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► **To cite this version:**

Pierre Brugière. Credit Risk, CDOs and Copulas. Doctoral. Risque de Credit CDOs CDS, Université Paris Dauphine, France. 2017, pp.89. cel-01511112v1

**HAL Id: cel-01511112**

**<https://hal.science/cel-01511112v1>**

Submitted on 27 Apr 2017 (v1), last revised 4 May 2020 (v2)

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# Credit Risk, CDOs and Copulas

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April 24, 2017

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## Structural and Reduced Form Models

# Financial Analysis for Credit Risk

Two problems to analyse "Credit Risk":

- capacity to pay financial flows
- capacity to reimburse debt at maturity or to refinance

To measure the capacity to pay interests people will look at the interest coverage ratio:

$$\text{Interest coverage ratios} = \frac{\text{interest expenses on debt}}{\text{Earnings Before Interests and Taxes}}$$

Altman's (1968) analysis based on some financial ratios :

- $X_1$  Working Capital / Total Assets.
- $X_2$  Retained Earnings / Total Assets
- $X_3$  Earnings Before Interest and Taxes / Total Assets
- $X_4$  Market Value / Book Value of Total Debt
- $X_4$  Sales / Total Assets (industry dependent)

**Remarks:** It seems Altman was the first to introduce statistical models to predict bankruptcy

## Reminders:

Working Capital = Current Assets - Current Liabilities

Current Assets = Cash + Account Receivables + Inventories

## Altman's Z-Score: with various revisions in 1983 and 1993

Based on historical studies of bankruptcies the Z-score was defined as:

$$Z = 0.012X_1 + 0.014X_2 + 0.033X_3 + 0.006X_4 + 0.999X_5$$

with the following predictions associated:

if  $Z < 1.81$  default within 1 year is predicted

if  $1.81 \leq Z \leq 2.67$  no prediction

if  $Z > 2.67$  prediction of no default within 1 year

The following results were obtained on the samples tested:

- 90.9% success rate in predicting bankruptcy
- 97% success rate in predicting non-bankruptcy

# Financial Analysis for Credit Risk

Some refinements have been done to the model in particular to define a probability of default.

Ohlson (1980) proposed the LOGIT Model:

$$P(\text{Default}|X_1, X_2, \dots, X_n) = \frac{1}{1 + \exp(-\beta_1 X_1 - \beta_2 X_2 - \dots - \beta_n X_n)}$$

where the parameters  $\beta_i$  are estimated on a sample by maximizing the likelihood.

Altman and Ohlson's models were developed principally for financial analysts with an estimation of the parameters based on bankruptcy historicals.

Since then other models have been developed which:

- are better suited for trading purposes
- are calibrated on corporate bonds and credit derivatives market prices instead of observed bankruptcies
- modelize the credit dynamics

# Structural and Reduced Form Models

For these new models two principal types:

- Structural Models: (Black-Scholes(1973), Merton(1974), Leland(1994), Schaeffer(2004))
- Reduced Form Models: (Jarrow-Turnbull(1997), Duffie-Singleton (1999))

## Definition: Structural Models

- model the dynamics of the Assets  $A_t$  and Liabilities  $L_t$
- assumption that the default happens iff  $A_t < L_t$
- in their simplest form:  $A_t = A_0 \exp(\sigma W_t - \frac{\sigma^2}{2} t)$  and  $L_t = L_0$
- approach adopted by Moody's KMV



## Definition: Distance to Default (DTD)

The Distance to Default is the number of standard deviations between the  $\ln$  of the current value of the company's assets (assumed to be normally distributed) and the log of its liabilities.

**Example:** We assume

- Assets:  $A_0 = EUR100$  and  $A_T = A_0 \exp(rT) \exp(\sigma W_T - \frac{\sigma^2}{2} T)$
- volatility of the assets:  $\sigma = 25\%$ ,  $r = 0$
- Liabilities:  $L_T = L_0 = EUR60$

then:  $A_T < L_T \iff \frac{W_T}{\sqrt{T}} < \frac{1}{\sigma\sqrt{T}} [\ln(\frac{L_0}{A_0}) + \frac{\sigma^2}{2} T] = DTD + \frac{\sigma}{2} \sqrt{T}$   
so here the 1 year Distance to Default is 2.

## Definition: Reduced Form models

- the probability of default is modelised directly
- in the simplest form:  
 $P(\tau > t) = \exp(-\lambda t)$  where  $P$  is the risk neutral probability
- in more complex forms (Cox Processes):  
 $P(\tau > t) = E[\exp(-\int_0^t \lambda(s, X_s) ds)]$  where  $(X_s)_{s \geq 0}$  is a multidimensional random process

**Remark 1:** to the simplest Structural Form model corresponds a complex Reduced Form Model

**Remark 2:** in the Reduced Form model with constant parameter  $\lambda$ , if the price of a bond without credit risk of Maturity  $T$  is  $100e^{-rT}$  the price of a risky Bond with the same Maturity paying zero in case of default will be  $e^{-(r+\lambda)T}$  as a result of the calculation of  $E[e^{-rT} 1_{\tau > T}]$ .

## Example: Construction of $P$ and $\lambda$

**Exemple:** we consider a two period economy  $0, T$  with:

- a risk free asset, a stock and a corporate bond of maturity  $T = 1$
- the risky asset is worth 100 with possible future values 130 and 70
- the risk free asset has a return of 5%

we assume that the corporate (risky) bond will be worth at maturity:

- 106 if the stock is worth 130 (economy in state 1)
- 84.8 if the stock is worth 70 (economy in state 2)

Then, if there is no arbitrage, we have the following results:

- a) the bond can be replicated by investing in the stock and risk-free bond
- b) the value of the risky bond is 92.54 today
- c) the risk neutral probability verifies  $p_1 = 58.33\%$  and  $p_2 = 41.67\%$
- d) according to the price derived for the risky bond we have  $\lambda = 53.90\%$

## Example: Construction of $P$ and $\lambda$

### Demonstration:

a) we search for  $a$  and  $b$  such that:

$$a105 + b130 = 106 \text{ and } a105 + b70 = 84.8$$

So,  $a = 0.57$  is the number of risk-free bonds to purchase and  $b = 0.35$  is the number of risky assets to purchase for the replicating portfolio.

b) if there is no arbitrage, the price of the risky bond is then

$$0.57 \times 100 + 0.35 \times 100 = 92.54$$

c) the corresponding risk neutral probability is such that

$$\frac{130}{1.05}p_1 + \frac{70}{1.05}(1 - p_1) = 100 \text{ so } p_1 = 58.33\% \text{ and } p_2 = 41.67\%.$$

$$d) P(\tau > 1) = p_1 \iff e^{-\lambda} = 58.33\% \iff \lambda = 53.90\%$$

**Remarks:** For the risky-Bond we have  $92.54 = \frac{106}{1.05} \times e^{-8.70\%}$  so the return of the bond will be 8.70% higher than the return of the risk-free bond if the bond does not default. This excess return is called the spread of the bond (calculated as a continuous rate).

## Remarks:

- when calibrating a Reduced Form model the risk free rate and the price of the risky bond are observed and from there  $\lambda$  can be deducted
- Duffie and others have compared the "implied"  $\lambda$  (under the risk neutral probability) for corporate Bonds derived from their prices and compared them to the "realized"  $\lambda$  (under the "real probability") derived from the defaults over the subsequent periods and found that  $\lambda_{implied} \sim 2 \times \lambda_{realized}$
- discrepancies between  $\lambda_{implied}$  under the risk neutral possibility and  $\lambda_{realized}$  under historical probability can be seen as similar issues to the discrepancies between "implied volatility" and "realized volatility"

## Theorem and Definition : Recovery Rate $R$ and Spread

The Recovery Rate  $R$  is the fraction of the amount due recovered if the counterparty defaults.

In practice  $R$  depends on the type of debt issued by the company (senior, junior, secured...)

- if  $R$  is the recovery rate of a zero coupon of maturity  $T$
- if  $r$  is the risk-free rate for the same maturity
- if  $S$  is the spread of the risky bond of maturity  $T$
- if  $\lambda$  is the constant default rate

then:  $S \sim (1 - R)\lambda$

### Demonstration:

Pricing the zero coupon with the risk neutral probability we have:

$$e^{-(r+S)T} = e^{-rT}(e^{-\lambda T} + R(1 - e^{-\lambda T})) \implies e^{-ST} = (1 - R)e^{-\lambda T} + R.$$

Developing to the first order we get the result.

## Remark:

In the previous example we have  $S = 8.70\%$ ,  $\lambda = 53.80\%$  and in case of default the recovery is  $\frac{84.8}{106} = 80\%$  so  $(1 - R)\lambda = 10.78\%$ . Here the approximation of  $S$  is not very good because  $\lambda$  is taking a quite extreme (large) value and corresponds to the rating of a company very distressed.

**Exercice 1:** if  $\lambda$  is deterministic and time dependant show that:

a)  $P(\tau > T) = \exp(-\Lambda_T T)$  where  $\Lambda_T = \frac{1}{T} \int_0^T \lambda_s ds$

b)  $\lim_{dt \rightarrow 0} \frac{P(\tau < t+dt | \tau > t)}{dt} = \lambda_t$

## Hint:

$$\begin{aligned} P(\tau < t + dt) - P(\tau < t) &= \exp\left(-\int_0^t \lambda_s ds\right) \left[1 - \exp\left(-\int_t^{t+dt} \lambda_s ds\right)\right] \\ &= P(\tau < t)(1 - \lambda_t dt + o(dt)) \end{aligned}$$

## Theorem and Definition: Cox Process

Let  $(\Omega, P)$  be a probability space.

Let  $\lambda$  be a positive function of  $\mathbb{R}^d$

Let  $(X_s)_{s \geq 0}$  be a random process of  $\mathbb{R}^d$  representing the factors which explain the intensity function  $\lambda(X_s)$  related to credit risk.

Let  $\tau_1$  be an exponential law of parameter 1 independent from the  $(X_s)_{s \geq 0}$

Let  $\tau_\lambda$  be defined by:

$$\tau_\lambda(\omega) = \inf \left\{ t, \int_0^t \lambda(X_s)(\omega) ds \geq \tau_1(\omega) \right\}$$

Then  $\tau_\lambda$  is called a Cox Process for default and we have

$$P(\tau_\lambda > t) = E \left[ \exp \left( - \int_0^t \lambda(X_s) ds \right) \right]$$



## Demonstration:

$$P(\tau_\lambda > t) = E[1_{\tau_\lambda > t}] = E\left(E[1_{\tau_\lambda > t} | \sigma(X_s)]\right)$$

$$E[1_{\tau_\lambda > t} | \sigma(X_s)] = P\left(\tau_1 > \int_0^t \lambda(X_s)(\omega) ds | \sigma(X_s)\right)$$

as  $(X_s)_{s \geq 0}$  and  $\tau_1$  are independent

$$P\left(\tau_1 > \int_0^t \lambda(X_s)(\omega) ds | \sigma(X_s)\right) = \exp\left(-\int_0^t \lambda(X_s)(\omega) ds\right)$$

$$\text{so, } P(\tau_\lambda > t) = E\left[\exp\left(-\int_0^t \lambda(X_s)(\omega) ds\right)\right] \text{ Q.E.D}$$

We assume now that we are in an economy where the instantaneous short term interest rate depends also on the factors  $(X_s)_{s \geq 0}$ . In the absence of arbitrage if we note  $\beta_t = \exp\left(-\int_0^t r(X_s) ds\right)$  then the price of a zero coupon bond of maturity  $T$ , nominal 1 with credit risk and zero recovery rate is  $E[\beta_T 1_{\tau > T}]$

## Proposition

For a risky zero coupon bond of Nominal 1, recovery rate zero, and maturity  $T$ :  $E[\beta_T 1_{\tau > T}] = E\left[\exp\left(-\int_0^T (r(X_s) + \lambda(X_s)) ds\right)\right]$   
so,  $\lambda(X_s)$  is the "instantaneous spread" at time  $s$

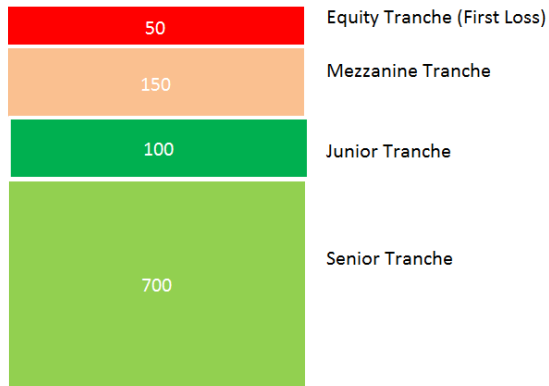
## Demonstration:

$$\begin{aligned} E[\beta_T 1_{\tau > T}] &= E\left[E[\beta_T 1_{\tau > T} | \sigma(X_s)]\right] = E\left[\beta_T E[1_{\tau > T} | \sigma(X_s)]\right] \\ &= E\left[\beta_T \exp\left(-\int_0^T \lambda(X_s) ds\right)\right] = E\left[\exp\left(-\int_0^T r(X_s) ds\right) \exp\left(-\int_0^T \lambda(X_s) ds\right)\right] \\ &= E\left[\exp\left(-\int_0^T (r(X_s) + \lambda(X_s)) ds\right)\right] \text{ Q.E.D} \end{aligned}$$

## Collateralized Debt Obligations

# Collateralized Debt Obligations

Collateralized Debt Obligations of Notional EUR 1000



10 Bonds of EUR 100 Nominal Each

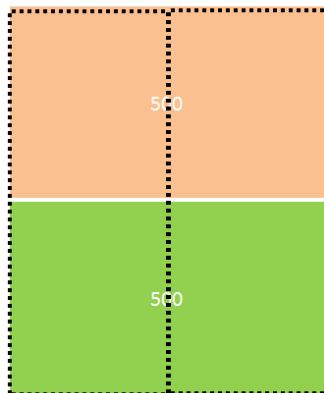
Rationale of the transaction:

- the risk is repackaged to be able to sell it better
- different type of investors can choose between different type of risks
- potentially Rating/ Pricing Arbitrage (up to 2008 too many senior tranches rated AAA)
- in the past potentially regulatory arbitrage (for keeping the risk on the equity tranche and deconsolidating)
- technology of packaging and tranching which can be applied to cash or synthetic underlyings

⇒ **Important to notice the importance of the "correlation" when pricing a CDO's tranche**

# Collateralized Debt Obligations

Example: two Tranches CDO made of two bonds



Bond 1  
probability of default  $p$

Bond 2  
probability of default  $q$

Case studies:  
correl = 1

correl = -1

2 Bonds of EUR 500 Notional Each

# Collateralized Debt Obligations

**Example:** we consider 2 Bonds, with zero Recovery rate, of EUR 500 Nominal each, packaged in a EUR 1000 Notional CDO and note  $Z_i = 1$  if the Bond  $i$  defaults before maturity and otherwise  $Z_i = 0$

a) if we assume that  $Z_1 = Z_2$  then:

- either the two bonds default together, resulting in a payout of zero for both tranches or
- none of the bonds defaults, resulting in a payout for both tranches of EUR 500

In this case, both tranches are the same, the senior tranche is not safer than the junior tranche and the correlation between the defaults is 100%.

b) if we assume that  $Z_2 = 1 - Z_1$  then:

there is always one bond which defaults so

- the junior tranche has always a payout of zero
- the senior tranche has always a payout of EUR 500

In this case the correlation between the defaults is  $-100\%$  and the two tranches have extremely different behaviours

⇒ Note that in this extreme example, the pricing of the two tranches does not depend on the probabilities of default but only on the correlation !

## Remarks:

- a low correlation between the bonds is good for senior tranche holders and bad for junior tranche holders
- a high correlation between the bonds is good for junior tranche holders and bad for senior tranche holders
- the impact of correlation is less clear for mezzanine tranches holders

To price CDOs we will need to simulate Bernoulli variables which are correlated



# Simulating Correlated Binomials

We construct here Bernoulli variables with the same parameter  $p$  which are correlated. The correlation is created through the default parameter in the following way.

## Theorem: Simulation of Correlated Bernoulli Variables

Let  $(Z_i)_{i \in \llbracket 1, n \rrbracket}$  be independent variables of uniform law in  $\llbracket 0, 1 \rrbracket$

Let  $\tilde{p}$  be a random variable in  $\llbracket 0, 1 \rrbracket$  with density  $f$

Let  $(X_i)_{i \in \llbracket 1, n \rrbracket}$  be Bernoulli variables defined by  $X_i = 1 \iff Z_i < \tilde{p}$

Then:

a) the  $(X_i)_{i \in \llbracket 1, n \rrbracket}$  are Bernoulli variables of parameters  $\bar{p} = E[\tilde{p}]$

b)  $\forall i \neq j, \rho(X_i, X_j) = \frac{\text{Var}(\tilde{p})}{\bar{p}(1-\bar{p})}$

## Demonstration :

$$a) E(X_i) = E\left(E(X_i|\tilde{p})\right) = E\left(E(1_{Z_i < \tilde{p}}|\tilde{p})\right) = E(\tilde{p})$$

$$b) E(X_i X_j) = E\left(E(X_i X_j|\tilde{p})\right) = E(\tilde{p}^2) \text{ as } X_i \text{ and } X_j \text{ are independent conditionnally on } \tilde{p}$$

so,  $Cov(X_i X_j) = E(\tilde{p}^2) - E(\tilde{p})^2 = Var(\tilde{p})$  and we know that for Bernouilli  $Var(X_i) = Var(X_j) = E(\tilde{p})(1 - E(\tilde{p}))$  Q.E.D

We consider now CDOs composed of bonds of the same notional with the same probabilities of default and same correlations and we are interested in calculating the law of the number of Bonds which default and therefore the

$$\text{law of } D_n = \sum_{i=1}^{i=n} X_i$$

## Remarks :

- the limitation of the model is that the resulting correlation between two bounds is always positive
- if  $\tilde{p}$  is constant the correlation between the bonds is zero
- if  $P(\tilde{p} = 0) = \frac{1}{2}$  and  $P(\tilde{p} = 1) = \frac{1}{2}$  the correlation between the bonds is 100% as  $\text{var}[\tilde{p}] = \frac{1}{4}$  and  $\bar{p}(1 - \bar{p}) = \frac{1}{4}$

## Exercice 1 :

Show that  $\forall X \in \llbracket 0, 1 \rrbracket, \text{Var}[X] \leq \frac{1}{4}$

**Hint :** 
$$\begin{aligned} \text{Var}[X] &= E[(X - E(X))^2] = E\left[\left((X - \frac{1}{2}) + (\frac{1}{2} - E(X))\right)^2\right] \\ &= E\left[\left(X - \frac{1}{2}\right)^2\right] + E\left[\left(\frac{1}{2} - E(X)\right)^2\right] + 2E\left[\left(X - \frac{1}{2}\right)\left(\frac{1}{2} - E(X)\right)\right] \\ &= E\left[\left(X - \frac{1}{2}\right)^2\right] + \left(\frac{1}{2} - E(X)\right)^2 - 2\left(\frac{1}{2} - E(X)\right)^2 \\ &= E\left[\left(X - \frac{1}{2}\right)^2\right] - \left(\frac{1}{2} - E(X)\right)^2 \leq E\left[\left(X - \frac{1}{2}\right)^2\right] \leq \frac{1}{4} \end{aligned}$$
 and the minimum is attained iff  $\forall \omega, |X(\omega) - \frac{1}{2}| = \frac{1}{2}$

## Exercise 2 :

Often in simulations  $\tilde{p} \sim B(\alpha, \beta)$  (beta law of parameters  $\alpha > 0$  and  $\beta > 0$ ) where the density is given by  $f_{\alpha, \beta}(x) \propto x^{\alpha-1}(1-x)^{\beta-1} 1_{x \in ]0, 1[}$

Show that:

a)  $E[\tilde{p}] = \frac{\alpha}{\alpha + \beta}$  noted  $(\bar{p})$

b)  $Var[\tilde{p}] = \frac{\bar{p}(1-\bar{p})}{\alpha + \beta + 1}$

c) simulating with  $\tilde{p}$  we have  $\forall i \neq j, \rho(X_i, X_j) = \frac{1}{\alpha + \beta + 1}$

d) show that  $\forall p, \rho \in ]0, 1[, \exists \alpha > 0, \beta > 0, \frac{\alpha}{\alpha + \beta} = \bar{p}$  and  $\frac{1}{\alpha + \beta + 1} = \rho$

**Remarks:** The Beta law is quite useful for the simulation of correlated Bernoulli variables as it is possible to choose  $\alpha$  and  $\beta$  to obtain any possible probability of default and (positive) correlation wanted in the model.

## Theorem: Law of $\frac{D_n}{n}$

$$a) E\left(\frac{D_n}{n}\right) = \bar{p}$$

$$b) \text{Var}\left(\frac{D_n}{n}\right) = \frac{\bar{p}(1-\bar{p})}{n} + \frac{n-1}{n} \text{Var}[\tilde{p}]$$

$$c) \frac{D_n}{n} \longrightarrow \mathcal{L}(\tilde{p}) \text{ (convergence in law)}$$

so, in practice the probability that less than  $k$  bonds over  $n$  default is approximated by  $P(\tilde{p} < \frac{k}{n})$

## demonstration

$$a) E\left(\frac{D_n}{n}\right) = E\left(\frac{1}{n} \sum_{i=1}^{i=n} X_i\right) = \frac{1}{n} \sum_{i=1}^{i=n} E(X_i) = E[\tilde{p}]$$

$$b) \text{Var}\left(\frac{D_n}{n}\right) = \frac{1}{n^2} \sum_{i=1}^{i=n} \text{Var}(X_i) + \frac{1}{n^2} \sum_{i \neq j} \text{Cov}(X_i, X_j)$$
$$= \frac{1}{n^2} \times n \times \bar{p}(1 - \bar{p}) + \frac{1}{n^2} \times n(n - 1) \times \text{var}[\tilde{p}]$$

# Simulating Correlated Binomials

c) to show the convergence in law we show the convergence of the distribution functions

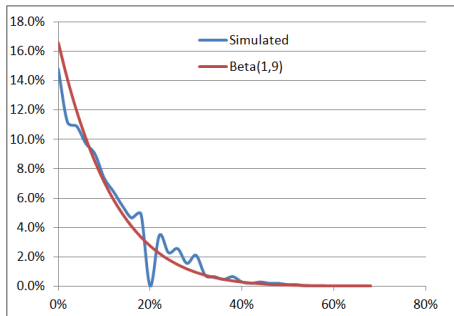
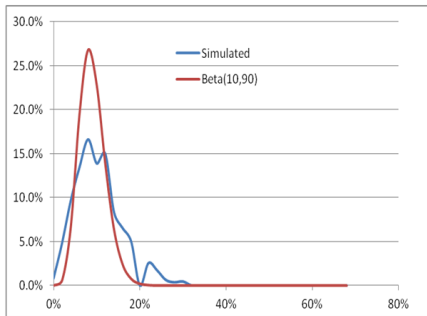
$$\begin{aligned}\lim_{n \rightarrow +\infty} P\left(\frac{D_n}{n} < t\right) &= \lim_{n \rightarrow +\infty} E(1_{\frac{D_n}{n} < t}) \\ &= \lim_{n \rightarrow +\infty} E(E(1_{\frac{D_n}{n} < t} | \tilde{p})) = E(E(\lim_{n \rightarrow +\infty} 1_{\frac{D_n}{n} < t} | \tilde{p}))\end{aligned}$$

but when  $\tilde{p}$  is known  $\frac{D_n}{n} \rightarrow \tilde{p}$  almost surely. so  
 $E(\lim_{n \rightarrow +\infty} 1_{\frac{D_n}{n} < t} | \tilde{p}) = 1_{\tilde{p} < t}$  so,

$$\lim_{n \rightarrow +\infty} P\left(\frac{D_n}{n} < t\right) = E[1_{\tilde{p} < t}] = P(\tilde{p} < t) \text{ Q.E.D}$$

**Remarks:** If the variables were not correlated in c) we would have convergence towards a single number (the mean) instead of a convergence to a distribution

# Histograms for $\frac{D_n}{n}$ for a CDO of 50 Bonds (1000 simulations)



Histograms are plotted by joining the values obtained for each 2% bucket

$$\bar{p} \sim \text{Beta}(10, 90) \implies E[\bar{p}] = 10\% \text{ and } \text{Var}[\bar{p}] = 0.99\%$$

$$\bar{p} \sim \text{Beta}(1, 9) \implies E[\bar{p}] = 10\% \text{ and } \text{Var}[\bar{p}] = 9.09\%$$

## Beta Law for $\tilde{p}$ and CDO Pricing

**Example:** we consider a CDO made of 50 Bonds of equal Notional 100 each. We assume that the probabilities of default of the Bonds are modeled by Binomial variables of expected default 10% and that the correlation of default between the bonds are  $\rho$ . We assume that the CDO has three tranches: Equity tranche (First 10% Loss), Junior Tranche (next 20% Loss), Senior Tranche (last 70% Loss). To calculate the price of the three tranches we use the approximation in Law  $\mathcal{L}(\frac{D_n}{n}) \sim \text{Beta}(\alpha, \beta)$ :

Table: Pricing as a function of  $\rho$

	i.i.d Bernouilli	Beta(10,90)	Beta(1,9)
$E[\tilde{p}]$	10%	10%	10%
$\rho$	0	0.99%	9.09%
Senior	100%	100%	99.45%
Junior	91.68%	89.85%	82.93%
Equity	16.64%	20.32%	38.08%



## Definition: Diversity Score (Moody's)

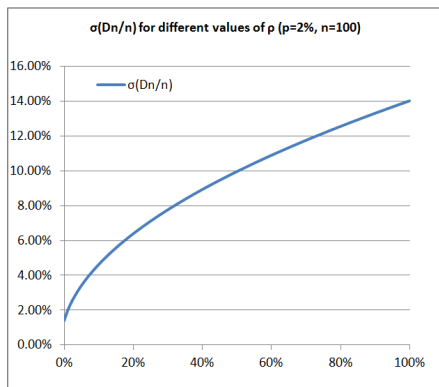
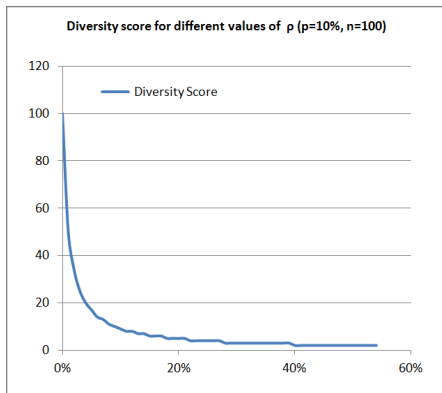
The Diversity Score is the number of uncorrelated bonds with the same probability of default  $\bar{p}$  for which the variance of the proportion of losses would be the closest to  $Var\left(\frac{D_n}{n}\right)$

**Remark:** The diversity score summarizes the real diversification effect created by Bonds which are correlated.

**Example:** for  $n$  bonds with probability of default  $p$  and correlation  $\rho$   
 $Var\left(\frac{D_n}{n}\right) = \frac{\bar{p}(1-\bar{p})}{n} + \frac{n-1}{n} Var[\tilde{p}]$  so we are searching for  $m$  such that  
 $\frac{\bar{p}(1-\bar{p})}{m} = \frac{\bar{p}(1-\bar{p})}{n} + \frac{n-1}{n} Var[\tilde{p}]$

N.A: for  $n = 100$ ,  $p = 2\%$  and  $\rho = 20\%$ ,  $\sigma_{20\%}\left(\frac{D_{100}}{100}\right) = 6.38\%$   
with 5 independent assets  $\sigma_{0\%}\left(\frac{D_5}{5}\right) = 6.26\%$  and with 4 independent assets  
 $\sigma_{0\%}\left(\frac{D_4}{4}\right) = 7.00\%$ . So we will take 5 as the Diversity Score.

# Diversity Score



Diversity Score and Standard Deviation for different values of the correlation

We make the following assumptions:

- bond  $i$  is in default at time  $T$  iif  $A_T^i < D^i$  where:  
$$dA_t^i = rA_t^i dt + \sigma^i A_t^i dW_t^i$$
- $\sigma^i$  is the same for all companies and is noted  $\sigma$
- the distance to default is the same for all companies and we note  
$$c = \frac{1}{\sigma\sqrt{T}} [\ln(\frac{D^i}{A_0^i}) - rT + \frac{\sigma^2}{2} T]$$
- we assume that the Brownian motions  $W_t^i$  verify  
$$dW_t^i = \rho dW_t + \sqrt{1 - \rho^2} dB_t^i$$
 where the  $B_t^i$  are brownian motions which are independent between them and independent from  $W_t$

## Remarks :

With the model  $\forall i \neq j, \rho(W_t^i, W_t^j) = \rho$ , and  $W_t$  is the common factor which creates correlation between the bonds.

## Proposition

Let  $Z_i$  be the Bernoulli random variable with value 1 if the company  $i$  defaults and 0 otherwise. Then  $Z_i = 1 \iff \frac{B_T^i}{\sqrt{T}} < \frac{c}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{W_T}{\sqrt{T}}$

**Demonstration** simple

**Remark 1** : Let  $\Phi$  be the repartition function of a normal law  $\mathcal{N}(0, 1)$ .

$$\frac{B_T^i}{\sqrt{T}} \sim \mathcal{N}(0, 1) \implies \Phi\left(\frac{B_T^i}{\sqrt{T}}\right) \sim \mathcal{N}(0, 1)$$

$$\text{so } X_i = 1 \iff \Phi\left(\frac{W_T^i}{\sqrt{T}}\right) < \Phi\left(\frac{c}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{W_T}{\sqrt{T}}\right)$$

so we end up simulating correlated Bernoulli variables with the function  $\tilde{p}$  having a law  $\tilde{p} \sim \Phi\left(\frac{c}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{W_T}{\sqrt{T}}\right)$

## Structural Models for $\tilde{p}$

**Remark 2 :** We have different alternatives for  $\tilde{p}$  to generate correlated binomials:

- to use a beta distribution  $B(\alpha, \beta)$
- to use the distribution of  $\Phi(\alpha + \beta Z)$  (where  $Z \sim \mathcal{N}(0, 1)$ )

In both cases:

- first we solve for  $\alpha$  and  $\beta$  to match the desired value for  $\bar{p}$  and  $\rho$
- then to price the CDO we approximate the law of  $\frac{D_n}{n}$  by the law of  $\tilde{p}$

### Proposition

If  $\tilde{p} \sim \Phi(\alpha + \beta Z)$  (where  $Z \sim \mathcal{N}(0, 1)$ ) then

a)  $E[\tilde{p}] = \Phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right)$  (that we note also  $\bar{p}$ )

b)  $Var[\tilde{p}] = \Phi_{2, \frac{\beta^2}{1+\beta^2}}\left(\frac{\alpha}{\sqrt{1+\beta^2}}, \frac{\alpha}{\sqrt{1+\beta^2}}\right)$

c)  $P(\tilde{p} < t) = \Phi\left(\frac{1}{\beta}[\Phi^{-1}(t) - \sqrt{1+\beta^2}\Phi^{-1}(\bar{p})]\right)$

# Structural Models for $\tilde{p}$

a)  $E[\tilde{p}] = E[\Phi(\alpha + \beta Z)] = E[E(1_{Z_0 < \alpha + \beta Z} | Z)]$  (with  $Z_0 \sim \mathcal{N}(0, 1)$  independent from  $Z$ )

$$= E[1_{Z_0 - \beta Z < \alpha}] = E\left[1_{\frac{Z_0 - \beta Z}{\sqrt{1 + \beta^2}} < \frac{\alpha}{\sqrt{1 + \beta^2}}}\right] = \Phi\left(\frac{\alpha}{\sqrt{1 + \beta^2}}\right)$$

b)  $E(\tilde{p}^2) = E[\Phi(\alpha + \beta Z)^2] = E[E(1_{Z_0 < \alpha + \beta Z} | Z)^2]$   
 $= E[E(1_{Z_0 < \alpha + \beta Z} | Z)E(1_{Z_1 < \alpha + \beta Z} | Z)]$  (with  $Z_0$  and  $Z_1 \sim \mathcal{N}(0, 1)$  independent from  $Z$ ) (Fubini's Trick)

$$= E\left[E\left(E(1_{Z_0 < \alpha + \beta Z} | Z)E(1_{Z_1 < \alpha + \beta Z} | Z) \mid Z_0, Z_1\right)\right]$$
$$= E\left[E\left(1_{Z_0 < \alpha + \beta Z} 1_{Z_1 < \alpha + \beta Z} \mid Z_0, Z_1\right)\right] = E\left[1_{\frac{Z_0 - \beta Z}{\sqrt{1 + \beta^2}} < \frac{\alpha}{\sqrt{1 + \beta^2}}} 1_{\frac{Z_1 - \beta Z}{\sqrt{1 + \beta^2}} < \frac{\alpha}{\sqrt{1 + \beta^2}}}\right]$$
$$= \Phi_{2, \frac{\beta^2}{1 + \beta^2}}\left(\frac{\alpha}{\sqrt{1 + \beta^2}}, \frac{\alpha}{\sqrt{1 + \beta^2}}\right) \text{ where } \Phi_{2, \gamma} \text{ is the repartition function of}$$

a bivariate normal variable  $\mathcal{N}\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}$

## Calibration of the two Models for $\tilde{p}$

$$c) P(\tilde{p} < t) = P(\Phi(\alpha + \beta Z) < t) = \Phi\left(\frac{\Phi^{-1}(t) - \alpha}{\beta}\right)$$

as  $\Phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right) = \bar{p}$  we have  $\alpha = \Phi^{-1}(\bar{p})\sqrt{1+\beta^2}$  so

$$P(\tilde{p} < t) = \Phi\left(\frac{1}{\beta}[\Phi^{-1}(t) - \sqrt{1+\beta^2}\Phi^{-1}(\bar{p})]\right). \text{ Q.E.D}$$

**Example** we consider a CDO with 100 Bonds of the same Notional and recovery rate of zero. The default of the bonds are modeled by Bernouilli variables  $X_i$  of parameter  $p$  and correlations  $\rho$ . We consider a junior tranche for the CDO which is exposed to the losses between above 10% and up to 30%. Price this junior tranche assuming  $\bar{p} = 2\%$  and  $\rho = 10\%$  with the two previous models:

a) assuming  $\tilde{p} \sim B(\alpha, \beta)$

b) assuming  $\tilde{p} \sim \Phi(\alpha + \beta Z)$  where  $Z \sim \mathcal{N}(0, 1)$

# Calibration of the two Models for $\tilde{p}$

## Results:

a) we solve and find  $\alpha = (\frac{1}{\rho} - 1)\bar{p}$  and  $\beta = (\frac{1}{\rho} - 1)(1 - \bar{p})$ . So, here  $\tilde{p} \sim B(0.18, 8.82)$ . Taking a risk free rate of zero we price the junior tranche in % of face value as  $\frac{1}{20} \sum_{i=11}^{i=30} P(\frac{D_{100}}{100} < \frac{i}{100})$  that we approximate by

$$\frac{1}{20} \sum_{i=11}^{i=30} P(\tilde{p} < \frac{i}{100}) = 98.12\%$$

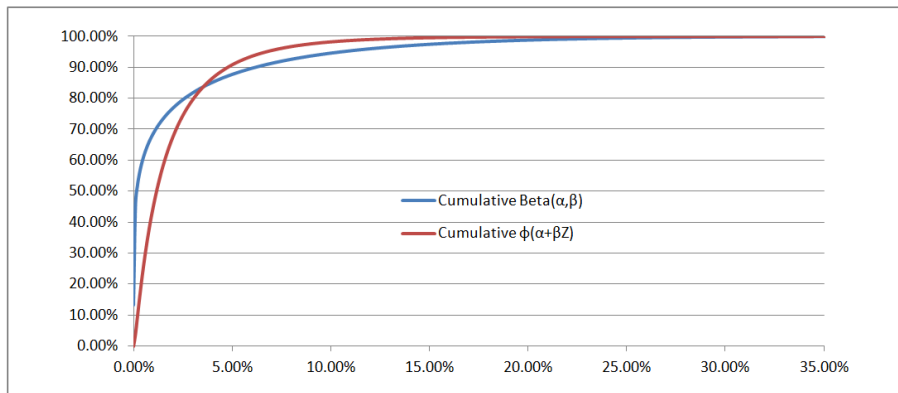
b) we solve  $\frac{\alpha}{\sqrt{1+\beta^2}} = -2.05375$  and (using a program to calculate the bivariate normal)  $\frac{\beta^2}{1+\beta^2} = 0.18$ . This implies  $\alpha = -2.2678$  and  $\beta^2 = 0.2195$ . The sign of  $\beta$  is not determined as both  $Z$  and  $-Z$  are  $\mathcal{N}(0, 1)$ , we will take  $\beta = 0.468521$ . Now,

$$\frac{1}{20} \sum_{i=11}^{i=30} P(\Phi(\alpha + \beta Z) < \frac{i}{100}) = \frac{1}{20} \sum_{i=11}^{i=30} \Phi(\frac{1}{\beta}[\Phi^{-1}(\frac{i}{100}) - \alpha]) = 99.66\%$$

**Remark :** the pricings are not exactly the same as the two laws produce the same expectations and correlations but are not the same.



# Calibration of the two Models for $\tilde{p}$



Modelizing  $\tilde{p}$  with Beta( $\alpha, \beta$ ) or  $\Phi(\alpha + \beta Z)$

## Proposition

We assume here that:

- the percentage lost for a bond which defaults (i.e  $1 - R$ ) is  $f(\tilde{p})$
- $f(p)$  is an increasing function of  $p$

If we note  $L_n^f = \frac{1}{n} \sum_{i=1}^{i=n} f(\tilde{p}) 1_{Z_i < \tilde{p}}$  the loss in percentage for the CDO  
we have :  $L_n^f \rightarrow \mathcal{L}(\tilde{p}f(\tilde{p}))$  (convergence in law).

## Demonstration :

to show the convergence in law we show the convergence of the distribution function

# Generalization to None Zero Recovery Rate

$$\begin{aligned}\lim_{n \rightarrow +\infty} P(L_n^f < t) &= \lim_{n \rightarrow +\infty} E(1_{L_n^f < t}) = \lim_{n \rightarrow +\infty} E(E(1_{L_n^f < t} | \tilde{p})) \\ &= E(E(\lim_{n \rightarrow +\infty} 1_{L_n^f < t} | \tilde{p}))\end{aligned}$$

but when  $\tilde{p}$  is known almost surely  $L_n^f \rightarrow E[f(\tilde{p})1_{Z_i < \tilde{p}}] = f(\tilde{p})\tilde{p}$   
so  $1_{L_n^f < t} \rightarrow 1_{f(\tilde{p})\tilde{p} < t}$  and

$$\lim_{n \rightarrow +\infty} P(L_n^f < t) = E(E(1_{\tilde{p}f(\tilde{p}) < t} | \tilde{p})) = P(\tilde{p}f(\tilde{p}) < t) \text{ Q.E.D}$$

**Remarks :** if  $R = 0$  then  $f$  is concentrated in 1 and we find the result we already demonstrated as in this case  $\mathcal{L}(\tilde{p}f(\tilde{p})) \sim \mathcal{L}(\tilde{p})$

## Definition: Infection Models

Let  $(Z_i)_{i \in \llbracket 1, n \rrbracket}$  and  $(Y_{i,j})_{i \neq j \in \llbracket 1, n \rrbracket}$  be independent variables, we assume  $Z_i \sim \mathcal{B}(p)$  and  $\forall i \neq j, Y_{i,j} \sim \mathcal{B}(q)$

Then we define in a contagion model the variables  $(X_i)_{i \in \llbracket 1, n \rrbracket}$  by :

$$X_i = Z_i + (1 - Z_i) \left[ 1 - \prod_{j \neq i} (1 - Z_j Y_{j,i}) \right]$$

### Remark:

The only possible values for  $X_i$  are 1 and 0.

$X_i = 1 \iff Z_i = 1$  or  $\exists i \neq j, Z_j = 1$  and  $Y_{j,i} = 1$  (i.e contamination)

We are now going to study the law of the  $X_i$  and their correlations.

## Proposition

$$X_i \sim \mathcal{B}(1 - (1 - p)(1 - pq)^{n-1})$$

**Demonstration** : because of independence

$$E(X_1) = E(Z_1) + (1 - E(Z_1))\left[1 - \prod_{j \neq 1} (1 - E(Z_j)E(Y_{j,1}))\right]$$

$$\begin{aligned} &= p + (1 - p)[1 - (1 - pq)^{n-1}] \\ &= p + 1 - p - (1 - p)(1 - pq)^{n-1} \\ &= 1 - (1 - p)(1 - pq)^{n-1} \text{ Q.E.D} \end{aligned}$$

**Remarks** :

$$\mathcal{L}(X_1) \xrightarrow[n \rightarrow +\infty]{} 1$$

## Proposition

$$E[X_1 X_2] = 1 - 2(1 - p)(1 - pq)^{n-1} + (1 - p)^2(1 - 2pq + pq^2)^{n-2}$$

## Demonstration :

$$E[X_1 X_2] = E \left[ \left( Z_1 + (1 - Z_1) \left[ 1 - \prod_{j \neq 1} (1 - Z_j Y_{j,1}) \right] \right) \left( Z_2 + (1 - Z_2) \left[ 1 - \prod_{j \neq 2} (1 - Z_j Y_{j,2}) \right] \right) \right]$$

we have 3 different type of terms:

a)  $E[Z_1 Z_2] = p^2$  (because  $Z_1$  and  $Z_2$  are independent)

b)  $E \left( Z_1 (1 - Z_2) \left[ 1 - \prod_{j \neq 2} (1 - Z_j Y_{j,2}) \right] \right)$  (this value will appear two times)

$$= E \left( Z_1 (1 - Z_2) \right) - E \left( Z_1 (1 - Z_1 Y_{1,2}) (1 - Z_2) \prod_{j \notin \{1,2\}} (1 - Z_j Y_{j,2}) \right)$$

$$= p(1 - p) - (p - pq)(1 - p)(1 - pq)^{n-2}$$

$$= p(1 - p) [1 - (1 - q)(1 - pq)^{n-2}]$$

$$c) E \left[ (1 - Z_1) (1 - Z_2) \left[ 1 - \prod_{j \neq 1} (1 - Z_j Y_{j,1}) \right] \left[ 1 - \prod_{j \neq 2} (1 - Z_j Y_{j,2}) \right] \right]$$

multiplying first the two terms on the right we get 3 different type of terms:

$$\circ E[(1 - Z_1)(1 - Z_2)] = (1 - p)^2$$

$$\begin{aligned}
 & \circ -E\left[(1 - Z_1)(1 - Z_2) \prod_{j \neq 2} (1 - Z_j Y_{j,2})\right] \text{ (this value will appear two times)} \\
 &= -E\left[(1 - Z_1)(1 - Z_1 Y_{1,2})(1 - Z_2) \prod_{j \notin \{1,2\}} (1 - Z_j Y_{j,2})\right] \\
 &= -(1 - pq - p + pq)(1 - p)(1 - pq)^{n-2} \\
 &= -(1 - p)^2(1 - pq)^{n-2} \\
 & \circ E\left[(1 - Z_1)(1 - Z_2) \prod_{j \neq 1} (1 - Z_j Y_{j,1}) \prod_{j \neq 2} (1 - Z_j Y_{j,2})\right] \\
 &= E\left[(1 - Z_1)(1 - Z_1 Y_{1,2})(1 - Z_2)(1 - Z_2 Y_{2,1}) \prod_{j \notin \{1,2\}} [(1 - Z_j Y_{j,1})(1 - Z_j Y_{j,2})]\right] \\
 &= (1 - pq - p + pq)^2(1 - pq - pq + pq^2)^{n-2} \\
 &= (1 - p)^2(1 - 2pq + pq^2)^{n-2}
 \end{aligned}$$

so at the end we obtain:

$$\begin{aligned}
 & p^2 + 2p(1 - p)[1 - (1 - q)(1 - pq)^{n-2}] + (1 - p)^2 \\
 & - 2(1 - p)^2(1 - pq)^{n-2} + (1 - p)^2(1 - 2pq + pq^2)^{n-2} \\
 &= 1 - 2(1 - p)(1 - pq)^{n-1} + (1 - p)^2(1 - 2pq + pq^2)^{n-2} \text{ Q.E.D}
 \end{aligned}$$

## Exercise :

Let  $D_n = \sum_{i=1}^{i=n} X_i$ , calculate as a function of  $p$  and  $q$  :

- a)  $E[D_n]$  and
- b)  $Var[D_n]$

## Hint :

a)  $E[D_n] = \sum_{i=1}^{i=n} E[X_i] = nE[X_1]$

b)  $Var[D_n] = E\left[\left(\sum_{i=1}^{i=n} X_i\right)^2\right] - (E[D_n])^2 = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} E[X_i X_j] - (E[D_n])^2$   
 $= n(n-1)E[X_1 X_2] + nE[X_1] - n^2 E[X_1]^2$

So we know how to calculate the first two moments of  $D_n$  as a function of  $p$  and  $q$  but in fact we can also calculate the law of  $D_n$



## Proposition :

$\forall k \in \llbracket 1, n \rrbracket$ ,

$$P(D_n = k) = C_n^k \sum_{i=1}^{i=k} C_k^i p^i (1-p)^{(n-i)} (1-q)^{i(n-k)} [1 - (1-q)^i]^{(k-i)}$$

## Demonstration :

$P(D_n = k) = C_n^k P(X_1 = 1, X_2 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0)$

and  $\{X_1 = 1, X_2 = 1, \dots, X_k = 1\}$  can be decomposed in  $k - 1$  cases depending on the number  $i$  of "direct" defaults. So

$P(X_1 = 1, X_2 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0)$

$$= \sum_{i=1}^{i=k} C_k^i P(Z_1 = 1, Z_2 = 1, \dots, Z_i = 1, (Z_{i+1} = 0, X_{i+1} = 1), \dots, (Z_k = 0, X_k = 1), X_{k+1} = 0, \dots, X_n = 0)$$

we can write each event as the intersection of three events

- $\{Z_1 = 1, Z_2 = 1, \dots, Z_i = 1, Z_{i+1} = 0, \dots, Z_n = 0\}$
- $\{\exists j \in \llbracket 1, i \rrbracket, Y_{j,i+1} = 1, \dots, \exists j \in \llbracket 1, i \rrbracket, Y_{j,k} = 1\}$
- $\{\forall j \in \llbracket 1, i \rrbracket, Y_{j,k+1} = 0, \dots, \forall j \in \llbracket 1, i \rrbracket, Y_{j,n} = 0\}$

the three events are independent.

- the probability of the first one is  $p^i(1-p)^{(n-i)}$
- the probability of the second one is  $[1 - (1-q)^i]^{(k-i)}$
- the probability of the third one is  $(1-q)^{i(n-k)}$  Q.E.D

## Example :

Table: Infection Models for  $n = 30$

$p = P(Z_i = 1)$	1%	1%	1%	1%	1%
$q$	0%	10%	20%	50%	100%
$p^* = P(X_i = 1)$	1%	3.83%	6.58%	14.39%	26.03%
Correlation	0%	12%	21%	50%	100%
Diversity Score	30	6.7	4.1	2	1

Remarks : if  $q = 100\%$

$$P(X_i = 1) = 1 - P(X_i = 0) = 1 - (P(Z_1 = 0))^{30} = 1 - (1 - p)^{30} = 26.03\%$$

# Copulas

## Definition : Copulas

$C : [0, 1]^d \rightarrow [0, 1]$  is a copula iff  $C$  is a multivariate cumulative distribution for a random vector of  $[0, 1]^d$  i.e

$\exists (U_1, U_2, \dots, U_d)$  r.v  $(\Omega, P) \rightarrow [0, 1]^d$  such that:

- $\forall i \in \llbracket 1, d \rrbracket, U_i \sim \mathcal{U}([0, 1])$
- $C(u_1, u_2, \dots, u_d) = P(U_1 \leq u_1, U_2 \leq u_2, \dots, U_d \leq u_d)$

**Notation** : we note

$F_U(u_1, u_2, \dots, u_d) = P(U_1 \leq u_1, U_2 \leq u_2, \dots, U_d \leq u_d)$  the multidimensional cumulative distribution function of  $U$

**Remark** : by definition for any copula  $C$  there is  $U = (U_1, U_2, \dots, U_d)$  with  $U_i \sim \mathcal{U}([0, 1])$  such that  $C = F_U$

**Exemples** : Let  $U = (U_1, U_2)$  where  $U_1$  and  $U_2 \sim \mathcal{U}([0, 1])$

a) if  $U_1$  and  $U_2$  are independent then  $F_U(u, v) = uv$

b) if  $U_2 = U_1$  then  $F_U(u, v) = \min(u, v)$

c) if  $U_2 = 1 - U_1$  then  $F_U(u, v) = \max(u + v - 1, 0)$

**Demonstration** : Let's show c)

$$\begin{aligned} P(U_1 \leq u, 1 - U_1 \leq v) &= P(U_1 \leq u, U_1 \geq 1 - v) = P(1 - v \leq U_1 \leq u) \\ &= \max(u + v - 1, 0) \text{ Q.E.D} \end{aligned}$$

## Theorem : Fréchet-Hoeffding Bounds

Let  $U = (U_1, U_2)$  be a r.v with  $U_i \sim \mathcal{U}([0, 1])$  then

$$\forall u, v \in [0, 1], \max(u + v - 1, 0) \leq F_U(u, v) \leq \min(u, v)$$

so the cases  $U_2 = U_1$  and  $U_2 = 1 - U_1$  represents the two extreme "correlation-structures".

## Demonstration :

$P(U_1 \leq u, U_2 \leq v) \leq P(U_1 \leq u)$  and  $P(U_1 \leq u, U_2 \leq v) \leq P(U_2 \leq v)$

implies  $P(U_1 \leq u, U_2 \leq v) \leq \min(P(U_1 \leq u), P(U_2 \leq v))$

$P(\{U_1 \leq u\} \cup \{U_2 \leq v\}) = P(U_1 \leq u) + P(U_2 \leq v) - P(U_1 \leq u, U_2 \leq v)$

implies  $P(U_1 \leq u, U_2 \leq v) \geq -1 + P(U_1 \leq u) + P(U_2 \leq v)$  Q.E.D

## Definition : Quantile (or Pseudo-Inverse)

We define  $F_X^+ : [0, 1] \rightarrow \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$  by

$$F_X^+(y) = \inf_{x \in \mathbb{R}} \{P(X \leq x) \geq y\}$$

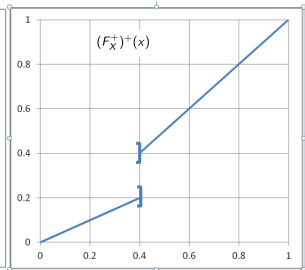
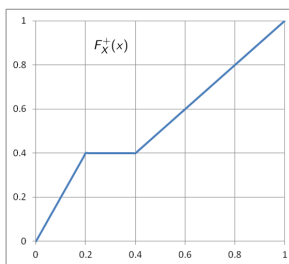
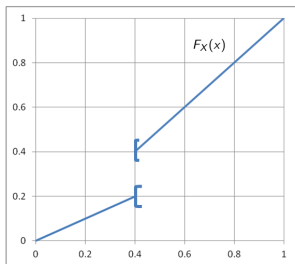
## Definition

$f$  is strictly increasing at  $x$  iff  $\forall x_1 < x < x_2, f(x_1) < f(x) < f(x_2)$

## Properties

- $F_X$  is increasing and right-continuous
- $F_X^+$  is increasing and left continuous
- $P(X = x) = 0 \iff F_X$  is continuous at  $x$
- $y \leq F_X(x) \iff F_X^+(y) \leq x$
- $F_X^+(F_X(x)) \leq x$
- $F_X(F_X^+(y)) \geq y$
- $F_X$  strictly increasing at  $x \iff F_X^+$  continuous at  $F_X(x)$
- $F_X^+$  strictly increasing at  $y \iff F_X$  continuous at  $F_X^+(y)$
- $F_X^+$  continuous at  $F_X(x) \iff F_X^+(F_X(x)) = x$
- $F_X$  continuous at  $F_X^+(y) \iff F_X(F_X^+(y)) = y$
- $F_X$  continuous and strictly increasing  $\iff F_X$  invertible and  $F_X^+ = F_X^{-1}$





Calculation of the Pseudo Inverse

**Demonstration :** Let as an exercise

## Proposition

a)  $F_X$  continuous  $\implies F_X(X) \sim \mathcal{U}([0, 1])$

b)  $F_X$  continuous and  $U \sim \mathcal{U}([0, 1]) \implies F_X^+(U) \sim X$  (same law)

**Demonstration :**

a) Let  $y \in ]0, 1[$

$$P(F_X(X) < y) = 1 - P(F_X(X) \geq y)$$

$$= 1 - P(X \geq F_X^+(y)) \text{ (according to the proposition)}$$

$$= P(X < F_X^+(y)) = P(X \leq F_X^+(y)) \text{ (because } F_X \text{ is continuous)}$$

$$= F_X(F_X^+(y)) = y \text{ (according to the proposition as } F_X \text{ is continuous)}$$

so  $F_X(X) \sim \mathcal{U}([0, 1])$  Q.E.D

b) Let  $x \in X(\Omega)$

$P(F_X^+(U) \leq x) = P(U \leq F_X(x))$  (according to the proposition)  
 $= F_X(x)$  so  $F_X^+(U) \sim X$  Q.E.D

## Sklar's Theorem: Multivariate with given Marginals and Copula

Let  $U = (U_1, U_2, \dots, U_d)$  with  $U_i \sim \mathcal{U}([0, 1])$  and  $C = F_U$  be a copula

Let  $F_1, F_2, \dots, F_d$  be  $d$  continuous cumulative distribution functions.

Let  $X = (F_1^+(U_1), F_2^+(U_2), \dots, F_d^+(U_d))$  and  $X_i = F_i^+(U_i)$  then:

- $F_X$  is a multidimensional distribution with marginals  $F_i$
- $F_i(X_i) = F_{X_i}(X_i) = U_i$
- $F_X(x_1, x_2, \dots, x_d) = C(F_1(x_1), F_2(x_2), \dots, F_d(x_d))$
- $C(u_1, u_2, \dots, u_d) = F_X(F_{X_1}^+(u_1), F_{X_2}^+(u_2), \dots, F_{X_d}^+(u_d))$

## Demonstration :

$F_i$  continuous  $\implies F_i$  is the cdf of  $F_i^+(U_i)$  (according to the proposition)

$F_i$  continuous  $\implies F_i(F_i^+)(U_i) = U_i$  (according to the properties)

$F_X(x_1, x_2, \dots, x_d) = P(F_1^+(U_1) \leq x_1, F_2^+(U_2) \leq x_2, \dots, F_d^+(U_d) \leq x_d)$

but  $F_i^+(U_i) \leq x_i \iff U_i \leq F_i(x_i)$  (according to properties) so

$= P(U_1 \leq F_1(x_1), U_2 \leq F_2(x_2), \dots, U_d \leq F_d(x_d))$  Q.E.D

## Theorem and Definition : Copula of a Multivariate Distribution

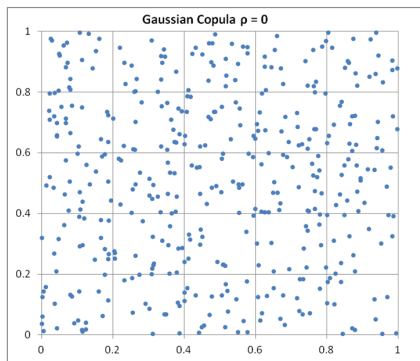
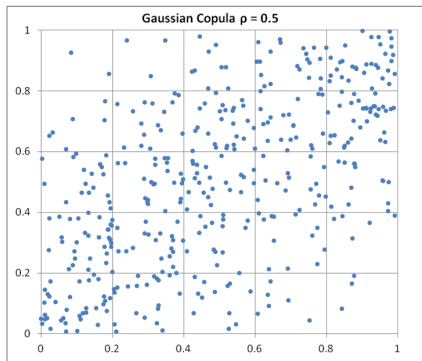
Let  $X = (X_1, X_2, \dots, X_d)$  with continuous marginal cdfs  $F_i$

Let  $U = (F_1(X_1), F_2(X_2), \dots, F_d(X_d))$

then

$F_i(X_i) \sim \mathcal{U}([0, 1])$  and we call copula of  $X$  and note  $C_X$  the cumulative multivariate distribution  $F_U$ .

# Copula



Simulations Gaussian Copula for various values of  $\rho$  : obtained by simulating  $X$  and calculating the  $F_i(X_i)$

**Remark 1** : According to Sklar's theorem:

- for any copula  $C$  and
- for any continuous cdfs  $(F_i)_{i \in \llbracket 1, d \rrbracket}$

we can find a multivariate random variable  $X$  such that:

- the  $F_i$  are the marginal cdfs of  $X$
- $C_X = C$

we will have  $F_X(x_1, x_2, \dots, x_d) = C_X(F_1(x_1), F_2(x_2), \dots, F_d(x_d))$

**Remark 2** : It is easy to simulate a Gaussian vector with a given correlation matrix and therefore easy to simulate Gaussian Copulas

**Conclusion:** Multivariate distributions with continuous marginal cdfs are determined by their marginals and their copula.

## Proposition : Invariance Properties of the Copulas

Let  $X = (X_1, X_2, \dots, X_d)$  with continuous marginal cdfs  $F_i$

Let  $T_1, T_2, \dots, T_d$  be strictly increasing real functions

Let  $Y = (Y_1, Y_2, \dots, Y_d)$  with  $Y_i = T_i(X_i)$

then  $C_Y = C_X$

So the Copula, which measures the association between the variables, is invariant by change of variables under strictly increasing functions (which is not the case for the correlation)

### Hint Demonstration :

$$P(T_i(X_i) \leq T(u_i)) = P(X_i \leq u_i) \text{ so } F_{T_i(X_i)}(T(u_i)) = F_{X_i}(u_i)$$

$$C_Y(u_1, u_2, \dots, u_d) = P(T(X_i) \leq F_{T_i(X_i)}^+(u_i), i \in \llbracket 1, d \rrbracket)$$

$$= P(F_{T_i(X_i)}(T(X_i)) \leq u_i, i \in \llbracket 1, d \rrbracket) = P(F_{X_i}(X_i) \leq u_i, i \in \llbracket 1, d \rrbracket)$$

$$= C_X(u_1, u_2, \dots, u_d) \text{ Q.E.D}$$

## Proposition : Copulas for Normalized Gaussian Vectors

Let  $X$  be a "normalized" Gaussian vector  $\mathcal{N}(0, R_d)$  with components  $X_i \sim \mathcal{N}(0, 1)$  and correlation matrix  $R_d$  invertible.

Let  $C_X$  the copula of  $X$  and  $c_X$  its density. Then:

- $C_X(x) = \Phi_{R_d}(\Phi^{-1}(x_1), \Phi^{-1}(x_2), \dots, \Phi^{-1}(x_d))$
- $c_X(x) = \frac{1}{|R|^{1/2}} \exp(-\frac{1}{2}x'(R_d^{-1} - I_d)x)$

where  $\Phi$  is the cdf of a  $\mathcal{N}(0, 1)$  and  $\Phi_{R_d}$  is the multivariate cumulative distribution function of a  $\mathcal{N}(0, R_d)$

**Demonstration** : from the property

$F_X(x_1, x_2, \dots, x_d) = C_X(F_1(x_1), F_2(x_2), \dots, F_d(x_d))$  we get

$\Phi_{R_d}(x_1, x_2, \dots, x_d) = C_X(\Phi(x_1), \Phi(x_2), \dots, \Phi(x_d))$  and

$\Phi_{R_d}(\Phi^{-1}(x_1), \Phi^{-1}(x_2), \dots, \Phi^{-1}(x_d)) = C_X(x_1, x_2, \dots, x_d)$  Q.E.D



Applying  $\frac{\partial}{\partial x_1 \partial x_2 \dots \partial x_d}$  to

$\Phi_{R_d}(x_1, x_2, \dots, x_d) = C_X(\Phi(x_1), \Phi(x_2), \dots, \Phi(x_d))$  we get

$\phi_{R_d}(x_1, x_2, \dots, x_d) = c_X(\Phi(x_1), \Phi(x_2), \dots, \Phi(x_d)) \phi(x_1) \phi(x_2) \dots \phi(x_d)$

the density  $\phi_{R_d}(x)$  equals  $(\frac{1}{\sqrt{2\pi}})^d \exp(-\frac{1}{2} x' R_d^{-1} x)$  and

$\phi(x_1) \phi(x_2) \dots \phi(x_d) = (\frac{1}{\sqrt{2\pi}})^d \exp(-\frac{1}{2} u' I_d^{-1} u)$  Q.E.D

## Proposition

The Copula of a Gaussian vector depends only on its correlation matrix  $R_d$

**Demonstration** : if  $X$  is a Gaussian vector of correlation matrix  $R_d$  we know (from the invariance property) that the normalized Gaussian vector  $U$  where  $U_i = T_i(X_i) = \frac{X_i - \mu_i}{\sigma_i}$  has the same copula as  $X$  and that  $U \sim \mathcal{N}(0, R_d)$  Q.E.D

## Rosenblatt's Theorem

Let  $X = (X_1, X_2, \dots, X_d)$  be a random vector of  $\mathbb{R}^d$   
we assume that the law of  $X$  has a density  $f_X(x_1, x_2, \dots, x_d)$

Let  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be defined by  $T(x) = y$  with

$$y_1 = P(X_1 \leq x_1)$$

$$y_2 = P(X_2 \leq x_2 | X_1 = x_1), \dots$$

$$y_d = P(X_d \leq x_d | X_1 = x_1, X_2 = x_2, \dots, X_{d-1} = x_{d-1})$$

Then :

$Y = T(X)$  follows a uniform law on  $[0, 1]^d$  i.e

- the  $Y_i$  are independent
- $Y_i \sim \mathcal{U}([0, 1])$

## Demonstration:

Let  $h$  be a measurable function from  $\mathbb{R}^d$  to  $\mathbb{R}$

$$E[h(Y)] = E[h(T(X))] = \int_{\mathbb{R}^d} h(T(x)) f_X(x) dx$$

We consider the change of variable  $y = T(x)$ .

The Jacobian matrix  $\left[\frac{dy}{dx}\right]$  is triangular and the diagonal elements are:

- $\frac{\partial}{\partial x_1} P(X_1 \leq x_1) = f_{X_1}(x_1)$
- $\frac{\partial}{\partial x_2} P(X_2 \leq x_2 | X_1 = x_1) = f_{X_2|X_1=x_1}(x_2) \cdots$
- $\frac{\partial}{\partial x_d} P(X_d \leq x_d | X_{d-1} = x_{d-1} \cdots, X_1 = x_1) = f_{X_d|(X_{d-1}=x_{d-1}\cdots)}(x_d)$

so, the determinant of the Jacobian Matrix equals  $f_X(x_1, x_2, \dots, x_d)$

so after the change of variable :  $E[h(Y)] = \int_{T(\mathbb{R}^d)} h(y) dy$

as the  $y_i$  are probabilities  $T(\mathbb{R}^d) \subset [0, 1]^d$  and by mass conservation

$$T(\mathbb{R}^d) = [0, 1]^d$$

so  $\forall h, E[h(Y)] = \int_{[0,1]^d} h(y) dy \implies Y \sim \mathcal{U}([0, 1]^d)$  Q.E.D

# Copulas : Example

We can create a correlation structure on  $d$  binomial variables  $Z_i \sim \mathcal{B}(p_i)$  by choosing a copula  $C$  and  $(\alpha_1, \alpha_2, \dots, \alpha_d)$  such that:

$$\forall i \in \llbracket 1, d \rrbracket, C(+\infty, \dots, \alpha_i, +\infty, \dots) = p_i$$

**Example:** Here  $d = 3$  and  $p_1 = 1\%$ ,  $p_2 = 2\%$  and  $p_3 = 3\%$ .

We consider a Gaussian Copula  $C$  with correlation  $\rho = 50\%$  between two variables .

We note  $X = (X_1, X_2, X_3)$  a Gaussian Vector having  $C$  as cumulative distribution function.

We solve  $P(X_i \leq \alpha_i) = p_i$  and find  $\alpha_1 = -2.326$   $\alpha_2 = -2.054$   
 $\alpha_3 = -1.881$

Here we did not try to calibrate a correlation matrix for  $X$  to match some input correlations between the  $Z_i$  but calculate the correlations between the defaults induced by the correlation matrix of  $X$

Here we get  $E(Z_1 Z_2) - E(Z_1)E(Z_2) = P(X_1 < \alpha_1, X_2 < \alpha_2) - p_1 p_2$  and  $\rho(Z_1, Z_2) = \frac{\text{cov}(Z_1, Z_2)}{\sigma(Z_1)\sigma(Z_2)} = 13.32\%$

and in the same way  $\rho(Z_1, Z_3) = 13.89\%$  and  $\rho(Z_2, Z_3) = 16.16\%$ .

To simulate  $Z$  we simulate  $X$  and then calculate  $(1_{X_1 < \alpha_1}, 1_{X_2 < \alpha_2}, 1_{X_3 < \alpha_3})$

## Remark:

The Structural Model for default is in fact a Copula model

## Exemples of Copula :

- Clayton  $C_n(u, v) = \max(u^{\frac{1}{n}} + v^{\frac{1}{n}}, 0)^{\frac{1}{n}}$
- Gumbel-Hougaard  $C(u) = \exp\left(\left[-\sum_{i=1}^{i=d} (-\log(u_i))\right]^\theta\right)^{\frac{1}{\theta}}$  with  $\theta > 1$

**Background** : Pearson's linear correlation  $\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma(X)\sigma(Y)}$  measures only the affine relationship between variables and presents some imperfections to measure the link between two variables. For example:

- if  $X \sim \mathcal{N}(0, 1)$  and  $Y = X^2$  then  $\rho(X, Y) = 0$  while there is a strong link between  $Y$  and  $X$  (we can indeed predict  $Y$  perfectly from  $X$ )
- if  $f$  and  $g$  are increasing in general  $\text{cov}(X, Y) \neq \text{cov}(f(X), g(Y))$

## Definition : Kendall's tau

Let  $(X, Y)$  be a random variable. Let  $(X_1, Y_1), (X_2, Y_2)$  be independent with the same law as  $(X, Y)$ . We call Kendall's tau and note  $\tau(X, Y)$  the quantity  $P((X_1 - X_2)(Y_1 - Y_2) \geq 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$

## Properties Kendall's $\tau$

- $-1 \leq \tau(X, Y) \leq 1$
- if  $f$  and  $g$  are strictly increasing  $\tau(f(X), g(Y)) = \tau(X, Y)$
- $\tau(F_X(X), F_Y(Y)) = \tau(X, Y)$
- $U \sim \mathcal{U}([0, 1]) \implies \tau(U, U) = 1$  and  $\tau(U, 1 - U) = -1$
- If  $(X, Y)$  has  $C$  for Copula then
$$\tau(X, Y) = -1 + 4 \int_{[0,1]^2} C(u, v) \frac{\partial^2 C}{\partial u \partial v} dudv$$

### Remark:

if  $X \sim \mathcal{N}(0, 1)$  and  $Y = X^2$  then  $\tau(X, Y) = 1$

**Demonstration :** let's show the last point

$$\begin{aligned}\tau(X, Y) &= P((X_1 - X_2)(Y_1 - Y_2) \geq 0) - (1 - P((X_1 - X_2)(Y_1 - Y_2) \geq 0)) \\ &= -1 + 2P((X_1 - X_2)(Y_1 - Y_2) \geq 0)\end{aligned}$$

$$= -1 + 2(P(X_1 - X_2 \leq 0, Y_1 - Y_2 \leq 0) + P(X_2 - X_1 \leq 0, Y_2 - Y_1 \leq 0))$$

as  $(X_1, Y_1)$  and  $(X_2, Y_2)$  have the same law, so we just need to calculate the first probability.

$$P(X_1 - X_2 \leq 0, Y_1 - Y_2 \leq 0) = E(E(1_{X_2 \geq X_1} 1_{Y_2 \geq Y_1}) | X_1, Y_1)$$

$$= E(C(X_1, Y_1)) = \int_{[0,1]^2} C(u, v) \frac{\partial^2 C}{\partial u \partial v} dudv$$

$$\text{so } \tau(X, Y) = -1 + 4 \int_{[0,1]^2} C(u, v) \frac{\partial^2 C}{\partial u \partial v} dudv \text{ Q.E.D}$$



## Definition : Spearman's correlation

If  $(X, Y)$  is a random variable with marginal laws  $F_X$  and  $F_Y$  then the Spearman's correlation  $\rho_S$  is defined by  $\rho_S(X, Y) = \rho(F_X(X), F_Y(Y))$

## Properties Spearman's correlation

- $-1 \leq \rho_S(X, Y) \leq 1$
- if  $f$  and  $g$  are strictly increasing  $\rho_S(f(X), g(Y)) = \rho_S(X, Y)$
- $\tau(F_X(X), F_Y(Y)) = \rho_S(X, Y)$
- $U \sim \mathcal{U}([0, 1]) \implies \rho_S(U, U) = 1$  and  $\rho_S(U, 1 - U) = -1$
- If  $(X, Y)$  has  $C$  for Copula then
$$\rho_S(X, Y) = -3 + 12 \int_{[0,1]^2} C(u, v) dudv$$

## Demonstration :

Let  $C$  be the copula of  $(X, Y)$  i.e the law of  $(U, V)$  where  $U = F_1(X)$  and  $V = F_2(Y)$ , let  $\rho_S(X, Y) = \text{cor}(U, V) = \frac{E(UV) - E(U)E(V)}{E(U)E(V)}$

$$\begin{aligned} E(UV) &= \int_0^1 \int_0^1 uv \frac{\partial^2 C}{\partial u \partial v} dudv = \int_0^1 u \left( \int_0^1 v \frac{\partial^2 C}{\partial u \partial v} dv \right) du \\ &= \int_0^1 u \left( [v \frac{\partial C}{\partial u}]_0^1 - \int_0^1 \frac{\partial C}{\partial u} dv \right) du = \int_0^1 u \left( f_U(u) - \int_0^1 \frac{\partial C}{\partial u} dv \right) du \\ &= E(U) - \int_0^1 \left( \int_0^1 u \frac{\partial C}{\partial u} du \right) dv = E(U) - \int_0^1 \left( [uC(u, v)]_0^1 - \int_0^1 C(u, v) du \right) dv \\ &= E(U) - \int_0^1 P(V \leq v) dv + \int_0^1 \int_0^1 C(u, v) dudv \\ &= E(U) - E(v) + \int_0^1 \int_0^1 C(u, v) dudv = \int_0^1 \int_0^1 C(u, v) dudv \\ \text{and } E(U)E(V) &= \frac{1}{4} \text{ and } \text{Var}(U) = \text{Var}(V) = \frac{1}{12} \text{ Q.E.D} \end{aligned}$$

## Appendix : Risk Neutral Probability

# Risk Neutral Probability (discrete case)

**Background:** We consider an economy with two instants  $\{0, 1\}$  where there are  $d$  assets whose vector of prices  $X$  is represented today by the vector  $X_0 = (X_0^1, X_0^2, \dots, X_0^d)'$ . We assume that at instant 1 there are  $n$  possible states for the economy and for each state  $i \in \llbracket 1, n \rrbracket$  the vector of the prices of the assets is  $X_i = (X_i^1, X_i^2, \dots, X_i^n)'$ . We assume that prices are all strictly positive.

## Definition: Absence of Arbitrage (AOA)

We say that there is no arbitrage in the economy iff:

$$\{w \in \mathbb{R}^d, \forall i \in \llbracket 1, n \rrbracket \langle w, X_i \rangle \geq 0\} \subset \{w \in \mathbb{R}^d, \langle w, X_0 \rangle \geq 0\}$$

## Remarks :

The definition means that it is not possible to receive money today to build a strategy which has positive values tomorrow in all cases.

## Theorem and Definition: Risk Neutral Probability

a) the two following propositions are equivalent:

- there is no arbitrage in the economy
- we can find  $(\lambda_i)_{i \in \llbracket 1, n \rrbracket}$ ,  $\lambda_i \geq 0$  such that  $X_0 = \sum_{i=1}^{i=n} \lambda_i X_i$

b) if there is a risk-free asset in the economy of return  $r$  over  $[0, 1]$  then:

- $\sum_{i=1}^{i=n} \lambda_i = \frac{1}{1+r}$
- if we define a probability  $\pi$ , over the  $n$  possible values of  $X$  at time 1, by  $\pi_i = \lambda_i(1+r)$  then  $X_0 = \frac{1}{1+r} E_\pi[X]$  and  $\pi$  is called the risk neutral probability for the economy

## Demonstration :

a) one of the implications is obvious.

We assume now that there is no arbitrage and define the cone

$$\mathcal{C} = \left\{ \sum_{i=1}^{i=n} \lambda_i X_i, \forall i \in \llbracket 1, n \rrbracket \lambda_i \geq 0 \right\}.$$

Then  $\mathcal{C}$  is convex and if  $X \notin \mathcal{C}$  we can separate  $X$  from  $\mathcal{C}$  by an hyperplane and find  $w \in \mathbb{R}^d$  such that:  $\langle w, X_0 \rangle < 0$  and for all  $Y$  in  $\mathcal{C}$   $\langle w, Y \rangle > 0$  but this would contradict the AOA hypothesis, so  $X_0 \in \mathcal{C}$ . Q.E.D

b) if we assume that the risk-free asset is component  $j$  then:

$$X_0^j = \sum_{i=1}^{i=n} \lambda_i X_i^j \text{ and } X_i^j = X_0^j(1+r) \implies \sum_{i=1}^{i=n} \lambda_i = \frac{1}{1+r} \text{ Q.E.D}$$

## Risk Neutral Probability (discrete case)

**Exercise :** we assume that there are 3 assets, of prices today  $X_0 = (1, 5, 10)'$  and 3 possible states of the economy tomorrow defined by the 3 vector of prices for the assets:

$X_1 = (1.03, 5, 11)'$ ,  $X_2 = (1.03, 5, 10)'$ ,  $X_3 = (1.03, 6, 10)'$ .

a) explain why the risk-free rate is 3%

b) show that:

$$\begin{cases} \frac{1}{1.03}(\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3) = X_0 \text{ (C1)} \\ \pi_1 + \pi_2 + \pi_3 = 1 \text{ (C2)} \end{cases} \implies \pi = (0.30, 0.55, 0.15)$$

c) explain why there is no arbitrage in this economy

If we assume now that  $X_2 = (1.03, 5, 12)$  show that:

d)  $\pi$  satisfies (C1) and (C2)  $\implies \pi = (1.4, -0.55, 0.15)$

e) explain why there are arbitrages in this case

f) exhibit an arbitrage for this economy

# Risk Neutral Probability (discrete case)

**Hint:** for f) as there is no risk neutral probability ( $\pi$  is not a probability), according to the AOA theorem, we can find a strategy  $w$  such that  $\langle w, X_0 \rangle < 0$  and  $\forall i \in \llbracket 1, 3 \rrbracket, \langle w, X_i \rangle > 0$ .

If we consider the strategy  $w = (-7.6, 0.5, 0.5)$  then,

At inception:

- the risk free asset is short-sold
- the two risk free assets are purchased in quantity of 0.5 and 0.5
- 0.10 of cash is generated ( $-7.60 + 0.5 \times 5 + 0.5 \times 10 = 0.10$ )

At maturity:

- for state 1: exiting the strategy earns 0.17  
( $-7.60 \times 1.03 + 0.5 \times 5 + 0.5 \times 11 = 0.17$ )
- for state 2: exiting the strategy earns 0.67
- for state 3: exiting the strategy earns 0.17



## Risk Neutral Probability (discrete case)

So here there is an arbitrage, as the strategy  $w$  enables to receive money at all times (0.10 at inception and at least 0.17 in the future) and therefore should be used in infinite quantity if it existed.

**Remark** : If we assume that the vector of prices today is  $X_0 = (1, 5, 10)'$  and that the 3 vectors of prices for tomorrow are :

$X_1 = (1.03, 5, 10)'$ ,  $X_2 = (1.03, 6, 12)'$ ,  $X_3 = (1.03, 6, 13)'$  then

a)  $\pi = (0.85, 0.15, 0)'$  is the risk neutral probability

b) according to a) there is no arbitrage

c) the strategy  $w = (0, -2, 1)'$  costs today zero and the possible outcomes tomorrow are 0 for the first two states and 1 for state 3, so it seems attractive to play it (as there is only upside) but strictly speaking this is not an arbitrage according to our definition.

# Risk Neutral Probability (continuous case)

## Background:

We consider a probability space  $(\Omega, \mathcal{F}, P)$  with  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$  where  $\mathcal{F}_t$  represents the information available at time  $t$ .

We assume that there are  $d$  financial assets following the equations:

$dX_s^i = \mu_s^i X_s^i ds + \sigma_s^i X_s^i dW_s^i$  where  $W_s = (W_s^1, W_s^2, \dots, W_s^d)$  is a  $d$ -dimensional Brownian motion

## Theorem and Definition : Risk Neutral Probability

Based on the previous hypotheses we can find a probability  $Q$  on  $(\Omega, \mathcal{F})$  such that:

$W^*$  defined by:  $dW_s^{i*} = (dW_s^i + \frac{\mu_s^i - r_s}{\sigma_s^i} ds)$  is a Brownian motion under  $Q$

We can then re-write the model:

$dX_s^i = r_s X_s^i ds + \sigma_s^i X_s^i dW_s^{i*}$  where  $W^*$  is a Brownian motion under  $Q$  and  $Q$  is called the risk neutral probability.

## Lemma and Definition

If  $(Z_s)_{s \geq 0}$  is a martingale under  $P$  with  $Z_s \geq 0$  and  $Z_0 = 1$  and if we define  $Q$  for any random variable  $Y_t$   $\mathcal{F}_t$ -measurable by  $E_Q[Y_t] = E_P[Y_t Z_t]$  then:

- $Q$  is a probability on  $(\Omega, \mathcal{F})$
- $E_Q[Y_T | \mathcal{F}_t] = E_P[Y_T \frac{Z_T}{Z_t} | \mathcal{F}_t]$

Usually we note  $(\frac{dQ}{dP})_t = Z_t$  and so we write  $E_Q[Y_t] = E_P[Y_t (\frac{dQ}{dP})_t]$

**Demonstration Lemma :** easy

**Demonstration Theorem :**

We note  $\Delta_s^i = \frac{\mu_s^i - r_s}{\sigma_s^i}$  and  $\Delta_s = (\Delta_s^1, \Delta_s^2, \dots, \Delta_s^d)'$ .

## Risk Neutral Probability (continuous case)

We search for a probability  $Q$  under which  $(W_s^*)_{s \geq 0}$  is a Brownian motion

For that we need  $E_Q[dW_t^* | \mathcal{F}_t] = 0$  and  $E_Q[dW_t^* (dW_t^*)' | \mathcal{F}_t] = Id_{\mathbb{R}^d} dt$

We note  $(\frac{dQ}{dP})_t = Z_t$  and so search for  $Z_t$

$$\begin{aligned} E_Q[dW_t^* | \mathcal{F}_t] &= E_P[dW_t^* \frac{Z_{t+dt}}{Z_t} | \mathcal{F}_t] = E_P[(dW_t + \Delta_t dt)(1 + \frac{dZ_t}{Z_t}) | \mathcal{F}_t] \\ &= E_P[dW_t | \mathcal{F}_t] + \frac{1}{Z_t} E_P[dW_t dZ_t | \mathcal{F}_t] + \Delta_t dt = \frac{1}{Z_t} E_P[dW_t dZ_t | \mathcal{F}_t] + \Delta_t dt \end{aligned}$$

If we search  $Z$  of the form  $dZ_s = \langle B_s, dW_s \rangle$  with  $B_s \in \mathbb{R}^d$  (no drift term as martingale) then:

$$\begin{aligned} E_P[dW_t dZ_t | \mathcal{F}_t] &= E_P[dW_t \langle dW_t, B_t \rangle | \mathcal{F}_t] = E_P[dW_t (dW_t)' B_t | \mathcal{F}_t] \\ &= E_P[dW_t (dW_t)' | \mathcal{F}_t] B_t = B_t dt \text{ so, } E_Q[dW_t^* | \mathcal{F}_t] = 0 \iff B_t = -\Delta_t Z_t \end{aligned}$$

Solving  $dZ_s = \langle -\Delta_s, dW_s \rangle Z_s$  and  $Z_0 = 1$  we get:

$$Z_t = \exp\left(\int_0^t -\langle \Delta_s, dW_s \rangle - \frac{1}{2} \int_0^t \|\Delta_s\|^2 ds\right) \text{ (we do not discuss here the}$$

conditions on  $\Delta_s$  for integrability that can be found in Girsanov's theorem)

The condition  $E_Q[dW_t^* (dW_t^*)' | \mathcal{F}_t] = Id_{\mathbb{R}^d} dt$  is easy to verify Q.E.D

**Remark 1:** for  $d = 1$  we get that:

- $X_T = X_0 e^{\mu T} e^{\sigma W_T - \frac{1}{2}\sigma^2 T}$  where  $(W_s)_{s \geq 0}$  is a Brownian under  $P$
- $X_T = X_0 e^{rT} e^{\sigma W_T^* - \frac{1}{2}\sigma^2 T}$  where  $(W_s^*)_{s \geq 0}$  is the Brownian under  $Q$  defined by  $W_T^* = W_T + \frac{\mu - r}{\sigma} T$
- $(\frac{dQ}{dP})_T = \exp(\frac{r - \mu}{\sigma} W_T - \frac{1}{2}(\frac{r - \mu}{\sigma})^2 T)$
- for any function  $h$ ,  $E_Q[h(X_T)] = E_P[h(X_T)(\frac{dQ}{dP})_T]$

To verify "manually" the last bullet point we just need to check that

$$E_Q[h(rT + \sigma W_T^*)] = E_P[h(\mu T + \sigma W_T)(\frac{dQ}{dP})_T]$$

So we calculate:

$$E_Q[h(rT + \sigma W_T^*)] = E_{Q^{W_T^*}}[h(rT + \sigma z)] = \int h(rT + \sigma z) \frac{1}{\sqrt{2\pi T}} \exp(-\frac{z^2}{2T}) dz$$

# Risk Neutral Probability (continuous case)

If we take the new variable  $u$  such that  $\mu T + \sigma u = rT + \sigma z$  we get :

$$\circ h(rT + \sigma z) = h(\mu T + \sigma u)$$

$$\circ \exp\left(-\frac{z^2}{2T}\right) = \exp\left(-\frac{(u + \frac{\mu-r}{\sigma} T)^2}{2T}\right) = \exp\left(-\frac{u^2}{2T}\right) \exp\left(-\frac{\mu-r}{\sigma} u - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right)$$

so,

$$\int h(rT + \sigma z) \frac{1}{2\pi\sqrt{T}} \exp\left(-\frac{z^2}{2T}\right) dz$$

$$= \int h(\mu T + \sigma u) \exp\left(-\frac{\mu-r}{\sigma} u - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right) \frac{1}{2\pi\sqrt{T}} \exp\left(-\frac{u^2}{2T}\right) du$$

$$= E_P[h(\mu T + \sigma W_T) \exp\left(-\frac{\mu-r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right)] \text{ Q.E.D}$$

## Risk Neutral Probability (continuous case)

**Remark 2:** we call  $u$  a utility function compatible with the price of asset  $X$ ,  $P$  the "real" probability and "Q" the risk neutral probability.

As  $X_0 = E_P[e^{-rT} u(X_T)]$  and  $X_0 = E_Q[e^{-rT} X_T]$  we have:

$E_P[e^{-rT} u(X_T)] = E_P[e^{-rT} X_T f(W_T)]$  with

$$f(w) = \exp\left(\frac{r-\mu}{\sigma} w - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right)$$

if we define  $g(x) = f\left(\frac{1}{\sigma}\left[\ln\left(\frac{x}{X_0}\right) + \left(\frac{\sigma^2}{2} - \mu\right)T\right]\right)$  then

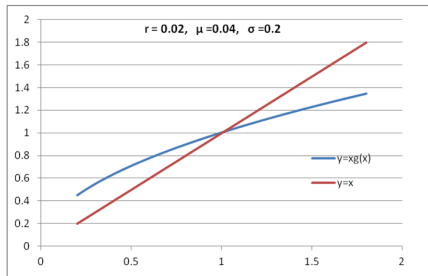
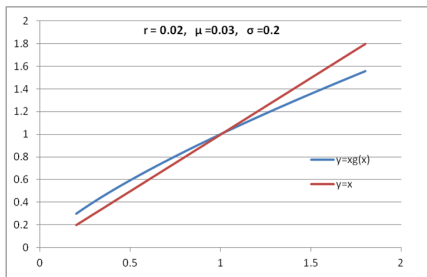
$$f(W_T) = g(X_T) \text{ and } E_P[e^{-rT} u(X_T)] = E_P[e^{-rT} X_T g(X_T)]$$

so,  $u(x) = xg(x)$  works to get the equality

In the following graph we price  $xg(x)$  for various values of  $r, \mu$  and  $\sigma$  with  $x_0 = 1$  and  $T = 1$

Note that depending on the value of the parameters  $xg(x)$  is not always increasing (which shows its limits in terms of admissible utility function..)

# Risk Neutral Probability (continuous case)



Utility functions derived from Girsanov's Theorem



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