

# Credit Risk, CDOs and Copulas (Draft)

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## Structural and Reduced Form Models

In this course we consider mainly credit risk analysis for structured financial products such as CDOs but credit risk arises in many different forms

**Credit risk problematics are embedded in all types of activities:**

- retail banking
- private banking
- commercial and investment banking with corporates
- lending between banks

**Credit risk is embedded in a wide variety of financing instruments:**

- Bonds, Commercial Papers
- Loans (syndicated, bilateral), Credit Lines
- Project Finance, Structured Finance
- Specialised finance, Equity Financing, Stock loan..etc

# Financial Analysis for Credit Risk

Two issues to analyse "Credit Risk" in traditional lending:

- capacity to pay financial flows
- capacity to reimburse debt at maturity or to refinance

To measure the capacity to pay interests people will look usually, amongst other things, at the interest coverage ratio:

$$\text{Interest coverage ratio} = \frac{\text{interest expenses on debt}}{\text{Earnings Before Interests and Taxes}}$$

- EBIT: Earnings Before Interests and Taxes
- EBITDA: Earnings Before Interests, Taxes, Depreciation and Amortization

In more structured financing a collateral/security can also be taken into account

It seems Altman was the first to introduce statistical models to predict bankruptcy and to quantify this risk

Altman's (1968) financial **score** based on some financial ratios:

- $X_1$  Working Capital / Total Assets.
- $X_2$  Retained Earnings / Total Assets
- $X_3$  Earnings Before Interest and Taxes / Total Assets
- $X_4$  Market Value / Book Value of Total Debt
- $X_4$  Sales / Total Assets (industry dependent)

## Reminders:

Working Capital = Current Assets - Current Liabilities

Current Assets = Cash + Account Receivables + Inventories

## Altman's Z-Score: with various revisions in 1983 and 1993

Based on historical studies of bankruptcies the Z-score was defined as:

$$Z = 0.012X_1 + 0.014X_2 + 0.033X_3 + 0.006X_4 + 0.999X_5$$

with the following predictions associated:

if  $Z < 1.81$  default within 1 year is predicted

if  $1.81 \leq Z \leq 2.67$  no prediction

if  $Z > 2.67$  prediction of no default within 1 year

The following results were obtained on the samples tested:

- 90.9% success rate in predicting bankruptcy
- 97% success rate in predicting non-bankruptcy

Some refinements have been done to the model in particular to define a probability of default.

Ohlson (1980) proposed the **LOGIT Model**:

$$P(\text{Default}|X_1, X_2, \dots, X_n) = \frac{1}{1 + \exp(-\beta_1 X_1 - \beta_2 X_2 - \dots - \beta_n X_n)}$$

where the parameters  $\beta_i$  are estimated on a sample by maximizing the likelihood.

**Remarks:**

- $P(\text{Default}|X) = \frac{1}{1 + \exp(-\langle \beta' X \rangle)} \iff \ln\left(\frac{P(\text{Default}|X)}{1 - P(\text{Default}|X)}\right) = \langle \beta' X \rangle$
- $f(p) = \ln\left(\frac{p}{1-p}\right)$  is called the logit function
- $g(x) = \frac{1}{1 + e^{-x}}$  is called the sigmoid function
- $f = g^{-1}$
- $f(0.5) = 0$



**Remarks:** Likelihood of a binomial model  $\mathcal{B}(p)$ :

For a single observation  $y_i \in \{0, 1\}$   $L(y_i) = p^{y_i}(1 - p)^{1-y_i}$

The Likelihood for the defaults or not of  $p$  companies according to

Ohlson's model is :  $\prod_{i=1}^{i=n} p_i^{y_i}(1 - p_i)^{1-y_i}$  (with  $y_i \in \{0, 1\}$  )

- the factors  $X_i$  which impact the probability of defaults  $p_i$  are observed for all these companies
- the Maximum Likelihood method consists in finding the  $\beta^i$  which maximise the likelihood (calibration of the model)
- when considering a new company, the probability of default for this company will be calculated by observing the variables  $X_i$  for this company an using the coefficients  $\beta^i$  calculated previously

Altman and Ohlson's models were developed principally for financial analysts with an estimation of the parameters based on bankruptcy historicals.

Since then other models have been developed which:

- are better suited for trading purposes
- are calibrated on corporate bonds and credit derivatives market prices instead of observed bankruptcies
- modelize the credit dynamics

# Structural and Reduced Form Models

For these new models two principal types:

- Structural Models: (Black-Scholes(1973), Merton(1974), Leland(1994), Schaeffer(2004))
- Reduced Form Models: (Jarrow-Turnbull(1997), Duffie-Singleton (1999))

## Definition: Structural Models - KMV

- model the dynamics of the Assets  $A_t$  and Liabilities  $L_t$
- assumption that the default happens iff  $A_t < L_t$
- in their simplest form:  $A_t = A_0 \exp(\sigma W_t - \frac{\sigma^2}{2} t)$  and  $L_t = L_0$
- approach adopted by Moody's KMV

## Definition: Distance to Default (DTD)

The Distance to Default is the number of standard deviations between the  $\ln$  of the current value of the company's assets (assumed to be normally distributed) and the log of its liabilities.

**Example:** We assume

- Assets:  $A_0 = EUR100$  and  $A_T = A_0 \exp(rT) \exp(\sigma W_T - \frac{\sigma^2}{2} T)$
- volatility of the assets:  $\sigma = 25\%$ ,  $r = 0$
- Liabilities:  $L_T = L_0 = EUR60$

then:  $A_T < L_T \iff \frac{W_T}{\sqrt{T}} < \frac{1}{\sigma\sqrt{T}} [\ln(\frac{L_0}{A_0}) + \frac{\sigma^2}{2} T] = DTD + \frac{\sigma}{2} \sqrt{T}$   
so here the 1 year Distance to Default is 2.

## Definition: Reduced Form models

- the instant of default  $\tau$  is modelised by an **Exponential law** of intensity  $\lambda$ ,  $\mathcal{E}(\lambda)$ , under the risk neutral probability  $P$
- when the intensity is constant :  
$$P(\tau > t) = \exp(-\lambda t)$$
- when the intensity is deterministic but time dependent :  
$$P(\tau > t) = \exp\left(-\int_0^t \lambda_s ds\right)$$
- when the intensity is stochastic  $\tau$  is a **Cox process** :  
$$P(\tau > t) = E\left[\exp\left(-\int_0^t \lambda(s, X_s) ds\right)\right]$$
 where  $(X_s)_{s \geq 0}$  is a stochastic process

## Remark:

$(X_s)$  can be taken as the short term interest rate if it is considered that monetary policy has a significant impact on the economy

## Proposition : variational definitions of exponential laws

Exponential laws can be defined equivalently if the following ways:

- $\tau \sim \mathcal{E}(\lambda) \iff P(t < \tau < t + dt | \tau > t) = \lambda dt$
- $\tau \sim \mathcal{E}(\lambda_t) \iff P(t < \tau < t + dt | \tau > t) = \lambda_t dt$
- $\tau \sim \mathcal{E}(\lambda(t, X_t)) \iff P(t < \tau < t + dt | \tau > t, \{X_s, s \in [0, t]\}) = \lambda(t, X_t) dt$

**Notation** We note  $\mathcal{X}_t = \{X_s, s \in [0, t]\}$

## Demonstration:

We just need to show the stochastic case as the other cases are particular cases. We note  $F(t) = P(\tau > t | \mathcal{X}_t)$ .

$$P(t < \tau < t + dt | \tau > t, \mathcal{X}_t) = \lambda(t, X_t) dt$$

$$\implies \frac{P(t < \tau < t + dt | \mathcal{X}_t)}{P(\tau > t | \mathcal{X}_t)} = \lambda(t, X_t) dt$$

$$\implies \frac{F(t) - F(t + dt)}{F(t)} = \lambda(t, X_t) dt$$

$$\implies d \ln F(t) = -\lambda(t, X_t) dt$$

$$\implies F(t) = F(0) \exp\left(-\int_0^t \lambda(s, X_s) ds\right)$$

but  $F(0) = 1$  and  $F(t) = E(1_{\tau > t} | \mathcal{X}_t)$

$$\text{so, } P(\tau > t) = E(1_{\tau > t}) = E[E(1_{\tau > t} | \mathcal{X}_t)] = E[F(t)]$$

$$= E\left[\exp\left(-\int_0^t \lambda(s, X_s) ds\right)\right]. \text{ Q.E.D.}$$

# Reduced Form models: Stochastic Intensity and Interest Rate

In an economy where the instantaneous short term interest rate depends on the factors  $(X_s)_{s \geq 0}$  if we note  $\beta_t = \exp(-\int_0^t r(X_s) ds)$  the actualisation factor then the price of a zero coupon bond of maturity  $T$  and nominal 1 with credit risk and zero recovery rate is  $E[\beta_T 1_{\tau > T}]$

## Proposition

$E[\beta_T 1_{\tau > T}] = E\left[\exp(-\int_0^T (r(X_s) + \lambda(s, X_s)) ds)\right]$   
so,  $\lambda(s, X_s)$  is the "instantaneous spread" at time  $s$

## Demonstration:

$$\begin{aligned} E[\beta_T 1_{\tau > T}] &= E\left[E[\beta_T 1_{\tau > T} | \mathcal{X}_t]\right] = E\left[\beta_T E[1_{\tau > T} | \mathcal{X}_t]\right] \\ &= E\left[\beta_T \exp(-\int_0^T \lambda(s, X_s) ds)\right] \\ &= E\left[\exp(-\int_0^T r(X_s) ds) \exp(-\int_0^T \lambda(s, X_s) ds)\right] \end{aligned}$$



$$= E \left[ \exp \left( - \int_0^T (r(X_s) + \lambda(s, X_s)) ds \right) \right] \text{ Q.E.D.}$$

## Corollary

If the short term interest rate  $r$  and the intensity of default  $\lambda$  are constant then the price of a zero coupon bond of maturity  $T$  and nominal 1 with credit risk and zero recovery rate is  $E[\beta_T 1_{\tau > T}] = \exp(-(r + \lambda)T)$ .  
As a consequence  $\lambda$  can be inferred from the price of a risky bond.

## Exemple :

If we assume that a one year zero coupon government bond is worth 100.10 % and that a one year zero coupon bond issued by risky issuer Zco is worth 99.50% then the intensity of default  $\lambda$  for the risky issuer is  $\ln\left(\frac{100.10}{99.50}\right) = 0.6\%$

## Remarks:

- a time dependent  $\lambda_t$  enables to calibrate a model to a term structure of spreads
- as we will show later a stochastic  $\lambda(t, X_t)$  enables **to modelize correlation between bonds**
- if  $\lambda$  is small then  $P(\tau < 1) \sim \lambda$ . So if  $\lambda = 2\%$  the probability of default within one year is approx 2%
- exponential laws are memoryless i.e :  
$$P(t < \tau < t + \delta | \tau > t) = P(\tau < \delta)$$
- exponential laws and normal laws are the two "benchmarks" in finance

# Pricing a new Issuance

To price a new issuance several methods can be considered at this stage, amongst them:

- if a rating already exists for the company and if the bond is vanilla price the spread based on this rating, the type of industry and the comparables
- analyse the fundamentals of the company, find a "comparable company" having a similar bond already issued and price by comparison
- analyse the fundamentals of the company and use for example Ohlson's model to calculate a probability of default and from there derive a price for the bond

If the bond is complex because :

- there are some conditional payouts
- there are some collaterals which guarantees it
- there are some specific optionalities embedded
- there are some hybrid issues involved

Then it it may be necessary to start with a full modelisation of all the stochastic elements involved before being able to be able to come up with a price.

The example below show how **credit risk modelisation can be embedded in a classic "Black and Scholes" modelisation framework** (which is based on the notion of non arbitrage possibility and risk neutral probability).

## Example: Construction of $P$ and $\lambda$ by arbitrage

**Exemple:** we consider a two period economy  $0, T$  with:

- a risk free asset, a company's stock and bond
- the stock is worth 100 with possible future values 130 and 0 (default)
- the risk free asset has a return of 5%

we assume that the company's bond will be worth at maturity:

- 106 if the company's stock is worth 130
- 84.8 if the company defaults (Recovery Rate 80%)

If there is no arbitrage, we have the following results:

- a) the bond can be replicated by investing in the stock and risk-free bond
- b) the value of the risky bond is 97.10 today
- c) the risk neutral probability verifies  $p = 80.77\%$  (probability no default)
- d) we have  $\lambda = 21.35\%$

## Example: Construction of $P$ and $\lambda$ by arbitrage

### Demonstration:

a) we search for  $a$  and  $b$  such that:

$$a105 + b130 = 106 \text{ and } a105 + b \times 0 = 84.8$$

So,  $a = 0.808$  is the number of risk-free bonds to purchase and  $b = 0.163$  is the number of stocks to purchase to replicate the corporate bond.

b) if there is no arbitrage, the price of the risky bond is then

$$0.808 \times 100 + 0.163 \times 100 = 97.10$$

c) the corresponding risk neutral probability is such that

$$\frac{130}{1.05}p + \frac{0}{1.05}(1 - p_1) = 100 \text{ so } p = 80.77\% .$$

$$d) P(\tau > 1) = p_1 \iff e^{-\lambda} = 19.23\% \iff \lambda = 21.35\%$$

**Remarks:** For the risky-Bond we have  $97.10 = \frac{106}{1.05} \times e^{-3.89\%}$  so the return of the bond will be 3.89% higher than the return of the risk-free bond if the bond does not default. This excess return is called the spread of the bond (calculated as a continuous rate).

## Remarks:

- when calibrating a Reduced Form model the risk free rate and the price of the risky bond are observed and from there  $\lambda$  can be deducted
- Duffie and others have compared the "implied"  $\lambda$  (under the risk neutral probability) for corporate Bonds derived from their prices and compared them to the "realized"  $\lambda$  (under the "real probability") derived from the defaults over the subsequent periods and found that  $\lambda_{implied} \sim 2 \times \lambda_{realized}$
- discrepancies between  $\lambda_{implied}$  under the risk neutral possibility and  $\lambda_{realized}$  under historical probability can be seen as similar issues to the discrepancies between "implied volatility" and "realized volatility"

## Theorem and Definition : Recovery Rate $R$ and Spread

The Recovery Rate  $R$  is the fraction of the amount due recovered if the counterparty defaults.

In practice  $R$  depends on the type of debt issued by the company (senior, junior, secured...)

- if  $R$  is the recovery rate of a zero coupon of maturity  $T$
- if  $r$  is the risk-free rate for the same maturity
- if  $S$  is the spread of the risky bond of maturity  $T$
- if  $\lambda$  is the constant default rate (under the risk neutral probability)

then:  $S \sim (1 - R)\lambda$



## Demonstration:

Pricing the zero coupon with the risk neutral probability we have:

$$e^{-(r+S)T} = e^{-rT}(e^{-\lambda T} + R(1 - e^{-\lambda T})) \implies e^{-ST} = (1 - R)e^{-\lambda T} + R.$$

Developing to the first order we get the result.

## Remark:

In the previous example we have  $S = 3.89\%$ ,  $\lambda = 21.35\%$  and the recovery is  $R = 80\%$  so  $(1 - R)\lambda = 4.27\%$ . Here the (first order) approximation of  $S$  is not very good because  $\lambda$  is taking a quite large value

## Exercise (alternative definition of a Cox process) :

Let  $\lambda(\cdot)$  be positive on  $\mathbb{R}^d$  and  $X = (X_s)_{s \geq 0}$  be a stochastic process of  $\mathbb{R}^d$

Let  $\tau_1$  be an exponential law of parameter 1 independent from  $X$

Let  $\tau_\lambda$  be defined by  $\tau_\lambda(\omega) = \inf\{t, \int_0^t \lambda(X_s)(\omega) ds \geq \tau_1(\omega)\}$

Show that  $\tau_\lambda \sim \mathcal{E}(\lambda(X))$

### Demonstration:

$$P(\tau_\lambda > t) = E[1_{\tau_\lambda > t}] = E\left(E[1_{\tau_\lambda > t} | X]\right)$$

$$E[1_{\tau_\lambda > t} | X] = P\left(\tau_1 > \int_0^t \lambda(X_s)(\omega) ds | X\right)$$

as  $X$  and  $\tau_1$  are independent

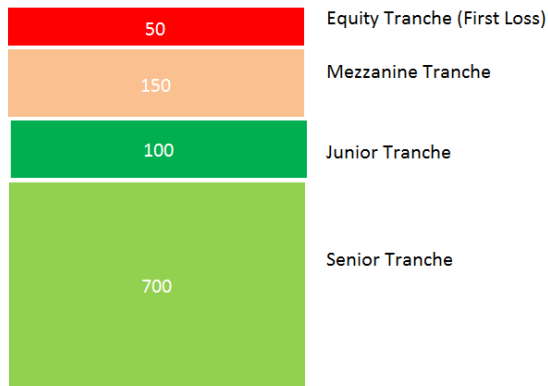
$$P\left(\tau_1 > \int_0^t \lambda(X_s)(\omega) ds | X\right) = \exp\left(-\int_0^t \lambda(X_s)(\omega) ds\right)$$

$$\text{so, } P(\tau_\lambda > t) = E\left[\exp\left(-\int_0^t \lambda(X_s)(\omega) ds\right)\right] \text{ Q.E.D.}$$

## Collateralized Debt Obligations

# Collateralized Debt Obligations

Collateralized Debt Obligations of Notional EUR 1000



10 Bonds of EUR 100 Nominal Each

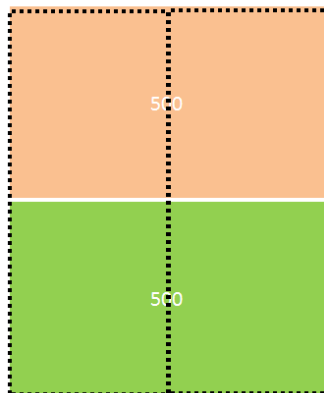
Rationale of the transaction:

- the risk is repackaged to be able to sell it better
- different type of investors can choose between different type of risks
- potentially Rating/ Pricing Arbitrage (up to 2008 too many senior tranches rated AAA)
- in the past potentially regulatory arbitrage (for keeping the risk on the equity tranche and deconsolidating)
- technology of packaging and tranching which can be applied to cash or synthetic underlyings

⇒ **Important to notice the importance of the "correlation" when pricing a CDO's tranche**

# Collateralized Debt Obligations

Example: two Tranches CDO made of two bonds



Bond 1  
probability of default  $p$

Bond 2  
probability of default  $q$

Case studies:  
correl = 1

correl = -1

2 Bonds of EUR 500 Notional Each

# Collateralized Debt Obligations

**Example:** we consider 2 Bonds, with zero Recovery rate, of EUR 500 Nominal each, packaged in a EUR 1000 Notional CDO and note  $Z_i = 1$  if the Bond  $i$  defaults before maturity and otherwise  $Z_i = 0$

a) if we assume that  $Z_1 = Z_2$  then:

- either the two bonds default together, resulting in a payout of zero for both tranches or
- none of the bonds defaults, resulting in a payout for both tranches of EUR 500

In this case, both tranches are the same, the senior tranche is not safer than the junior tranche and the correlation between the defaults is 100%.

b) if we assume that  $Z_2 = 1 - Z_1$  then:

there is always one bond which defaults so

- the junior tranche has always a payout of zero
- the senior tranche has always a payout of EUR 500

In this case the correlation between the defaults is  $-100\%$  and the two tranches have extremely different behaviours

⇒ Note that in this extreme example, the pricing of the two tranches does not depend on the probabilities of default (which nevertheless have to add up to 100% here) but only on the correlation !

## Remarks:

- a low correlation between the bonds is good for senior tranche holders and bad for junior tranche holders
- a high correlation between the bonds is good for junior tranche holders and bad for senior tranche holders
- the impact of correlation is less clear for mezzanine tranches holders

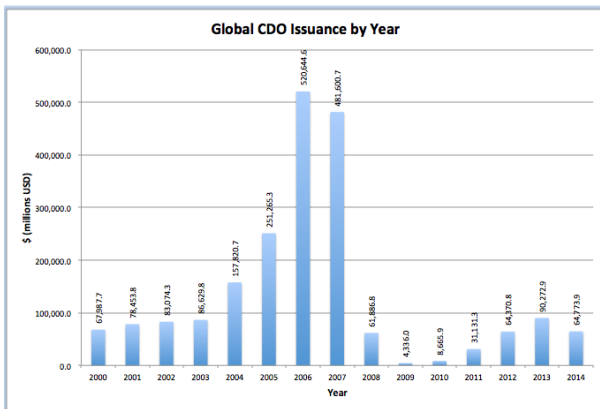
**To price CDOs we will need to simulate Bernoulli variables which are correlated**



- CLOs package together high-risk corporate debt and are then sold to institutional investors seeking potentially substantial returns
- CDOs (collateralized debt obligations), were comprised mostly of subprime mortgages and were blamed for the financial meltdown a decade ago
- The global CLO market is in 2019 between 1.4 and 2 trillion. In 2007, CDOs were 1.2 to 2.4 trillion
- Today's CLOs usually comprise corporate loans across a diversified set of industries

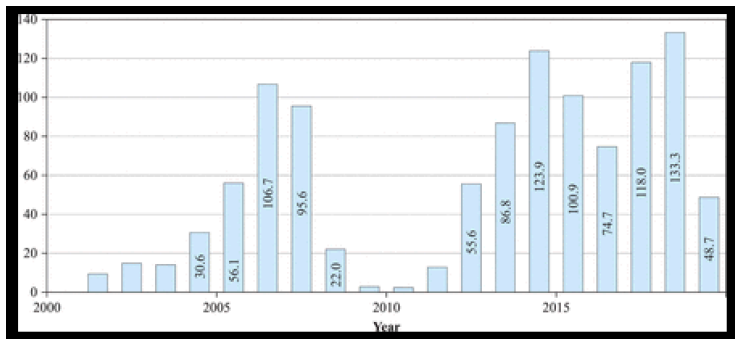
source <https://www.intralinks.com/blog/2019/11/clos-should-they-stay-or-should-they-go>

# Collateralized Debt Obligations

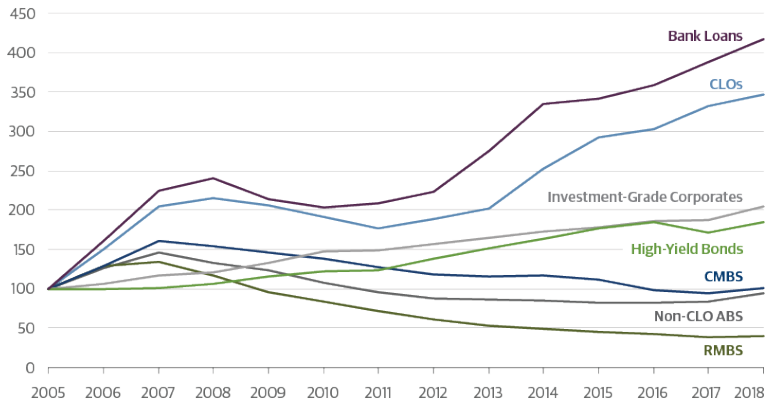


Source: Securities Industry and Financial Markets Association (SIFMA)

# Collateralized Loan Obligations



# Collateralized Loan Obligations



Fixed-Income instruments issuance; Index 2005 = 100, Source: Guggenheim May 2019

<https://www.guggenheiminvestments.com/perspectives/portfolio-strategy/collateralized-loan-obligations-clo>

# Simulating Correlated Binomials

We construct here Bernoulli variables with the same parameter  $p$  which are correlated. The correlation is created through the default parameter in the following way.

## Theorem: Simulation of Correlated Bernoulli Variables

Let  $(Z_i)_{i \in \llbracket 1, n \rrbracket}$  be independent variables of uniform law in  $\llbracket 0, 1 \rrbracket$

Let  $\tilde{p}$  be a random variable in  $\llbracket 0, 1 \rrbracket$  with density  $f$

Let  $(X_i)_{i \in \llbracket 1, n \rrbracket}$  be Bernoulli variables defined by  $X_i = 1 \iff Z_i < \tilde{p}$

Then:

a) the  $(X_i)_{i \in \llbracket 1, n \rrbracket}$  are Bernoulli variables of parameters  $\bar{p} = E[\tilde{p}]$

b)  $\forall i \neq j, \rho(X_i, X_j) = \frac{\text{Var}(\tilde{p})}{\bar{p}(1-\bar{p})}$

## Demonstration :

$$a) E(X_i) = E\left(E(X_i|\tilde{p})\right) = E\left(E(1_{Z_i < \tilde{p}}|\tilde{p})\right) = E(\tilde{p})$$

$$b) E(X_i X_j) = E\left(E(X_i X_j|\tilde{p})\right) = E(\tilde{p}^2) \text{ as } X_i \text{ and } X_j \text{ are independent conditionnally on } \tilde{p}$$

so,  $Cov(X_i X_j) = E(\tilde{p}^2) - E(\tilde{p})^2 = Var(\tilde{p})$  and we know that for Bernouilli  $Var(X_i) = Var(X_j) = E(\tilde{p})(1 - E(\tilde{p}))$  Q.E.D.

We consider now CDOs composed of bonds of the same notional with the same probabilities of default and same correlations and we are interested in calculating the law of the number of Bonds which default and therefore the

$$\text{law of } D_n = \sum_{i=1}^{i=n} X_i$$

## Remarks :

- the limitation of the model is that the resulting correlation between two bounds is always positive
- if  $\tilde{p}$  is constant the correlation between the bonds is zero
- if  $P(\tilde{p} = 0) = \frac{1}{2}$  and  $P(\tilde{p} = 1) = \frac{1}{2}$  the correlation between the bonds is 100% as  $\text{var}[\tilde{p}] = \frac{1}{4}$  and  $\bar{p}(1 - \bar{p}) = \frac{1}{4}$

## exercise 1 :

Show that  $\forall X$  random variable in  $[[0, 1]]$ ,  $\text{Var}[X] \leq \frac{1}{4}$

**Hint :** 
$$\begin{aligned}\text{Var}[X] &= E[(X - E(X))^2] = E\left[\left(\left(X - \frac{1}{2}\right) + \left(\frac{1}{2} - E(X)\right)\right)^2\right] \\ &= E\left[\left(X - \frac{1}{2}\right)^2\right] + E\left[\left(\frac{1}{2} - E(X)\right)^2\right] + 2E\left[\left(X - \frac{1}{2}\right)\left(\frac{1}{2} - E(X)\right)\right] \\ &= E\left[\left(X - \frac{1}{2}\right)^2\right] + \left(\frac{1}{2} - E(X)\right)^2 - 2\left(\frac{1}{2} - E(X)\right)^2 \\ &= E\left[\left(X - \frac{1}{2}\right)^2\right] - \left(\frac{1}{2} - E(X)\right)^2 \leq E\left[\left(X - \frac{1}{2}\right)^2\right] \leq \frac{1}{4} \text{ and the minimum is} \\ &\text{attained iff } \forall \omega, |X(\omega) - \frac{1}{2}| = \frac{1}{2}\end{aligned}$$

## exercise 2 :

Often in simulations  $\tilde{p} \sim B(\alpha, \beta)$  (beta law of parameters  $\alpha > 0$  and  $\beta > 0$ ) where the density is given by  $f_{\alpha, \beta}(x) \propto x^{\alpha-1}(1-x)^{\beta-1}1_{x \in ]0,1[}$

Show that:

a)  $E[\tilde{p}] = \frac{\alpha}{\alpha+\beta}$  noted  $(\bar{p})$

b)  $Var[\tilde{p}] = \frac{\bar{p}(1-\bar{p})}{\alpha+\beta+1}$

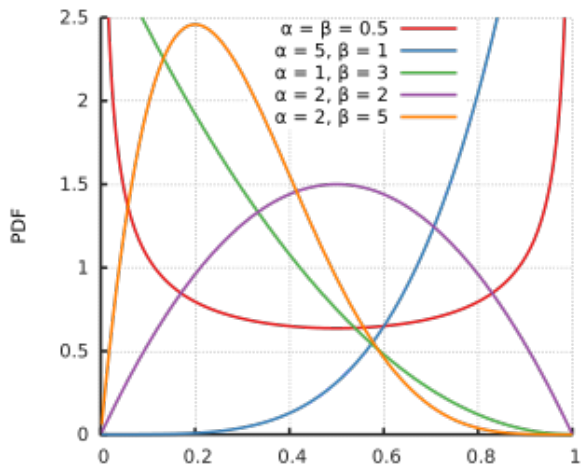
c) simulating with  $\tilde{p}$  we have  $\forall i \neq j, \rho(X_i, X_j) = \frac{1}{\alpha+\beta+1}$

d) show that  $\forall p, \rho \in ]0, 1[, \exists \alpha > 0, \beta > 0, \frac{\alpha}{\alpha+\beta} = \bar{p}$  and  $\frac{1}{\alpha+\beta+1} = \rho$

**Remarks:** The Beta law is quite useful for the simulation of correlated Bernoulli variables as it is possible to choose  $\alpha$  and  $\beta$  to obtain any possible probability of default and (positive) correlation wanted in the model.



# Beta distribution



Beta distribution - source wikipedia

## Theorem: Law of $\frac{D_n}{n}$

a)  $E\left(\frac{D_n}{n}\right) = \bar{p}$

b)  $Var\left(\frac{D_n}{n}\right) = \frac{\bar{p}(1-\bar{p})}{n} + \frac{n-1}{n} Var[\tilde{p}]$

c)  $\frac{D_n}{n} \longrightarrow \mathcal{L}(\tilde{p})$  (convergence in law)

so, in practice the probability that less than  $k$  bonds over  $n$  default is approximated by  $P(\tilde{p} < \frac{k}{n})$

### demonstration

a)  $E\left(\frac{D_n}{n}\right) = E\left(\frac{1}{n} \sum_{i=1}^{i=n} X_i\right) = \frac{1}{n} \sum_{i=1}^{i=n} E(X_i) = E[\tilde{p}]$

b)  $Var\left(\frac{D_n}{n}\right) = \frac{1}{n^2} \sum_{i=1}^{i=n} Var(X_i) + \frac{1}{n^2} \sum_{i \neq j} Cov(X_i, X_j)$   
 $= \frac{1}{n^2} \times n \times \bar{p}(1 - \bar{p}) + \frac{1}{n^2} \times n(n - 1) \times var[\tilde{p}]$

# Simulating Correlated Binomials

c) to show the convergence in law we show the convergence of the distribution functions

$$\begin{aligned}\lim_{n \rightarrow +\infty} P\left(\frac{D_n}{n} < t\right) &= \lim_{n \rightarrow +\infty} E(1_{\frac{D_n}{n} < t}) \\ &= \lim_{n \rightarrow +\infty} E(E(1_{\frac{D_n}{n} < t} | \tilde{p})) = E(E(\lim_{n \rightarrow +\infty} 1_{\frac{D_n}{n} < t} | \tilde{p}))\end{aligned}$$

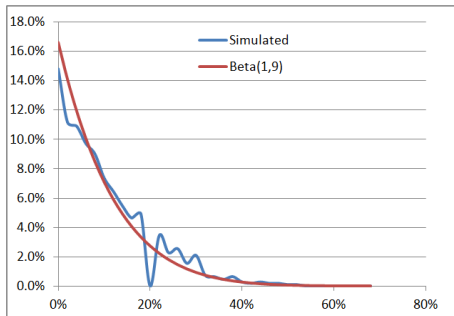
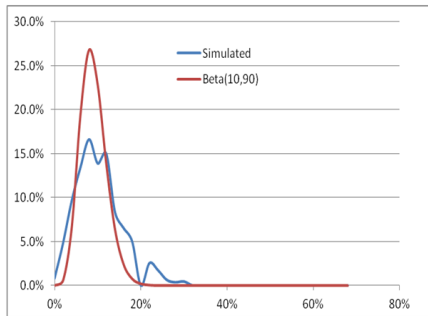
but when  $\tilde{p}$  is known  $\frac{D_n}{n} \rightarrow \tilde{p}$  almost surely. so

$$E(\lim_{n \rightarrow +\infty} 1_{\frac{D_n}{n} < t} | \tilde{p}) = 1_{\tilde{p} < t} \text{ so,}$$

$$\lim_{n \rightarrow +\infty} P\left(\frac{D_n}{n} < t\right) = E[1_{\tilde{p} < t}] = P(\tilde{p} < t) \text{ Q.E.D.}$$

**Remarks:** If the variables were not correlated in c) we would have convergence towards a single number, the mean, according to the Law of Large Numbers, instead of a convergence to a distribution

# Histograms for $\frac{D_n}{n}$ for a CDO of 50 Bonds (1000 simulations)



Histograms are plotted by joining the values obtained for each 2% bucket

$\bar{p} \sim \text{Beta}(10, 90) \implies E[\bar{p}] = 10\%$  and  $\text{Var}[\bar{p}] = 0.99\%$

$\bar{p} \sim \text{Beta}(1, 9) \implies E[\bar{p}] = 10\%$  and  $\text{Var}[\bar{p}] = 9.09\%$

## Beta Law for $\tilde{p}$ and CDO Pricing

**Example:** we consider a CDO made of 50 Bonds of equal Notional 100 each. We assume that the probabilities of default of the Bonds is 10% and note  $\rho$  the correlation of default between the bonds. We assume that the CDO has three tranches: Equity tranche (First 10% Loss), Junior Tranche (next 20% Loss), Senior Tranche (last 70% Loss). To calculate the price of the three tranches we use the approximation in Law  $\mathcal{L}(\frac{D_n}{n}) \sim \text{Beta}(\alpha, \beta)$ :

Table: Pricing as a function of  $\rho$

	i.i.d Bernouilli	Beta(10,90)	Beta(1,9)
$E[\tilde{p}]$	10%	10%	10%
$\rho$	0	0.99%	9.09%
Senior	100%	100%	99.45%
Junior	91.68%	89.85%	82.93%
Equity	16.64%	20.32%	38.08%

## Definition: Diversity Score (Moody's)

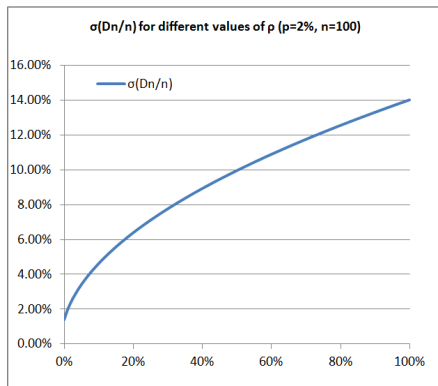
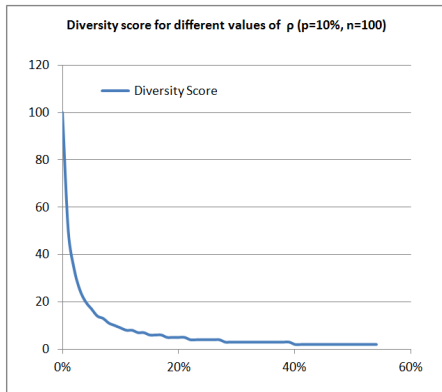
The Diversity Score is the number of uncorrelated bonds with the same probability of default  $\bar{p}$  for which the variance of the proportion of losses would be the closest to  $Var\left(\frac{D_n}{n}\right)$

**Remark:** The diversity score summarizes the real diversification effect created by Bonds which are correlated.

**Example:** for  $n$  bonds with probability of default  $p$  and correlation  $\rho$   
 $Var\left(\frac{D_n}{n}\right) = \frac{\bar{p}(1-\bar{p})}{n} + \frac{n-1}{n} Var[\tilde{p}]$  so we are searching for  $m$  such that  
 $\frac{\bar{p}(1-\bar{p})}{m} = \frac{\bar{p}(1-\bar{p})}{n} + \frac{n-1}{n} Var[\tilde{p}]$

N.A: for  $n = 100$ ,  $p = 2\%$  and  $\rho = 20\%$ ,  $\sigma_{20\%}\left(\frac{D_{100}}{100}\right) = 6.38\%$   
with 5 independent assets  $\sigma_{0\%}\left(\frac{D_5}{5}\right) = 6.26\%$  and with 4 independent assets  
 $\sigma_{0\%}\left(\frac{D_4}{4}\right) = 7.00\%$ . So we will take 5 as the Diversity Score.

# Diversity Score



Diversity Score and Standard Deviation for different values of the correlation

# Structural Models for $\tilde{p}$

We analyse here how Structural Models enable to create correlation in the modellisation (instead of creating it ex-nihilo in the model)

We make the following assumptions :

- bond  $i$  is in default at time  $T$  iif  $A_T^i < D^i$  where:  
$$dA_t^i = rA_t^i dt + \sigma^i A_t^i dW_t^i$$
- $\sigma^i$  is the same for all companies and is noted  $\sigma$
- the distance to default is the same for all companies and we note  
$$c = \frac{1}{\sigma\sqrt{T}} [\ln(\frac{D^i}{A_0^i}) - rT + \frac{\sigma^2}{2} T]$$
- we assume that the Brownian motions  $W_t^i$  verify  
$$dW_t^i = \rho dW_t + \sqrt{1 - \rho^2} dB_t^i$$
 where the  $B_t^i$  are brownian motions which are independent between them and independent from  $W_t$

## Remarks :

With the model  $\forall i \neq j, \rho(W_t^i, W_t^j) = \rho$ , and  $W_t$  is the common factor which creates correlation between the  $A_t^i$  and the default of the bonds.



## Proposition

Let  $Z_i$  be the Bernoulli random variable with value 1 if the company  $i$  defaults and 0 otherwise. Then  $Z_i = 1 \iff \frac{B_T^i}{\sqrt{T}} < \frac{c}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{W_T}{\sqrt{T}}$

**Demonstration** simple

**Remark 1** : Let  $\Phi$  be the repartition fonction of a normal law  $\mathcal{N}(0, 1)$ .

$$\frac{B_T^i}{\sqrt{T}} \sim \mathcal{N}(0, 1) \implies \Phi\left(\frac{B_T^i}{\sqrt{T}}\right) \sim \mathcal{U}(0, 1)$$

$$\text{so } X_i = 1 \iff \Phi\left(\frac{W_T^i}{\sqrt{T}}\right) < \Phi\left(\frac{c}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{W_T}{\sqrt{T}}\right)$$

so we end up simulating (as previously) correlated Bernoulli variables with the function  $\tilde{p}$  having a law  $\tilde{p} \sim \Phi\left(\frac{c}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{W_T}{\sqrt{T}}\right)$

# Structural Models for $\tilde{p}$

**Remark 2 :** We have different alternatives for  $\tilde{p}$  to generate correlated binomials:

- to use a beta distribution  $B(\alpha, \beta)$  (as seen previously)
- to use the distribution of  $\Phi(\alpha + \beta Z)$  (where  $Z \sim \mathcal{N}(0, 1)$ )

In both cases:

- first we solve for  $\alpha$  and  $\beta$  to match the desired value for  $\bar{p}$  and  $\rho$
- then to price the CDO we approximate the law of  $\frac{D_n}{n}$  by the law of  $\tilde{p}$

## Proposition

If  $\tilde{p} \sim \Phi(\alpha + \beta Z)$  (where  $Z \sim \mathcal{N}(0, 1)$ ) then

a)  $E[\tilde{p}] = \Phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right)$  (that we note also  $\bar{p}$ )

b)  $E[\tilde{p}^2] = \Phi_{2, \frac{\beta^2}{1+\beta^2}}\left(\frac{\alpha}{\sqrt{1+\beta^2}}, \frac{\alpha}{\sqrt{1+\beta^2}}\right)$

c)  $P(\tilde{p} < t) = \Phi\left(\frac{1}{\beta}[\Phi^{-1}(t) - \sqrt{1+\beta^2}\Phi^{-1}(\bar{p})]\right)$

# Structural Models for $\tilde{\rho}$

$$\begin{aligned} \text{a) } E[\tilde{\rho}] &= E[\Phi(\alpha + \beta Z)] = E[E(1_{Z_0 < \alpha + \beta Z} | Z)] \\ &\text{(with } Z_0 \sim \mathcal{N}(0, 1) \text{ independent from } Z) \\ &= E[1_{Z_0 - \beta Z < \alpha}] = E\left[1_{\frac{Z_0 - \beta Z}{\sqrt{1 + \beta^2}} < \frac{\alpha}{\sqrt{1 + \beta^2}}}\right] = \Phi\left(\frac{\alpha}{\sqrt{1 + \beta^2}}\right) \end{aligned}$$

$$\begin{aligned} \text{b) } E(\tilde{\rho}^2) &= E[\Phi(\alpha + \beta Z)^2] \\ &\text{and } \Phi(\alpha + \beta Z)^2 = E[1_{Z_0 < \alpha + \beta Z} 1_{Z_1 < \alpha + \beta Z} | Z] \\ &\text{(with } Z_0, Z_1, Z \text{ independent } \mathcal{N}(0, 1)) \\ \text{so, } E[\Phi(\alpha + \beta Z)^2] &= E[1_{Z_0 < \alpha + \beta Z} 1_{Z_1 < \alpha + \beta Z}] \\ &= E\left[1_{\frac{Z_0 - \beta Z}{\sqrt{1 + \beta^2}} < \frac{\alpha}{\sqrt{1 + \beta^2}}} 1_{\frac{Z_1 - \beta Z}{\sqrt{1 + \beta^2}} < \frac{\alpha}{\sqrt{1 + \beta^2}}}\right] \\ &= \Phi_{2, \frac{\beta^2}{1 + \beta^2}}\left(\frac{\alpha}{\sqrt{1 + \beta^2}}, \frac{\alpha}{\sqrt{1 + \beta^2}}\right) \end{aligned}$$

with  $\Phi_{2, \gamma}$  repartition function of a bivariate normal variable  $\mathcal{N}\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}$

## Calibration of the two Models for $\tilde{p}$

$$c) P(\tilde{p} < t) = P(\Phi(\alpha + \beta Z) < t) = \Phi\left(\frac{\Phi^{-1}(t) - \alpha}{\beta}\right)$$

as  $\Phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right) = \bar{p}$  we have  $\alpha = \Phi^{-1}(\bar{p})\sqrt{1+\beta^2}$  so

$$P(\tilde{p} < t) = \Phi\left(\frac{1}{\beta}[\Phi^{-1}(t) - \sqrt{1+\beta^2}\Phi^{-1}(\bar{p})]\right). \text{ Q.E.D.}$$

**Example** we consider a CDO with 100 Bonds of the same Notional and recovery rate of zero. The default of the bonds are modeled by Bernoulli variables  $X_i$  of parameter  $p$  and correlations  $\rho$ . We consider a junior tranche for the CDO which is exposed to the losses between above 10% and up to 30%. Price this junior tranche assuming  $\bar{p} = 2\%$  and  $\rho = 10\%$  with the two previous models:

a) assuming  $\tilde{p} \sim B(\alpha, \beta)$

b) assuming  $\tilde{p} \sim \Phi(\alpha + \beta Z)$  where  $Z \sim \mathcal{N}(0, 1)$

# Calibration of the two Models for $\tilde{p}$

## Results:

a) we solve and find  $\alpha = (\frac{1}{\rho} - 1)\bar{p}$  and  $\beta = (\frac{1}{\rho} - 1)(1 - \bar{p})$ . So, here  $\tilde{p} \sim B(0.18, 8.82)$ . Taking a risk free rate of zero we price the junior tranche in % of face value as  $\frac{1}{20} \sum_{i=11}^{i=30} P(\frac{D_{100}}{100} < \frac{i}{100})$  that we approximate by

$$\frac{1}{20} \sum_{i=11}^{i=30} P(\tilde{p} < \frac{i}{100}) = 98.12\%$$

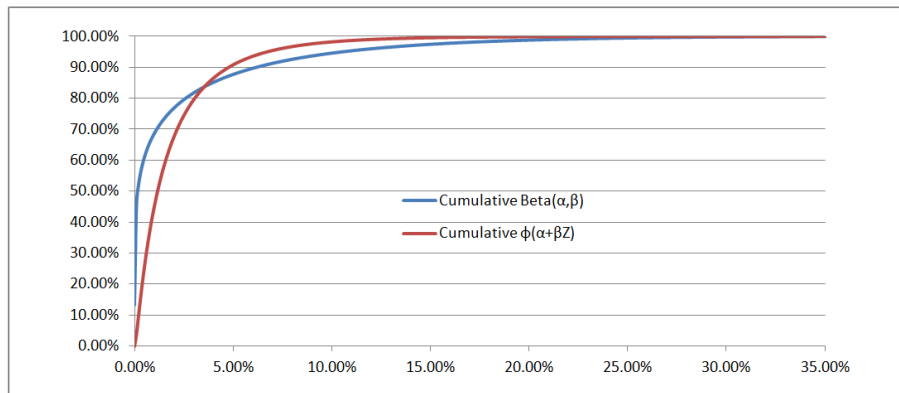
b) we solve  $\frac{\alpha}{\sqrt{1+\beta^2}} = -2.05375$  and (using a program to calculate the bivariate normal)  $\frac{\beta^2}{1+\beta^2} = 0.18$ . This implies  $\alpha = -2.2678$  and  $\beta^2 = 0.2195$ . The sign of  $\beta$  is not determined as both  $Z$  and  $-Z$  are  $\mathcal{N}(0, 1)$ , we will take  $\beta = 0.468521$ . Now,

$$\frac{1}{20} \sum_{i=11}^{i=30} P(\Phi(\alpha + \beta Z) < \frac{i}{100}) = \frac{1}{20} \sum_{i=11}^{i=30} \Phi(\frac{1}{\beta}[\Phi^{-1}(\frac{i}{100}) - \alpha]) = 99.66\%$$

## Remarks:

- the pricings for a) and b) are not exactly the same as the two laws used for  $\tilde{p}$  produce the same expectations and correlations between the default events (the Bernoulli variables) but not exactly the same joint distributions. Also they do not generate the same laws for  $\frac{D_n}{n}$  and therefore not the same pricing.
- the fact that the two laws generated for  $\frac{D_n}{n}$  are different is also put in evidence by the fact that (as it has been demonstrated previously)  $\frac{D_n}{n}$  converge here towards two different distributions which are the two distinct laws of  $\tilde{p}$  that we use.
- the choice of the distribution  $\tilde{p}$  used to create the correlation structure is therefore important and it is exactly the aim of the study of copulas to create adequate correlation structures. Transformations of normal variables  $\Phi(\alpha + \beta Z)$  to create correlation structures have been criticised after 2008.

# Calibration of the two Models for $\tilde{p}$



Modelizing  $\tilde{p}$  with Beta( $\alpha, \beta$ ) or  $\Phi(\alpha + \beta Z)$

## Proposition

We assume here that:

- the percentage lost for a bond which defaults (i.e  $1 - R$ ) is  $f(\tilde{p})$
- $f(p)$  is an increasing function of  $p$

If we note  $L_n^f = \frac{1}{n} \sum_{i=1}^{i=n} f(\tilde{p}) 1_{Z_i < \tilde{p}}$  the loss in percentage for the CDO  
we have :  $L_n^f \rightarrow \mathcal{L}(\tilde{p}f(\tilde{p}))$  (convergence in law).

## Demonstration :

to show the convergence in law we show the convergence of the distribution function



# Generalization to None Zero Recovery Rate

$$\begin{aligned}\lim_{n \rightarrow +\infty} P(L_n^f < t) &= \lim_{n \rightarrow +\infty} E(1_{L_n^f < t}) = \lim_{n \rightarrow +\infty} E(E(1_{L_n^f < t} | \tilde{\rho})) \\ &= E(E(\lim_{n \rightarrow +\infty} 1_{L_n^f < t} | \tilde{\rho}))\end{aligned}$$

when  $\tilde{\rho}$  is known then according to the law of large numbers

$$L_n^f \rightarrow E[f(\tilde{\rho})1_{Z_i < \tilde{\rho}}] = f(\tilde{\rho})\tilde{\rho} \text{ and so } 1_{L_n^f < t} \rightarrow 1_{f(\tilde{\rho})\tilde{\rho} < t}$$

from there  $\lim_{n \rightarrow +\infty} P(L_n^f < t) = E(E(1_{\tilde{\rho}f(\tilde{\rho}) < t} | \tilde{\rho})) = P(\tilde{\rho}f(\tilde{\rho}) < t)$  Q.E.D.

**Remarks:** if  $R = 0$  then  $f(\tilde{\rho})$  is always 1 and we find the result we already demonstrated that  $\mathcal{L}(\tilde{\rho}f(\tilde{\rho})) \sim \mathcal{L}(\tilde{\rho})$

**Remark:** In the Cox model a default occurs for company  $i$  iff

$$\mathcal{E}_i < \int_0^t \lambda(X_s) ds$$

where the  $\mathcal{E}_i$  are independent and independent from  $X$ .

Let  $F(t) = P(\mathcal{E}_i \leq t) = 1 - \exp(-t)$ . Then the  $Z_i = F^{-1}(\mathcal{E}_i)$  are independent  $\mathcal{U}([0, 1])$  and a default occurs for company  $i$  iff

$$Z_i < F^{-1}\left(\int_0^t \lambda(X_s) ds\right)$$

$$\iff Z_i < \tilde{p}$$

where  $\tilde{p} = 1 - \exp(-\int_0^t \lambda(X_s) ds)$ .

## Definition: Infection Models

Let  $(Z_i)_{i \in \llbracket 1, n \rrbracket}$  and  $(Y_{i,j})_{i \neq j \in \llbracket 1, n \rrbracket}$  be independent variables, we assume  $Z_i \sim \mathcal{B}(p)$  and  $\forall i \neq j, Y_{i,j} \sim \mathcal{B}(q)$

Then we define in a contagion model the variables  $(X_i)_{i \in \llbracket 1, n \rrbracket}$  by :

$$X_i = Z_i + (1 - Z_i) \left[ 1 - \prod_{j \neq i} (1 - Z_j Y_{j,i}) \right]$$

### Remark:

The only possible values for  $X_i$  are 1 and 0.

$X_i = 1 \iff Z_i = 1$  or  $\exists i \neq j, Z_j = 1$  and  $Y_{j,i} = 1$  (i.e contamination)

We are now going to study the law of the  $X_i$  and their correlations.

## Proposition

$$X_i \sim \mathcal{B}(1 - (1 - p)(1 - pq)^{n-1})$$

**Demonstration :** because of independence

$$E(X_1) = E(Z_1) + (1 - E(Z_1))\left[1 - \prod_{j \neq 1} (1 - E(Z_j)E(Y_{j,1}))\right]$$

$$\begin{aligned} &= p + (1 - p)[1 - (1 - pq)^{n-1}] \\ &= p + 1 - p - (1 - p)(1 - pq)^{n-1} \\ &= 1 - (1 - p)(1 - pq)^{n-1} \text{ Q.E.D.} \end{aligned}$$

**Remarks :**

$$\mathcal{L}(X_1) \xrightarrow[n \rightarrow +\infty]{} 1$$

## Proposition

$$E[X_1 X_2] = 1 - 2(1 - p)(1 - pq)^{n-1} + (1 - p)^2(1 - 2pq + pq^2)^{n-2}$$

## Demonstration :

$$E[X_1 X_2] \\ = E \left[ \left( Z_1 + (1 - Z_1) \left[ 1 - \prod_{j \neq 1} (1 - Z_j Y_{j,1}) \right] \right) \left( Z_2 + (1 - Z_2) \left[ 1 - \prod_{j \neq 2} (1 - Z_j Y_{j,2}) \right] \right) \right]$$

we have 3 different type of terms:

a)  $E[Z_1 Z_2] = p^2$  (because  $Z_1$  and  $Z_2$  are independent)

b)  $E \left( Z_1 (1 - Z_2) \left[ 1 - \prod_{j \neq 2} (1 - Z_j Y_{j,2}) \right] \right)$  (this value will appear two times)

$$= E \left( Z_1 (1 - Z_2) \right) - E \left( Z_1 (1 - Z_1 Y_{1,2}) (1 - Z_2) \prod_{j \notin \{1,2\}} (1 - Z_j Y_{j,2}) \right)$$

$$= p(1 - p) - (p - pq)(1 - p)(1 - pq)^{n-2}$$

$$= p(1 - p) [1 - (1 - q)(1 - pq)^{n-2}]$$

$$c) E \left[ (1 - Z_1) (1 - Z_2) \left[ 1 - \prod_{j \neq 1} (1 - Z_j Y_{j,1}) \right] \left[ 1 - \prod_{j \neq 2} (1 - Z_j Y_{j,2}) \right] \right]$$

multiplying first the two terms on the right we get 3 different type of terms:

$$\circ E[(1 - Z_1)(1 - Z_2)] = (1 - p)^2$$

# Infection Models

$$\begin{aligned} & \circ -E\left[(1 - Z_1)(1 - Z_2) \prod_{j \neq 2} (1 - Z_j Y_{j,2})\right] \text{ (this value will appear two times)} \\ &= -E\left[(1 - Z_1)(1 - Z_1 Y_{1,2})(1 - Z_2) \prod_{j \notin \{1,2\}} (1 - Z_j Y_{j,2})\right] \\ &= -(1 - pq - p + pq)(1 - p)(1 - pq)^{n-2} \\ &= -(1 - p)^2(1 - pq)^{n-2} \\ & \circ E\left[(1 - Z_1)(1 - Z_2) \prod_{j \neq 1} (1 - Z_j Y_{j,1}) \prod_{j \neq 2} (1 - Z_j Y_{j,2})\right] \\ &= E\left[(1 - Z_1)(1 - Z_1 Y_{1,2})(1 - Z_2)(1 - Z_2 Y_{2,1}) \prod_{j \notin \{1,2\}} [(1 - Z_j Y_{j,1})(1 - Z_j Y_{j,2})]\right] \\ &= (1 - pq - p + pq)^2(1 - pq - pq + pq^2)^{n-2} \\ &= (1 - p)^2(1 - 2pq + pq^2)^{n-2} \end{aligned}$$

so at the end we obtain:

$$\begin{aligned} & p^2 + 2p(1 - p)[1 - (1 - q)(1 - pq)^{n-2}] + (1 - p)^2 \\ & - 2(1 - p)^2(1 - pq)^{n-2} + (1 - p)^2(1 - 2pq + pq^2)^{n-2} \\ &= 1 - 2(1 - p)(1 - pq)^{n-1} + (1 - p)^2(1 - 2pq + pq^2)^{n-2} \text{ Q.E.D.} \end{aligned}$$

**exercise :**

Let  $D_n = \sum_{i=1}^{i=n} X_i$ , calculate as a function of  $p$  and  $q$  :

a)  $E[D_n]$  and

b)  $Var[D_n]$

**Hint :**

a)  $E[D_n] = \sum_{i=1}^{i=n} E[X_i] = nE[X_1]$

b)  $Var[D_n] = E\left[\left(\sum_{i=1}^{i=n} X_i\right)^2\right] - (E[D_n])^2 = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} E[X_i X_j] - (E[D_n])^2$   
 $= n(n-1)E[X_1 X_2] + nE[X_1] - n^2 E[X_1]^2$

So we know how to calculate the first two moments of  $D_n$  as a function of  $p$  and  $q$  but in fact we can also calculate the law of  $D_n$

## Proposition :

$\forall k \in \llbracket 1, n \rrbracket,$

$$P(D_n = k) = C_n^k \sum_{i=1}^{i=k} C_k^i p^i (1-p)^{(n-i)} (1-q)^{i(n-k)} [1 - (1-q)^i]^{(k-i)}$$

## Demonstration :

$P(D_n = k) = C_n^k P(X_1 = 1, X_2 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0)$

and  $\{X_1 = 1, X_2 = 1, \dots, X_k = 1\}$  can be decomposed in  $k - 1$  cases depending on the number  $i$  of "direct" defaults. So

$P(X_1 = 1, X_2 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0)$

$$= \sum_{i=1}^{i=k} C_k^i P(Z_1 = 1, Z_2 = 1, \dots, Z_i = 1, (Z_{i+1} = 0, X_{i+1} = 1), \dots, (Z_k = 0, X_k = 1), X_{k+1} = 0, \dots, X_n = 0)$$

we can write each event as the intersection of three events



- $\{Z_1 = 1, Z_2 = 1, \dots, Z_i = 1, Z_{i+1} = 0, \dots, Z_n = 0\}$
- $\{\exists j \in \llbracket 1, i \rrbracket, Y_{j,i+1} = 1, \dots, \exists j \in \llbracket 1, i \rrbracket, Y_{j,k} = 1\}$
- $\{\forall j \in \llbracket 1, i \rrbracket, Y_{j,k+1} = 0, \dots, \forall j \in \llbracket 1, i \rrbracket, Y_{j,n} = 0\}$

the three events are independent.

- the probability of the first one is  $p^i(1-p)^{(n-i)}$
- the probability of the second one is  $[1 - (1-q)^i]^{(k-i)}$
- the probability of the third one is  $(1-q)^{i(n-k)}$  Q.E.D.

## Example :

Table: Infection Models for  $n = 30$

$p = P(Z_i = 1)$	1%	1%	1%	1%	1%
$q$	0%	10%	20%	50%	100%
$p^* = P(X_i = 1)$	1%	3.83%	6.58%	14.39%	26.03%
Correlation	0%	12%	21%	50%	100%
Diversity Score	30	6.7	4.1	2	1

Remarks : if  $q = 100\%$

$$P(X_i = 1) = 1 - P(X_i = 0) = 1 - (P(Z_1 = 0))^{30} = 1 - (1 - p)^{30} = 26.03\%$$

# Copulas

## Definition : Copulas

$C : [0, 1]^d \rightarrow [0, 1]$  is a copula iff  $C$  is a multivariate cumulative distribution for a random vector of  $[0, 1]^d$  i.e

$\exists (U_1, U_2, \dots, U_d)$  r.v  $(\Omega, P) \rightarrow [0, 1]^d$  such that:

- $\forall i \in \llbracket 1, d \rrbracket, U_i \sim \mathcal{U}([0, 1])$
- $C(u_1, u_2, \dots, u_d) = P(U_1 \leq u_1, U_2 \leq u_2, \dots, U_d \leq u_d)$

**Notation** : we note

$F_U(u_1, u_2, \dots, u_d) = P(U_1 \leq u_1, U_2 \leq u_2, \dots, U_d \leq u_d)$  the multidimensional cumulative distribution function of  $U$ .

By definition for any copula  $C$  there is  $U = (U_1, U_2, \dots, U_d)$  with  $U_i \sim \mathcal{U}([0, 1])$  such that  $C = F_U$

**Exemples** : Let  $U = (U_1, U_2)$  where  $U_1$  and  $U_2 \sim \mathcal{U}([0, 1])$

a) if  $U_1$  and  $U_2$  are independent then  $F_U(u, v) = uv$

b) if  $U_2 = U_1$  then  $F_U(u, v) = \min(u, v)$

c) if  $U_2 = 1 - U_1$  then  $F_U(u, v) = \max(u + v - 1, 0)$

**Demonstration** : Let's show c)

$$\begin{aligned} P(U_1 \leq u, 1 - U_1 \leq v) &= P(U_1 \leq u, U_1 \geq 1 - v) = P(1 - v \leq U_1 \leq u) \\ &= \max(u + v - 1, 0) \text{ Q.E.D.} \end{aligned}$$

## Theorem : Frchet-Hoeffding Bounds

Let  $U = (U_1, U_2)$  be a r.v with  $U_i \sim \mathcal{U}([0, 1])$  then

$$\forall u, v \in [0, 1], \max(u + v - 1, 0) \leq F_U(u, v) \leq \min(u, v)$$

so the cases  $U_2 = U_1$  and  $U_2 = 1 - U_1$  represents the two extreme "correlation-structures".

## Demonstration :

$P(U_1 \leq u, U_2 \leq v) \leq P(U_1 \leq u)$  and  $P(U_1 \leq u, U_2 \leq v) \leq P(U_2 \leq v)$

implies  $P(U_1 \leq u, U_2 \leq v) \leq \min(P(U_1 \leq u), P(U_2 \leq v))$

$P(\{U_1 \leq u\} \cup \{U_2 \leq v\}) = P(U_1 \leq u) + P(U_2 \leq v) - P(U_1 \leq u, U_2 \leq v)$

implies  $P(U_1 \leq u, U_2 \leq v) \geq P(U_1 \leq u) + P(U_2 \leq v) - 1$  Q.E.D.

## Definition : Quantile (or Pseudo-Inverse)

We define  $F_X^+ : [0, 1] \rightarrow \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$  by

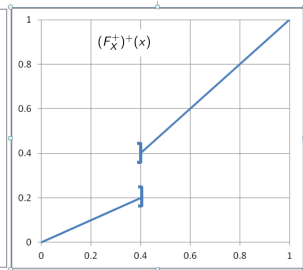
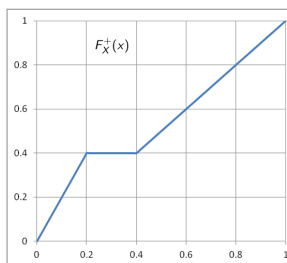
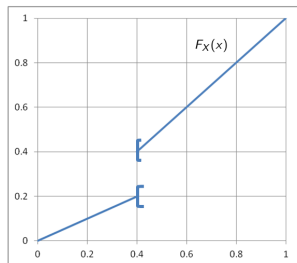
$$F_X^+(y) = \inf_{x \in \mathbb{R}} \{P(X \leq x) \geq y\}$$

## Definition

$f$  is strictly increasing at  $x$  iff  $\forall x_1 < x < x_2, f(x_1) < f(x) < f(x_2)$

## General Properties

- $F_X$  is increasing and right-continuous
- $F_X^+$  is increasing and left continuous
- $F_X$  is continuous at  $x \iff P(X = x) = 0$
- $F_X(x) \geq y \iff x \geq F_X^+(y)$
- $F_X^+(F_X(x)) \leq x$
- $F_X(F_X^+(y)) \geq y$
- $F_X^+$  continuous at  $F_X(x) \iff F_X$  strictly increasing at  $x$
- $F_X^+$  continuous at  $F_X(x) \iff F_X^+(F_X(x)) = x$
- $F_X$  continuous at  $F_X^+(y) \iff F_X^+$  strictly increasing at  $y$
- $F_X$  continuous at  $F_X^+(y) \iff F_X(F_X^+(y)) = y$
- $F_X$  and  $F_X^+$  continuous  $\iff F_X$  invertible and  $F_X^{-1} = F_X^+$



Calculation of the Pseudo Inverse



**Demonstration :** Let as an exercise

## Proposition and Definition. Copula $C_X$ of a Random Vector

If  $X$  r.v taking values  $\in \mathbb{R}$

a)  $F_X$  continuous  $\implies F_X(X) \sim \mathcal{U}([0, 1])$

if  $X = (X_1, \dots, X_d)$  with cdfs  $F_{X_1}, \dots, F_{X_d}$  continuous and  $U = (F_{X_1}(X_1), \dots, F_{X_d}(X_d))$  then

b)  $\forall i \in \llbracket 1, d \rrbracket, F_{X_i}(X_i) \sim \mathcal{U}([0, 1])$

We call copula of  $X$  and note  $C_X$  the function  $F_U$   
(that we can also note  $C_U$ )

## Demonstration:

a) let  $y \in ]0, 1[$

$$P(F_X(X) < y) = 1 - P(F_X(X) \geq y)$$

$$= 1 - P(X \geq F_X^+(y)) \text{ (according to the general properties)}$$

$$= P(X < F_X^+(y)) = P(X \leq F_X^+(y)) \text{ (because } F_X \text{ is continuous)}$$

$$= F_X(F_X^+(y)) = y \text{ (according to the proposition as } F_X \text{ is continuous)}$$

so  $F_X(X) \sim \mathcal{U}([0, 1])$  Q.E.D.

b) direct consequence of a)

## Remarks :

if  $U = (U_1, U_2, \dots, U_d)$  with  $U_i \sim \mathcal{U}([0, 1])$  then  $C_U = F_U$ .

Sklar's theorem enable to build a random vector with given continuous cdf marginals and copula.

## Sklar's Theorem: Multivariate with given Marginals and Copula

Let  $U = (U_1, U_2, \dots, U_d)$  with  $U_i \sim \mathcal{U}([0, 1])$

Let  $F_1, F_2, \dots, F_d$  be continuous cumulative distribution functions.

Let  $X = (F_1^+(U_1), F_2^+(U_2), \dots, F_d^+(U_d))$

Then,

- $F_{X_i} = F_i$  and
- $C_X = C_U$

## Demonstration :

$$\begin{aligned} \text{a) } P(X_i \leq x) &= P(F_i^+(U) \leq x) \\ &= P(U \leq F_i(x)) \text{ (according to the general properties)} \\ &= F_i(x) \text{ Q.E.D.} \end{aligned}$$

$$\begin{aligned} \text{b) } C_X(u_1, u_2, \dots, u_d) &= P(F_1(X_1) \leq u_1, \dots, F_d(X_d) \leq u_d) \text{ (by definition)} \\ &= P((F_1 \circ F_1^+)(U_1) \leq u_1, \dots, (F_d \circ F_d^+)(U_d) \leq u_d) \\ &\text{but } F_i \text{ continuous} \implies F_i \circ F_i^+ = Id \text{ (according to the general properties)} \\ &\text{so,} \\ &= P(U_1 \leq u_1, \dots, U_d \leq u_d) \\ &= C_U(u_1, \dots, u_d) \text{ Q.E.D.} \end{aligned}$$

**Remark 1** : According to Sklar's theorem:

- for any copula  $C$  and
- for any continuous cdfs  $(F_i)_{i \in \llbracket 1, d \rrbracket}$

we can find a multivariate random variable  $X$  such that:

- the  $F_i$  are the marginal cdfs of  $X$
- $C_X = C$

we will have  $F_X(x_1, x_2, \dots, x_d) = C(F_1(x_1), F_2(x_2), \dots, F_d(x_d))$

**Remark 2** : It is easy to simulate a Gaussian vector  $Z$  with a given correlation matrix and therefore easy to simulate variables  $U$  with marginals  $\mathcal{U}([0, 1])$  and with  $C_U = C_Z$  by calculating for each value of  $Z$  the vector  $U = (F_{Z_1}(Z_1), \dots, F_{Z_d}(Z_d))$

**Exercise:** Let  $Z$  be a Gaussian vector with a given correlation matrix and  $C_Z$  be its copula. Let  $F_1, \dots, F_d$  be continuous cdfs.

Show that we can simulate a r.v  $X$  of copula  $C_Z$  and marginals  $F_i$  by

- simulating  $Z$  and
- calculating for each value of  $Z$  the vector
$$X = ((F_1^+ \circ F_{Z_1})(Z_1), \dots, (F_d^+ \circ F_{Z_d})(Z_d))$$

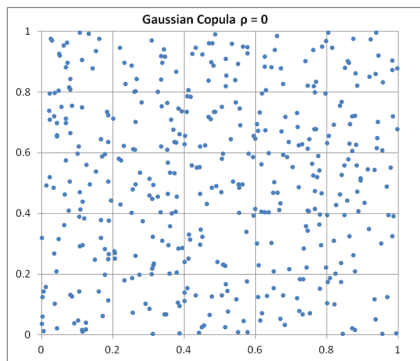
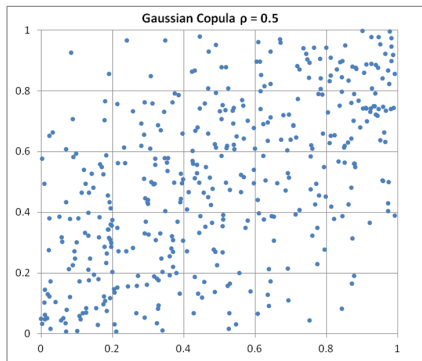
**Solution:**

$F_{Z_i}(Z_i) \sim \mathcal{U}([0, 1]) \implies F_i^+(F_{Z_i}(Z_i))$  has for cdf  $F_i$  (according to Sklar's theorem). So, we get the right marginals for  $X$

$$\begin{aligned} C_X(u_1, \dots, u_d) &= P(F_1(X_1) \leq u_1, \dots, F_d(X_d) \leq u_d) \\ &= P(F_1 \circ (F_1^+ \circ F_{Z_1})(Z_1) \leq u_1, \dots, F_d \circ (F_d^+ \circ F_{Z_d})(Z_d) \leq u_d) \\ &= P(F_{Z_1}(Z_1) \leq u_1, \dots, F_{Z_d}(Z_d) \leq u_d) = C_Z(u_1, \dots, u_d) \end{aligned}$$

so we get the right copula for  $X$ . Q.E.D.

# Copula



Simulations Gaussian Copula for various values of  $\rho$

## Remarks:

In the SRM model  $X_i = 1 \iff \frac{B_T^i}{\sqrt{T}} < \frac{c}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{W_T}{\sqrt{T}}$

but  $Z^i < \alpha + \beta Z \iff \Phi\left(\frac{Z^i - \beta Z}{\sqrt{1+\beta^2}}\right) < \Phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right)$

Let

$U = (U_1, \dots, U_d)$  with  $U_i = \Phi\left(\frac{Z^i - \beta Z}{\sqrt{1+\beta^2}}\right)$  and

$G = \left(\frac{Z^1 - \beta Z}{\sqrt{1+\beta^2}}, \dots, \frac{Z^d - \beta Z}{\sqrt{1+\beta^2}}\right)$  then

$U_i \sim \mathcal{N}(0, 1)$  and  $C_U = C_G$  and therefore  $C_U$  is a Gaussian Copula

The fact that the defaults  $(U_i < \Phi\left(\frac{\alpha}{\sqrt{1+\beta^2}}\right))$  are correlated implies, when simulating  $U$ , the formation of a cluster of points (density of points higher than the average) in the region near 0 as the probability for this event is higher than what would be expected for a product of the marginals.



## Proposition : Invariance Properties of the Copulas

Let  $X = (X_1, X_2, \dots, X_d)$  with continuous marginal cdfs  $F_i$

Let  $T_1, T_2, \dots, T_d$  be strictly increasing real functions

Let  $Y = (Y_1, Y_2, \dots, Y_d)$  with  $Y_i = T_i(X_i)$

then  $C_Y = C_X$

So, the Copula, which measures the association between the variables, is invariant by change of variables under strictly increasing functions (which is not the case for the linear correlation).

**Demonstration :**  $C_Y(u_1, \dots, u_d) = P(F_{Y_1}(Y_1) \leq u_1, \dots, F_{Y_d}(Y_d) \leq u_d)$

but  $F_{Y_i}(y) = P(Y_i \leq y) = P(T_i(X_i) \leq y)$  so

$F_{Y_i}(T_i(x)) = P(T_i(X_i) \leq T_i(x)) = P(X_i \leq x)$  (as  $T_i$  is strictly increasing)

so,  $F_{Y_i}(T_i(x)) = F_{X_i}(x)$  and in particular  $F_{Y_i}(Y_i) = F_{Y_i}(T_i(X_i)) = F_{X_i}(X_i)$

so,

$C_Y(u_1, \dots, u_d) = P(F_{X_1}(X_1) \leq u_1, \dots, F_{X_d}(X_d) \leq u_d) = C_X(u_1, \dots, u_d).$

## Proposition : Copulas for Normalized Gaussian Vectors

Let  $X$  be a "normalized" Gaussian vector  $\mathcal{N}(0, \Sigma)$  with components  $X_i \sim \mathcal{N}(0, 1)$  and correlation matrix  $\Sigma$  invertible.

Let  $C_X$  the copula of  $X$  and  $c_X$  its density. Then:

- $C_X(x) = F_X(\Phi^{-1}(x_1), \Phi^{-1}(x_2), \dots, \Phi^{-1}(x_d))$
- $c_X(x) = \frac{1}{|R|^{\frac{1}{2}}} \exp(-\frac{1}{2}x'(\Sigma^{-1} - I_d)x)$

where  $\Phi$  is the cdf of a  $\mathcal{N}(0, 1)$

### Demonstration :

$$\begin{aligned} F_X(x_1, x_2, \dots, x_d) &= P(X_1 \leq x_1, \dots, X_d \leq x_d) \\ &= P(\Phi(X_1) \leq \Phi(x_1), \dots, \Phi(X_d) \leq \Phi(x_d)) \text{ (as } \Phi \text{ is strictly increasing)} \\ &= C_X(\Phi(x_1), \dots, \Phi(x_d)) \text{ so,} \end{aligned}$$

$$C_X(x_1, x_2, \dots, x_d) = F_X(\Phi^{-1}(x_1), \Phi^{-1}(x_2), \dots, \Phi^{-1}(x_d)) = \text{Q.E.D.}$$

Applying  $\frac{\partial}{\partial x_1 \partial x_2 \dots \partial x_d}$  to  $F_X(x_1, x_2, \dots, x_d) = C_X(\Phi(x_1), \Phi(x_2), \dots, \Phi(x_d))$  we get

$$f_X(x_1, x_2, \dots, x_d) = c_X(\Phi(x_1), \Phi(x_2), \dots, \Phi(x_d))\phi(x_1)\phi(x_2)\cdots\phi(x_d)$$

the density  $f_X(x)$  equals  $(\frac{1}{\sqrt{2\pi}})^d \exp(-\frac{1}{2}x'\Sigma^{-1}x)$  and

$$\phi(x_1)\phi(x_2)\cdots\phi(x_d) = (\frac{1}{\sqrt{2\pi}})^d \exp(-\frac{1}{2}x'x) \text{ Q.E.D.}$$

## Proposition

The Copula of a Gaussian vector depends only on its correlation matrix  $\Lambda$

**Demonstration** : if  $X$  is a Gaussian vector of correlation matrix  $\Sigma$  we know (from the invariance property) that the normalized Gaussian vector  $Y$  where  $Y_i = T_i(X_i) = \frac{X_i - \mu_i}{\sigma_i}$  has the same copula as  $X$  and that  $Y \sim \mathcal{N}(0, \Lambda)$  where  $\Lambda$  is the correlation matrix of  $X$ . Q.E.D.

**Remark:** We can create a correlation structure on  $d$  binomial variables  $X_i \sim \mathcal{B}(p_i)$  by choosing a copula  $C$  and  $(\alpha_1, \alpha_2, \dots, \alpha_d)$  such that:

$\forall i \in \llbracket 1, d \rrbracket, C(1, \dots, \alpha_i, 1 \dots) = p_i$

**Example:** Here  $d = 3$  and  $p_1 = 1\%$ ,  $p_2 = 2\%$  and  $p_3 = 3\%$ .

$Z = (Z_1, Z_2, Z_3)$  is a Gaussian vector with correlation  $\rho = 50\%$  between two variables and we assume that  $X_i = 1$  ( $i$  defaults)  $\iff Z_i \leq \alpha_i$

Solving  $P(Z_i \leq \alpha_i) = p_i$  we find:  $\alpha_1 = -2.326$   $\alpha_2 = -2.054$   $\alpha_3 = -1.881$

Here we do not try to calibrate a correlation matrix for  $Z$  to match some input correlations between the  $X_i$  but calculate the correlations between the defaults induced by the correlation matrix of  $Z$ .

Here we get  $E(X_1 X_2) - E(X_1)E(X_2) = P(Z_1 < \alpha_1, Z_2 < \alpha_2) - p_1 p_2$  and

$$\rho(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sigma(X_1)\sigma(X_2)} = 13.32\%$$

and in the same way  $\rho(X_1, X_3) = 13.89\%$  and  $\rho(X_2, X_3) = 16.16\%$ .

To simulate  $Z$  we simulate  $X$  and then calculate  $(1_{X_1 < \alpha_1}, 1_{X_2 < \alpha_2}, 1_{X_3 < \alpha_3})$

## Rosenblatt's Theorem

Let  $X = (X_1, X_2, \dots, X_d)$  be a random vector of  $\mathbb{R}^d$   
we assume that the law of  $X$  has a density  $f_X(x_1, x_2, \dots, x_d)$  strictly positive

Let  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be defined by  $T(x) = y$  with

$$y_1 = P(X_1 \leq x_1)$$

$$y_2 = P(X_2 \leq x_2 | X_1 = x_1), \dots$$

$$y_d = P(X_d \leq x_d | X_1 = x_1, X_2 = x_2, \dots, X_{d-1} = x_{d-1})$$

Then,  $T(Y) \sim \mathcal{U}([0, 1]^d)$

## Demonstration:

Let  $h$  be a measurable function from  $\mathbb{R}^d$  to  $\mathbb{R}$

$$E[h(Y)] = E[h(T(X))] = \int_{\mathbb{R}^d} h(T(x))f_X(x)dx$$

We consider the change of variable  $y = T(x)$ .

The Jacobian matrix  $[\frac{dy}{dx}]$  is triangular and the diagonal elements are:

- $\frac{\partial}{\partial x_1} P(X_1 \leq x_1) = f_{X_1}(x_1)$
- $\frac{\partial}{\partial x_2} P(X_2 \leq x_2 | X_1 = x_1) = f_{X_2|X_1=x_1}(x_2) \cdots$
- $\frac{\partial}{\partial x_d} P(X_d \leq x_d | X_{d-1} = x_{d-1} \cdots, X_1 = x_1) = f_{X_d|(X_{d-1}=x_{d-1}\cdots)}(x_d)$

so, the determinant of the Jacobian Matrix equals  $f_X(x_1, x_2, \cdots, x_d)$

so after the change of variable :  $E[h(Y)] = \int_{T(\mathbb{R}^d)} h(y)dy$

as the  $y_i$  are probabilities  $T(\mathbb{R}^d) \subset [0, 1]^d$  and by mass conservation

$$T(\mathbb{R}^d) = [0, 1]^d$$

so  $\forall h, E[h(Y)] = \int_{[0,1]^d} h(y)dy \implies Y \sim \mathcal{U}([0, 1]^d)$  Q.E.D.

In some situations a Copula  $C$  is defined analytically as any function satisfying the properties of a cdf of a variable taking its values in  $[0, 1]^d$  and whose marginals are  $\mathcal{U}([0, 1])$

## Exemples of Copula :

- Clayton  $C(u, v) = \max(u^{-\theta} + v^{-\theta} - 1, 0)^{-\frac{1}{\theta}}$  with  $\theta > -1$  and  $\theta \neq 0$
- Gumbel-Hougaard  $C(u) = \exp\left(\left[-\sum_{i=1}^{i=d} (-\ln(u_i))^\theta\right]^{\frac{1}{\theta}}\right)$   
with  $\theta > 1$  and the conventions  $\ln(0) = -\infty$  and  $\exp(-\infty) = 0$
- Archimedean  $C(u) = \psi\left(\sum_{i=1}^{i=d} \psi^{-1}(u_i)\right)$   
with  $\psi : [0, \infty] \rightarrow [0, 1]$  satisfying (among other things)  
 $\psi(0) = 1$  and  $\psi(\infty) = 0$

## Remarks

- there are some conditions  $C$  must satisfy to be a copula. For a two dimension copula we need at least  $C(u, 0) = C(0, v) = 0$  and  $C(u, 1) = u$  and  $C(1, v) = v$
- there are some conditions the function  $\psi$  must satisfy for the Archimedean expression to be a copula
- the Clayton and Gumbel-Hougaard copulas are two particular cases of Archimedean copulas. For the Clayton Copula  $\psi(\theta) = \frac{1}{\theta}(u^{-\theta} - 1)$



## Theorem (admitted)

If  $Z$  is a random variable with  $Z > 0$  and if

$\psi_Z(s) = E[\exp(-sZ)]$  for  $s \in [0, \infty]$  (Laplace transform)

Then,

the Archimedean function defined by  $\psi_Z$  is a copula

**exercise:**

Show that if  $Z \sim \text{Gamma}(\frac{1}{\theta}, 1)$  with  $0 < \theta < +\infty$  then the Archimedean Copula generated by  $\psi_Z$  is the Clayton Copula of parameter  $\theta$

**Solution:**

$$\psi_Z(s) = \int_0^{+\infty} \frac{z^{\frac{1}{\theta}-1} e^{-z}}{\Gamma(\frac{1}{\theta})} e^{-sz} dz = (1+z)^{-\frac{1}{\theta}}$$

**Remarks:** Archimedean Copulas can be created "on demand" by calculating the Laplace transform of any arbitrary random variable  $Z$

Even if the simulation of Gaussian Copulas is easy it may be more complicated to simulate arbitrary copulas.

The Rosenblatt Theorem provides an easy way to simulate copulas in dimension 2.

## Proposition

let  $U = (U_1, U_2)$  of copula  $C$  with  $U_i \sim \mathcal{U}([0, 1])$

If  $T$  is the Rosenblatt's transformation

- $T(U) = (U_1, \frac{\partial C}{\partial u_1}(U_1, U_2))$  and the components are i.i.d  $\mathcal{U}([0, 1])$

So, if we take  $W \sim \mathcal{U}([0, 1])$  independent from  $U_1$  then

- $T^{-1}(U_1, W) \sim U$

**Remark:** in practice for each simulation  $(u_1, w)$  we find  $u_2$  the solution of  $\frac{\partial C}{\partial u_1}(u_1, u_2) = w$  and by doing so we simulate  $U$  of Copula  $C$ .

**Background** : Pearson's linear correlation  $\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma(X)\sigma(Y)}$  measures only the affine relationship between variables and presents some imperfections to measure the "link" between two variables. For example:

- if  $X \sim \mathcal{N}(0, 1)$  and  $Y = X^2$  then  $\rho(X, Y) = 0$  while there is a strong link between  $Y$  and  $X$  (we can indeed predict  $Y$  perfectly from  $X$ )
- if  $f$  and  $g$  are increasing in general  $\text{cov}(X, Y) \neq \text{cov}(f(X), g(Y))$

## Definition : Kendall's tau

Let  $(X, Y)$  be a random variable. Let  $(X_1, Y_1), (X_2, Y_2)$  be independent with the same law as  $(X, Y)$ . We call Kendall's tau and note  $\tau(X, Y)$  the quantity  $P((X_1 - X_2)(Y_1 - Y_2) \geq 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$

## Properties Kendall's $\tau$

- $-1 \leq \tau(X, Y) \leq 1$
- if  $f$  and  $g$  are strictly increasing  $\tau(f(X), g(Y)) = \tau(X, Y)$
- $\tau(F_X(X), F_Y(Y)) = \tau(X, Y)$
- $U \sim \mathcal{U}([0, 1]) \implies \tau(U, U) = 1$  and  $\tau(U, 1 - U) = -1$
- If  $(X, Y)$  has  $C$  for Copula then
$$\tau(X, Y) = -1 + 4 \int_{[0,1]^2} C(u, v) \frac{\partial^2 C}{\partial u \partial v} dudv$$

### Remark:

if  $X \sim \mathcal{N}(0, 1)$  and  $Y = X^2$  then  $\tau(X, Y) = 1$

**Demonstration:** let's show the last point

as  $\tau(F_X(X), F_Y(Y)) = \tau(X, Y)$  we can show it for  $X$  and  $Y \sim \mathcal{U}([0, 1])$

$$\begin{aligned}\tau(X, Y) &= P((X_1 - X_2)(Y_1 - Y_2) \geq 0) - (1 - P((X_1 - X_2)(Y_1 - Y_2) \geq 0)) \\ &= -1 + 2P((X_1 - X_2)(Y_1 - Y_2) \geq 0) \\ &= -1 + 2(P(X_1 - X_2 \leq 0, Y_1 - Y_2 \leq 0) + P(X_2 - X_1 \leq 0, Y_2 - Y_1 \leq 0))\end{aligned}$$

as  $(X_1, Y_1)$  and  $(X_2, Y_2)$  have the same law, so we just need to calculate the first probability.

$P(X_1 - X_2 \leq 0, Y_1 - Y_2 \leq 0) = E(E(1_{X_1 \leq X_2} 1_{Y_1 \leq Y_2}) | X_2, Y_2)$  and  
 $E(1_{X_1 \leq X_2} 1_{Y_1 \leq Y_2}) = P(X_1 \leq x_2, Y_1 \leq y_2) = C_X(x_2, y_2)$

so we have to calculate  $E(C(X_2, Y_2)) = \int_{[0,1]^2} C(u, v) \frac{\partial^2 C}{\partial u \partial v} dudv$

so  $\tau(X, Y) = -1 + 4 \int_{[0,1]^2} C(u, v) \frac{\partial^2 C}{\partial u \partial v} dudv$  Q.E.D.

## Remark Kendall's tau:

Based on the observations  $(x_i, y_i)_{i \in \llbracket 1, n \rrbracket}$  the Kendall's tau is estimated by the quantity

$$\frac{2}{n(n-1)} \sum_{i < j} \text{sgn}(x_i - x_j) \text{sgn}(y_i - y_j)$$

## Definition : Spearman's correlation

If  $(X, Y)$  is a random variable with marginal laws  $F_X$  and  $F_Y$  then the Spearman's correlation  $\rho_S$  is defined by  $\rho_S(X, Y) = \rho(F_X(X), F_Y(Y))$

## Properties Spearman's correlation

- $-1 \leq \rho_S(X, Y) \leq 1$
- if  $f$  and  $g$  are strictly increasing  $\rho_S(f(X), g(Y)) = \rho_S(X, Y)$
- $\tau(F_X(X), F_Y(Y)) = \rho_S(X, Y)$
- $U \sim \mathcal{U}([0, 1]) \implies \rho_S(U, U) = 1$  and  $\rho_S(U, 1 - U) = -1$
- If  $(X, Y)$  has  $C$  for Copula then
$$\rho_S(X, Y) = -3 + 12 \int_{[0,1]^2} C(u, v) dudv$$

## Demonstration :

Let  $C$  be the copula of  $(X, Y)$  i.e the cdf of  $(U, V)$  where  $U = F_1(X)$  and  $V = F_2(Y)$ , let  $\rho_S(X, Y) = \frac{E(UV) - E(U)E(V)}{E(U)E(V)}$

$$\begin{aligned} E(UV) &= \int_0^1 \int_0^1 uv \frac{\partial^2 C}{\partial u \partial v} dudv = \int_0^1 u \left( \int_0^1 v \frac{\partial^2 C}{\partial u \partial v} dv \right) du \\ &= \int_0^1 u \left( [v \frac{\partial C}{\partial u}]_0^1 - \int_0^1 \frac{\partial C}{\partial u} dv \right) du = \int_0^1 u \left( f_U(u) - \int_0^1 \frac{\partial C}{\partial u} dv \right) du \\ &= E(U) - \int_0^1 \left( \int_0^1 u \frac{\partial C}{\partial u} du \right) dv = E(U) - \int_0^1 \left( [uC(u, v)]_0^1 - \int_0^1 C(u, v) du \right) dv \\ &= E(U) - \int_0^1 P(V \leq v) dv + \int_0^1 \int_0^1 C(u, v) dudv \\ &= E(U) - E(V) + \int_0^1 \int_0^1 C(u, v) dudv = \int_0^1 \int_0^1 C(u, v) dudv \\ \text{and } E(U)E(V) &= \frac{1}{4} \text{ and } \text{Var}(U) = \text{Var}(V) = \frac{1}{12} \text{ Q.E.D.} \end{aligned}$$



## Remark: Spearman's correlation.

Based on the observations  $(x_i, y_i)_{i \in \llbracket 1, n \rrbracket}$  the Spearman's correlation is estimated by calculating the correlations of the

$$(F_{x[n]}(x_i), F_{y[n]}(y_i))$$

where  $F_{x[n]}(x_i)$  is the quantile for  $x_i$  amongst  $x_1, x_2, \dots, x_n$  and  $F_{y[n]}(y_i)$  is the quantile for  $y_i$  amongst  $y_1, y_2, \dots, y_n$ .

## Appendix : Risk Neutral Probability and Utility functions

# Risk Neutral Probability (discrete case)

**Background:** We consider an economy with two instants  $\{0, 1\}$  where there are  $d$  assets whose vector of prices  $X$  is represented today by the vector  $X_0 = (X_0^1, X_0^2, \dots, X_0^d)'$ . We assume that at instant 1 there are  $n$  possible states for the economy and for each state  $i \in \llbracket 1, n \rrbracket$  the vector of the prices of the assets is  $X_i = (X_i^1, X_i^2, \dots, X_i^n)'$ . We assume that prices are all strictly positive.

## Definition: Absence of Arbitrage (AOA)

We say that there is no arbitrage in the economy iff:

$$\{w \in \mathbb{R}^d, \forall i \in \llbracket 1, n \rrbracket \langle w, X_i \rangle \geq 0\} \subset \{w \in \mathbb{R}^d, \langle w, X_0 \rangle \geq 0\}$$

## Remarks :

The definition means that it is not possible to receive money today to build a strategy which has positive values tomorrow in all cases.

## Theorem and Definition: Risk Neutral Probability

a) the two following propositions are equivalent:

- there is no arbitrage in the economy
- we can find  $(\lambda_i)_{i \in \llbracket 1, n \rrbracket}$ ,  $\lambda_i \geq 0$  such that  $X_0 = \sum_{i=1}^{i=n} \lambda_i X_i$

b) if there is a risk-free asset in the economy of return  $r$  over  $[0, 1]$  then:

- $\sum_{i=1}^{i=n} \lambda_i = \frac{1}{1+r}$
- if we define a probability  $\pi$ , over the  $n$  possible values of  $X$  at time 1, by  $\pi_i = \lambda_i(1+r)$  then  $X_0 = \frac{1}{1+r} E_\pi[X]$  and  $\pi$  is called the risk neutral probability for the economy.

## Demonstration :

a) one of the implications is obvious.

We assume now that there is no arbitrage and define the cone

$$\mathcal{C} = \left\{ \sum_{i=1}^{i=n} \lambda_i X_i, \forall i \in \llbracket 1, n \rrbracket \lambda_i \geq 0 \right\}.$$

Then  $\mathcal{C}$  is convex and if  $X \notin \mathcal{C}$  we can separate  $X$  from  $\mathcal{C}$  by an hyperplane and find  $w \in \mathbb{R}^d$  such that:  $\langle w, X_0 \rangle < 0$  and for all  $X_i$  in  $\mathcal{C}$   $\langle w, X_i \rangle > 0$  but this would contradict the AOA hypothesis, so  $X_0 \in \mathcal{C}$ . Q.E.D.

b) if we assume that the risk-free asset is component  $j$  then:

$$X_0^j = \sum_{i=1}^{i=n} \lambda_i X_i^j \text{ and } X_i^j = X_0^j(1+r) \implies \sum_{i=1}^{i=n} \lambda_i = \frac{1}{1+r} \text{ Q.E.D.}$$

**Exercise :** we assume that there are 3 assets, of prices today  $X_0 = (1, 5, 10)'$  and 3 possible states of the economy tomorrow defined by the 3 vector of prices for the assets:

$X_1 = (1.03, 5, 11)'$ ,  $X_2 = (1.03, 5, 10)'$ ,  $X_3 = (1.03, 6, 10)'$ .

- explain why the risk-free rate is 3%
- show that  $\pi = (0.30, 0.55, 0.15)'$
- explain why there is no arbitrage in this economy.

## Risk Neutral Probability (discrete case)

**Remark :** If we assume that the vector of prices today is  $X_0 = (1, 5, 10)'$  and that the 3 vectors of prices for tomorrow are :

$X_1 = (1.03, 5, 10)'$ ,  $X_2 = (1.03, 6, 12)'$ ,  $X_3 = (1.03, 6, 13)'$  then

a)  $\pi = (0.85, 0.15, 0)'$  is the risk neutral probability

b) according to a) there is no arbitrage

c) the strategy  $w = (0, -2, 1)'$  costs today zero and the possible outcomes tomorrow are 0 for the first two states and 1 for state 3, so it seems attractive to play it (as there is only upside) but strictly speaking this is not an arbitrage according to our definition.

# Risk Neutral Probability (continuous case)

## Background:

We consider a probability space  $(\Omega, \mathcal{F}, P)$  with  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$  where  $\mathcal{F}_t$  represents the information available at time  $t$ .

We assume that there are  $d$  financial assets following the equations:

$dX_s^i = \mu_s^i X_s^i ds + \sigma_s^i X_s^i dW_s^i$  where  $W_s = (W_s^1, W_s^2, \dots, W_s^d)$  is a  $d$ -dimensional Brownian motion.

## Theorem and Definition : Risk Neutral Probability

We can find a probability  $Q$  on  $(\Omega, \mathcal{F})$  such that:

$W^*$  defined by:  $dW_s^{i*} = (dW_s^i + \frac{\mu_s^i - r_s}{\sigma_s^i} ds)$  is a Brownian motion under  $Q$

We can then re-write the model:

$dX_s^i = r_s X_s^i ds + \sigma_s^i X_s^i dW_s^{i*}$  where  $W^*$  is a Brownian motion under  $Q$  and  $Q$  is called the risk neutral probability.



## Lemma and Definition

If  $(Z_s)_{s \geq 0}$  is a martingale under  $P$  with  $Z_s \geq 0$  and  $Z_0 = 1$  and if we define  $Q$  for any random variable  $Y_t$   $\mathcal{F}_t$ -measurable by  $E_Q[Y_t] = E_P[Y_t Z_t]$  then:

- $Q$  is a probability on  $(\Omega, \mathcal{F})$
- $E_Q[Y_T | \mathcal{F}_t] = E_P[Y_T \frac{Z_T}{Z_t} | \mathcal{F}_t]$

Usually we note  $(\frac{dQ}{dP})_t = Z_t$  and so we write  $E_Q[Y_t] = E_P[Y_t (\frac{dQ}{dP})_t]$

**Demonstration Lemma :** easy

**Demonstration Theorem (hint) :**

We note  $\Delta_s^i = \frac{\mu_s^i - r_s}{\sigma_s^i}$  and  $\Delta_s = (\Delta_s^1, \Delta_s^2, \dots, \Delta_s^d)'$ .

# Risk Neutral Probability (continuous case)

We search for a probability  $Q$  under which  $(W_s^*)_{s \geq 0}$  is a Brownian motion

For that we need  $E_Q[dW_t^* | \mathcal{F}_t] = 0$  and  $E_Q[dW_t^* (dW_t^*)' | \mathcal{F}_t] = Id_{\mathbb{R}^d} dt$

We note  $(\frac{dQ}{dP})_t = Z_t$  and so search for  $Z_t$ .

$$\begin{aligned} E_Q[dW_t^* | \mathcal{F}_t] &= E_P[dW_t^* \frac{Z_{t+dt}}{Z_t} | \mathcal{F}_t] = E_P[(dW_t + \Delta_t dt)(1 + \frac{dZ_t}{Z_t}) | \mathcal{F}_t] \\ &= E_P[dW_t | \mathcal{F}_t] + \frac{1}{Z_t} E_P[dW_t dZ_t | \mathcal{F}_t] + \Delta_t dt = \frac{1}{Z_t} E_P[dW_t dZ_t | \mathcal{F}_t] + \Delta_t dt \end{aligned}$$

If we search  $Z$  of the form  $dZ_s = \langle B_s, dW_s \rangle$  with  $B_s \in \mathbb{R}^d$  (no drift term as martingale) then:

$$\begin{aligned} E_P[dW_t dZ_t | \mathcal{F}_t] &= E_P[dW_t \langle dW_t, B_t \rangle | \mathcal{F}_t] = E_P[dW_t (dW_t)' B_t | \mathcal{F}_t] \\ &= E_P[dW_t (dW_t)' | \mathcal{F}_t] B_t = B_t dt \text{ so, } E_Q[dW_t^* | \mathcal{F}_t] = 0 \iff B_t = -\Delta_t Z_t \end{aligned}$$

Solving  $dZ_s = \langle -\Delta_s, dW_s \rangle Z_s$  and  $Z_0 = 1$  we get:

$$Z_t = \exp\left(\int_0^t -\langle \Delta_s, dW_s \rangle - \frac{1}{2} \int_0^t \|\Delta_s\|^2 ds\right) \text{ (we do not discuss here the}$$

conditions on  $\Delta_s$  for integrability that can be found in Girsanov's theorem)

The condition  $E_Q[dW_t^* (dW_t^*)' | \mathcal{F}_t] = Id_{\mathbb{R}^d} dt$  is easy to verify Q.E.D.

**Remark 1:** for  $d = 1$  we get that:

- $X_T = X_0 e^{\mu T} e^{\sigma W_T - \frac{1}{2}\sigma^2 T}$  where  $(W_s)_{s \geq 0}$  is a Brownian under  $P$
- $X_T = X_0 e^{rT} e^{\sigma W_T^* - \frac{1}{2}\sigma^2 T}$  where  $(W_s^*)_{s \geq 0}$  is the Brownian under  $Q$  defined by  $W_T^* = W_T + \frac{\mu - r}{\sigma} T$
- $(\frac{dQ}{dP})_T = \exp(\frac{r - \mu}{\sigma} W_T - \frac{1}{2}(\frac{r - \mu}{\sigma})^2 T)$
- for any function  $h$ ,  $E_Q[h(X_T)] = E_P[h(X_T)(\frac{dQ}{dP})_T]$

**Exercise:** verify by calculations for  $d = 1$  that

$$E_Q[h(rT + \sigma W_T^*)] = E_P[h(\mu T + \sigma W_T)(\frac{dQ}{dP})_T]$$

**Solution:**  $E_Q[h(rT + \sigma W_T^*)] = E_{Q^{W_T^*}}[h(rT + \sigma z)]$

$$= \int h(rT + \sigma z) \frac{1}{\sigma \sqrt{2\pi T}} \exp(-\frac{z^2}{2T}) dz$$

# Risk Neutral Probability (continuous case)

If we take the new variable  $u$  such that  $\mu T + \sigma u = rT + \sigma z$  we get :

- $h(rT + \sigma z) = h(\mu T + \sigma u)$

- $\exp(-\frac{z^2}{2T}) = \exp(-\frac{(u + \frac{\mu-r}{\sigma} T)^2}{2T}) = \exp(-\frac{u^2}{2T}) \exp(-\frac{\mu-r}{\sigma} u - \frac{1}{2}(\frac{\mu-r}{\sigma})^2 T)$

so,

$$\int h(rT + \sigma z) \frac{1}{2\pi\sqrt{T}} \exp(-\frac{z^2}{2T}) dz$$

$$= \int h(\mu T + \sigma u) \exp(-\frac{\mu-r}{\sigma} u - \frac{1}{2}(\frac{\mu-r}{\sigma})^2 T) \frac{1}{2\pi\sqrt{T}} \exp(-\frac{u^2}{2T}) du$$

$$= E_P[h(\mu T + \sigma W_T) \exp(-\frac{\mu-r}{\sigma} W_T - \frac{1}{2}(\frac{\mu-r}{\sigma})^2 T)] \text{ Q.E.D.}$$

**Remark 2:** we call  $u$  a utility function compatible with the price of asset  $X$ ,  $P$  the "real" probability and "Q" the risk neutral probability.

As  $X_0 = E_P[e^{-rT} u(X_T)]$  and  $X_0 = E_Q[e^{-rT} X_T]$  we have under the previous assumptions concerning the law of  $X$  under  $P$  :

$$E_P[e^{-rT} u(X_T)] = E_P[e^{-rT} X_T f(W_T)] \text{ with}$$

$$f(w) = \exp\left(\frac{r-\mu}{\sigma} w - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right)$$

if we define  $g(x) = f\left(\frac{1}{\sigma}\left[\ln\left(\frac{x}{X_0}\right) + \left(\frac{\sigma^2}{2} - \mu\right)T\right]\right)$  then

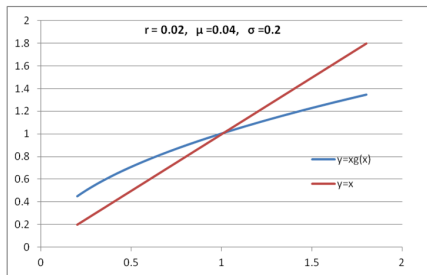
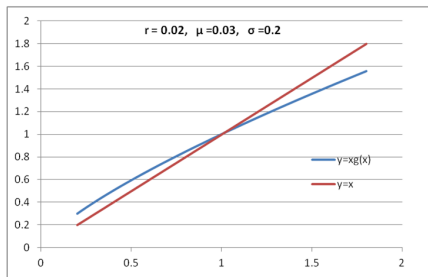
$$f(W_T) = g(X_T) \text{ and } E_P[e^{-rT} u(X_T)] = E_P[e^{-rT} X_T g(X_T)]$$

so,  $u(x) = xg(x)$  is an adequate utility function for this modelisation of  $X$ .

In the following graph we represent  $xg(x)$  for various values of  $r, \mu$  and  $\sigma$  with  $x_0 = 1$  and  $T = 1$ .

Note that depending on the value of the parameters  $xg(x)$  is not always increasing (which shows its limits in terms of admissible utility function..)

# Risk Neutral Probability (continuous case)



Utility functions derived from Girsanov's Theorem

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