall n \in \mathbb{Z}, \quad u_n = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) e^{-in\theta} d\theta.$$

We define $P : H \rightarrow H$ by, for all $u \in H$, $(Pu)_n = u_n$ if $n \in \mathbb{N}$ and $(Pu)_n = 0$ if $n < 0$.

The range of P is called the Hardy space and denoted by \mathcal{H}^2 .

Definition 5.12. Let $a \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$. We let $T(a) = PM_aP : \mathcal{H}^2 \rightarrow \mathcal{H}^2$, where $M_a : H \rightarrow H$ is the multiplication by a . $T(a)$ is the Toeplitz operator of symbol a .

Lemma 5.13. Let $a \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$. We have $T(a) \in \mathcal{L}(\mathcal{H}^2)$ and $\|T\| \leq \|a\|_\infty$.

Lemma 5.14. Let $n \in \mathbb{Z}$. Then, $[T(e_n), P]$ is a finite rank operator (and thus it is compact).

Proposition 5.15. Let $a \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$. Then, $[M_a, P]$ is a compact operator.

Proof. By the Fejér theorem, a can be approximated by trigonometric polynomials in $\|\cdot\|_\infty$ -norm. \square

Proposition 5.16. Let $a, b \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$. Then, there exists $K \in \mathcal{K}(\mathcal{H}^2)$ such that

$$T(a)T(b) = T(ab) + K.$$

Proof. It is sufficient to use Proposition 5.15. \square

Proposition 5.17. *Let $a \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$. Assume that a does not vanish. Then, $T(a)$ is a Fredholm operator.*

Proof. It is a consequence of Proposition 5.16 with $b = a^{-1}$. \square

Lemma 5.18. *Let $a \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$. Assume that a vanishes on a non-empty open set. Then, $T(a)$ is not a Fredholm operator.*

Proof. Let us consider a closed bounded interval $[\gamma_1, \gamma_2] \subset [0, 2\pi]$ with $\gamma_1 < \gamma_2$ and on which a is zero. If $\alpha \in \mathbb{R}$ and if ρ_α is the translation by α defined by $\rho_\alpha u(\theta) = u(\theta - \alpha)$, we have $[\rho_\alpha, P] = 0$. We choose $\alpha = \gamma_2 - \gamma_1$. Then, there exists $n \in \mathbb{N}$ such that $(\rho_\alpha M_a)^n = 0$.

By using commutators (see Proposition 5.15), we see that $(\rho_\alpha T(a))^n$ is compact. If $T(a)$ were Fredholm so would be $(\rho_\alpha T(a))^n$ (see Proposition 5.9) and there would exist $S \in \mathcal{L}(\mathcal{H}^2)$ and $K \in \mathcal{K}(\mathcal{H}^2)$ (see Proposition 5.8) such that

$$S(\rho_\alpha T(a))^n = \text{Id}_{\mathcal{H}^2} + K,$$

and thus $\text{Id}_{\mathcal{H}^2}$ would be compact. This would be a contradiction. Therefore $T(a)$ is not Fredholm. \square

Proposition 5.19. *Let $a \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$. Assume that there exists $\theta_0 \in \mathbb{S}^1$ such that $a(\theta_0) = 0$. Then, $T(a)$ is not a Fredholm operator.*

Proof. For all $\varepsilon > 0$, there exists $\tilde{a} \in \mathcal{C}^0(\mathbb{S}^1, \mathbb{C})$ such that $\|a - \tilde{a}\|_\infty \leq \varepsilon$ and \tilde{a} vanishes in a neighborhood of θ_0 . If a were Fredholm, so would be \tilde{a} by Lemma 5.2. With Lemma 5.18, this would be a contradiction. \square

Proposition 5.20. *Let $a \in \mathcal{C}^1(\mathbb{S}^1, \mathbb{C})$. Assume that a does not vanish. We can write $a(\theta) = r(\theta)e^{i\alpha(\theta)}$, with $r > 0, \alpha$ of class \mathcal{C}^1 . Then*

$$\text{ind } T(a) = \text{ind } T(e^{i\alpha}) = k := \frac{\alpha(2\pi) - \alpha(0)}{2\pi} = \frac{1}{2i\pi} \int_0^{2\pi} \frac{a'}{a} d\theta.$$

Proof. Let us consider the following continuous family $(a_t)_{t \in [0,1]}$:

$$a_t(\theta) = ((1-t)r(\theta) + t)e^{i\alpha(\theta)}.$$

For all $t \in [0, 1]$, the function a_t does not vanish. We see that $(T(a_t))_{t \in [0,1]}$ is a continuous family of Fredholm operators. The index being preserved by perturbation, we get the first equality. For the second one, we consider

$$f_t(\theta) = e^{(1-t)i\alpha(\theta) + ikt\theta} = e^{i\alpha(\theta) + it \int_0^\theta (k - \alpha'(u)) du}.$$

It defines a continuous 2π -periodic function. We get

$$\text{ind } T(e^{i\alpha}) = \text{ind } T(e^{ik\cdot}) = k.$$

\square

6. SPECTRUM OF SELF-ADJOINT OPERATORS

6.1. Compact normal operators.

Lemma 6.1. *Let $T \in \mathcal{T}(\mathbb{H})$ be a normal operator.*

- i. *If $V \subset \mathbb{H}$ is a subspace such that $T(V) \subset V$, then $T^*(V^\perp) \subset V^\perp$.*
- ii. *We have $\ker(T) = \ker(T^*)$.*

Proof. Assume that V is a subspace such that $T(V) \subset V$ and take $u \in V^\perp$ and $v \in V$. We have

$$\langle T^*u, v \rangle = \langle u, Tv \rangle = 0.$$

For the second point, note that, for all $x \in H$,

$$\|Tx\|^2 = \langle T^*Tx, x \rangle = \langle TT^*x, x \rangle = \|T^*x\|^2.$$

□

Theorem 6.2. *Assume that H is infinite dimensional. Let us consider $T \in \mathcal{L}(H)$ be a compact normal operator. Then, its non zero spectrum is discrete and 0 belongs to the spectrum. Let us consider the sequence of the distinct non zero eigenvalues $(\lambda_j)_{1 \leq j \leq k}$ (with k being possibly $+\infty$) and let $\lambda_0 = 0$. Then, we have the Hilbertian decomposition*

$$H = \bigoplus_{j=0}^k \ker(T - \lambda_j),$$

and

$$T = \sum_{j=0}^k \lambda_j P_j,$$

where P_j is the orthogonal projection on $\ker(T - \lambda_j)$.

Proof. If $\lambda, \mu \in \text{sp}(T) \setminus \{0\}$, then the corresponding eigenspaces are orthogonal. Indeed, if $u \in \ker(T - \lambda)$ and $v \in \ker(T - \mu)$, then, with Lemma 6.1,

$$0 = \langle (T - \lambda)u, v \rangle = \langle u, (T^* - \bar{\lambda})v \rangle = (\bar{\mu} - \bar{\lambda})\langle u, v \rangle.$$

We consider the Hilbertian sum

$$V = \bigoplus_{j=1}^k \ker(T - \lambda_j).$$

V is stable under T so that, V^\perp is stable under T^* . Thus, we can consider $T_{|V^\perp}^* \in \mathcal{L}(V^\perp)$. It is a compact normal operator on V^\perp . Its non zero spectrum does not exist. Therefore $T_{|V^\perp}^* \in \mathcal{L}(V^\perp)$ is a normal operator with zero spectrum and $T_{|V^\perp}^* = 0$. Thus $V^\perp \subset \ker T^* = \ker(T)$ and then $V^\perp = \ker T$.

Let us deal with the case $k = +\infty$. For $u \in H$, we write

$$u = \sum_{j=0}^{+\infty} u_j,$$

so that, for all $N \geq 0$ and $u \in H$,

$$\left\| \left(T - \sum_{j=0}^N \lambda_j P_j \right) u \right\|^2 = \sum_{j=N+1}^{+\infty} \lambda_j^2 \|P_j u\|^2 \leq \lambda_{N+1}^2 \|u\|^2.$$

□

Proposition 6.3 (Self-adjoint operators with compact resolvent). *Assume that H is infinite dimensional. Let us consider a self-adjoint operator T with compact resolvent. Then, its spectrum is real, discrete and can be written as a sequence tending to $+\infty$ in absolute value.*

Proof. We notice that the resolvent set is not empty (it contains i) since T is self-adjoint. Moreover, the spectrum is real. We can use Proposition 5.5 to see that the spectrum of T is discrete. The question is to know if the sequence of the eigenvalues of T tends to $+\infty$, or, equivalently, if 0 is an accumulation point in the spectrum of the normal operator $(T + i)^{-1}$. If it is not the case, by using the Hilbert space decomposition, we see that the resolvent has finite rank and thus $\text{Dom}(T)$ is finite dimensional and then $\text{Dom}(T) = H$. This is not possible if H is infinite dimensional. \square

Exercise 6.4. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set.

- i. Prove that the spectrum of the Dirichlet (resp. Neumann) Laplacian on Ω is real, discrete and can be written as a sequence tending to $+\infty$.
- ii. If $d = 1$ and $\Omega = (0, 1)$, exhibit a Hilbertian basis of $L^2(\Omega)$ made of functions in $H_0^1(\Omega)$.

Exercise 6.5. Prove the statement in Remark 4.19.

6.2. About the harmonic oscillator. Let us discuss the properties of a very important operator.

6.2.1. Definition of the harmonic oscillator and domain considerations. Let us consider the operator

$$\mathcal{H}_0 = (\mathcal{C}_0^\infty(\mathbb{R}), -\partial_x^2 + x^2).$$

This operator is essentially self-adjoint as we have seen in Example 2.61. Let us denote by \mathcal{H} its closure.

We have

$$\text{Dom}(\mathcal{H}) = \text{Dom}(\mathcal{H}_0^*) = \{\psi \in L^2(\mathbb{R}) : (-\partial_x^2 + x^2)\psi \in L^2(\mathbb{R})\}.$$

By using the results of Section 2.5.3, we also see that \mathcal{H} is the operator associated with the sesquilinear form defined by

$$\forall \varphi, \psi \in B^1(\mathbb{R}), \quad Q(\varphi, \psi) = \int_{\mathbb{R}} (\varphi' \overline{\psi'} + x^2 \varphi \overline{\psi}) \, dx.$$

We can prove the following separation property.

Proposition 6.6. *We have*

$$\text{Dom}(\mathcal{H}) = \{\psi \in H^2(\mathbb{R}) : x^2 \psi \in L^2(\mathbb{R})\}.$$

Proof. The proof is an illustration of the difference quotient method (see [Bre83, Section IX.6]).

Let $\psi \in \text{Dom}(\mathcal{H})$. It is sufficient to prove that $\psi'' \in L^2(\mathbb{R})$. There exists $f \in L^2(\mathbb{R})$ such that

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}), \quad \langle \partial_x \psi, \partial_x \varphi \rangle + \langle x \psi, x \varphi \rangle = \langle f, \varphi \rangle,$$

where the bracket is now the L^2 -bracket. Since $\psi \in B^1(\mathbb{R})$ and $\mathcal{C}_0^\infty(\mathbb{R})$ is dense in $B^1(\mathbb{R})$, we can extend this equality and get

$$\forall \varphi \in B^1(\mathbb{R}), \quad \langle \partial_x \psi, \partial_x \varphi \rangle + \langle x \psi, x \varphi \rangle = \langle f, \varphi \rangle.$$

Let us define the difference quotient

$$Q_h \varphi(x) = \frac{\varphi(x+h) - \varphi(x)}{h}, \quad x \in \mathbb{R}, \quad h \neq 0.$$

If $\varphi \in B^1(\mathbb{R})$, then $Q_h \varphi \in B^1(\mathbb{R})$. We get

$$\forall \varphi \in B^1(\mathbb{R}), \quad \langle \partial_x \psi, \partial_x Q_h \varphi \rangle + \langle x \psi, x Q_h \varphi \rangle = \langle f, Q_h \varphi \rangle.$$

It follows that

$$\langle \partial_x \psi, \partial_x Q_h \varphi \rangle = -\langle \partial_x Q_{-h} \psi, \partial_x \varphi \rangle$$

and

$$\langle x\psi, xQ_h \varphi \rangle = -\langle xQ_{-h} \psi, x\varphi \rangle - \langle \psi(x-h), x\varphi \rangle - \langle x\psi, \varphi(x+h) \rangle.$$

We find, for all $\varphi \in \mathcal{B}^1(\mathbb{R})$ and $h \neq 0$,

$$\langle \partial_x Q_{-h} \psi, \partial_x \varphi \rangle + \langle xQ_{-h} \psi, x\varphi \rangle = -\langle f, Q_h \varphi \rangle - \langle \psi(x-h), x\varphi \rangle - \langle x\psi, \varphi(x+h) \rangle.$$

Applying this equality to $\varphi = Q_{-h} \psi$, we get

$$\begin{aligned} & \langle \partial_x Q_{-h} \psi, \partial_x Q_{-h} \psi \rangle + \langle xQ_{-h} \psi, xQ_{-h} \psi \rangle \\ &= -\langle f, Q_h Q_{-h} \psi \rangle - \langle \psi(x-h), xQ_{-h} \psi \rangle - \langle x\psi, Q_{-h} \psi(x+h) \rangle. \end{aligned}$$

Then we notice that

$$\begin{aligned} |\langle f, Q_h Q_{-h} \psi \rangle| &\leq \|f\|_{L^2(\mathbb{R})} \|Q_h Q_{-h} \psi\|_{L^2(\mathbb{R})} \\ &\leq \|f\|_{L^2(\mathbb{R})} \|\partial_x Q_{-h} \psi\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{2} \left(\|f\|_{L^2(\mathbb{R})}^2 + \|\partial_x Q_{-h} \psi\|_{L^2(\mathbb{R})}^2 \right), \end{aligned}$$

where we have used Proposition 4.10. We can deal with the other terms in the same way and thus get

$$\begin{aligned} & \|\partial_x Q_{-h} \psi\|_{L^2(\mathbb{R})}^2 + \|xQ_{-h} \psi\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{1}{2} \left(\|f\|_{L^2(\mathbb{R})}^2 + \|\partial_x Q_{-h} \psi\|_{L^2(\mathbb{R})}^2 + \|\psi\|_{L^2(\mathbb{R})}^2 + \|xQ_{-h} \psi\|_{L^2(\mathbb{R})}^2 + \|\psi\|_{\mathcal{B}^1(\mathbb{R})}^2 + |h| \|\psi\|_{H^1(\mathbb{R})}^2 \right). \end{aligned}$$

We deduce that

$$\|Q_{-h} \partial_x \psi\|_{L^2(\mathbb{R})}^2 + \|xQ_{-h} \psi\|_{L^2(\mathbb{R})}^2 \leq \|f\|_{L^2(\mathbb{R})}^2 + \|\psi\|_{L^2(\mathbb{R})}^2 + \|\psi\|_{\mathcal{B}^1(\mathbb{R})}^2 + |h| \|\psi\|_{H^1(\mathbb{R})}^2.$$

We may again use Proposition 4.10 and we conclude that $\partial_x \psi \in H^1(\mathbb{R})$ and $x\psi \in H^1(\mathbb{R})$. \square

6.2.2. Spectrum of the harmonic oscillator. We have seen in Exercise 4.21 that \mathcal{H} has compact resolvent. Actually, one could also directly use Propositions 6.6 and 4.17.

Thus, the spectrum is real, discrete and it is a non-decreasing sequence $(\lambda_n)_{n \geq 1}$ tending to $+\infty$ (we repeat the eigenvalue according to its multiplicity). We would like to compute these eigenvalues.

Let us consider the following differential operators (acting on $\mathcal{S}(\mathbb{R})$)

$$a = \frac{1}{\sqrt{2}}(\partial_x + x), \quad c = \frac{1}{\sqrt{2}}(-\partial_x + x).$$

We have

$$2ca = -\partial_x^2 + x^2 - 1, \quad [a, c] = 1.$$

Lemma 6.7. For all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$, we have

$$\langle a\varphi, \psi \rangle_{L^2(\mathbb{R})} = \langle \varphi, c\psi \rangle_{L^2(\mathbb{R})}.$$

Lemma 6.8. For all $n \in \mathbb{N} \setminus \{0\}$,

$$ac^n = nc^{n-1} + c^n a.$$

Proposition 6.9. For all $n \geq 1$, we have $\lambda_n = 2n - 1$. In particular, the eigenvalues are simple.

Proof. We let $g_0(x) = e^{-x^2/2}$. We check that $ag = 0$. In particular, we have $1 \in \text{sp}(\mathcal{H})$.

For $n \in \mathbb{N}$, we let $g_n = c^n g_0$. By induction, we see that $g_n = H_n g_0$ where H_n is a polynomial of degree n . In particular, the functions g_n are in the domain of the harmonic oscillator.

Let us notice that

$$cac^n = nc^n + c^{n+1}a.$$

We get that

$$\mathcal{H}g_n = (2n + 1)g_n.$$

In particular, $\{2n + 1, n \in \mathbb{N}\} \subset \text{sp}(\mathcal{H})$.

Let us check that $(g_n)_{n \in \mathbb{N}}$ is an orthogonal family. Let $n, m \in \mathbb{N}$ with $n < m$. Let us consider

$$\langle g_n, g_m \rangle_{L^2(\mathbb{R})} = \langle c^n g_0, c^m g_0 \rangle_{L^2(\mathbb{R})} = \langle a^m c^n g_0, g_0 \rangle_{L^2(\mathbb{R})} = 0,$$

where we used Lemmas 6.7 and 6.8, $ag_0 = 0$, and an induction procedure.

Let us check that the family is total. Take $f \in L^2(\mathbb{R})$ such that, for all $n \in \mathbb{N}$, $\langle f, g_n \rangle_{L^2(\mathbb{R})} = 0$. It follows that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} x^n f(x) e^{-x^2/2} dx = 0.$$

For all $\xi \in \mathbb{R}$, we let

$$F(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) e^{-x^2/2} dx.$$

The function F is well defined. Now, we notice that

$$F(\xi) = \int_{\mathbb{R}} \sum_{k=0}^{+\infty} f(x) \frac{(-ix\xi)^k}{k!} e^{-x^2/2} dx.$$

We can apply the Fubini theorem to get

$$F(\xi) = \sum_{k=0}^{+\infty} \xi^k \int_{\mathbb{R}} f(x) \frac{(-ix)^k}{k!} e^{-x^2/2} dx = 0.$$

Therefore, the Fourier transform of $f e^{-x^2/2}$ is 0 and $f = 0$

If we denote by $(f_n)_{n \in \mathbb{N}}$ the L^2 -normalization of the family $(g_n)_{n \in \mathbb{N}}$, $(f_n)_{n \in \mathbb{N}}$ is a Hilbertian basis of $L^2(\mathbb{R})$ such that $\mathcal{H}f_n = (2n + 1)f_n$.

Since the spectrum of \mathcal{H} is discrete, we only have to care about the eigenvalues. Let us solve $\mathcal{H}\psi = \lambda\psi$ with $\lambda \in \mathbb{R}$ and $\psi \in \text{Dom}(\mathcal{H})$. We write the following decomposition, converging in $L^2(\mathbb{R})$,

$$\psi = \sum_{n \in \mathbb{N}} \langle \psi, f_n \rangle_{L^2(\mathbb{R})} f_n.$$

For all $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$\langle \psi, (\mathcal{H} - \lambda)\varphi \rangle_{L^2(\mathbb{R})} = 0.$$

Thus, by convergence in $L^2(\mathbb{R})$, for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\sum_{n \in \mathbb{N}} \langle \psi, f_n \rangle_{L^2(\mathbb{R})} \langle f_n, (\mathcal{H} - \lambda)\varphi \rangle_{L^2(\mathbb{R})} = 0.$$

We choose $\varphi = f_k$ to see that

$$\sum_{n \in \mathbb{N}} \langle \psi, f_n \rangle_{L^2(\mathbb{R})} \langle f_n, ((2k + 1) - \lambda)f_k \rangle_{L^2(\mathbb{R})} = \langle \psi, f_k \rangle_{L^2(\mathbb{R})} ((2k + 1) - \lambda) = 0.$$

If, for all $k \in \mathbb{N}$, $\langle \psi, f_k \rangle_{L^2(\mathbb{R})} = 0$, then $\psi = 0$. Therefore, there exists $k \in \mathbb{N}$ such that $(2k + 1) - \lambda = 0$.

We have proved that

$$\text{sp}(\mathcal{H}) = \{2n - 1, n \in \mathbb{N} \setminus \{0\}\}.$$

Let us now prove the statement about the multiplicity. Consider a solution $\psi \in \text{Dom}(\mathcal{H})$ of $\mathcal{H}\psi = (2n + 1)\psi$. For all $k \in \mathbb{N}$, we get

$$\langle \psi, f_k \rangle_{L^2(\mathbb{R})} ((2k + 1) - (2n + 1)) = 0.$$

Thus, for $k \neq n$, $\langle \psi, f_k \rangle_{L^2(\mathbb{R})} = 0$. Thus, ψ is proportional to f_n . \square

6.3. Characterization of the different spectra.

6.3.1. Properties.

Lemma 6.10. *If T is self-adjoint, we have the equivalence: $\lambda \in \text{sp}(T)$ if and only if there exists a sequence $(u_n) \in \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$, and $(T - \lambda)u_n \xrightarrow{n \rightarrow +\infty} 0$ in \mathbb{H} .*

Proof. Let us notice that if there exists a sequence $(u_n) \in \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$, (u_n) and $(T - \lambda)u_n \xrightarrow{n \rightarrow +\infty} 0$ then $\lambda \in \text{sp}(T)$ (see Lemma 3.12).

If $\lambda \notin \mathbb{R}$, then since T is self-adjoint, $T - \lambda$ is invertible (with bounded inverse because T is closed). Now, for $\lambda \in \mathbb{R}$, if there is no sequence $(u_n) \subset \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$, (u_n) and $(T - \lambda)u_n \xrightarrow{n \rightarrow +\infty} 0$, then we can find $c > 0$ such that

$$\|(T - \lambda)u\| \geq c\|u\|, \quad \forall u \in \text{Dom}(T).$$

Therefore $T - \lambda$ is injective with closed range. But, since $T - \lambda = (T - \lambda)^*$, the range of $T - \lambda$ is dense in \mathbb{H} and so $T - \lambda$ is surjective. \square

Lemma 6.11 (Weyl criterion). *If T is self-adjoint, then $\lambda \in \text{sp}_{\text{ess}}(T)$ if and only if there exists a sequence $(u_n) \subset \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$, (u_n) has no subsequence converging in \mathbb{H} , and $(T - \lambda)u_n \xrightarrow{n \rightarrow +\infty} 0$ in \mathbb{H} .*

Proof. If $\lambda \in \text{sp}(T) \setminus \text{sp}_{\text{ess}}(T)$, the operator $T - \lambda$ is Fredholm. Let $(u_n) \subset \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$ and $\lim_{n \rightarrow +\infty} (T - \lambda)u_n = 0$.

The operator $T - \lambda : \ker(T - \lambda)^\perp \rightarrow \text{ran}(T - \lambda)$ is injective with closed range. Therefore, there exists $c > 0$ such that, for all $w \in \ker(T - \lambda)^\perp$, $\|(T - \lambda)w\| \geq c\|w\|$. We write $u_n = v_n + w_n$, with $v_n \in \ker(T - \lambda)$ and $w_n \in \ker(T - \lambda)^\perp$. We have

$$\|(T - \lambda)u_n\|^2 = \|(T - \lambda)v_n\|^2 + \|(T - \lambda)w_n\|^2,$$

and we deduce that $w_n \rightarrow 0$. Moreover, (v_n) is bounded in a finite dimensional space, thus there exists a converging subsequence of (u_n) .

Conversely, let us assume that $\lambda \in \text{sp}(T)$ and that any sequence $(u_n) \subset \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$ and $\lim_{n \rightarrow +\infty} (T - \lambda)u_n = 0$ has a converging subsequence.

The kernel $\ker(T - \lambda)$ is finite dimensional. Indeed, if it were of infinite dimension, one could construct a infinite orthonormal family (u_n) in $\ker(T - \lambda)$ and in particular we would get $u_n \rightarrow 0$, which is a contradiction. Let us now check that

$$\exists c > 0, \forall u \in \ker(T - \lambda)^\perp, \quad \|(T - \lambda)u\| \geq c\|u\|.$$

If not, there exists a normalized sequence (u_n) in $\ker(T - \lambda)^\perp$ such that $\|(T - \lambda)u_n\| \rightarrow 0$. By assumption, we may assume that (u_n) converges towards some u_∞ that necessarily belongs to $\ker(T - \lambda)^\perp$. But since $T - \lambda$ is closed (it is self-adjoint), we have $(T - \lambda)u_\infty = 0$ so that $u_\infty = 0$, and this is a contradiction.

We deduce that the range of $T - \lambda$ is closed. \square

The following lemma is a slight improvement of Lemma 6.11.

Lemma 6.12. *Assume that T is self-adjoint. Then $\lambda \in \text{sp}_{\text{ess}}(T)$ if and only if there exists a sequence $(u_n) \subset \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$, (u_n) converges weakly to 0, and*

$$\lim_{n \rightarrow +\infty} (T - \lambda)u_n = 0, \quad \text{in } \mathbb{H}.$$

Proof. Let $\lambda \in \text{sp}_{\text{ess}}(T)$. If $\dim \ker(T - \lambda \text{Id}) = +\infty$, then (by considering a Hilbertian basis of the kernel) we can easily construct a orthonormal sequence (v_n) weakly converging to 0 such that $(T - \lambda)v_n = 0$.

Therefore, we consider the case when $\dim \ker(T - \lambda) < +\infty$. By Lemma 6.11, there exists a sequence $(u_n) \subset \text{Dom}(T)$ such that $\|u_n\|_{\mathbb{H}} = 1$ with no converging subsequence such that we have $\lim_{n \rightarrow +\infty} (T - \lambda)u_n = 0$ in \mathbb{H} .

We can write

$$u_n = \tilde{u}_n + k_n, \quad \text{with} \quad \tilde{u}_n \in \ker(T - \lambda)^\perp, \quad k_n \in \ker(T - \lambda).$$

We have $(T - \lambda)\tilde{u}_n \xrightarrow{n \rightarrow +\infty} 0$ and we may assume (up to a subsequence extraction) that (k_n) converges to k . Since (u_n) has no converging subsequence, (\tilde{u}_n) does not converge, and so it does not go to 0. Therefore, up to another extraction, we may assume that

$$\exists \varepsilon_0 > 0, \forall n \in \mathbb{N}, \quad \|\tilde{u}_n\| \geq \varepsilon_0.$$

Now set $\hat{u}_n = \frac{\tilde{u}_n}{\|\tilde{u}_n\|}$; then $(T - \lambda)\hat{u}_n \xrightarrow{n \rightarrow +\infty} 0$. Up to another extraction, we may assume that (\hat{u}_n) converges weakly to some $\hat{u} \in \ker(T - \lambda)^\perp$. We have

$$\forall v \in \text{Dom}(T), \quad \langle \hat{u}, (T - \lambda)v \rangle_{\mathbb{H}} = 0.$$

We deduce that $\hat{u} \in \text{Dom}(T^*) = \text{Dom}(T)$ and that $(T - \lambda)\hat{u} = 0$. Thus $\hat{u} = 0$.

In any case, we have found a sequence with the required property. For the converse, it is just an application of Lemma 6.11. \square

Definition 6.13. We call Fredholm spectrum of T the complement of the essential spectrum of T in the spectrum of T .

Lemma 6.14. *Let T be self-adjoint. We have the following properties.*

- i. *If $\lambda \in \text{sp}(T)$ is not isolated, then $\lambda \in \text{sp}_{\text{ess}}(T)$.*
- ii. *The Fredholm spectrum is formed by isolated eigenvalues of finite multiplicity.*
- iii. *If $\lambda \in \text{sp}(T)$ is isolated, then it is an eigenvalue.*
- iv. *All isolated eigenvalues of finite multiplicity belong to the Fredholm spectrum.*

In particular, the discrete spectrum coincides with the Fredholm spectrum.

Proof. Let us prove (i) and (ii). Let $\lambda \in \text{sp}(T) \setminus \text{sp}_{\text{ess}}(T)$.

There exists a Weyl sequence (u_n) of unit vectors such that $(T - \lambda)u_n \rightarrow 0$. We may assume that (u_n) converges to some u (of norm 1) and we get $(T - \lambda)u = 0$. The eigenvalue λ has finite multiplicity. Let us prove that it is isolated. If this were not the case, then one could consider a non-constant sequence λ_n tending to λ . Moreover, one could find a sequence (u_n) of unit vectors such that

$$\|(T - \lambda_n)u_n\| \leq \frac{|\lambda - \lambda_n|}{n}.$$

We may assume that (u_n) converges to some $u \in \text{Dom}(T)$ and thus one would get $(T - \lambda)u = 0$, and so

$$\langle (T - \lambda_n)u, u_n \rangle = (\lambda - \lambda_n)\langle u, u_n \rangle.$$

By the Cauchy-Schwarz inequality, $\langle u_n, u \rangle \rightarrow 0$ and we get $u = 0$, which is a contradiction.

Let us now prove (iii). Consider an isolated point $\lambda \in \text{sp}(T)$. By definition, this means that there exists $\varepsilon_0 > 0$ such that, for all $\mu \neq \lambda$ such that $|\mu - \lambda| \leq \varepsilon_0$, we have $\mu \notin \text{sp}(T)$. For all $\varepsilon \in (0, \varepsilon_0)$, we introduce

$$P_\varepsilon = \frac{1}{2i\pi} \int_{\Gamma_\varepsilon} (\zeta - T)^{-1} d\zeta = P,$$

where Γ_ε is the circle of radius ε centered at λ .

Since T is closed (and using Riemannian sums), P_λ is valued in $\text{Dom}(T)$ and

$$(T - \lambda)P = \frac{1}{2i\pi} \int_{\Gamma_\varepsilon} (T - \lambda)(\zeta - T)^{-1} d\zeta = \frac{1}{2i\pi} \int_{\Gamma_\varepsilon} (\zeta - \lambda)(\zeta - T)^{-1} d\zeta.$$

Now, we use the resolvent bound to get (as soon as ε_0 is chosen small enough):

$$\|(T - \zeta)^{-1}\| \leq \frac{1}{|\lambda - \zeta|}.$$

Thus, we infer that $\|(T - \lambda)P\| \leq \varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$. Therefore, P is valued in $\ker(T - \lambda)$. It remains to apply Lemma 3.30.

Let us now consider (iv). Since λ is isolated, (iii) shows that it cannot belong to the spectrum of the restriction $T|_{\ker(T - \lambda)^\perp}$. Thus, there exists $c > 0$ such that

$$\forall u \in \ker(T - \lambda)^\perp, \quad \|(T - \lambda)u\| \geq c\|u\|.$$

We deduce that the range of $T - \lambda$ is closed and that $T - \lambda$ is Fredholm, since we have $\dim \ker(T - \lambda) < +\infty$. \square

Finally, let us prove another useful property.

Lemma 6.15. *Let T be self-adjoint. Consider $\lambda \in \text{sp}_{\text{ess}}(T)$. Then, for all $N \in \mathbb{N}^*$ and $\varepsilon > 0$, there exists an orthonormal family $(u_n^\varepsilon)_{1 \leq n \leq N}$ such that, for all $n \in \{1, \dots, N\}$,*

$$\|(T - \lambda)u_n^\varepsilon\| \leq \varepsilon.$$

Proof. If λ is isolated, then it is an eigenvalue of infinite multiplicity (see Lemma 6.14) and the conclusion follows. Let $\varepsilon \in (0, 1)$. If λ is not isolated, we may consider a sequence of distinct numbers of the spectrum $(\lambda_n)_{n \in \mathbb{N}}$ tending to λ and such that, for all $j, k \in \mathbb{N}$, we have $|\lambda_j - \lambda_k| \leq \frac{\varepsilon}{2}$. If $N = 1$, by the Weyl criterion, we get the existence of u_1^ε such that $\|(T - \lambda_1)u_1^\varepsilon\| \leq \frac{\varepsilon}{2}$. The conclusion follows for $N = 1$ since $|\lambda - \lambda_1| \leq \frac{\varepsilon}{2}$. Let us now only treat the case when $N = 2$. By the Weyl criterion, we can find u_1^ε and \tilde{u}_2^ε of norm 1 such that

$$\|(T - \lambda_1)u_1^\varepsilon\| \leq \frac{\varepsilon}{2}|\lambda_1 - \lambda_2|, \quad \|(T - \lambda_2)\tilde{u}_2^\varepsilon\| \leq \frac{\varepsilon}{2}|\lambda_1 - \lambda_2|.$$

Since T is self-adjoint and by computing $\langle (T - \lambda_1)u_1^\varepsilon, \tilde{u}_2^\varepsilon \rangle$, we find $|\langle u_1^\varepsilon, \tilde{u}_2^\varepsilon \rangle| \leq \varepsilon$. Setting

$$u_2^\varepsilon = \tilde{u}_2^\varepsilon - \langle \tilde{u}_2^\varepsilon, u_1^\varepsilon \rangle u_1^\varepsilon,$$

we have

$$\|(T - \lambda_2)u_2^\varepsilon\| \leq \frac{\varepsilon}{2}|\lambda_1 - \lambda_2| + \varepsilon \left(|\lambda_1 - \lambda_2| + \frac{\varepsilon}{2}|\lambda_1 - \lambda_2| \right).$$

Moreover, $\|u_2^\varepsilon\| \geq \sqrt{1 - \varepsilon^2}$. Up to changing ε , the conclusion follows for $N = 2$. We leave the case $N \geq 3$ to the reader. \square

6.3.2. *Determining the essential spectrum: an example.* As in Exercises 4.12 and 5.10, we consider a function $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ such that ∇V is bounded and $\lim_{|x| \rightarrow +\infty} V(x) = 0$. We are interested in the essential spectrum of the operator $\mathcal{L} + V$ with domain $H^2(\mathbb{R}^d)$. This operator is self-adjoint. Therefore its spectrum is real. Moreover, with Exercise 5.10, we have $\text{sp}_{\text{ess}}(\mathcal{L} + V) \subset [0, +\infty)$.

Let us prove that $\text{sp}_{\text{ess}}(\mathcal{L} + V) = [0, +\infty)$. Let us start by showing that $0 \in \text{sp}_{\text{ess}}(\mathcal{L} + V)$. For that purpose, we use Lemma 6.12. Let us consider $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\|\chi\|_{L^2(\mathbb{R}^d)} = 1$. For $n \in \mathbb{N}$, we consider $\chi_n(x) = n^{-\frac{d}{2}} \chi(n^{-1}x - ne_1)$. The sequence (χ_n) is L^2 -normalized and converges to 0 weakly. For n large enough, we have

$$\|(\mathcal{L} + V)\chi_n\| = \|\mathcal{L}\chi_n\| = \mathcal{O}(n^{-2}).$$

Let us now consider $k \in \mathbb{R}$ and the sequence $\chi_{n,k} = e^{ik \cdot} \chi_n$. We have

$$\|(\mathcal{L} + V - k^2)\chi_{n,k}\| = \|e^{ik \cdot}(\mathcal{L} + V - k^2)\chi_n + [\mathcal{L}, e^{ik \cdot}]\chi_n\|.$$

But,

$$e^{-ik \cdot}[\mathcal{L}, e^{ik \cdot}] = k^2 - 2ik \cdot \nabla,$$

and we deduce that $k^2 \in \text{sp}_{\text{ess}}(\mathcal{L} + V)$, for all $k \in \mathbb{R}$.

6.4. Min-max principle.

6.4.1. *Statement and proof.* We now give a standard method to estimate the discrete spectrum and the bottom of the essential spectrum of a self-adjoint operator T on an Hilbert space H . We recall first the definition of the Rayleigh quotients of a self-adjoint operator T .

Definition 6.16. The Rayleigh quotients associated with the self-adjoint operator (semi-bounded from below) T on H with domain $\text{Dom}(T)$ are defined for all positive natural number n by

$$\mu_n(T) = \sup_{\psi_1, \dots, \psi_{n-1}} \inf_{\substack{u \in \text{span}(\psi_1, \dots, \psi_{n-1})^\perp \\ u \in \text{Dom}(T), u \neq 0}} \frac{\langle Tu, u \rangle_H}{\langle u, u \rangle_H}.$$

Remark 6.17. Note that, if T comes from a quadratic form Q via a representation theorem à la Lax-Milgram, we can replace $u \in \text{Dom}(T)$ by $u \in \text{Dom}(Q)$ and $\langle Tu, u \rangle$ by $Q(u)$.

Lemma 6.18. *If T is self-adjoint with non negative spectrum, then $\mu_1(T) \geq 0$.*

Proof. Let us assume that $\mu_1(T) < 0$. We define the sesquilinear form

$$Q(u, v) = \langle (T - \mu_1(T))^{-1}u, v \rangle$$

on H ; Q is non-negative. Thus, the Cauchy-Schwarz inequality provides, for $u, v \in H$,

$$|\langle (T - \mu_1(T))^{-1}u, v \rangle| \leq \langle (T - \mu_1(T))^{-1}u, u \rangle^{\frac{1}{2}} \langle (T - \mu_1(T))^{-1}v, v \rangle^{\frac{1}{2}}.$$

We take $v = (T - \mu_1(T))^{-1}u$ and deduce for all $u \in H$,

$$\|(T - \mu_1(T))^{-1}u\| \leq \|(T - \mu_1(T))^{-1}\|^{\frac{1}{2}} \langle (T - \mu_1(T))^{-1}u, u \rangle^{\frac{1}{2}},$$

and thus, for all $v \in \text{Dom}(T)$,

$$\|v\| \leq \|(T - \mu_1(T))^{-1}\|^{\frac{1}{2}} \langle v, (T - \mu_1(T))v \rangle^{\frac{1}{2}}.$$

By the definition of $\mu_1(T)$, there is a sequence (v_n) , $\|v_n\| = 1$, such that $\langle Tv_n, v_n \rangle \rightarrow \mu_1(T)$ and we get a contradiction. \square

The following statement gives the relation between Rayleigh quotients and eigenvalues.

Theorem 6.19. Let T be a self-adjoint operator with domain $\text{Dom}(T)$. We assume that T is semi-bounded from below. Then the Rayleigh quotients μ_n of T form a non-decreasing sequence and one of the following holds

- i. $\mu_n(T)$ is the n -th eigenvalue counted with multiplicity of T and T has only discrete spectrum in $(-\infty, \mu_n(T)]$.
- ii. $\mu_n(T)$ is the bottom of the essential spectrum and, for all $j \geq n$, $\mu_j(T) = \mu_n(T)$.

Proof. Let us provide an elementary proof which does not use the spectral projections. First it is easy to see that the sequence (μ_n) is non-decreasing. Then, we notice that

$$(6.1) \quad a < \mu_n \implies (-\infty, a) \cap \text{sp}_{\text{ess}}(T) = \emptyset.$$

Indeed, if $\lambda \in (-\infty, a)$ were in the essential spectrum, by Lemma 6.15, for all $N \geq 1$ and $\varepsilon > 0$, we could find an orthonormal family $(u_j)_{j \in \{1, \dots, N\}}$ such that $\|(T - \lambda)u_j\| \leq \frac{\varepsilon}{\sqrt{N}}$. Then, given $n \geq 1$ and taking $N \geq n$, for all $(\psi_1, \dots, \psi_{n-1}) \in \mathbb{H}$, there exists a non-zero u in the intersection $\text{span}(u_1, \dots, u_N) \cap \text{span}(\psi_1, \dots, \psi_{n-1})^\perp$. We write $u = \sum_{j=1}^N \alpha_j u_j$ and notice that

$$\frac{\langle Tu, u \rangle_{\mathbb{H}}}{\langle u, u \rangle_{\mathbb{H}}} \leq \lambda + \frac{\|(T - \lambda)u\|}{\|u\|} \leq \lambda + \left(\sum_{j=1}^N \|(T - \lambda)u_j\|^2 \right)^{\frac{1}{2}} \leq \lambda + \varepsilon,$$

and thus $\mu_n \leq \lambda + \varepsilon$. For ε small enough, we get $\mu_n \leq a$, which is a contradiction. If γ is the infimum of the essential spectrum (suppose that it is not empty), we have $\mu_n \leq \gamma$. Note also that if $\mu_n = +\infty$ for some n , then the essential spectrum is empty. This implies the second assertion.

It remains to prove the first assertion. Thus, we assume that $\mu_n < \gamma$. By the same considerations as above, if $a < \mu_n$, the number of eigenvalues (with multiplicity) lying in $(-\infty, a)$ is less than $n - 1$. Let us finally show that, if $a \in (\mu_n, \gamma)$, then the number of eigenvalues in $(-\infty, a)$ is at least n . If not, the direct sum of eigenspaces associated with eigenvalues below a would be spanned by $\psi_1, \dots, \psi_{n-1}$ and

$$\mu_n \geq \inf_{\substack{u \in \text{span}(\psi_1, \dots, \psi_{n-1})^\perp \\ u \in \text{Dom}(T), u \neq 0}} \frac{\langle Tu, u \rangle_{\mathbb{H}}}{\langle u, u \rangle_{\mathbb{H}}} \geq a,$$

where we have used Lemma 6.18 and the fact that $\text{sp}(T|_F) \subset [a, +\infty)$, with

$$F = \text{span}(\psi_1, \dots, \psi_{n-1})^\perp.$$

□

An often used consequence of this theorem (or of its proof) is the following proposition.

Proposition 6.20. Suppose that there exists $a \in \mathbb{R}$ with $a < \inf \text{sp}_{\text{ess}}(T)$ and an n -dimensional space $V \subset \text{Dom } T$ such that

$$\langle T\psi, \psi \rangle_{\mathbb{H}} \leq a \|\psi\|^2, \quad \forall \psi \in V,$$

Then, the n -th eigenvalue exists and satisfies

$$\lambda_n(T) \leq a.$$

Exercise 6.21. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Prove that there exists $c(\Omega) > 0$ such that, for all $\psi \in H_0^1(\Omega)$,

$$\int_{\Omega} |\nabla \psi|^2 dx \geq c(\Omega) \|\psi\|^2.$$

What is the optimal $c(\Omega)$? We will consider the Dirichlet Laplacian on Ω .

Exercise 6.22. Consider the self-adjoint operator \mathcal{L} associated with the quadratic form

$$\forall \psi \in \mathbf{H}^1(\mathbb{R}), \quad Q(\psi) = \int_{\mathbb{R}} |\psi'|^2 + V(x)|\psi|^2 dx,$$

where $V \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R})$.

- i. What is the essential spectrum?
- ii. We assume that $\int_{\mathbb{R}} V(x) dx < 0$. Prove that the discrete spectrum is not empty.

7. EXAMPLES

7.1. Sturm-Liouville's oscillation theorem. We consider the operator $\mathcal{L} = -\partial_x^2 + V(x)$, with $V \in \mathcal{C}^\infty([0, 1])$, on $[0, 1]$ and domain

$$\text{Dom}(\mathcal{L}) = \{\psi \in \mathbf{H}_0^1((0, 1)) : (-\partial_x^2 + V(x))\psi \in \mathbf{L}^2((0, 1))\}.$$

\mathcal{L} is a self-adjoint operator with compact resolvent. Therefore, we may consider the non-decreasing sequence of its eigenvalues $(\lambda_n)_{n \geq 1}$.

Lemma 7.1. *The eigenvalues of \mathcal{L} are simple.*

Proof. It follows from the Cauchy-Lipschitz theorem. □

For all $n \geq 1$, let us consider an \mathbf{L}^2 -normalized eigenfunction u_n associated with λ_n . Notice that $\langle u_n, u_m \rangle = 0$ if $n \neq m$ and that the zeros of u_n are simple and thus isolated.

Theorem 7.2. *For all $n \geq 1$, the function u_n admits exactly $n - 1$ zeros in $(0, 1)$.*

Proof. Let us denote by Z_n the number of zeros of u_n in $(0, 1)$.

Let us prove that $Z_n \leq n - 1$. If the eigenfunction u_n admits at least n zeros in $(0, 1)$, denoted by z_1, \dots, z_n . We let $z_0 = 0$ and $z_{n+1} = 1$. We define $(u_{n,j})_{j=0, \dots, n}$ by $u_{n,j}(x) = u_n(x)$ for $x \in [z_j, z_{j+1}]$ and $u_{n,j}(x) = 0$ elsewhere. It is clear that these functions belong to the form domain of \mathcal{L} and that they form an orthogonal family. By integrating by parts, we get

$$\forall v \in \text{span}_{j \in \{0, \dots, n\}} u_{n,j}, \quad Q(v, v) \leq \lambda_n \|v\|_{\mathbf{L}^2((0,1))}^2.$$

By the min-max principle, we get $\lambda_{n+1} \leq \lambda_n$ and this contradicts the simplicity of the eigenvalues.

Let us now prove that $Z_n \geq Z_{n-1} + 1$. It is sufficient to show that if u_{n-1} is zero in z_0 and z_1 (two consecutive zeros, for example u_{n-1} is positive on (z_0, z_1)), then u_n vanishes in (z_0, z_1) . Indeed, this would imply that u_n vanishes at least $Z_{n-1} + 1$ times. For that purpose we introduce $W(f_1, f_2) = f_1' f_2 - f_1 f_2'$ and compute

$$W(u_{n-1}, u_n)' = (\lambda_n - \lambda_{n-1})u_{n-1}u_n.$$

Assume that u_n does not vanish on (z_0, z_1) . For instance $u_n > 0$ on (z_0, z_1) . Then, we get $W(u_{n-1}, u_n)' > 0$. We have $W(u_{n-1}, u_n)(z_0) \geq 0$ and $W(u_{n-1}, u_n)(z_1) \leq 0$, and thus we get a contradiction.

The conclusion follows easily. □

7.2. Weyl's law in one dimension.

7.2.1. *Two examples.*

Definition 7.3. If $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ is a self-adjoint operator and $E \in \mathbb{R}$, $\mathbf{N}(\mathcal{L}, E)$ denotes the number of eigenvalues of \mathcal{L} below E .

Let $\mathcal{H}_h^{\text{Dir}} = h^2 D_x^2$ be the Dirichlet Laplacian on $(0, 1)$. Its domain is given by

$$\text{Dom}(\mathcal{H}_h^{\text{Dir}}) = \mathbf{H}^2(0, 1) \cap \mathbf{H}_0^1(0, 1),$$

and $\mathcal{H}_h^{\text{Dir}}$ has compact resolvent. We can easily compute the eigenvalues:

$$\lambda_n(\mathcal{H}_h^{\text{Dir}}) = h^2 n^2 \pi^2, \quad n \in \mathbb{N} \setminus \{0\},$$

so that, for $E > 0$,

$$\mathbf{N}(\mathcal{H}_h^{\text{Dir}}, E) \underset{h \rightarrow 0}{\sim} \frac{\sqrt{E}}{\pi h} = \frac{1}{2\pi h} \int_{\{(x, \xi) \in (0, 1) \times \mathbb{R} : \xi^2 \leq E\}} dx d\xi.$$

In the same way, we can explicitly compute the eigenvalues when $\mathcal{H}_h = h^2 D_x^2 + x^2$. We have

$$\lambda_n(\mathcal{H}_h) = (2n - 1)h, \quad n \in \mathbb{N} \setminus \{0\},$$

so that, for $E > 0$,

$$\mathbf{N}(\mathcal{H}_h, E) \underset{h \rightarrow 0}{\sim} \frac{E}{2h} = \frac{1}{2\pi h} \int_{\{(x, \xi) \in \mathbb{R}^2 : \xi^2 + x^2 \leq E\}} dx d\xi.$$

From these examples, one could guess the more general formula

$$\mathbf{N}(\mathcal{H}_h, E) \underset{h \rightarrow 0}{\sim} \frac{1}{2\pi h} \int_{\{(x, \xi) \in \mathbb{R}^2 : \xi^2 + V(x) \leq E\}} dx d\xi = \frac{1}{\pi h} \int_{\mathbb{R}} \sqrt{(E - V)_+} dx.$$

7.2.2. *Statement in one dimension.* We propose to prove the following version of the Weyl law in dimension one. It generalizes the previous two asymptotic formulas. For a more general presentation, one can read [RS, Vol. IV, Section XIII.15].

Proposition 7.4. *Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be a piecewise Lipschitzian function with a finite number of discontinuities and which satisfies:*

- i. $V \rightarrow \ell_{\pm\infty}$ when $x \rightarrow \pm\infty$ with $\ell_{+\infty} \leq \ell_{-\infty}$;
- ii. $\sqrt{(\ell_{+\infty} - V)_+}$ belongs to $L^1(\mathbb{R})$.

Consider the operator $\mathcal{H}_h = h^2 D_x^2 + V(x)$ and assume that the function $(0, 1) \ni h \mapsto E(h) \in (-\infty, \ell_{+\infty})$ satisfies

- i. *for any $h \in (0, 1)$, $\{x \in \mathbb{R} : V(x) \leq E(h)\} = [x_{\min}(E(h)), x_{\max}(E(h))]$;*
- ii. $h^{1/3}(x_{\max}(E(h)) - x_{\min}(E(h))) \xrightarrow{h \rightarrow 0} 0$;
- iii. $E(h) \xrightarrow{h \rightarrow 0} E_0 \leq \ell_{+\infty}$.

Then

$$\mathbf{N}(\mathcal{H}_h, E(h)) \underset{h \rightarrow 0}{\sim} \frac{1}{\pi h} \int_{\mathbb{R}} \sqrt{(E_0 - V)_+} dx.$$

7.2.3. *Proof.* The following lemma is a consequence of the definition of the Rayleigh quotients.

Lemma 7.5 (Dirichlet-Neumann bracketing). *Let $(s_j)_{j \in \mathbb{Z}}$ be a subdivision of \mathbb{R} and consider the operators (with Dirichlet or Neumann conditions on the points of the subdivision)*

$$\mathcal{H}_h^{\text{Dir/Neu}} = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}_{h,j}^{\text{Dir/Neu}},$$

where $\mathcal{H}_{h,j}^{\text{Dir/Neu}}$ is the Dir/Neu realization of $h^2 D_x^2 + V(x)$ on (s_j, s_{j+1}) . We have, in terms of the domains of the quadratic forms,

$$\text{Dom}(\mathcal{Q}_h^{\text{Dir}}) \subset \text{Dom}(\mathcal{Q}_h) \subset \text{Dom}(\mathcal{Q}_h^{\text{Neu}}),$$

and the Rayleigh quotients satisfy, for all $n \geq 1$,

$$\mu_n(\mathcal{H}_h^{\text{Neu}}) \leq \mu_n(\mathcal{H}_h) \leq \mu_n(\mathcal{H}_h^{\text{Dir}}).$$

We can now start the proof of Proposition 7.4.

We consider a subdivision of the real axis $(s_j(h^\alpha))_{j \in \mathbb{Z}}$, which contains the discontinuities of V , for which there exist $c > 0$, $C > 0$ such that, for all $j \in \mathbb{Z}$ and $h > 0$, $ch^\alpha \leq s_{j+1}(h^\alpha) - s_j(h^\alpha) \leq Ch^\alpha$, where $\alpha > 0$ is to be determined. Denote

$$J_{\min}(h^\alpha) = \min\{j \in \mathbb{Z} : s_j(h^\alpha) \geq x_{\min}(E(h))\},$$

$$J_{\max}(h^\alpha) = \max\{j \in \mathbb{Z} : s_j(h^\alpha) \leq x_{\max}(E(h))\}.$$

For $j \in \mathbb{Z}$ we introduce the Dirichlet (resp. Neumann) realization on $(s_j(h^\alpha), s_{j+1}(h^\alpha))$ of $h^2 D_x^2 + V(x)$ denoted by $\mathcal{H}_{h,j}^{\text{Dir}}$ (resp. $\mathcal{H}_{h,j}^{\text{Neu}}$). The Dirichlet-Neumann bracketing implies that

$$\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} \mathbf{N}(\mathcal{H}_{h,j}^{\text{Dir}}, E(h)) \leq \mathbf{N}(\mathcal{H}_h, E(h)) \leq \sum_{j=J_{\min}(h^\alpha)-1}^{J_{\max}(h^\alpha)+1} \mathbf{N}(\mathcal{H}_{h,j}^{\text{Neu}}, E(h)).$$

Let us estimate $\mathbf{N}(\mathcal{H}_{h,j}^{\text{Dir}}, E(h))$. If $Q_{h,j}^{\text{Dir}}$ denotes the quadratic form of $\mathcal{H}_{h,j}^{\text{Dir}}$, we have

$$Q_{h,j}^{\text{Dir}}(\psi) \leq \int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)} h^2 |\psi'(x)|^2 + V_{j,\text{sup},h} |\psi(x)|^2 dx, \quad \forall \psi \in \mathcal{C}_0^\infty((s_j(h^\alpha), s_{j+1}(h^\alpha))),$$

where

$$V_{j,\text{sup},h} = \sup_{x \in (s_j(h^\alpha), s_{j+1}(h^\alpha))} V(x).$$

We infer that

$$\mathbf{N}(\mathcal{H}_{h,j}^{\text{Dir}}, E(h)) \geq \#\left\{n \geq 1 : n \leq \frac{1}{\pi h} (s_{j+1}(h^\alpha) - s_j(h^\alpha)) \sqrt{(E(h) - V_{j,\text{sup},h})_+}\right\},$$

so that

$$\mathbf{N}(\mathcal{H}_{h,j}^{\text{Dir}}, E(h)) \geq \frac{1}{\pi h} (s_{j+1}(h^\alpha) - s_j(h^\alpha)) \sqrt{(E(h) - V_{j,\text{sup},h})_+} - 1,$$

and thus

$$\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} \mathbf{N}(\mathcal{H}_{h,j}^{\text{Dir}}, E(h)) \geq \frac{1}{\pi h} \sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} (s_{j+1}(h^\alpha) - s_j(h^\alpha)) \sqrt{(E(h) - V_{j,\text{sup},h})_+} - (J_{\max}(h^\alpha) - J_{\min}(h^\alpha) + 1).$$

Let us consider the function

$$f_h(x) = \sqrt{(E(h) - V(x))_+}$$

and analyze

$$\begin{aligned} & \left| \sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} (s_{j+1}(h^\alpha) - s_j(h^\alpha)) \sqrt{(E(h) - V_{j,\text{sup},h})_+} - \int_{\mathbb{R}} f_h(x) \, dx \right| \\ & \leq \left| \sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} \int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)} \sqrt{(E(h) - V_{j,\text{sup},h})_+} - f_h(x) \, dx \right| \\ & \quad + \int_{s_{J_{\max}(h^\alpha)}}^{x_{\max}(E(h))} f_h(x) \, dx + \int_{x_{\min}(E(h))}^{s_{J_{\min}(h^\alpha)}} f_h(x) \, dx \\ & \leq \left| \sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} \int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)} \sqrt{(E(h) - V_{j,\text{sup},h})_+} - f_h(x) \, dx \right| + \tilde{C}h^\alpha. \end{aligned}$$

Using the trivial inequality $|\sqrt{a_+} - \sqrt{b_+}| \leq \sqrt{|a - b|}$, we get

$$\left| f_h(x) - \sqrt{(E(h) - V_{j,\text{sup},h})_+} \right| \leq \sqrt{|V(x) - V_{j,\text{sup},h}|}.$$

Since V is Lipschitzian on $(s_j(h^\alpha), s_{j+1}(h^\alpha))$, we get:

$$\left| \sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} \int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)} \sqrt{(E(h) - V_{j,\text{sup},h})_+} - f_h(x) \, dx \right| \leq (J_{\max}(h^\alpha) - J_{\min}(h^\alpha) + 1) \tilde{C}h^\alpha h^{\alpha/2}.$$

This leads to the optimal choice $\alpha = \frac{2}{3}$ and we obtain the lower bound

$$\sum_{j=J_{\min}(h^{2/3})}^{J_{\max}(h^{2/3})} \mathbf{N}(\mathcal{H}_{h,j}^{\text{Dir}}, E(h)) \geq \frac{1}{\pi h} \left(\int_{\mathbb{R}} f_h(x) \, dx - \tilde{C}h(J_{\max}(h^{2/3}) - J_{\min}(h^{2/3}) + 1) \right).$$

It follows that

$$\mathbf{N}(\mathcal{H}_h, E(h)) \geq \frac{1}{\pi h} \left(\int_{\mathbb{R}} f_h(x) \, dx - \tilde{C}h^{1/3}(x_{\max}(E(h)) - x_{\min}(E(h)) - \tilde{C}h) \right).$$

Note that $f_h(x) \leq \sqrt{(\ell_{+\infty} - V(x))_+}$, so that we can apply the dominated convergence theorem. We can deal with the Neumann realizations in the same way.

8. HILLE-YOSIDA'S THEOREM

For this section, the Reader might want to consult [Yos95, Chapter IX] or [Bre83, Chapter VII].

8.1. Semi-groups.

Definition 8.1. Let E be a Banach space. A \mathcal{C}^0 -semigroup is family $(T_t)_{t \geq 0}$ of bounded operators on E such that

- i. for all $s, t \geq 0$, $T_t T_s = T_{t+s}$,
- ii. $T_0 = \text{Id}$,
- iii. for all $x \in E$, $\mathbb{R}_+ \ni x \mapsto T_t x$ is continuous.

Lemma 8.2. Let $(T_t)_{t \geq 0}$ be a \mathcal{C}^0 -semigroup. Then, there exist $M \geq 0$ and $c \geq 0$ such that

$$\forall t \geq 0, \quad \|T_t\| \leq M e^{ct}.$$

Proof. For all $t \geq 0$, we have

$$\|T_t\| \leq \|T_1\|^{[t]} \sup_{s \in [0,1]} \|T_s\|.$$

Now, for all $x \in E$, the family $(\|T_s x\|)_{s \in [0,1]}$ is bounded (by continuity of the semi-group on the compact $[0, 1]$). Since E is a Banach space, we can use the Banach-Steinhaus theorem to deduce that $(T_s)_{s \in [0,1]}$ is bounded.

The conclusion follows with $c = \ln \|T_1\|$ and $M = \sup_{s \in [0,1]} \|T_s\|$. \square

Definition 8.3. Let $(T_t)_{t \geq 0}$ be a \mathcal{C}^0 -semigroup. We let

$$\text{Dom}(A) = \left\{ x \in E : \lim_{t \rightarrow 0^+} t^{-1}(T_t - \text{Id})x \text{ exists} \right\},$$

and, for all $x \in \text{Dom}(A)$, we let $Ax = \lim_{t \rightarrow 0^+} t^{-1}(T_t - \text{Id})x$. The operator A is called the infinitesimal generator of the semigroup. Let us discuss some properties of A . In the following the integrals can be understood in the Riemannian sense.

Proposition 8.4. Let $(T_t)_{t \geq 0}$ be a \mathcal{C}^0 -semigroup and A its generator. Then,

- (i) for all $x \in E$ and $t \geq 0$, $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} T_s x \, ds = T_t x$,
- (ii) for all $x \in E$ and $t \geq 0$, $\int_0^t T_s x \, ds \in \text{Dom}(A)$ and $A \int_0^t T_s x \, ds = (T_t - \text{Id})x$,
- (iii) for all $x \in \text{Dom}(A)$ and $t \geq 0$, $T_t x \in \text{Dom}(A)$ and $t \mapsto T_t x$ is of class \mathcal{C}^1 and

$$\frac{d(T_t x)}{dt} = A T_t x = T_t A x,$$

- (iv) for all $x \in \text{Dom}(A)$, for all $s, t \geq 0$, $(T_t - T_s)x = \int_s^t A T_\tau x \, d\tau$.

Proof. The point (i) follows from the continuity. From the point (ii), we write, for all $\varepsilon > 0$,

$$\varepsilon^{-1}(T_\varepsilon - \text{Id}) \int_0^t T_s x \, ds = \varepsilon^{-1} \int_0^t T_{s+\varepsilon} x \, ds - \varepsilon^{-1} \int_0^t T_s x \, ds$$

and thus

$$\varepsilon^{-1}(T_\varepsilon - \text{Id}) \int_0^t T_s x \, ds = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} T_u x \, du - \frac{1}{\varepsilon} \int_0^\varepsilon T_u x \, du.$$

Thus, we can take the limit $\varepsilon \rightarrow 0$ and the equality follows. Let us consider (iii). Let $\varepsilon > 0$, $x \in \text{Dom}(A)$, and $t \geq 0$. We have

$$\frac{1}{\varepsilon}(T_\varepsilon - \text{Id})T_t x = T_t \left(\frac{1}{\varepsilon}(T_\varepsilon - \text{Id})x \right).$$

The right-hand-side has a limit, when $\varepsilon \rightarrow 0^+$, $T_t(Ax)$ by definition of $\text{Dom}(A)$ and continuity of T_t . Thus, by definition of $\text{Dom}(A)$, we get $T_t x \in \text{Dom}(A)$ and $AT_t x = T_t Ax$. We have to check the derivability on the left at $t > 0$. We write

$$\varepsilon^{-1}(T_t x - T_{t-\varepsilon} x) = T_{t-\varepsilon}(\varepsilon^{-1}(T_\varepsilon x - x)) = T_{t-\varepsilon}(Ax) + T_{t-\varepsilon}(\varepsilon^{-1}(T_\varepsilon x - x) - Ax).$$

Since $t \mapsto \|T_t\|$ is locally bounded (by Lemma 8.2), the conclusion follows. The point (iv) follows from the point (iii). \square

Proposition 8.5. *Let $(T_t)_{t \geq 0}$ be a \mathcal{C}^0 -semigroup and A its generator. Then, $\text{Dom}(A)$ is dense and A is closed.*

Proof. For $\varepsilon > 0$, we let $R_\varepsilon = \varepsilon^{-1} \int_0^\varepsilon T_s x ds$. Let $x \in E$. We have $R_\varepsilon x \in \text{Dom}(A)$ and $\lim_{\varepsilon \rightarrow 0} R_\varepsilon x = x$. Thus, $\text{Dom}(A)$ is dense.

Then, we consider $(x_n) \in \text{Dom}(A)^\mathbb{N}$ such that $x_n \rightarrow x$ and $Ax_n \rightarrow y$. For all $t \geq 0$, we have

$$(T_t - \text{Id})x_n = \int_0^t T_s Ax_n ds,$$

and thus, since $s \mapsto \|T_s\|$ is locally bounded,

$$(T_t - \text{Id})x = \int_0^t T_s y ds.$$

Dividing by t and taking the limit $t \rightarrow 0^+$ we find that $x \in \text{Dom}(A)$ and $y = Ax$. \square

8.2. Hille-Yosida's theorem.

Definition 8.6. A contraction on E is a linear map such that $\|T\| \leq 1$.

Theorem 8.7 (Hille-Yosida's theorem). *An operator A is the infinitesimal generator of a contraction semigroup $(T_t)_{t \geq 0}$ if and only if*

- i. *A is closed and $\text{Dom}(A)$ is dense,*
- ii. *$(0, +\infty) \subset \rho(A)$ and, for all $\lambda > 0$, $\|(A - \lambda)^{-1}\| \leq \lambda^{-1}$.*

8.2.1. *Necessary condition.* If A is the infinitesimal generator of a contraction semigroup $(T_t)_{t \geq 0}$, we have already seen that A is closed and $\text{Dom}(A)$ is dense.

Then, for $\lambda > 0$ and $x \in E$, we define

$$R_\lambda x = \int_0^{+\infty} e^{-\lambda t} T_t x dt.$$

It is indeed well defined since $\|e^{-\lambda t} T_t x\| \leq e^{-\lambda t} \|x\|$. Note that

$$\|R_\lambda x\| \leq \lambda^{-1} \|x\|.$$

Let us check that $R_\lambda x \in \text{Dom}(A)$. For all $\varepsilon > 0$, we write, by continuity,

$$\varepsilon^{-1}(T_\varepsilon - \text{Id})R_\lambda x = \varepsilon^{-1} \int_0^{+\infty} e^{-\lambda t} (T_{t+\varepsilon} x - T_t x) dt.$$

Thus,

$$\varepsilon^{-1}(T_\varepsilon - \text{Id})R_\lambda x = \varepsilon^{-1} e^{\varepsilon \lambda} \int_\varepsilon^{+\infty} e^{-\lambda t} T_t x dt - \varepsilon^{-1} \int_0^{+\infty} e^{-\lambda t} T_t x dt,$$

so that

$$\varepsilon^{-1}(T_\varepsilon - \text{Id})R_\lambda x = \varepsilon^{-1}(e^{\varepsilon \lambda} - 1) \int_0^{+\infty} e^{-\lambda t} T_t x dt - \varepsilon^{-1} \int_0^\varepsilon e^{-\lambda t} T_t x dt.$$

This proves that $R_\lambda x \in \text{Dom}(A)$ and $AR_\lambda x = \lambda R_\lambda x - x$. Thus, $(\lambda - A)R_\lambda = \text{Id}$. We can also check that $R_\lambda(\lambda - A) = \text{Id}_{\text{Dom}(A)}$.

8.2.2. *Sufficient condition.* Let us now assume that A is closed and $\text{Dom}(A)$ is dense and that $(0, +\infty) \subset \rho(A)$ and, for all $\lambda > 0$, $\|(A - \lambda)^{-1}\| \leq \lambda^{-1}$.

The idea is to approximate A by a bounded operator and use the exponential. For $\lambda > 0$, we let $S_\lambda = \lambda(\lambda - A)^{-1}$ and $A_\lambda = AS_\lambda$.

For $x \in \text{Dom}(A)$, we have

$$\lambda(\lambda - A)^{-1}x - (\lambda - A)(\lambda - A)^{-1}x = (\lambda - A)^{-1}Ax,$$

so that

$$\lim_{\lambda \rightarrow +\infty} S_\lambda x = x.$$

Note that $\|S_\lambda\| \leq 1$ and $\text{Dom}(A)$ is dense. Thus, for all $x \in E$,

$$\lim_{\lambda \rightarrow +\infty} S_\lambda x = x.$$

Since $S_\lambda A = AS_\lambda$ on $\text{Dom}(A)$, we deduce that

$$\forall x \in \text{Dom}(A), \quad \lim_{\lambda \rightarrow +\infty} A_\lambda x = Ax.$$

Note that A_λ is a bounded operator. We have, for all $t \geq 0$ and $\lambda > 0$,

$$e^{tA_\lambda} = e^{-t\lambda + t\lambda^2 R_\lambda},$$

so that

$$\|e^{tA_\lambda}\| \leq 1.$$

Then, we write

$$e^{tA_\lambda}x - e^{tA_\mu}x = \int_0^1 \frac{d}{ds} e^{tsA_\lambda} e^{t(1-s)A_\mu} x \, ds,$$

so that, for all $t \geq 0$, $\lambda > 0$ and $x \in E$,

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\|.$$

Therefore, for all $x \in \text{Dom}(A)$, $\lim_{\lambda \rightarrow +\infty} e^{tA_\lambda}x$ exists locally uniformly in t . By density of $\text{Dom}(A)$ and $\|e^{tA_\lambda}\| \leq 1$, this limit exists for all $x \in E$ and we let

$$T_t x = \lim_{\lambda \rightarrow +\infty} e^{tA_\lambda}x.$$

We can check that $(T_t)_{t \geq 0}$ is a contraction \mathcal{C}^0 -semigroup. Let us consider B its generator. Let $x \in \text{Dom}(A)$ and $\varepsilon > 0$. We have

$$\varepsilon^{-1}(T_\varepsilon - \text{Id})x = \lim_{\lambda \rightarrow +\infty} \varepsilon^{-1}(e^{\varepsilon A_\lambda} - \text{Id})x = \lim_{\lambda \rightarrow +\infty} \varepsilon^{-1} \int_0^\varepsilon e^{sA_\lambda} A_\lambda x \, ds = \varepsilon^{-1} \int_0^\varepsilon T_s Ax \, ds.$$

We deduce that $x \in \text{Dom}(B)$ and $Bx = Ax$. Thus $A \subset B$.

Since $1 \in \rho(A)$, we have $(1 - A)\text{Dom}(A) = E$. But, $1 \in \rho(B)$ so that $(1 - B)\text{Dom}(B) = E$. It follows that $\text{Dom}(B) \subset \text{Dom}(A)$.

8.3. Stone's theorem.

Theorem 8.8 (Stone's theorem). *Let H be a self-adjoint operator. There exists a unique \mathcal{C}^0 -unitary group $(U_t)_{t \in \mathbb{R}}$ such that*

- (i) $U_t : \text{Dom}(H) \rightarrow \text{Dom}(H)$,
- (ii) for all $u \in \text{Dom}(H)$, $U_t u \in \mathcal{C}^1(\mathbb{R}, H) \cap \mathcal{C}^0(\mathbb{R}, \text{Dom}(H))$,
- (iii) for all $u \in \text{Dom}(H)$, $\frac{d}{dt} U_t u = iH U_t u = iU_t H u$,
- (iv) $U_0 = \text{Id}$.

We let $U_t = e^{itH}$ for all $t \in \mathbb{R}$.

Conversely, if $(U_t)_{t \in \mathbb{R}}$ is a \mathcal{C}^0 -unitary group, then, there exists a unique self-adjoint operator H such that, for all $t \in \mathbb{R}$, $U_t = e^{itH}$. The domain is

$$\text{Dom}(H) = \left\{ u \in \mathbb{H} : \sup_{0 < t \leq 1} t^{-1} \|U_t u - u\| < +\infty \right\}.$$

8.3.1. Necessary condition. Let H be a self-adjoint operator. H is closed with dense domain. For all $\lambda > 0$, we have already seen that $\pm iH - \lambda$ is bijective and that $\|(\pm iH - \lambda)^{-1}\| \leq \lambda^{-1}$. Therefore the operators $\pm iH$ are the generators of \mathcal{C}^0 -semigroups $(U_t^\pm)_{t \geq 0}$. We have $\frac{d}{dt} U_t^- U_t^+ u = -iH U_t^- U_t^+ u + U_t^- iH U_t^+ u = 0$. We get that, for all $t \geq 0$, $U_t^- U_t^+ u = u$. We let $U_t = U_t^+$ for $t \geq 0$ and $U_t = U_{-t}^-$ for $t < 0$. $(U_t)_{t \in \mathbb{R}}$ is a \mathcal{C}^0 -group. We have, for all $t \in \mathbb{R}$, $U_t' = iH U_t$. For all $u \in \text{Dom}(H)$, we have

$$\frac{d}{dt} \|U_t u\|^2 = \langle iH U_t u, U_t u \rangle + \langle U_t u, iH U_t u \rangle = 0.$$

Thus, $(U_t)_{t \in \mathbb{R}}$ is unitary.

8.3.2. Sufficient condition. Let $(U_t)_{t \in \mathbb{R}}$ be a \mathcal{C}^0 -unitary group. Let us write the generator of $(U_t)_{t \geq 0}$ as iH . H is closed and has a dense domain. Moreover $\text{ran}(H + i) = \mathbb{H}$. Differentiating $U_t U_{-t} = \text{Id}$, we get that $-iH$ is the generator of $(U_{-t})_{t \geq 0}$. In particular $\text{ran}(H - i) = \mathbb{H}$. Differentiating $\|U_t u\|^2 = \|u\|^2$, we get that H is symmetric. We deduce that H is self-adjoint.

We have

$$\text{Dom}(H) \subset \left\{ u \in \mathbb{H} : \sup_{0 < t \leq 1} t^{-1} \|U_t u - u\| < +\infty \right\}.$$

Then, take $u \in \mathbb{H}$ such that

$$\sup_{0 < t \leq 1} t^{-1} \|U_t u - u\| < +\infty,$$

and consider $v \in \text{Dom}(H)$. We have

$$|\langle u, H v \rangle| = \lim_{t \rightarrow 0^+} |\langle u, U_t v - v \rangle| = \lim_{t \rightarrow 0^+} |\langle U_{-t} u - u, v \rangle| \leq C \|v\|.$$

This shows that $u \in \text{Dom}(H^*) = \text{Dom}(H)$.

9. ABOUT THE SPECTRAL MEASURE

9.1. A functional calculus based on the Fourier transform. The aim of this section is to introduce the Reader to the notion of spectral measure associated with a self-adjoint operator. One can consult [RS, Vol. I, Chapter VII] for an alternative presentation or the older references [Sto90, Hal98].

If H is a self-adjoint operator we would like to define functions of H such that

- (i) $f(H) : \text{Dom}(H) \rightarrow \text{Dom}(H)$,
- (ii) $[f(H), H] = 0$, on $\text{Dom}(H)$,
- (iii) $f(H) + g(H) = (f + g)(H)$, on $\text{Dom}(f(H)) \cap \text{Dom}(g(H))$,
- (iv) $f(H)g(H) = (fg)(H)$, on $\{u \in \text{Dom}(g(H)) : g(H)u \in \text{Dom}(f(H))\}$,
- (v) $f(H)^* = \bar{f}(H)$.

We will construct such a functional calculus by using the inverse Fourier transform. We will denote by \mathcal{F} the Fourier transform, defined on $\mathcal{S}(\mathbb{R})$ by

$$\mathcal{F}\psi(\xi) = \int_{\mathbb{R}} \psi(x) e^{-ix\xi} dx.$$

Definition 9.1. Let H be a self-adjoint operator. For all $f \in \mathcal{S}(\mathbb{R})$ and $u \in \mathbf{H}$, we let

$$f(H)u = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F} f(t) e^{itH} u \, dt.$$

Note that $f(H) \in \mathcal{L}(\mathbf{H})$.

Proposition 9.2. For all $f, g \in \mathcal{S}(\mathbb{R})$, we have (i)–(v).

Proof. We leave this proof to the reader. It uses the fact that the Fourier transform of a convolution of two functions is the product of their Fourier transforms. \square

We introduce $\mathcal{A} = \mathcal{S}(\mathbb{R}) \oplus \mathbb{C}$ and we extend our functional calculus (we add the constants!) by letting, for all $f \in \mathcal{A}$,

$$f(H) = f_0(H) + \lambda_0 \text{Id} \in \mathcal{L}(\mathbf{H}),$$

where $(f_0, \lambda_0) \in \mathcal{S}(\mathbb{R}) \times \mathbb{C}$ is defined by $f = f_0 + \lambda_0$.

Proposition 9.3. We have, for all $f, g \in \mathcal{A}$, (i)–(v).

Lemma 9.4. Let $f \in \mathcal{A}$ with $f \geq 0$. Then, we have, for all $u \in \mathbf{H}$,

$$\langle f(H)u, u \rangle \geq 0.$$

Proof. Let $\varepsilon > 0$. The function $(\varepsilon + f)^{\frac{1}{2}}$ belongs to \mathcal{A} . We have

$$(\varepsilon + f)^{\frac{1}{2}}(H)(\varepsilon + f)^{\frac{1}{2}}(H) = (\varepsilon + f)(H).$$

Thus, since $(\varepsilon + f)^{\frac{1}{2}}(H)$ is symmetric, for all $u \in \mathbf{H}$,

$$\langle u, (\varepsilon + f)(H)u \rangle = \|(\varepsilon + f)^{\frac{1}{2}}(H)u\|^2 \geq 0.$$

Then, we take the limit $\varepsilon \rightarrow 0$. \square

Lemma 9.5. For all $f \in \mathcal{A}$, we have $\|f(H)\| \leq \|f\|_{\infty}$.

Proof. Let us consider $g = \|f\|_{\infty}^2 - |f|^2 \in \mathcal{A}$. We get, for all $u \in \mathbf{H}$,

$$\langle g(H)u, u \rangle \geq 0,$$

so that

$$0 \leq \langle |f|^2(H)u, u \rangle \leq \|f\|_{\infty}^2 \|u\|^2.$$

But, we have

$$\langle |f|^2(H)u, u \rangle = \langle (\bar{f}f)(H)u, u \rangle = \langle \bar{f}(H)f(H)u, u \rangle = \langle f(H)^* f(H)u, u \rangle = \|f(H)u\|^2.$$

\square

Lemma 9.6. Consider $\chi \in \mathcal{C}_0^{\infty}(\mathbb{R}, \mathbb{R})$ such that $0 \leq \chi \leq 1$ equal to 1 in a neighborhood of 0. For $R > 0$, we let $\chi_R(\cdot) = \chi(R^{-1}\cdot)$. Then, for all $u \in \mathbf{H}$,

$$\lim_{R \rightarrow +\infty} \chi_R(H)u = u.$$

Proof. By definition, we have

$$2\pi \chi_R(H)u = \int_{\mathbb{R}} \mathcal{F} \chi_R(t) e^{itH} u \, dt = \int_{\mathbb{R}} R(\mathcal{F} \chi)(Rt) e^{itH} u \, dt = \int_{\mathbb{R}} (\mathcal{F} \chi)(t) e^{itH/R} u \, dt.$$

We have, by continuity of the group, for all $t \in \mathbb{R}$,

$$\lim_{R \rightarrow +\infty} e^{itH/R} u = u.$$

Moreover,

$$\|(\mathcal{F} \chi)(t) e^{itH/R} u\| \leq |(\mathcal{F} \chi)(t)| \|u\|.$$

Therefore, we can use the dominated convergence theorem in the context of the Riemann integration (or notice directly that the convergence is uniform on the compacts) and we get

$$\lim_{R \rightarrow +\infty} \chi_R(H)u = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F} \chi(t)u dt = \chi(0)u = u.$$

□

9.2. Where the spectral measure comes into play.

9.2.1. Extending a map.

Definition 9.7. For all $f \in \mathcal{S}(\mathbb{R})$ and $u, v \in \mathbf{H}$, we let

$$\omega_{u,v}(f) = (f(H)u, v).$$

Lemma 9.8. *The following holds.*

- i. For all $f \in \mathcal{S}(\mathbb{R})$, $\omega_{\cdot,\cdot}(f)$ is a continuous sesquilinear form on \mathbf{H} and

$$\|\omega_{\cdot,\cdot}(f)\| \leq \|f\|_{\infty}.$$

- ii. For all $u \in \mathbf{H}$, the linear form $\omega_{u,u} : \mathcal{S}(\mathbb{R}) \ni f \mapsto \omega_{u,u}(f) \in \mathbb{C}$ is non-negative and continuous for the topology of $\|\cdot\|_{\infty}$.

- iii. If $\mathcal{S}(\mathbf{H} \times \mathbf{H}, \mathbb{C})$ denotes the set of the continuous sesquilinear form on \mathbf{H} , the map

$$(\mathcal{S}(\mathbb{R}), \|\cdot\|_{\infty}) \ni f \mapsto \omega_{\cdot,\cdot}(f) \in (\mathcal{S}(\mathbf{H} \times \mathbf{H}, \mathbb{C}), \|\cdot\|)$$

is linear and continuous. It can be uniquely extended as a continuous linear map on $(\mathcal{C}_{\rightarrow 0}^0(\mathbb{R}), \|\cdot\|_{\infty})$. Keeping the same notation $\omega_{\cdot,\cdot}(f)$ for the extended map, we have

$$\forall f \in \mathcal{C}_{\rightarrow 0}^0(\mathbb{R}), \quad \|\omega_{\cdot,\cdot}(f)\| \leq \|f\|_{\infty},$$

and, for all $f \in \mathcal{C}_{\rightarrow 0}^0(\mathbb{R})$, with $f \geq 0$, we have $\omega_{\cdot,\cdot}(f) \geq 0$.

By the Riesz representation theorem, we can extend the functional calculus to continuous functions tending to zero at infinity.

Proposition 9.9. *Let $f \in \mathcal{C}_{\rightarrow 0}^0(\mathbb{R})$. There exists a unique bounded operator, denoted by $f(H)$, such that, for all $u, v \in \mathbf{H}$,*

$$\langle f(H)u, v \rangle = \omega_{u,v}(f).$$

We have (i)–(v). Moreover, we have

$$\|f(H)\| \leq \|f\|_{\infty}.$$

9.2.2. *About the Riesz-Markov theorem and definition of the spectral measure.* Let us now recall a classical representation theorem (see [Rud80, Théorème 2.14]).

Theorem 9.10. *Let X be a separated and locally compact topological space. Let ω be a non-negative form on $\mathcal{C}_0^0(X)$.*

Then, there exists a σ -algebra \mathcal{M} containing the Borelian sets of X and a unique non-negative measure μ on \mathcal{M} such that

$$\forall f \in \mathcal{C}_0^0(\mathbb{R}), \quad \omega(f) = \int_X f d\mu.$$

Moreover, this measure μ is regular.

We apply this theorem to $\omega_{u,u}$ and we get a measure $\mu_{u,u}$ (the spectral measure associated with u) and a σ -algebra $\mathcal{M}_{u,u}$. Now, we let

$$\mathcal{M} = \bigcap_{u \in \mathbf{H}} \mathcal{M}_{u,u}.$$

It is still a σ -algebra containing the Borelian sets.

Lemma 9.11. For all $u \in \mathbb{H}$, the measure $\mu_{u,u}$ is finite, and $\mu_u(\mathbb{R}) = \|u\|^2$.

Proof. We recall Lemma 9.6. Let $u \in \mathbb{H}$. We use the function χ_R . We have, for all $R > 0$,

$$\omega_{u,u}(\chi_R) \leq \|u\|^2,$$

and

$$\lim_{R \rightarrow +\infty} \omega_{u,u}(\chi_R) = \|u\|^2.$$

Moreover, we have

$$\omega_{u,u}(\chi_R) = \int_{\mathbb{R}} \chi_R(\lambda) d\mu_{u,u}(\lambda).$$

With the Fatou Lemma, we get

$$\mu_{u,u}(\mathbb{R}) \leq \liminf_{R \rightarrow +\infty} \int_{\mathbb{R}} \chi_R(\lambda) d\mu_{u,u}(\lambda) \leq \|u\|^2 < +\infty.$$

Thus, the measure $\mu_{u,u}$ is finite. It remains to use the dominated convergence theorem to see that

$$\|u\|^2 = \lim_{R \rightarrow +\infty} \omega_{u,u}(\chi_R) = \mu_{u,u}(\mathbb{R}).$$

□

Definition 9.12. For Ω is a Borelian set, we consider the application $q : \mathbb{H} \rightarrow \mathbb{R}_+$ defined by

$$\mathbb{H} \ni u \mapsto \int_{\mathbb{R}} \mathbb{1}_{\Omega} d\mu_{u,u} = \mu_{u,u}(\Omega).$$

Lemma 9.13. q_{Ω} is a continuous quadratic form.

Proof. Note that $0 \leq \mu_{u,u}(\Omega) \leq \|u\|^2$. In particular, once we will have proved that q_{Ω} is a quadratic form, it will be a continuous quadratic form (by using the polarization formula).

Since, for all $u \in \mathbb{H}$, $\mu_{u,u}$ is a measure, we only have to prove the result when Ω is an open set and even when Ω is an interval in the form $[a, b]$. In this case, we introduce the sequence of continuous and piecewise affine functions (f_n) such that $f_n(x) = 1$ on $[a, b]$, $f_n(x) = 0$ for $x \leq a - \frac{1}{n}$ and $x \geq b + \frac{1}{n}$. By dominated convergence, we have

$$\lim_{n \rightarrow +\infty} \langle f_n(H)u, u \rangle = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}} f_n d\mu_{u,u} = \mu_{u,u}(\Omega),$$

and the conclusion follows from the polarization formula. □

Proposition 9.14. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a bounded Borelian function. Then there exists a unique continuous sesquilinear form $\tilde{\omega}_{\cdot, \cdot}(f)$ on \mathbb{H} such that

$$\forall u \in \mathbb{H}, \quad \tilde{\omega}_{u,u}(f) = \int_{\mathbb{R}} f d\mu_{u,u}.$$

Proof. With Lemma 9.13, this result is known for $f = \mathbb{1}_{\Omega}$, for all Borelian set Ω . From the measure theory, one knows that all bounded Borelian function is a uniform limit of step functions. This implies that $u \mapsto \int_{\mathbb{R}} f d\mu_{u,u}$ is a quadratic form. It is continuous since $|\int_{\mathbb{R}} f d\mu_{u,u}| \leq \|f\|_{\infty} \|u\|^2$. □

From this proposition, we can define $f(H)$ via the Riesz representation theorem.

Proposition 9.15. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a bounded Borelian function. There exists a unique bounded operator, denoted by $f(H)$, such that, for all $u \in \mathbb{H}$,

$$\langle f(H)u, u \rangle = \int_{\mathbb{R}} f d\mu_{u,u}.$$

When $f \in \mathcal{C}_{\rightarrow 0}^0$ or $f \in \mathcal{A}$, we recover the same $f(H)$.

Proposition 9.16. *Let $t \in \mathbb{R}$ and consider $f(\cdot) = e^{it\cdot}$. We have $f(H) = e^{itH}$. In particular,*

$$(9.1) \quad \forall u \in \mathbb{H}, \forall t \in \mathbb{R}, \quad \langle e^{itH}u, u \rangle = \int_{\mathbb{R}} e^{it\lambda} d\mu_{u,u}(\lambda).$$

Proof. Let us consider $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $\text{supp}(\rho) \subset [-1, 1]$ and $\int_{\mathbb{R}} \rho(x) dx = 2\pi$. We introduce $\chi \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}\chi = \rho$. For all $n \in \mathbb{N}^*$, we let

$$\rho_n(\cdot) = n\rho(n\cdot) = \mathcal{F}(\chi(n^{-1}\cdot)).$$

Note that

$$\chi(n^{-1}x) = (2\pi)^{-1} \int_{\mathbb{R}} \rho_n(x) e^{ix\xi} d\xi = (2\pi)^{-1} \int_{\mathbb{R}} \rho(x) e^{i\xi \frac{x}{n}} d\xi.$$

Thus, $\lim_{n \rightarrow +\infty} \chi(n^{-1}x) = 1$ and $\|\chi(n^{-1}\cdot)\|_\infty \leq 1$.

Let us consider $f_n(\cdot) = \chi(n^{-1}\cdot)e^{it\cdot} \in \mathcal{S}(\mathbb{R})$. For all $u \in \mathbb{H}$, we have

$$\langle f_n(H)u, u \rangle = \int_{\mathbb{R}} f_n d\mu_{u,u}.$$

By the dominated convergence theorem, we have

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}} f_n d\mu_{u,u} = \int_{\mathbb{R}} e^{it\lambda} d\mu_{u,u}(\lambda).$$

But, we also have

$$f_n(H)u = (2\pi)^{-1} \int_{\mathbb{R}} \mathcal{F}f_n(\lambda) e^{i\lambda H} u d\lambda = (2\pi)^{-1} \int_{\mathbb{R}} \rho_n(\lambda - t) e^{i\lambda H} u d\lambda,$$

and then

$$f_n(H)u = (2\pi)^{-1} e^{itH} \int_{\mathbb{R}} \rho(\lambda) e^{in^{-1}\lambda H} u d\lambda,$$

so that

$$\lim_{n \rightarrow +\infty} f_n(H)u = e^{itH}u.$$

Therefore, we have, for all $u \in \mathbb{H}$,

$$\langle e^{itH}u, u \rangle = \int_{\mathbb{R}} e^{it\lambda} d\mu_{u,u}(\lambda).$$

□

9.3. Spectral projections.

9.3.1. Properties.

Definition 9.17. Let Ω be a Borelian set. We let $E_\Omega = \mathbb{1}_\Omega(H)$.

Proposition 9.18. *There holds:*

- (i) $E_\emptyset = 0$ and $E_{\mathbb{R}} = \text{Id}$.
- (ii) For all Borelian set Ω , E_Ω is an orthogonal projection.
- (iii) For all Borelian sets Ω_1 and Ω_2 , $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$.
- (iv) Let $\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j$ be a Borelian partition. Then, for all $u \in \mathbb{H}$,

$$\lim_{N \rightarrow +\infty} \sum_{j=0}^N E_{\Omega_j} u = E_\Omega u.$$

Proof. For the first point, we use Lemma 9.11. The proof of the other points uses the proof of the Riesz-Markov theorem (see [Rud80, p. 51]). Let $V \subset \mathbb{R}$ be an open set. We have

$$\mu_{u,u}(V) = \sup\{\langle f(H)u, u \rangle, \quad f \in \mathcal{C}_0^0(\mathbb{R}), \quad 0 \leq f \leq \mathbb{1}_V\}.$$

By using an exhaustion by compact sets of V and Urysohn's lemma (see [Rud80, Lemme 2.12]), we can construct a non decreasing sequence $(f_n) \subset \mathcal{C}_0^0(\mathbb{R})$ such that $f_n f_m = f_n$ for all $m \geq n$ and $\lim_{n \rightarrow +\infty} f_n = \mathbb{1}_V$. For all $u \in \mathbb{H}$, we have

$$\langle f_n(H)u, u \rangle = \int_{\mathbb{R}} |f_n|^2 d\mu_{u,u},$$

and thus, by Beppo Levi's theorem,

$$\lim_{n \rightarrow +\infty} \langle f_n(H)u, u \rangle = \langle \mathbb{1}_V(H)u, u \rangle.$$

This implies that, for all $u, v \in \mathbb{H}$,

$$\lim_{n \rightarrow +\infty} \langle f_n(H)u, v \rangle = \langle \mathbb{1}_V(H)u, v \rangle.$$

We have, for all $m \geq n$,

$$\langle f_m(H)u, f_n(H)^*u \rangle = \langle (f_n f_m)(H)u, u \rangle = \langle f_n(H)u, u \rangle.$$

Taking the limit $m \rightarrow +\infty$, we get

$$\langle f_n(H)\mathbb{1}_V(H)u, u \rangle = \langle \mathbb{1}_V(H)u, f_n(H)^*u \rangle = \langle \mathbb{1}_V(H)u, u \rangle,$$

so that, for all $u \in \mathbb{H}$,

$$\langle \mathbb{1}_V(H)^2 u, u \rangle = \langle \mathbb{1}_V(H)u, u \rangle.$$

Thus $\mathbb{1}_V(H)^2 = \mathbb{1}_V(H)$ and it is clear that the operator $\mathbb{1}_V(H)$ is self-adjoint (by using that $\overline{f_n} = f_n$). If V_1 and V_2 are two open sets, we easily get, by considering associated sequences of functions,

$$\mathbb{1}_{V_1}(H)\mathbb{1}_{V_2}(H) = \mathbb{1}_{V_1 \cap V_2}(H).$$

Easy manipulations allow to extend this to Borelian sets.

Let us prove (iv). Take $u \in \mathbb{H}$. For all $n \geq p$, we have

$$\left\| \sum_{j=p}^n \mathbb{1}_{\Omega_j}(H)u \right\|^2 = \left\langle \sum_{j=p}^n \mathbb{1}_{\Omega_j}(H)u, \sum_{j=p}^n \mathbb{1}_{\Omega_j}(H)u \right\rangle = \left\langle \sum_{j=p}^n \mathbb{1}_{\Omega_j}(H)u, u \right\rangle = \int_{\mathbb{R}} \sum_{j=p}^n \mathbb{1}_{\Omega_j} d\mu_{u,u},$$

we get the desired convergence by the Cauchy criterion. \square

Corollary 9.19. *For all bounded step function $f : \mathbb{R} \rightarrow \mathbb{C}$, we have (iii)–(v), and*

$$\|f(H)\| \leq \|f\|_{\infty}.$$

Proof. The last inequality comes from the fact that, for all $u \in \mathbb{H}$,

$$\|f(H)u\|^2 = \langle f(H)^* f(H)u, u \rangle = \langle \overline{f}(H) f(H)u, u \rangle = \langle (\overline{f}f)(H)u, u \rangle = \int_{\mathbb{R}} |f|^2 d\mu_{u,u}.$$

\square

All bounded Borelian function can be uniformly approximated by a sequence of step functions.

Proposition 9.20. *For all bounded Borelian functions, we have (i)–(v), and*

$$\|f(H)\| \leq \|f\|_{\infty}.$$

Proof. Points (iii)-(v) are obtained by considering uniform limits of step functions.

We have just to check (i) and (ii). Let $u \in \text{Dom}(H)$ and $\varepsilon > 0$. Then, we have, with the multiplication property (iv) and Proposition 9.16,

$$\frac{e^{i\varepsilon H} - \text{Id}}{\varepsilon} f(H)u = f(H) \frac{e^{i\varepsilon H} - \text{Id}}{\varepsilon} u.$$

The conclusion follows by taking the limit $\varepsilon \rightarrow 0$. \square

Proposition 9.21. *Let Ω be a bounded Borelian set. Then, for all $u \in \mathbb{H}$, we have $\mathbb{1}_\Omega(H)u \in \text{Dom}(H)$.*

Proof. For all $\varepsilon > 0$ and $u \in \mathbb{H}$, we have, by Propositions 9.16 and 9.20,

$$\left\| \frac{e^{i\varepsilon H} - \text{Id}}{\varepsilon} \mathbb{1}_\Omega(H)u \right\|^2 = \int_\Omega \left| \frac{e^{i\varepsilon\lambda} - 1}{\varepsilon} \right|^2 d\mu_{u,u} \leq \int_\Omega |\lambda|^2 d\mu_{u,u} < +\infty.$$

\square

9.3.2. Extension to unbounded functions.

Definition 9.22. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a Borelian function. We let

$$\text{Dom}(f(H)) = \left\{ u \in \mathbb{H} : \int_{\mathbb{R}} |f|^2 d\mu_{u,u} < +\infty \right\}.$$

For all $u \in \text{Dom}(H)$, we let

$$f(H)u = \lim_{n \rightarrow +\infty} f_n(H)u,$$

with $f_n(\lambda) = f(\lambda) \mathbb{1}_{|f| \leq n}(\lambda)$.

Note that this definition is consistent since, for all $u \in \text{Dom}(H)$, and all $m \geq n$,

$$\|(f_n(H) - f_m(H))u\|^2 = \int_{\mathbb{R}} |f_n - f_m|^2 d\mu_{u,u} = \int_{\{|f| > n\}} |f|^2 d\mu_{u,u}.$$

Lemma 9.23. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a Borelian function. Then $\text{Dom}(f(H))$ is dense.*

Proof. For all $\varphi \in \mathbb{H}$, we let $\varphi_n = \mathbb{1}_{|f| \leq n}(H)\varphi$. The sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges to φ .

For all $k \in \mathbb{N}$, we have

$$\|f_k(H)\varphi_n\|^2 = \int_{\mathbb{R}} |f_k|^2 d\mu_{\varphi_n, \varphi_n} = \int_{\mathbb{R}} |f_k|^2 \mathbb{1}_{|f| \leq n} d\mu_{\varphi, \varphi} = \int_{\mathbb{R}} |f|^2 \mathbb{1}_{|f| \leq k} \mathbb{1}_{|f| \leq n} d\mu_{\varphi, \varphi}.$$

Thus, for $k \geq n$, we have

$$\int_{\mathbb{R}} |f_k|^2 d\mu_{\varphi_n, \varphi_n} \leq n^2 \|\varphi\|^2.$$

By the Fatou lemma, it follows

$$\int_{\mathbb{R}} |f|^2 d\mu_{\varphi_n, \varphi_n} \leq n^2 \|\varphi\|^2 < +\infty.$$

The density follows.

Let us explain why $f(H)\varphi_n = f_n(H)\varphi$. We have $f_k(H)\varphi_n = (f \mathbb{1}_{|f| \leq k} \mathbb{1}_{|f| \leq n})(H)\varphi = f_n(H)\varphi_k$. We can take the limit $k \rightarrow +\infty$ and we find $f(H)\varphi_n = f_n(H)\varphi$. \square

Proposition 9.24. *Let us consider $f = \text{Id}_{\mathbb{R}}$. We have $f(H) = H$.*

Proof. We must check that

$$\text{Dom}(H) = \left\{ u \in \mathsf{H} : \int_{\mathbb{R}} |\lambda|^2 d\mu_{u,u} < +\infty \right\}.$$

Thanks to Proposition 9.16, we have, for all $u \in \mathsf{H}$,

$$\left\| \frac{e^{i\varepsilon H} - \text{Id}}{\varepsilon} u \right\|^2 = \int_{\mathbb{R}} \left| \frac{e^{i\varepsilon\lambda} - 1}{\varepsilon} \right|^2 d\mu_{u,u}.$$

If $u \in \text{Dom}(H)$, we have $\lim_{\varepsilon \rightarrow 0} \frac{e^{i\varepsilon H} - \text{Id}}{\varepsilon} u = Hu$. Thus, by the Fatou lemma, it follows that

$$\|Hu\|^2 \geq \int_{\mathbb{R}} |\lambda|^2 d\mu_{u,u}.$$

Conversely, if $\int_{\mathbb{R}} |\lambda|^2 d\mu_{u,u} < +\infty$, and noticing that

$$\left| \frac{e^{i\varepsilon\lambda} - 1}{\varepsilon} \right|^2 \leq |\lambda|^2,$$

we get that $\left\| \frac{e^{i\varepsilon H} - \text{Id}}{\varepsilon} u \right\|^2$ is bounded for $\varepsilon \in (0, 1]$. Thus, $u \in \text{Dom}(H)$. Note that this implies that

$$\|Hu\|^2 = \int_{\mathbb{R}} |\lambda|^2 d\mu_{u,u}.$$

Then, we consider $f_n(\lambda) = \lambda \mathbb{1}_{|\lambda| \leq n}(\lambda)$ and we write, for all $u \in \mathsf{H}$,

$$\langle f_n(H)u, u \rangle = \int_{\mathbb{R}} \lambda \mathbb{1}_{|\lambda| \leq n}(\lambda) d\mu_{u,u}.$$

By the Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}} |\lambda| d\mu_{u,u} \leq \left(\int_{\mathbb{R}} |\lambda|^2 d\mu_{u,u} \right)^{\frac{1}{2}} \|u\|,$$

and thus, we can use the dominated convergence theorem to get, for all $u \in \text{Dom}(H)$,

$$\langle f(H)u, u \rangle = \int_{\mathbb{R}} \lambda d\mu_{u,u} = \langle Hu, u \rangle,$$

where we used the derivative of (9.1) for the last equality. The conclusion follows. \square

Proposition 9.25. *If Ω is a bounded Borelian, we have, for all $u \in \text{Dom}(H)$,*

$$\|\mathbb{1}_{\Omega}(H)Hu\| \leq \sup_{\lambda \in \Omega} |\lambda| \|u\|.$$

In particular, $\mathbb{1}_{\Omega}(H)H$ can be extended as a bounded operator on H .

Proof. For all $n \in \mathbb{N}^*$, we let $f_n(\lambda) = \lambda \chi(n^{-1}\lambda)$. For all $u \in \text{Dom}(H)$, we have, for all $m \geq n$,

$$\|(f_n(H) - f_m(H))u\|^2 = \int_{\mathbb{R}} |f_n(\lambda) - f_m(\lambda)|^2 d\mu_{u,u} \leq 4 \int_{n \leq |\lambda| \leq m} |\lambda|^2 d\mu_{u,u}.$$

Thus, $(f_n(H)u)_{n \in \mathbb{N}^*}$ is a Cauchy sequence and it converges. By considering $\langle f_n(H)u, u \rangle$, we deduce that

$$\forall u \in \text{Dom}(H), \quad \lim_{n \rightarrow +\infty} f_n(H)u = Hu.$$

Now, for all $n \in \mathbb{N}^*$ and $u \in \mathsf{H}$,

$$\|\mathbb{1}_{\Omega}(H)f_n(H)u\| \leq \sup_{\lambda \in \Omega} |\lambda| \|u\|.$$

Taking the limit for $u \in \text{Dom}(H)$, we get the result. \square

Proposition 9.26. *In the class of Borelian functions, we have (iii)–(v). $f(T)$ is closed with dense domain.*

Proof. The density of the domain comes from Lemma 9.23. For all $u, v \in \text{Dom}(f(H)) = \text{Dom}(\overline{f}(H))$, we have

$$\langle f(H)u, v \rangle = \lim_{n \rightarrow +\infty} \langle f_n(H)u, v \rangle = \lim_{n \rightarrow +\infty} \langle u, \overline{f}_n(H)v \rangle = \langle u, \overline{f}(H)v \rangle.$$

This shows that $\overline{f}(H) \subset f(H)^*$. Let us now take $v \in \text{Dom}(f(H)^*)$. We have, for all $u \in \text{Dom}(f(H))$,

$$\langle f(H)u, v \rangle = \langle u, f(H)^*v \rangle,$$

so that

$$|\langle f(H)u, v \rangle| \leq \|f(H)^*v\| \|u\|.$$

For all $n \in \mathbb{N}$, we take $u = u_n = \mathbb{1}_{|f| \leq n} \varphi$ with $\varphi \in \mathbb{H}$ (see the proof of Lemma 9.23). We get, for all $n \in \mathbb{N}$ and $\varphi \in \mathbb{H}$,

$$|\langle f_n(H)\varphi, v \rangle| \leq \|f(H)^*v\| \|\varphi\|,$$

and thus

$$|\langle \varphi, \overline{f}_n(H)v \rangle| \leq \|f(H)^*v\| \|\varphi\|.$$

We deduce that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} |f_n|^2 d\mu_{v,v} = \|\overline{f}_n(H)v\|^2 \leq \|f(H)^*v\|^2.$$

By the Fatou lemma, we get that $v \in \text{Dom}(\overline{f}(H))$. This proves that $f(H)^* = \overline{f}(H)$. In particular, this establishes that $f(H)$ is closed as the adjoint of $\overline{f}(H)$.

It remains to prove (iv). We have, for all $u \in \mathbb{H}$,

$$f_m(H)g_n(H)u = (f_m g_n)(H)u.$$

Then,

$$\|f_m(H)g_n(H)u\|^2 = \int_{\mathbb{R}} |f_m|^2 |g_n|^2 d\mu_{u,u},$$

so that, for all $u \in \{v \in \text{Dom}(g(H)) : g(H)v \in \text{Dom}(f(H))\}$,

$$\liminf_{m \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}} |f_m|^2 |g_n|^2 d\mu_{u,u} \leq \|f(H)g(H)u\|^2.$$

By the Fatou lemma, it follows that $u \in \text{Dom}((fg)(H))$. We have

$$f_m(H)g_n(H)u = (f_m g_n)(H)u,$$

and it remains to take the limits. \square

9.3.3. Characterization of the spectra.

Proposition 9.27. $\lambda \in \text{sp}(H)$ if and only if, for all $\varepsilon > 0$, $\mathbb{1}_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H) \neq 0$. In particular, for all $u \in \mathbb{H}$, the support of $\mu_{u,u}$ is contained in $\text{sp}(H)$.

Proof. Assume that, for all $\varepsilon > 0$, $\mathbb{1}_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H) \neq 0$. We can consider $u_\varepsilon \in \mathbb{H}$ such that $\|u_\varepsilon\| = 1$ and

$$\mathbb{1}_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H)u_\varepsilon = u_\varepsilon \in \text{Dom}(H).$$

We write

$$\|(H - \lambda)u_\varepsilon\|^2 = \|\mathbb{1}_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H)(H - \lambda)u_\varepsilon\|^2 \leq \varepsilon^2.$$

Thus, $\lambda \in \text{sp}(H)$.

Conversely, assume that there exists $\varepsilon_0 > 0$ such that $\mathbb{1}_{(\lambda-\varepsilon_0, \lambda+\varepsilon_0)}(H) = 0$. Let us consider the bounded operator R_λ defined via

$$\forall u \in \mathbf{H}, \quad \langle R_\lambda u, u \rangle = \int_{|\mu-\lambda| \geq \varepsilon_0} (\mu - \lambda)^{-1} d\mu_{u,u}.$$

By considering, for all $t \in (0, 1]$ and all $u \in \mathbf{H}$,

$$\left\| \frac{e^{itH} - \text{Id}}{t} R_\lambda u \right\|^2 = \int_{|\mu-\lambda| \geq \varepsilon_0} (\mu - \lambda)^{-2} \left| \frac{e^{it\lambda} - 1}{t} \right|^2 d\mu_{u,u} \leq \int_{|\mu-\lambda| \geq \varepsilon_0} \lambda^2 (\mu - \lambda)^{-2} d\mu_{u,u} < +\infty,$$

we get that $R_\lambda u \in \text{Dom}(H)$. With Proposition 9.26, we write, for all $u \in \mathbf{H}$,

$$\langle (H - \lambda) R_\lambda u, u \rangle = \int_{|\mu-\lambda| \geq \varepsilon_0} d\mu_{u,u} = \|u\|^2.$$

This shows that $(H - \lambda)R_\lambda = \text{Id}$. In the same way, we get that $R_\lambda(H - \lambda) = \text{Id}_{\text{Dom}(H)}$. Thus, $\lambda \in \rho(H)$. \square

Exercise 9.28. For $z \notin \text{sp}(H)$, we introduce the Borelian function $f_z(x) = (x - z)^{-1}$. Show that $f_z(H) = (H - z)^{-1}$.

Let us give a useful example of application of the functional calculus.

Proposition 9.29 (Stone's formula). *Consider $a, b \in \mathbb{R}$ such that $a < b$. We have, for all $u \in \mathbf{H}$,*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{2i\pi} \int_{[a,b]} ((H - (\lambda + i\varepsilon))^{-1} - (H - (\lambda - i\varepsilon))^{-1}) u d\lambda = \frac{1}{2} (\mathbb{1}_{[a,b]}(H) + \mathbb{1}_{(a,b)}(H)) u.$$

Proof. For $\varepsilon > 0$, we introduce, for all $x \in [a, b]$,

$$f_\varepsilon(x) = \frac{1}{2i\pi} \int_{[a,b]} ((x - (\lambda + i\varepsilon))^{-1} - (x - (\lambda - i\varepsilon))^{-1}) d\lambda,$$

and we notice that, for all $x \in [a, b]$,

$$f_\varepsilon(x) = \frac{1}{\pi} \left(\arctan\left(\frac{b-x}{\varepsilon}\right) - \arctan\left(\frac{a-x}{\varepsilon}\right) \right),$$

so that

$$\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(x) = \frac{1}{2} (\mathbb{1}_{[a,b]}(x) + \mathbb{1}_{(a,b)}(x)) = g(x),$$

and $|f_\varepsilon(x)| \leq 1$. Since, for all $u \in \mathbf{H}$,

$$\|(f_\varepsilon(H) - g(H))u\|^2 = \int_{\mathbb{R}} |f_\varepsilon(\lambda) - g(x)|^2 d\mu_{u,u},$$

we get, by dominated convergence,

$$\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(H)u = \frac{1}{2} (\mathbb{1}_{[a,b]}(H) + \mathbb{1}_{(a,b)}(H)) u.$$

By using Riemannian sums and Exercise 9.28, we get, for all $\varepsilon > 0$,

$$f_\varepsilon(H) = \frac{1}{2i\pi} \int_{[a,b]} ((H - (\lambda + i\varepsilon))^{-1} - (H - (\lambda - i\varepsilon))^{-1}) d\lambda,$$

and the conclusion follows. \square

Lemma 9.30. *Let f be a Borelian function. If $u \in \text{Dom}(H)$ satisfies $Hu = \lambda u$, then $f(H)u = f(\lambda)u$.*

Proof. We have, for all $t \in \mathbb{R}$, $e^{itH}u = e^{it\lambda}u$. Thus, for all $f \in \mathcal{S}(\mathbb{R})$, by the inverse Fourier transform, we have $f(H)u = f(\lambda)u$. This can be extended to $f \in \mathcal{C}_{\rightarrow 0}^0(\mathbb{R})$ by density and then to all Borelian function. \square

Proposition 9.31. λ belongs to the point spectrum if and only if $\mathbb{1}_{\{\lambda\}}(H) \neq 0$. Moreover, $\mathbb{1}_{\{\lambda\}}(H)$ is the orthogonal projection on $\ker(H - \lambda)$.

Proof. If there exists $u \in \text{Dom}(H)$ with $u \neq 0$ such that $Hu = \lambda u$, then $\mathbb{1}_{\{\lambda\}}(H)u = u \neq 0$.

Conversely, assume that $\mathbb{1}_{\{\lambda\}}(H) \neq 0$. Then, take $u \neq 0$ such that $\mathbb{1}_{\{\lambda\}}(H)u = u$. We get $H\mathbb{1}_{\{\lambda\}}(H)u = Hu$ and thus $\lambda u = Hu$. \square

Proposition 9.32. $\lambda \in \text{sp}_{\text{ess}}(H)$ if and only if, for all $\varepsilon > 0$, $\dim \text{ran } \mathbb{1}_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H) = +\infty$.

Proof. If $\lambda \notin \text{sp}_{\text{ess}}(H)$, it is isolated with finite multiplicity. Then, for some $\varepsilon > 0$, we have $\mathbb{1}_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H) = \mathbb{1}_{\{\lambda\}}(H)$. We have $\text{ran } \mathbb{1}_{\{\lambda\}}(H) = \ker(H - \lambda)$.

Conversely, if λ is not isolated with finite multiplicity, $\dim \text{ran } \mathbb{1}_{(\lambda-\varepsilon, \lambda+\varepsilon)}(H) = +\infty$ (we have an infinite orthonormal family in the range of the projector). \square

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