



# ON THE ABSOLUTE CONTINUOUS SPECTRUM OF DISCRETE OPERATORS

Sylvain Golenia

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# ON THE ABSOLUTE CONTINUOUS SPECTRUM OF DISCRETE OPERATORS

*CIMPA SCHOOL: THÉORIE SPECTRALE DES GRAPHS ET DES VARIÉTÉS*

*TO THE MEMORY OF AHMAD EL SOUFI*

SYLVAIN GOLÉNIA

ABSTRACT. This aim of this course is to give an overview to the study of the continuous spectrum of bounded self-adjoint operators and especially those coming from the setting of graphs. For the sake of completeness, a short course in spectral theory is given with proofs. The continuous and Borelian functional calculi are also developed.

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## 1. INTRODUCTION

These notes are a detailed version of the 7.5 hour long course that I gave in Kairouan from the 7th to the 16th of November 2016. This idea of this course is to provide in a fast and motivated way the tools to young researchers that are starting in spectral theory or graph theory. For simplicity we shall concentrate on the analysis of bounded self-adjoint operators.

After recalling the main properties of Hilbert spaces, we give a short course in general spectral theory. Then we start to develop a series of examples coming from the study of graphs. Next, we enter in the world of functional calculus. We first construct the continuous functional calculus with detailed proofs. Subsequently we browse few famous approaches to continuous functional calculus. In the next section we present the theory of the essential spectrum with full proofs. After this done, we go even further and construct the Borelian functional calculus. We discuss the nature of the spectral measure, the Putnam theorem and the stability of the a.c. spectrum. Finally, still motivated by the examples given on graphs, we give an introduction to the Mourre theory.

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## 2. SPECTRAL PROPERTIES OF OPERATORS

We start with few notation. Given  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$  and  $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$  two Banach spaces, we denote by  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  the set of continuous linear maps acting from  $\mathcal{X}$  to  $\mathcal{Y}$ . Endowed with the norm

$$\|T\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} := \sup_{x \in \mathcal{X}, \|x\|_{\mathcal{X}}=1} \|Tx\|_{\mathcal{Y}},$$

we have that  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  is a Banach space. When  $\mathcal{X} = \mathcal{Y}$  we set  $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$ .

We denote by  $\mathbb{N}$  the set of non-negative integers (be careful  $0 \in \mathbb{N}$ ), by  $\mathbb{N}^*$  the set of positive integers and  $\mathbb{Z}$  is the set of integers. We set  $\delta_{a,b} := 1$  if  $a = b$  and 0 otherwise.

**2.1. Hilbert spaces.** This section is a compilation of well-known results in the theory of Hilbert spaces. We focus on the study of complex Hilbert spaces.

**Definition 2.1.** *Given a complex vector space  $\mathcal{X}$ , a scalar product is a map  $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$  such that for all  $x, y, z \in \mathcal{X}$  and  $\lambda \in \mathbb{C}$  :*

- 1)  $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$ ,
- 2)  $\langle z, x + \lambda y \rangle = \langle z, x \rangle + \lambda \langle z, y \rangle$ ,
- 3)  $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- 4)  $\langle x, x \rangle = 0$  if and only if  $x = 0$ .

*A vector space  $\mathcal{X}$  endowed with a scalar product is a pre-Hilbert space.*

Note that the third line gives  $\langle x, x \rangle \geq 0$ .

**Remark 2.2.** *Here we take the convention to be anti-linear with respect to the first variable. It is a choice.*

**Proposition 2.3.** *Let  $(\mathcal{X}, \langle \cdot, \cdot \rangle)$  be a pre-Hilbert space. We set  $\|x\| := \sqrt{\langle x, x \rangle}$ . We have that  $\|\cdot\|$  is a norm, i.e., for all  $x, y \in \mathcal{X}$  and  $\lambda \in \mathbb{C}$*

- 1)  $\|x\| = 0$  if and only if  $x = 0$ ,

- 2)  $\|\lambda x\| = |\lambda| \cdot \|x\|$ ,  
 3)  $\|x + y\| \leq \|x\| + \|y\|$ .

If  $(\mathcal{X}, \|\cdot\|)$  is complete, we say that  $\mathcal{X}$  is a Hilbert space.

A central result is the existence of a orthogonal complementary subspace.

**Theorem 2.4.** *Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{F} \subset \mathcal{H}$  be a closed subspace. Then*

- 1)  $\mathcal{F}^\perp := \{y \in \mathcal{H}, \langle x, y \rangle = 0, \forall x \in \mathcal{F}\}$  is a closed subspace of  $\mathcal{H}$ .  
 2) We have

$$\mathcal{H} = \mathcal{F} \oplus \mathcal{F}^\perp.$$

- 3) Moreover, given  $x = x_{\mathcal{F}} + x_{\mathcal{F}^\perp}$  in this decomposition we have:

$$\|x\|^2 = \|x_{\mathcal{F}}\|^2 + \|x_{\mathcal{F}^\perp}\|^2.$$

We turn to the properties of Hilbert basis.

**Definition 2.5.** *Given a Hilbert space  $(\mathcal{H}, \|\cdot\|)$ , we say that  $(e_n)_{n \in \mathbb{N}}$  is a Hilbert basis, if*

- 1)  $\langle e_n, e_m \rangle = \delta_{n,m}$  for all  $n, m \in \mathbb{N}$ . In particular,  $\|e_n\| = 1$  for all  $n \in \mathbb{N}$ ,  
 2)  $\sum_{n \in \mathbb{N}} \mathbb{C}e_n = \mathcal{H}$ .

**Remark 2.6.** *Sometimes it is useful to take  $\mathbb{Z}$  or  $\mathbb{N}^*$  instead of  $\mathbb{N}$  in this definition.*

**Definition 2.7.** *A metric space  $(\mathcal{X}, d)$  is separable if and only if there is a countable set  $\mathcal{F} \subset \mathcal{X}$  such that  $\mathcal{F}$  is dense in  $\mathcal{X}$ .*

**Proposition 2.8.** *Given a Hilbert space  $(\mathcal{H}, \|\cdot\|)$ . The following statements are equivalent:*

- 1)  $\mathcal{H}$  is separable,  
 2)  $\mathcal{H}$  has a Hilbert basis.

*Proof.* 2)  $\implies$  1): Given  $(e_n)_n$  a Hilbert basis, take  $\mathcal{F} := \cup_n (\mathbb{Q} + i\mathbb{Q})e_n$ .

1)  $\implies$  2): We have  $\mathcal{F} = \cup_n f_n$  with  $f_n \in \mathcal{H}$ . Use Gram-Schmidt on  $(f_n)_n$ . □

**Remark 2.9.** *From now on:*

**All the Hilbert spaces are complex and separable.**

We give the two main examples:

**Example 2.10.** *Set  $\mathcal{H} := \ell^2(\mathbb{N}; \mathbb{C}) := \{f : \mathbb{N} \rightarrow \mathbb{C}, \text{ such that } \sum_n |f_n|^2 < \infty\}$  endowed with*

$$\langle f, g \rangle := \sum_{n \in \mathbb{N}} \overline{f_n} g_n,$$

*for  $f, g \in \ell^2(\mathbb{N}; \mathbb{C})$ . For all  $n \in \mathbb{N}$ , set  $e_n : \mathbb{N} \rightarrow \mathbb{C}$  given by  $e_n(m) := \delta_{n,m}$ . We have that  $(e_n)_{n \in \mathbb{N}}$  is a Hilbert basis.*

**Example 2.11.** Set  $\mathcal{H} := L^2([-\pi, \pi]; \mathbb{C})$ , endowed with

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx,$$

with  $f, g \in \mathcal{H}$ . For all  $n \in \mathbb{Z}$ , set  $e_n(x) := e^{inx}$ . We have that  $(e_n)_{n \in \mathbb{Z}}$  is a Hilbert basis.

**2.2. Polarisation.** We turn to the polarisation properties.

**Proposition 2.12.** Let  $\mathcal{X}$  be  $\mathbb{C}$ -vector space. We take  $\mathcal{Q} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$  to be a sesquilinear form which is linear on the right and anti-linear on the left, i.e.,

- 1)  $\mathcal{Q}(x, y + \lambda z) = \mathcal{Q}(x, y) + \lambda \mathcal{Q}(x, z)$ ,
- 2)  $\mathcal{Q}(x + \lambda y, z) = \mathcal{Q}(x, z) + \bar{\lambda} \mathcal{Q}(y, z)$ ,

for all  $x, y, z \in \mathcal{X}$  et  $\lambda \in \mathbb{C}$ . Set  $\mathcal{Q}(x) := \mathcal{Q}(x, x)$  (because this is not necessarily real!). We have the following identity of polarisation:

$$\mathcal{Q}(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k \mathcal{Q}(i^k x + y).$$

*Proof.* Develop the right hand side. □

**Remark 2.13.** In particular we get:

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \|i^k x + y\|^2.$$

In other words, given a norm that comes from a scalar product, we can recover the scalar product. To recall this formula, note that  $i^k$  lies next to the anti-linear part of the scalar product.

**Remark 2.14.** When the vector space is real a bilinear form  $\mathcal{Q}$  satisfies:

$$\mathcal{Q}(x, y) = \frac{1}{4} (\mathcal{Q}(x + y) - \mathcal{Q}(x - y)),$$

for all  $x, y \in \mathcal{X}$ .

**Corollary 2.15.** Given  $\mathcal{H}$  a Hilbert space and  $S, T$  two bounded operators. If

$$\langle x, Sx \rangle = \langle x, Tx \rangle, \text{ pour tout } x \in \mathcal{X}$$

then  $S = T$ .

*Proof.* Set  $\mathcal{Q}_1(x, y) := \langle x, Sy \rangle$  and  $\mathcal{Q}_2(x, y) := \langle x, Ty \rangle$  for all  $x, y \in \mathcal{H}$ . There are sesquilinear forms. By hypothesis we have  $\mathcal{Q}_1(x) = \mathcal{Q}_2(x)$  for all  $x \in \mathcal{H}$ . In particular we have:

$$\langle x, Sy \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \mathcal{Q}_1(i^k x + y) = \frac{1}{4} \sum_{k=0}^3 i^k \mathcal{Q}_2(i^k x + y) = \langle x, Ty \rangle.$$

which is the result. □

**2.3. Spectral properties of bounded operators.** We start with the notion of spectrum

**Definition 2.16.** Let  $T \in \mathcal{L}(\mathcal{H})$ .

1) The resolvent set of  $T$  is:

$$\rho(T) := \{\lambda \in \mathbb{C}, \lambda \text{Id} - T \text{ is invertible}\}.$$

2) If  $\lambda \in \rho(T)$ , we define the resolvent  $R_\lambda(T)$  (or simply  $R_\lambda$ ) of  $T$  at  $\lambda$  by

$$R_\lambda(T) := (\lambda \text{Id} - T)^{-1}.$$

3) The spectrum of  $T$  is

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

4) We say that  $\lambda \in \mathbb{C}$  is an eigenvalue of  $T$  if  $\lambda \text{Id} - T$  is not injectif, i.e.,  $\ker(\lambda \text{Id} - T) \neq \{0\}$ . The point spectrum is given by:

$$\sigma_p(T) := \overline{\{\lambda \in \mathbb{C}, \ker(\lambda \text{Id} - T) \neq \{0\}\}}.$$

**Remark 2.17.** If  $\lambda \in \rho(T)$ ,  $R_\lambda(T) \in \mathcal{L}(\mathcal{H})$  (Banach's Theorem).

**Remark 2.18.** Set  $T \in \mathcal{L}(\mathcal{H})$ . Let  $U \in \mathcal{L}(\mathcal{H})$ , be unitary. Then,  $\sigma(T) = \sigma(UTU^*)$ .

**Remark 2.19.** We have:

1) When  $\mathcal{H}$  is of finite dimension and  $T \in \mathcal{L}(\mathcal{H})$ , the rank theorem states that  $T$  is surjective if and only if  $T$  is injective if and only if it is bijective. In particular

$$\sigma_p(T) = \sigma(T), \text{ when } \dim \mathcal{H} < \infty$$

The situation is very different in infinite dimension.

2) The point spectrum is usually different from the set of eigenvalues.

**Proposition 2.20.** Let  $T \in \mathcal{L}(\mathcal{H})$ .

1) If  $|\lambda| > \|T\|$  then  $\lambda \in \rho(T)$ . In particular  $\sigma(T) \subset \overline{D(0, \|T\|)}$ . Moreover,

$$\|(\lambda - T)^{-1}\| \leq \frac{1}{|\lambda| - \|T\|}.$$

2)  $\rho(T)$  is open and non-empty in  $\mathbb{C}$ .

3)  $\sigma(T)$  is compact in  $\mathbb{C}$ .

4)  $\sigma_p(T) \subset \sigma(T)$ .

*Proof.* 1) We have  $\lambda \neq 0$ . Then  $\|\lambda^{-1}T\| < 1$ . Moreover since

$$\lambda \text{Id} - T = \lambda(\text{Id} - \lambda^{-1}T),$$

we deduce that  $\text{Id} - \lambda^{-1}T$  is invertible. In particular  $\lambda \in \rho(T)$ . Furthermore,

$$\|(\lambda \text{Id} - T)^{-1}\| = \frac{1}{|\lambda|} \|(\text{Id} - \lambda^{-1}T)^{-1}\| \leq \frac{1}{|\lambda|} \sum_{k=0}^{\infty} \left\| \frac{T}{\lambda} \right\|^k = \frac{1}{|\lambda| - \|T\|}.$$

2) First we prove that the group invertible linear continuous maps is open in  $\mathcal{L}(\mathcal{H})$ .

Let  $H$  be invertible. Let  $V \in \mathcal{L}(\mathcal{H})$  such that  $\|V\| < \|H^{-1}\|^{-1}$ .

$$\|H^{-1}V\| \leq \|H^{-1}\| \cdot \|V\| < 1.$$

We have:

$$H + V = H \underbrace{(\text{Id} - H^{-1}V)}_{\text{invertible by 1)}$$

By composition this yields that  $H + V$  is invertible as product of invertible maps. In particular,  $B(H, \|H^{-1}\|^{-1}) \subset \mathcal{GL}(\mathcal{H})$ . In other words,  $\mathcal{GL}(\mathcal{H})$  is open.

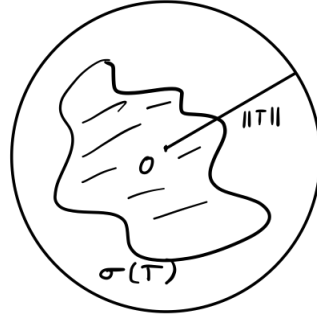
Let  $\varphi : \mathbb{C} \rightarrow \mathcal{L}(\mathcal{X})$  be given by  $\varphi := \lambda \mapsto \lambda \text{Id} - T$ . We have that  $\varphi$  is continuous because it is Lipschitz.

$$\|\varphi(\lambda) - \varphi(\mu)\| = \|(\lambda - \mu)\text{Id}\| = |\lambda - \mu|.$$

Note that  $\rho(T) = \varphi^{-1}(\mathcal{GL}(\mathcal{X}))$ . The resolvent set is the reciprocal image of an open set by a continuous function. Hence, it is open.

3) By 2)  $\sigma(T)$  is closed and by 1) it is bounded. It is therefore compact.

4) The eigenvalues are part of  $\sigma(T)$  by definition. Since  $\sigma(T)$  is closed then by taking the closure  $\sigma_p(T) \subset \sigma(T)$ .  $\square$



**Definition 2.21.** Given  $T \in \mathcal{L}(\mathcal{H})$ . We call spectral radius:

$$\text{rad}(T) := \inf\{r, \sigma(T) \subset \overline{B}(0, r)\}.$$

**Proposition 2.22.** Let  $T \in \mathcal{L}(\mathcal{H})$ , we have

$$\text{rad}(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}.$$

*Proof.* If  $T = 0$ , the result is trivial. Set  $T \neq 0$ .

1) We start with the equality of the right hand side. Fix  $m \in \mathbb{N}$ . For all  $n \in \mathbb{N}$ , there exist  $p, q \in \mathbb{N}$  with  $q \in [0, m[$  such that  $n = pm + q$ . If  $\|T\| > 1$ , we have  $\|T^q\| \leq \|T\|^p \leq \|T\|^n$ . Moreover if  $\|T\| \leq 1$ ,  $\|T^p\| \leq 1$ . This yields

$$\|T^n\| \leq \|T^m\|^p \cdot \|T^q\| \leq \max(1, \|T\|^m) \cdot \|T^m\|^p.$$

Then note that  $\|T\|^{m/n}$  tends to 1 and that  $p/n$  tends to  $1/m$  as  $n$  goes to  $\infty$ . We infer

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq \limsup_{n \rightarrow \infty} \left( (\max(1, \|T\|^m))^{1/n} \cdot \|T^m\|^{p/n} \right) \leq \|T^m\|^{1/m}.$$

for all  $m \in \mathbb{N}$ . Therefore:

$$\limsup_{n \rightarrow \infty} \|T^n\|^{1/n} \leq \inf_{m \in \mathbb{N}} \|T^m\|^{1/m} \leq \liminf_{m \rightarrow \infty} \|T^m\|^{1/m}.$$



In particular, the limit  $\lim_{m \rightarrow \infty} \|T^m\|^{1/m}$  exists and is equal to  $\inf_{m \in \mathbb{N}} \|T^m\|^{1/m}$ .

2) We now show that the limit is equal to the spectral radius. The radius of convergence of the series  $\sum_n z^{n+1} T^n$  is  $(\limsup_n \|T^n\|^{1/n})^{-1}$ . Thus, if we take  $|\lambda| > \limsup_n \|T^n\|^{1/n}$  the series  $S := \sum_n T^n / \lambda^{n+1}$  converges. Easily we see that  $S(\lambda \text{Id} - T) = (\lambda \text{Id} - T)S = \text{Id}$ . In particular  $\lambda \in \rho(T)$  and  $S = R_\lambda(T)$ . This yields

$$r(T) \leq \limsup_{n \rightarrow \infty} \|T^n\|^{1/n} = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}.$$

Next, we prove that the spectral radius cannot be smaller. If  $\sigma(T) \cap B(0, r)^c \neq \emptyset$  for some  $r < \lim_n \|T^n\|^{1/n}$  then

$$R_\lambda(T) = (\lambda \text{Id} - T)^{-1} = \frac{1}{\lambda} \left( \text{Id} - \frac{1}{\lambda} T \right)^{-1}$$

is analytic for  $|\lambda| > r$ . We then have  $z \rightarrow (1 - zT)^{-1}$  is analytic on  $B(0, r^{-1})$  and then the radius of convergence of the power series  $\sum_n z^n T^n$ , that is equal to  $\limsup_n \|T^n\|^{1/n}$  would be greater than  $r^{-1}$ . But by hypothesis  $r^{-1} > (\lim_n \|T^n\|^{1/n})^{-1}$ . Contradiction.  $\square$

**Remark 2.23.** *Because of possible nilpotent component, the spectral radius is not equal to the norm in general. For instance for*

$$H := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*we have  $0 = \text{rad}(H) < \|H\|$ .*

**Proposition 2.24** (Identities of the resolvent). *Let  $S, T \in \mathcal{L}(\mathcal{H})$ .*

1) *Suppose that  $\lambda \in \rho(S) \cap \rho(T)$ . We have:*

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T - S)R_\lambda(S).$$

2) *Suppose that  $\lambda, \mu \in \rho(T)$ , then*

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda)R_\lambda(T)R_\mu(T) = (\mu - \lambda)R_\mu(T)R_\lambda(T).$$

*In particular  $R_\lambda$  and  $R_\mu$  commute.*

3) *The map  $R_\cdot(T) := \lambda \mapsto R_\lambda(T)$  acting from  $\rho(T)$  into  $\mathcal{GL}(\mathcal{H})$  is differentiable with derivative:*

$$\frac{dR_\lambda}{d\lambda} = -R_\lambda^2.$$

*Proof.* We have:

$$\begin{aligned} (\lambda \text{Id} - T)^{-1} - (\lambda \text{Id} - S)^{-1} &= \\ &= (\lambda \text{Id} - T)^{-1}(\lambda \text{Id} - S)(\lambda \text{Id} - S)^{-1} - (\lambda \text{Id} - T)^{-1}(\lambda \text{Id} - T)(\lambda \text{Id} - S)^{-1} \\ &= (\lambda \text{Id} - T)^{-1}((\lambda \text{Id} - S) - (\lambda \text{Id} - T))(\lambda \text{Id} - S)^{-1} \\ &= (\lambda \text{Id} - T)^{-1}(T - S)(\lambda \text{Id} - S)^{-1}. \end{aligned}$$

2) similar.

3) By the previous proposition we have that  $T \mapsto T^{-1}$  is continuous. This implies that  $\lambda \mapsto R_\lambda(T)$  is continuous on  $\rho(T)$ . We therefore use 2):

$$\frac{R_{\lambda_0}(T) - R_\lambda(T)}{\lambda_0 - \lambda} = -R_{\lambda_0}(T)R_\lambda(T).$$

and let  $\lambda$  tend to  $\lambda_0$  to conclude.  $\square$

By bootstrap we see that the third property implies that the resolvent is infinitely differentiable on the resolvent set. We have actually a stronger result.

**Proposition 2.25** (Analyticity of the resolvent). *Let  $T \in \mathcal{L}(\mathcal{H})$ . The map  $R_\cdot(T) := \lambda \mapsto R_\lambda(T)$  is analytic on  $\rho(T)$ . Given  $\lambda_0 \in \rho(T)$ , then for all  $\lambda \in D(\lambda_0, \|R_{\lambda_0}\|^{-1}) \subset \rho(T)$  we have:*

$$R_\lambda = \sum_{n \geq 0} (-1)^n R_{\lambda_0}^{n+1} (\lambda - \lambda_0)^n.$$

Moreover, we have:

$$(2.3.1) \quad \|R_{\lambda_0}\| \geq \frac{1}{d(\sigma(T), \lambda_0)},$$

where  $d(\sigma(T), \lambda_0) := \inf\{|z - \lambda_0|, z \in \sigma(T)\}$ .

*Proof.* We have  $\|(\lambda - \lambda_0)R_{\lambda_0}\| < 1$  and the power series

$$\sum_{n \geq 0} (-R_{\lambda_0}(\lambda - \lambda_0))^n$$

converges normally in  $\mathcal{L}(\mathcal{H})$  and therefore converges (because  $\mathcal{H}$  is a Hilbert). Set

$$S_\lambda := R_{\lambda_0} \sum_{n \geq 0} (-R_{\lambda_0}(\lambda - \lambda_0))^n = \sum_{n \geq 0} (-1)^n R_{\lambda_0}^{n+1} (\lambda - \lambda_0)^n.$$

Hence we have:

$$\begin{aligned} (\lambda \text{Id} - T)S_\lambda &= (\lambda_0 \text{Id} - T)S_\lambda + (\lambda - \lambda_0)S_\lambda \\ &= \sum_{n \geq 0} (-1)^n R_{\lambda_0}^n (\lambda - \lambda_0)^n - \sum_{n \geq 0} (-1)^{n+1} R_{\lambda_0}^{n+1} (\lambda - \lambda_0)^{n+1} = \text{Id}. \end{aligned}$$

Similarly  $S_\lambda(\lambda \text{Id} - T) = \text{Id}$ . Therefore we have  $\lambda \in D(\lambda_0, \|R_{\lambda_0}\|^{-1})$ ,  $R_\lambda = S_\lambda$ . Moreover, as  $\|(\lambda_0 - \lambda)R_{\lambda_0}\| < 1$  implies  $\lambda \in \rho(T)$ , by taking the contraposition we obtain that if  $\lambda \in \sigma(T)$  then

$$\|R_{\lambda_0}\| \geq \frac{1}{|\lambda - \lambda_0|} \geq \frac{1}{d(\sigma(T), \lambda_0)}.$$

This ends the proof.  $\square$

**Proposition 2.26.** *Let  $T \in \mathcal{L}(\mathcal{H})$  where  $\mathcal{H}$  is a complex Hilbert space. Then  $\sigma(T) \neq \emptyset$ .*

**Remark 2.27.** *This proposition is wrong when  $\mathcal{H}$  is real. Take for instance*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

*It has no eigenvalue.*

*Proof.* If  $\sigma(T) = \emptyset$ , then  $\lambda \mapsto R_\lambda(T)$  is analytic on  $\mathbb{C}$ . For  $|\lambda| \geq \|T\|$ , we have:

$$R_\lambda(T) = \frac{1}{\lambda} \sum_{n \geq 0} \left( \frac{1}{\lambda} T \right)^n.$$

Then, we see that  $R_\lambda$  tends to 0 when  $|\lambda|$  tends to  $\infty$ .

In particular,  $\sup_{\lambda \in \mathbb{C}} \|R_\lambda\| < \infty$ . For all  $x, y \in \mathcal{H}$  we have that  $\lambda \mapsto \langle x, R_\lambda y \rangle$  is an analytic and bounded function on  $\mathbb{C}$ . Liouville theorem ensures that  $\langle x, R_\lambda y \rangle = 0$  for all  $\lambda \in \mathbb{C}$ . We conclude that  $R_\lambda = 0$  for all  $\lambda \in \mathbb{C}$ . This is a contradiction with the definition of invertibility.  $\square$

**2.4. Adjoint of an operator.** We first recall the Riesz's isomorphism.

**Proposition 2.28** (Riesz's isomorphism). *Set  $\phi \in \mathcal{H}'$ , where  $\mathcal{H}'$  is the set of anti-linear continuous forms defined on  $\mathcal{H}$ . Then there exists a unique  $x_\phi \in \mathcal{H}$  such that*

$$\phi(x) = \langle x, x_\phi \rangle,$$

*for all  $x \in \mathcal{H}$ . Moreover  $\|x_\phi\|_{\mathcal{H}} = \|\phi\|_{\mathcal{H}'}$ .*

**Remark 2.29.** *Here we have chosen the space of anti-linear forms instead of the space of linear forms. It seems a bit peculiar but this provides that*

$$\Phi : \begin{cases} \mathcal{H}' & \rightarrow \mathcal{H} \\ \phi & \mapsto x_\phi \end{cases}$$

*is a (linear) isomorphism of Hilbert spaces.*

*Proof.* First we check that  $\|x \mapsto \langle x, x_\phi \rangle\| = \|x_\phi\|$ . If  $x_\phi = 0$ , this is clear. Suppose that it is not zero. On one side we have:

$$|\langle x, x_\phi \rangle| \leq \|x\| \cdot \|x_\phi\|,$$

for all  $x \in \mathcal{H}$ . On the other side:

$$\left| \left\langle \frac{x_\phi}{\|x_\phi\|}, x_\phi \right\rangle \right| = \|x_\phi\|.$$

We turn to unicity. If  $x_\phi$  and  $x'_\phi$  are both solutions, then

$$\langle x, x_\phi - x'_\phi \rangle = 0,$$

for all  $x \in \mathcal{H}$ . By taking  $x = x_\phi - x'_\phi$ , we obtain that  $\|x_\phi - x'_\phi\| = 0$  and therefore that  $x_\phi = x'_\phi$ .

We turn to the construction. If  $\phi = 0$  set  $x_\phi = 0$ . Set  $\phi \in \mathcal{H}'$  non-zero. Since  $\phi$  is continuous, its kernel is of co-dimension 1. Let  $Y$  be the orthogonal of  $\ker(\phi)$  in  $\mathcal{H}$ , i.e.,

$$\mathcal{H} = Y \perp \ker(\phi).$$

Let  $y \in Y$  such that  $\|y\| = 1$ . Set

$$\psi(x) := \langle x, \phi(y) \cdot y \rangle.$$

First note that  $\psi$  and  $\phi$  are 0 on  $\ker(\phi)$ , because  $Y \perp \ker(\phi)$ . Then

$$\psi(y) = \langle y, \phi(y) \cdot y \rangle = \phi(y) \cdot \|y\|^2 = \phi(y)$$

and therefore  $\psi = \phi$  on  $Y$ . We conclude that  $\phi = \psi$  on  $\mathcal{H}$ .  $\square$

In this course we focus on the study of bounded operators. In particular, we shall not discuss the notion of adjoint of a unbounded self-adjoint operator.

**Proposition 2.30.** *Set  $T \in \mathcal{L}(\mathcal{H})$ . There is a unique  $S \in \mathcal{L}(\mathcal{H})$  so that*

$$\langle x, Ty \rangle = \langle Sx, y \rangle,$$

*for all  $x, y \in \mathcal{H}$ . We denote it by  $T^* := S$ . Moreover, we have:*

$$\|T\| = \|T^*\|.$$

*Proof.* Given  $x \in \mathcal{H}$ , by Riesz, there is  $\tilde{x} \in \mathcal{H}$  such that

$$\langle x, Ty \rangle = \langle \tilde{x}, y \rangle, \quad \text{for all } y \in \mathcal{H}.$$

Note that  $S := x \mapsto \tilde{x}$  is linear. Moreover,

$$\|Sx\| = \|\tilde{x}\| = \sup_{\|y\|=1} \langle \tilde{x}, y \rangle = \sup_{\|y\|=1} \langle x, Ty \rangle \leq \|x\| \cdot \|T\|.$$

Therefore  $S \in \mathcal{L}(\mathcal{H})$  and  $\|S\| \leq \|T\|$ . Finally, since  $\langle Sx, y \rangle = \langle x, Ty \rangle$ , we also have:

$$\|Ty\| = \sup_{\|x\|=1} \langle \tilde{x}, Ty \rangle = \sup_{\|x\|=1} \langle \tilde{S}x, y \rangle \leq \|S\| \cdot \|y\|.$$

Hence  $\|S\| = \|T\|$ . □

**Remark 2.31.** *We have  $T^{**} = T$ .*

**Proposition 2.32.** *Given  $T \in \mathcal{L}(X)$ , we have:*

$$\|TT^*\| = \|T^*T\| = \|T\|^2.$$

*Proof.* First,  $\|TT^*\| \leq \|T^*\| \cdot \|T\| = \|T\|^2$ . Then, we have

$$\|Tx\|^2 = |\langle Tx, Tx \rangle| = |\langle x, T^*Tx \rangle| \leq \|x\|^2 \|T^*T\|,$$

for all  $x \in \mathcal{H}$ . In particular  $\|T\|^2 \leq \|T^*T\|$ . This gives the last equality. For the first one, use the last one with  $T^*$  instead of  $T$  and recall that  $\|T\| = \|T^*\|$ . □

**Definition 2.33.** *Let  $T \in \mathcal{L}(\mathcal{H})$ ,*

- 1)  *$T$  is normal if  $T^*T = TT^*$ .*
- 2)  *$T$  is self-adjoint if  $T = T^*$ .*
- 3)  *$T$  is unitary if  $T^*T = TT^* = \text{Id}$ .*

**Exercise 2.1.** *Set  $T \in \mathcal{L}(\mathcal{H})$ . Using the polarisation identity, prove that  $T$  is unitary if and only if  $T$  is surjective and is an isometry, i.e.,  $\|Tx\| = \|x\|$ , for all  $x \in \mathcal{H}$ .*

**Proposition 2.34.** *Let  $T \in \mathcal{L}(\mathcal{H})$ . For all  $\lambda \in \mathbb{C}$ ,*

- 1)  $\ker(\lambda \text{Id} - T) = \text{Im}(\lambda \text{Id} - T^*)^\perp$ ,
- 2)  $\ker(\lambda \text{Id} - T)^\perp = \overline{\text{Im}(\lambda \text{Id} - T^*)}$ .

*Proof.* For 1), we have:

$$\begin{aligned} x \in \ker(\lambda \text{Id} - T) &\Leftrightarrow \lambda x - Tx = 0 \Leftrightarrow \langle \lambda x - Tx, y \rangle = 0, \text{ for all } y \in \mathcal{H}, \\ &\Leftrightarrow \langle x, \bar{\lambda}y - T^*y \rangle = 0, \text{ for all } y \in \mathcal{H} \\ &\Leftrightarrow x \in \text{Im}(\bar{\lambda} \text{Id} - T^*)^\perp. \end{aligned}$$

Then, for 2), recall that for a vector space  $X$ , we have  $\overline{X} = (X^\perp)^\perp$ .  $\square$

**Proposition 2.35.** *Let  $T \in \mathcal{L}(\mathcal{H})$  be self-adjoint. Then*

- 1)  $\sigma(T) \subset \mathbb{R}$ .
- 2) For  $z \in \mathbb{C} \setminus \mathbb{R}$ , we have  $z \notin \sigma(T)$  and

$$\|(z\text{Id} - T)^{-1}\| \leq \frac{1}{\Im(z)}.$$

- 3) Let  $\lambda_1$  and  $\lambda_2$  two distinct eigenvalues of  $T$ , Then  $\ker(\lambda_1\text{Id} - T) \perp \ker(\lambda_2\text{Id} - T)$ .
- 4)  $T$  has at most a countable number of eigenvalues.

*Proof.* 1) Let  $z = a + ib$  with  $a, b \in \mathbb{R}$ . For all  $x \in \mathcal{H}$ , we have:

$$\begin{aligned} \|(z\text{Id} - T)x\|^2 &= \|(a\text{Id} - T)x + ibx\|^2 \\ &= \|(a\text{Id} - T)x\|^2 + \|bx\|^2 + \underbrace{\langle (a\text{Id} - T)x, ibx \rangle + \langle ibx, (a\text{Id} - T)x \rangle}_{=0} \\ (2.4.2) \quad &\geq |\Im(z)|^2 \cdot \|x\|^2. \end{aligned}$$

Therefore  $z\text{Id} - T$  is injective. Similarly,  $\bar{z}\text{Id} - T^*$  is also injective. We deduce from the previous proposition that  $z\text{Id} - T$  has dense image. Using (2.4.2) and some Cauchy sequence sequences we infer that the image is closed. The operator  $z\text{Id} - T$  is bijective. In particular  $\sigma(T) \subset \mathbb{R}$ .

- 2) We apply (2.4.2) with  $x = (z\text{Id} - T)^{-1}y$  to get

$$\|(z\text{Id} - T)^{-1}x\| \leq \frac{1}{|\Im z|} \|x\|,$$

for all  $x \in \mathcal{H}$ .

- 3) Let  $x_1 \in \ker(\lambda_1\text{Id} - T) \setminus \{0\}$  and  $x_2 \in \ker(\lambda_2\text{Id} - T) \setminus \{0\}$ . Since  $\lambda_1 \neq \lambda_2$ , one of them is non-zero, say  $\lambda_1 \neq 0$ . We have:

$$\langle x_1, x_2 \rangle = \frac{1}{\lambda_1} \langle Tx_1, x_2 \rangle = \frac{1}{\lambda_1} \langle x_1, Tx_2 \rangle = \frac{\lambda_2}{\lambda_1} \langle x_1, x_2 \rangle.$$

If  $\langle x_1, x_2 \rangle \neq 0$ , then  $\lambda_1 = \lambda_2$ . Contradiction.

- 4) By 3), the eigenspaces are two by two orthogonal. Suppose that the algebraical dimension of the eigenspaces is non-countable. We take  $x_i$ , with norm being 1 in every subspaces. This is a uncountable Hilbert basis for the Hilbert space given by the closure of the direct sum of the eigenspaces. Hence, the initial Hilbert space is not separable. Contradiction.  $\square$

**Proposition 2.36.** *Let  $T \in \mathcal{L}(\mathcal{H})$  be self-adjoint. Then*

$$\|A\| = \sup_{\|x\|=\|y\|=1} |\langle x, Ay \rangle| = \sup_{\|x\|=1} |\langle x, Ax \rangle|.$$

*Proof.* First we have:

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle y, Ax \rangle|,$$

where we used the Riesz's isomorphism in the second inequality. Indeed, the anti-linear form  $y \mapsto \langle y, Ax \rangle$  has norm  $\|Ax\|$  which is in turn equal to  $\sup_{\|y\|=1} |\langle y, Ax \rangle|$ .

We turn to the second equality. Set  $a := \sup_{\|x\|=\|y\|=1} |\langle x, Ay \rangle|$ . It is enough to show that  $|\langle x, Ay \rangle| \leq a$  for all  $x$  and  $y$  such that  $\|x\| = \|y\| = 1$ . If  $\langle x, Ay \rangle = 0$  there is nothing to do so we suppose it is non-zero. Set

$$b := \frac{\overline{\langle x, Ay \rangle}}{|\langle x, Ay \rangle|},$$

whose norm is 1. By polarisation, we have:

$$\begin{aligned} |\langle x, Ay \rangle| &= \langle x, A\lambda y \rangle = \Re \langle x, A\lambda y \rangle = \Re \left( \frac{1}{4} \sum_{k=0}^3 i^k \underbrace{\langle i^k x + \lambda y, A(i^k x + \lambda y) \rangle}_{\in \mathbb{R}} \right) \\ &= \frac{1}{4} (\langle x + \lambda y, A(x + \lambda y) \rangle - \langle -x + \lambda y, A(-x + \lambda y) \rangle) \\ &\leq \frac{a}{4} (\|x + \lambda y\|^2 + \|-x + \lambda y\|^2) \leq a, \end{aligned}$$

where we used in the last equality that  $\|x\| = \|y\| = |b| = 1$ .  $\square$

**Proposition 2.37.** *Let  $T \in \mathcal{L}(\mathcal{H})$  be self-adjoint. Let*

$$\begin{aligned} m &:= \inf \{ \langle x, Tx \rangle, x \in \mathcal{H} \text{ with } \|x\| = 1 \} \\ M &:= \sup \{ \langle x, Tx \rangle, x \in \mathcal{H} \text{ with } \|x\| = 1 \}. \end{aligned}$$

*Then  $\sigma(T) \subset [m, M]$ . Moreover,  $m$  and  $M$  belong to  $\sigma(T)$ . In particular*

$$\|T\| = \max(|m|, |M|)$$

*and  $\|T\| \in |\sigma(T)|$ .*

*Proof.* We have  $\langle x, Tx \rangle \geq m\|x\|^2$ , for all  $x \in \mathcal{H}$ . Therefore for  $\varepsilon > 0$ , we get

$$\langle x, (T - (m - \varepsilon)\text{Id})x \rangle \geq \varepsilon\|x\|^2,$$

for all  $x \in \mathcal{H}$ . Using Cauchy sequences, we deduce that  $(T - (m - \varepsilon)\text{Id})$  is injective and with closed range. Taking the adjoint, we also infer that the range is dense in  $\mathcal{H}$ . Then  $(T - (m - \varepsilon)\text{Id})$  is bijective and therefore  $]-\infty, m[ \subset \rho(T)$ . Repeating with  $-T$  we also get  $]M, \infty[ \subset \rho(T)$ .

Set now  $S := m\text{Id} + T$ . We have that  $\langle x, Sx \rangle \geq 0$ . By the previous proposition,  $\|S\| = M + m$ . Then since the spectral radius is equal to the norm in the case of a self-adjoint operator, we infer that,  $M + m \in \sigma(S) = \sigma(T) + m$ . Then  $M \in \sigma(T)$ . Repeat the proof with  $S := M\text{Id} - T$  for  $m$ .

The link with  $\|T\|$  is given by Proposition 2.36.  $\square$

We can also compute the spectrum with the help of *approximate eigenvalues*:

**Proposition 2.38.** *Let  $H \in \mathcal{L}(\mathcal{H})$  be self-adjoint. Then  $\lambda \in \sigma(H)$  if and only if*

$$\exists f_n \in \mathcal{H}, \|f_n\| = 1 \text{ and } \|(H - \lambda)f_n\| \rightarrow 0.$$

*Proof.* Take  $\lambda \in \sigma(H)$ . Since  $\sigma(H) \subset \mathbb{R}$ , (2.3.1) ensures that there are  $\lambda_n \in \rho(H)$  such that  $\lambda_n \rightarrow \lambda$  and such that  $\|(H - \lambda_n\text{Id})^{-1}\| \rightarrow \infty$ , when  $n \rightarrow \infty$ . For all  $\varepsilon > 0$ , there exist  $y_n \in \mathcal{H}$  with norm 1 such that

$$\|(H - \lambda_n\text{Id})^{-1}y_n\| \geq (1 - \varepsilon)\|(H - \lambda_n\text{Id})^{-1}\|.$$

Then, set

$$f_n := \frac{(H - \lambda_n \text{Id})^{-1} y_n}{\|(H - \lambda_n \text{Id})^{-1} y_n\|}.$$

Note that  $\|f_n\| = 1$ . Next, remark that

$$\begin{aligned} \|(H - \lambda \text{Id})f_n\| &\leq |\lambda - \lambda_n| + \|(H - \lambda_n)f_n\| \\ &\leq |\lambda - \lambda_n| + \frac{\|y_n\|}{(1 - \varepsilon)\|(H - \lambda_n \text{Id})^{-1}\|} \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . Therefore  $\lambda$  is an approximate eigenvalue for  $H$ .

Suppose now that  $\lambda$  is an approximate eigenvalue for  $H$ . Suppose also that  $\lambda \in \rho(H)$ . As  $(H - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ , there is  $c > 0$  such that

$$\|(H - \lambda)^{-1}x\| \leq c\|x\|, \text{ for all } x \in \mathcal{H}.$$

In particular, by taking  $x = (H - \lambda)f_n$ , we infer

$$1 = \|f_n\| \leq c\|(H - \lambda)f_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

This a contradiction. Hence,  $\lambda \in \sigma(H)$ .  $\square$

**2.5. Few words about compact operators.** We recall the main properties of compact operators. We refer for instance to [RS1, Section VI.5] for the proofs.

**Definition 2.39.** Let  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ , where  $\mathcal{X}$  and  $\mathcal{Y}$  are two Banach spaces. We say that  $T$  is a compact operator if  $TB_{\mathcal{X}}(0, 1)$  is relatively compact. We denote the set of compact operators by  $\mathcal{K}(\mathcal{X}, \mathcal{Y})$  and simply by  $\mathcal{K}(\mathcal{X})$  if  $\mathcal{X} = \mathcal{Y}$ .

**Theorem 2.40.** Let  $\mathcal{H}$  be a Hilbert space with  $\dim(\mathcal{H}) = \infty$  and let  $K \in \mathcal{K}(\mathcal{H})$ .

- 1)  $0 \in \sigma(K)$ .
- 2)  $\sigma(K) \setminus \{0\}$  is exclusively constituted of eigenvalues of finite multiplicity. They can accumulate solely in 0.
- 3) Assume also that  $K$  is self-adjoint. Let  $\{0\} \cup \{\lambda_n\}_{n \in I}$  be the spectrum of  $K$ . Denoting by  $f_n$  the eigenfunctions of  $K$  associated to  $\lambda_n$  we have that

$$\mathcal{H} = \ker(K) \bigoplus \bigoplus_{n \in I} \mathbb{C}f_n.$$

The compact operators have the property to improve the convergence.

**Proposition 2.41.** Let  $K \in \mathcal{K}(\mathcal{H})$ . Let  $f_n$  that tends weakly to  $f$  in  $\mathcal{H}$ , i.e.,  $\langle f_n, g \rangle \rightarrow \langle f, g \rangle$ , for all  $g \in \mathcal{H}$ , as  $n \rightarrow \infty$ . Then,

- 1) There is  $M > 0$  such that  $\|f_n\| \leq M$ , for all  $n \in \mathbb{N}$ .
- 2)  $Kf_n$  tends in norm to  $Kf$ , i.e.,  $\|K(f_n - f)\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proposition 2.42.** Let  $K \in \mathcal{K}(\mathcal{H})$ . Let  $T_n \in \mathcal{L}(\mathcal{H})$  for  $n \in \mathbb{N}$  and  $T \in \mathcal{L}(\mathcal{H})$ .

- 1) Assume that  $T_n$  tends weakly to  $T$ , i.e.,

$$\langle T_n f, g \rangle \rightarrow \langle T f, g \rangle,$$

for all  $f, g \in \mathcal{H}$ , as  $n \rightarrow \infty$ . Then there is  $M > 0$  such that  $\|T_n\| \leq M$ , for all  $n \in \mathbb{N}$ . Moreover,  $KT_n$  tends strongly to  $KT$ , i.e.,

$$\|KT_n f\| \rightarrow \|KT f\|,$$

for all  $f \in \mathcal{H}$ , as  $n \rightarrow \infty$ .

- 2) Assume that  $KT_n$  tends strongly to  $T$ , then there is  $M > 0$  such that  $\|T_n\| \leq M$ , for all  $n \in \mathbb{N}$ . Moreover,  $KT_n$  tends in norm to  $KT$ , i.e.  $\|KT_n - KT\| \rightarrow 0$ , as  $n \rightarrow \infty$ .

Note that

$$\text{norm convergence} \Rightarrow \text{strong convergence} \Rightarrow \text{weak convergence}.$$

**2.6. Operators of multiplication.** We start with the discrete case.

**Proposition 2.43.** Let  $\mathcal{H} := \ell^2(\mathbb{N}; \mathbb{C})$ . Let  $F \in \ell^\infty(\mathbb{N}; \mathbb{C})$ . We denote by  $F(Q)$  the operator of multiplication by  $F$ , i.e., for all  $f \in \mathcal{H}$ ,

$$(F(Q)f)(n) := F(n)f(n), \quad \text{for all } n \in \mathbb{N}.$$

- 1)  $F(Q)$  is bounded.
- 2)  $F(Q)$  is normal.
- 3)  $F(Q)$  is self-adjoint if and only if  $F(n) \in \mathbb{R}$ , for all  $n \in \mathbb{N}$ .
- 4)  $F(Q)$  is unitary if and only if  $|F(n)| = 1$ , for all  $n \in \mathbb{N}$ .
- 5) Given  $\lambda \in \mathbb{C}$ , we have:

$$\ker(\lambda \text{Id} - F(Q)) = \{f \in \ell^2(\mathbb{N}; \mathbb{C}) \text{ with support in } F^{-1}(\{\lambda\})\}.$$

In particular  $\cup_{n \in \mathbb{N}} \{F(n)\}$  is the set of eigenvalues of  $F(Q)$ .

- 6)  $\sigma(F(Q)) = \overline{\cup_{n \in \mathbb{N}} \{F(n)\}}$ .
- 7)  $F(Q)$  is compact if and only if  $\lim_{n \rightarrow \infty} F(n) = 0$ .
- 8)  $F(Q)$  is of finite rank if and only if  $F$  has finite support.

*Proof.* 1) The operator is bounded because:

$$\|F(Q)f\|^2 = \sum_{n \in \mathbb{N}} |F(n)f(n)|^2 \leq \|F\|_\infty^2 \|f\|^2.$$

Then note that  $(F(Q))^* = \overline{F}(Q)$ . Therefore, 2), 3) and 4) follow directly.

5) Set  $\lambda \in \mathbb{C}$ . Let  $f$  be with support in  $F^{-1}(\{\lambda\})$ . We obtain

$$(2.6.3) \quad F(n)f(n) = \lambda f(n), \quad \forall n \in \mathbb{N}.$$

because  $f(n) = 0$  when  $F(n) \neq \lambda$ . Hence  $f \in \ker(\lambda \text{Id} - F(Q))$ . Reciprocally, suppose that  $f \in \ker(\lambda \text{Id} - F(Q))$ . This yields (2.6.3). Given  $n$  such that  $F(n) \neq \lambda$  we infer from  $(F(n) - \lambda)f(n) = 0$  that  $f(n) = 0$ . In particular,  $f$  is with support in  $F^{-1}(\{\lambda\})$ .

6) By 5), since the spectrum is closed, we obtain  $\overline{\cup_{n \in \mathbb{N}} \{F(n)\}} \subset \sigma(F(Q))$ . Let  $\lambda \notin \overline{\cup_{n \in \mathbb{N}} \{F(n)\}}$ . Set

$$G(n) := \frac{1}{\lambda - F(n)}.$$

Since  $\lambda \notin \overline{\cup_{n \in \mathbb{N}} \{F(n)\}}$ , note that  $G \in \ell^\infty(\mathbb{N}; \mathbb{C})$ . In particular, by 1),  $G(Q)$  is bounded. Finally observe that

$$(F(Q) - \lambda \text{Id})G = G(F(Q) - \lambda \text{Id}) = \text{Id}.$$

Hence  $\lambda \notin \sigma(F(Q))$ . By contraposition, this yields  $\sigma(F(Q)) \subset \overline{\cup_{n \in \mathbb{N}} \{F(n)\}}$ .

7) Suppose that  $F(Q)$  is compact. Therefore the spectrum of  $F(Q)$  is the union of  $\{0\}$  and of a sequence (that could be finite or empty) of non-zero eigenvalues



that converges to 0. Recalling 5) we obtain that  $\lim_{n \rightarrow \infty} F(n) = 0$ . Suppose now that  $(F(n))_{n \in \mathbb{N}}$  does not converges to 0, since the sequence is bounded there is a subsequence  $(F(\varphi(n)))_{n \in \mathbb{N}}$  that converges to some  $\lambda \neq 0$ , where  $\varphi$  is an increasing function. Suppose that  $F(Q)$  is compact. We have that  $\|\delta_{\varphi(n)}\| = 1$  and that  $(\delta_{\varphi(n)})_n$  tends weakly to 0. By compactness of  $F(Q)$ , as  $n \rightarrow \infty$ ,  $F(Q)\delta_{\varphi(n)}$  tends to 0. However,  $\|F(Q)\delta_{\varphi(n)}\|$  tends to  $|\lambda| \neq 0$ . Contradiction.

8) Note that  $\text{ran}(F(Q)) = \{F(Q)f, f \in \ell^2(\mathbb{N}; \mathbb{C})\}$ . If  $F$  is with finite support, clearly  $\dim \text{ran}(F(Q))$  is a finite dimension. Suppose that  $F(Q)$  is of finite rank, then the spectrum of  $F(Q)$  is given by  $\{0\}$  and a finite number of eigenvalue of finite multiplicity. Using 5), we infer that  $F$  is with finite support.  $\square$

**Exercise 2.2.** Let  $\mathcal{H} := \ell^2(\mathbb{N}; \mathbb{C})$ . Give  $F \in \ell^\infty(\mathbb{N}; \mathbb{C})$  such that  $\sigma(F(Q)) = [0, 2]$ .

We now turn to the continuous case. Here  $[0, 1]$  plays no special rôle but is here to fix ideas. One could redo the proof with  $X$  being a compact set of  $\mathbb{R}^n$  with almost no change.

**Proposition 2.44.** Let  $\mathcal{H} := L^2([0, 1]; \mathbb{C})$ . Let  $F \in \mathcal{C}^0([0, 1]; \mathbb{C})$ . We denote by  $F(Q)$  the operator of multiplication by  $F$ , i.e., for all  $f \in \mathcal{H}$ ,

$$(F(Q)f)(x) := F(x)f(x), \quad \text{for all } x \in [0, 1].$$

- 1)  $F(Q)$  is bounded.
- 2)  $F(Q)$  is normal.
- 3)  $F(Q)$  is self-adjoint if and only if  $F(x) \in \mathbb{R}$ , for all  $x \in [0, 1]$ .
- 4)  $F(Q)$  is unitary if and only if  $|F(x)| = 1$ , for all  $x \in [0, 1]$ .
- 5)  $\lambda$  is an eigenvalue of  $F(Q)$  if and only if  $F^{-1}(\{\lambda\})$  is of non-empty interior. It this holds,  $\lambda$  is an eigenvalue of infinite multiplicity.
- 6)  $\sigma(F(Q)) = F([0, 1])$ .
- 7)  $F(Q)$  is compact if and only if  $F \equiv 0$

*Proof.* Since  $F$  is continuous and  $[0, 1]$  compact,  $\|F\|_\infty$  is finite. Then, for all  $f \in L^2([0, 1])$ , we have:

$$\|F(Q)f\|^2 \leq \int_0^1 |f(x)|^2 \cdot |F(x)|^2 dx \leq \|F\|_\infty^2 \|f\|^2.$$

Then note that  $(F(Q))^* = \overline{F}(Q)$ . Therefore, 2), 3) and 4) follow directly.

5) If  $g \in \ker(F(Q) - \lambda \text{Id}) \setminus \{0\}$ , we have that:

$$(F(x) - \lambda)g(x) = 0, \quad \text{pour tout } x \in [0, 1].$$

On the other hand,  $g^{-1}(\{0\})$  is closed, because  $g$  is continuous. Then, for all  $x_0 \in g^{-1}(\mathbb{C} \setminus \{0\})$ , there exists an open neighbourhood  $\mathcal{V}_{x_0}$  of  $x_0$  such that  $\mathcal{V}_{x_0} \subset g^{-1}(\mathbb{C} \setminus \{0\})$ . On  $\mathcal{V}_{x_0}$ , we can divide by  $g(x)$  and infer:

$$(F(x) - \lambda) = 0, \quad \text{for all } x \in \mathcal{V}_{x_0}.$$

We conclude that  $\mathcal{V}_{x_0} \subset f^{-1}(\{\lambda\})$  has non-empty interior.

Suppose now that  $F^{-1}(\{\lambda\})$  has non-empty interior. There exists an open and non-empty interval  $\mathcal{O}$  such that  $\mathcal{O} \subset F^{-1}(\{\lambda\})$ . For all  $g \in \mathcal{C}([0, 1]; \mathbb{C})$  non-zero and with support in  $\mathcal{O}$ , we have

$$(F(x) - \lambda)g(x) = 0, \quad \text{for all } x \in [0, 1].$$

Thus,  $g \in \ker(F(Q) - \lambda \text{Id}) \setminus \{0\}$ . Moreover, for all  $N \in \mathbb{N}^*$  and all  $i \in \{1, \dots, N\}$ , we can construct  $g_i \in \mathcal{C}([0, 1]; \mathbb{C}) \setminus \{0\}$  with support in  $\mathcal{O}$  and such that the supports are two by two disjoint. The function  $g_i$  are linearly independent and eigenvalues of  $F(Q)$  associated to  $\lambda$ . Therefore, for all  $N \in \mathbb{N}^*$ , we have  $\dim \ker(T_f - \lambda \text{Id}) \geq N$ .

6) If  $F^{-1}(\lambda) = \emptyset$ , we have that  $F(x) - \lambda \neq 0$ , for all  $x \in [0, 1]$ . In particular, since  $F$  is continuous,  $\inf |f(x) - \lambda| > 0$  and  $x \mapsto 1/(f(x) - \lambda)$  is continuous. Therefore  $1/(F(Q) - \lambda)$  is bounded and

$$(F(Q) - \lambda) \frac{1}{F(Q) - \lambda} = \frac{1}{F(Q) - \lambda} (F(Q) - \lambda) = \text{Id}.$$

Hence,  $\lambda \in \rho(F(Q) - \lambda)$ .

Suppose now that there is  $x_0 \in [0, 1]$  such that  $F(x_0) = \lambda$ . If there is an open neighbourhood  $\mathcal{O}$  of  $x_0$  such that  $F(\mathcal{O}) = \lambda$ , then  $\lambda$  is an eigenvalue. If this is not the case, for all  $\varepsilon > 0$  there exists an open interval  $\mathcal{I} = ]x_0, x_0 + \eta[$  or  $]x_0 - \eta, x_0[$  for some  $\eta > 0$  such that  $0 < |F(x) - \lambda| < \varepsilon$  for all  $x \in \mathcal{I}$ . Suppose that  $F(Q) - \lambda$  is invertible. There is a bounded operator  $B$  such that, for all  $g \in L^2([0, 1]; \mathbb{C})$ . We infer that

$$(F(x) - \lambda)(Bg)(x) = g(x), \text{ a.e.}$$

In particular, for almost  $x \in \mathcal{I}$ , we have:

$$(Bg)(x) = \frac{1}{F(x) - \lambda} g(x), \text{ a.e. in } \mathcal{I}.$$

To fix ideas, take  $\mathcal{I} = ]x_0, x_0 + \eta[$ . Set  $g_n(x) := 1_{[x_0, x_0 + 1/n]} \sqrt{n}$ . We have  $\|g_n\| = 1$  and for  $n$  large enough

$$|(Bg_n)(x)| \geq \frac{1}{\varepsilon} |g_n(x)|, \text{ a.e. in } \mathcal{I}.$$

In particular

$$\|B\| \geq \frac{1}{\varepsilon}.$$

By letting  $\varepsilon$  goes to 0, this yields that the norm of  $B$  is not finite. We conclude that  $\lambda \in \sigma(F(Q))$  is not invertible.

Finally we turn to 7). If  $F \equiv 0$  this is clear. Suppose that  $F(Q)$  is compact. Then recalling that  $\sigma(F(Q)) \setminus \{0\}$  is constituted of eigenvalues of finite multiplicity, 5) implies that  $\sigma(F(Q)) = \{0\}$ . By 6) we conclude that  $F \equiv 0$ .  $\square$

**Exercise 2.3.** State this result for  $F \in L^\infty([0, 1]; \mathbb{C})$ .

### 3. EXAMPLES OF SELF-ADJOINT OPERATORS ACTING ON THE DISCRETE LINE

We now develop the spectral analysis of some basic but important models.

**3.1. The adjacency matrix acting on  $\mathbb{Z}$ .** Let  $\mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C})$ . We define the adjacency matrix by:

$$(\mathcal{A}f)(n) := f(n-1) + f(n+1), \quad \text{for } f \in \mathcal{H}.$$

It is a self-adjoint operator. Indeed we have for all  $g, f \in \mathcal{H}$ :

$$\langle f, \mathcal{A}g \rangle = \sum_{n \in \mathbb{Z}} \overline{f(n)} (g(n+1) + g(n-1)) = \sum_{n \in \mathbb{Z}} \overline{f(n+1) + f(n-1)} g(n) = \langle \mathcal{A}f, g \rangle$$

The Fourier transform  $\mathcal{F} : \ell^2(\mathbb{Z}) \rightarrow L^2([-\pi, \pi])$  is defined by

$$(\mathcal{F}f)(x) := \frac{1}{\sqrt{2\pi}} \sum_n f(n) e^{-inx}, \text{ for all } f \in \ell^2(\mathbb{Z}) \text{ and } x \in [-\pi, \pi].$$

It is unitary and its inverse is given by:

$$(\mathcal{F}^{-1}f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{ikx} dx, \text{ for all } f \in L^2([-\pi, \pi]) \text{ and } k \in \mathbb{Z}.$$

We take advantage of the Fourier Transform to study  $\mathcal{A}$  and set:

$$\tilde{\mathcal{A}} := \mathcal{F} \mathcal{A} \mathcal{F}^{-1}.$$

Let  $f \in L^2([-\pi, \pi])$ . We have:

$$\begin{aligned} (\tilde{\mathcal{A}}f)(x) &= \mathcal{F}(\mathcal{A} \mathcal{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \sum_n e^{-inx} (\mathcal{A} \mathcal{F}^{-1}f)(n) \\ &= \frac{1}{\sqrt{2\pi}} \sum_n e^{-inx} ((\mathcal{F}^{-1}f)(n+1) + (\mathcal{F}^{-1}f)(n-1)) \\ &= \frac{1}{2\pi} \sum_n e^{-inx} \int_{-\pi}^{\pi} (e^{i(n+1)t} f(t) + e^{i(n-1)t} f(t)) dt \\ &= \frac{1}{2\pi} \sum_n e^{-inx} \int_{-\pi}^{\pi} e^{int} 2 \cos(t) f(t) dt = 2 \cos(x) f(x). \end{aligned}$$

Therefore

$$\tilde{\mathcal{A}} := \mathcal{F} \mathcal{A} \mathcal{F}^{-1} = 2 \cos(Q).$$

In particular by Proposition 2.44

$$\sigma(\mathcal{A}) = [-2, 2]$$

and  $\mathcal{A}$  has no eigenvalue.

**Exercise 3.1.** *Compute the spectrum of  $\mathcal{A}$  using the approximate eigenvalues approach.*

**3.2. The adjacency matrix acting on  $\mathbb{N}$ .** We turn to the half-line version. Let  $\mathcal{H} := \ell^2(\mathbb{N}; \mathbb{C})$ . For  $f \in \ell^2(\mathbb{N})$ , we define the adjacency matrix by:

$$(\mathcal{A}f)(n) := \begin{cases} f(n-1) + f(n+1), & \text{if } n \geq 1, \\ f(1), & \text{if } n = 0. \end{cases}$$

The Fourier transform  $\mathcal{F} : \ell^2(\mathbb{N}) \rightarrow L^2_{\text{odd}}([-\pi, \pi])$  is defined by

$$(\mathcal{F}f)(x) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{N}} f(n+1) \sin((n+1)x), \text{ for all } f \in \ell^2(\mathbb{N}) \text{ and } x \in [-\pi, \pi].$$

It is unitary.

We take advantage of this Fourier transform and obtain similarly

$$\tilde{\mathcal{A}} := \mathcal{F} \mathcal{A} \mathcal{F}^{-1} = 2 \cos(Q).$$

In particular:

$$\sigma(\mathcal{A}) = [-2, 2],$$

and  $\mathcal{A}$  has no eigenvalue.

**Exercise 3.2.** Compute  $\mathcal{F}^{-1}$  and show that  $\tilde{\mathcal{A}} = 2 \cos(Q)$ .

**3.3. The discret Dirac operator acting on  $\mathbb{Z}$ .** Next, we study properties of relativistic (massive or not) charged particles with spin-1/2. We follow the Dirac formalism, see [Di]. We shall focus on the 1-dimensional discrete version of the problem to simplify. The mass of the particle is given by  $m \geq 0$ . For simplicity, we re-normalize the speed of light and the reduced Planck constant by 1. Let  $\mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C}^2)$ , endowed with the scalar product

$$\langle f, g \rangle = \sum_{n \in \mathbb{Z}} \langle f(n), g(n) \rangle_{\mathbb{C}^2} = \sum_{n \in \mathbb{Z}} \overline{f_1(n)} g_1(n) + \overline{f_2(n)} g_2(n).$$

where  $f, g \in \mathcal{H}$ ,  $f(n) = \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix}$  and  $g(n) = \begin{pmatrix} g_1(n) \\ g_2(n) \end{pmatrix}$ .

The *Dirac discrete operator*, acting on  $\ell^2(\mathbb{Z}; \mathbb{C}^2)$ , is defined by

$$D_m := \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix},$$

where  $d := \text{Id} - \tau$  and  $\tau$  is the right shift, defined by

$$\tau f(n) = f(n+1), \text{ for all } f \in \ell^2(\mathbb{Z}; \mathbb{C}).$$

Note that  $\tau^* f(n) = f(n-1)$ , for all  $f \in \ell^2(\mathbb{Z}; \mathbb{C})$ .

The operator  $D_m$  is self-adjoint and we have:

$$D_m^2 = \begin{pmatrix} \Delta + m^2 & 0 \\ 0 & \Delta + m^2 \end{pmatrix},$$

where  $\Delta = 2 - \mathcal{A}_{\mathbb{Z}}$ . Recall that  $\sigma(\Delta) = 2 - \sigma(\mathcal{A}_{\mathbb{Z}}) = [0, 4]$ .

Since we have a direct sum, we have:

$$\sigma(D_m^2) = [m^2, m^2 + 4].$$

To remove the square above  $D_m$ , we define the symmetry  $S$  on  $\ell^2(\mathbb{Z}; \mathbb{C})$  by

$$Sf(n) = f(-n)$$

and the unitary operator on  $\ell^2(\mathbb{Z}; \mathbb{C}^2)$

$$U := \begin{pmatrix} 0 & iS \\ -iS & 0 \end{pmatrix}.$$

Clearly  $U = U^* = U^{-1}$ . We have that

$$UD_mU = -D_m.$$

In particular, we have

$$\sigma(D_m) = \sigma(-D_m) = \left[ -\sqrt{m^2 + 4}, -m \right] \cup \left[ m, \sqrt{m^2 + 4} \right]$$

and  $D_m$  has no eigenvalue. We refer to [GH] for a further spectral analysis of this model and references therein.

**Exercise 3.3.** Show that  $D_m$  is unitarily equivalent to

$$\begin{pmatrix} \sqrt{m^2 + 2 - 2\cos(Q)} & 0 \\ 0 & -\sqrt{m^2 + 2 - 2\cos(Q)} \end{pmatrix},$$

which acts in  $L^2([-\pi, \pi]; \mathbb{C}^2)$ . Compute the spectrum in an alternative way.

#### 4. OPERATOR ACTING ON GRAPHS

**4.1. Boundedness.** There are many type of discret graphs. Here we stick to the two of them. Let  $\mathcal{V}$  be a finite or countable set and let  $\mathcal{E} := \mathcal{V} \times \mathcal{V} \rightarrow \{0, 1\}$  such that

$$\mathcal{E}(x, y) = \mathcal{E}(y, x), \quad \text{for all } x, y \in \mathcal{V}.$$

We say that  $\mathcal{G} := (\mathcal{V}, \mathcal{E})$  is a non-oriented graph with *edges*  $\mathcal{E}$  and *vertices*  $\mathcal{V}$ .

We say that  $x, y \in \mathcal{V}$  are *neighbours* if  $\mathcal{E}(x, y) = 1$ . We write  $x \sim y$  and  $\mathcal{N}(x) := \{y \in \mathcal{V}, x \sim y\}$ .

The *degree* of  $x \in \mathcal{V}$  is given by:

$$\deg_G(x) := |\{y \in \mathcal{E} \mid x \sim y\}|.$$

In the sequel we suppose:

**Hypotheses:**  $\deg_G(x) < \infty$  and  $\mathcal{E}(x, x) = 0$  for all  $x \in \mathcal{V}$ .

Set

$$\mathcal{H} := \ell^2(\mathcal{V}; \mathbb{C}) := \{f : \mathcal{V} \rightarrow \mathbb{C}, \sum_{x \in \mathcal{V}} |f(x)|^2 < \infty\},$$

endowed with  $\langle f, g \rangle := \sum_{x \in \mathcal{V}} \overline{f(x)}g(x)$ . We set also

$$\mathcal{C}_c(\mathcal{V}) := \{f : \mathcal{V} \rightarrow \mathbb{C} \text{ with finite support}\}.$$

The *Laplacian* is given by:

$$\Delta f(x) = \sum_{y \sim x} (f(x) - f(y)), \quad \text{for all } f \in \mathcal{C}_c(\mathcal{V}).$$

The *adjacency matrix* is given by

$$\mathcal{A}f(x) = \sum_{y \sim x} f(y), \quad \text{for all } f \in \mathcal{C}_c(\mathcal{V}).$$

Note that  $\Delta = \deg_G(Q) - \mathcal{A}$ . They are both symmetric on  $\mathcal{C}_c(\mathcal{V})$ .

**Proposition 4.1.** *We have:*

1) *The following equivalences hold true:*

$$\Delta \text{ bounded} \iff \mathcal{A} \text{ bounded} \iff \deg(\cdot) \text{ bounded}.$$

*In particular, if  $\deg(\cdot)$  is bounded then  $\Delta$  and  $\mathcal{A}$  are self-adjoint.*

2) *The following estimate hold true:*

$$0 \leq \langle f, \Delta f \rangle \leq 2\langle f, \deg(Q)f \rangle, \text{ for all } f \in \mathcal{C}_c(\mathcal{V}).$$

*In particular,  $\sigma(\Delta) \subset [0, 2 \sup_{x \in \mathcal{V}} \deg(x)]$ .*

*Proof.* We start with the second point.

$$\begin{aligned} \langle f, \Delta f \rangle &= \frac{1}{2} \sum_{x \in \mathcal{V}} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) |f(x) - f(y)|^2 \\ &\leq \sum_{x \in \mathcal{V}} \sum_{y \sim x} (|f(x)|^2 + |f(y)|^2) = 2 \langle f, \deg(Q) f \rangle, \end{aligned}$$

for  $f \in \mathcal{C}_c(\mathcal{V})$ .

We turn to the first point. For  $\Delta$ , using 2) and that  $\langle \delta_x, \Delta \delta_x \rangle = \deg(x)$  we have the equivalence between  $\Delta$  and  $\deg$ .

We focus on  $\mathcal{A}$ .

$$|\langle f, \mathcal{A} f \rangle| = \left| \sum_{x \in \mathcal{V}} \overline{f(x)} \sum_{y \sim x} f(y) \right| \leq \frac{1}{2} \sum_x \sum_{y \sim x} (|f(x)|^2 + |f(y)|^2) = \langle f, \deg(Q) f \rangle.$$

Therefore, by Proposition 2.36, if  $\deg$  is bounded,  $\mathcal{A}$  is bounded. We turn to the other implication, since  $\mathcal{E}(x, y) \in \{0, 1\}$ , we have:

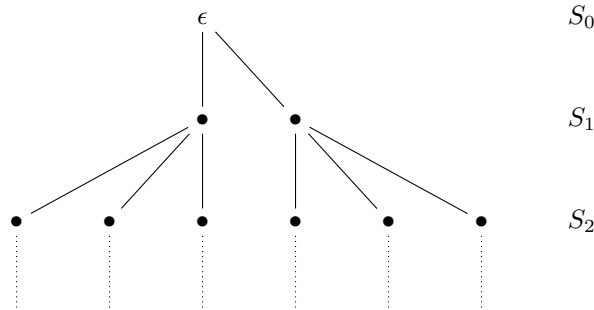
$$\begin{aligned} \|\mathcal{A} f\|^2 &= \sum_x \left| \sum_{y \sim x} \mathcal{E}(x, y) f(y) \right|^2 \geq \sum_x \sum_{y \sim x} \mathcal{E}(x, y) |f(y)|^2 = \sum_x \sum_{y \sim x} \mathcal{E}(x, y) |f(x)|^2 \\ &= \langle f, \deg(Q) f \rangle. \end{aligned}$$

Therefore,

$$\sup_{\|f\|=1} |\langle f, \deg(Q) f \rangle| \leq \sup_{\|f\|=1} \|\mathcal{A} f\|^2 \leq \|\mathcal{A}\|^2.$$

Again Proposition 2.36 concludes that  $\deg(Q)$  is bounded. It ends the proof.  $\square$

**4.2. Case of a tree.** Consider a tree  $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ , a connected graph with no cycle. Due to its structure, one can take any point of  $V$  to be a root. We denote it by  $\epsilon$ .



We define inductively the *spheres*  $S_n$  by  $S_{-1} = \emptyset$ ,  $S_0 := \{\epsilon\}$ , and  $S_{n+1} := \mathcal{N}(S_n) \setminus S_{n-1}$ . Given  $n \in \mathbb{N}$ ,  $x \in S_n$ , and  $y \in \mathcal{N}(x)$ , one sees that  $y \in S_{n-1} \cup S_{n+1}$ .

We write  $x \succ y$  and say that  $x$  is a *son* of  $y$ , if  $y \in S_{n-1}$ , while we write  $x \prec y$  and say that  $x$  is a *father* of  $y$ , if  $y \in S_{n+1}$ .

Notice that  $\epsilon$  has no father.

Given  $x \neq \epsilon$ , note that there is a unique  $y \in V$  with  $x \sim > y$ , i.e., everyone apart from  $\epsilon$  has one and only one father. We denote the father of  $x$  by  $\overleftarrow{x}$ .

Given  $x \in S_n$ , we set  $\ell(x) := n$ , the *length* of  $x$ . The *offspring* of an element  $x$  is given by

$$\text{off}(x) := |\{y \in \mathcal{N}(x), y \sim > x\}|,$$

i.e., it is the number of sons of  $x$ . When  $\ell(x) \geq 1$ , note that  $\text{off}(x) = \deg(x) - 1$ .

We consider the tree  $\mathcal{G} = (\mathcal{E}, \mathcal{V})$  with uniform offspring sequence  $(b_n)_{n \in \mathbb{N}}$ , i.e., every  $x \in S_n$  has  $b_n$  sons. We define:

$$(Uf)(x) := \mathbf{1}_{\{\cup_{n \geq 1} S_n\}}(x) \frac{1}{\sqrt{b_{\ell(\overleftarrow{x})}}} f(\overleftarrow{x}), \text{ for } f \in \ell^2(\mathcal{V}).$$

Easily, one get  $\|Uf\| = \|f\|$ , for all  $f \in \ell^2(\mathcal{V})$ . The adjoint  $U^*$  of  $U$  is given by

$$(U^*f)(x) := \frac{1}{\sqrt{b_{\ell(x)}}} \sum_{y \sim > x} f(y), \text{ for } f \in \ell^2(\mathcal{V}).$$

Note that one has:

$$(\mathcal{A}_G f)(x) = \sqrt{b_{\ell(\overleftarrow{x})}} (Uf)(x) + \sqrt{b_{\ell(x)}} (U^*f)(x), \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

Supposing now that  $b_n \geq 1$  for all  $n \in \mathbb{N}$ , we construct invariant subspaces for  $\mathcal{A}_G$ .

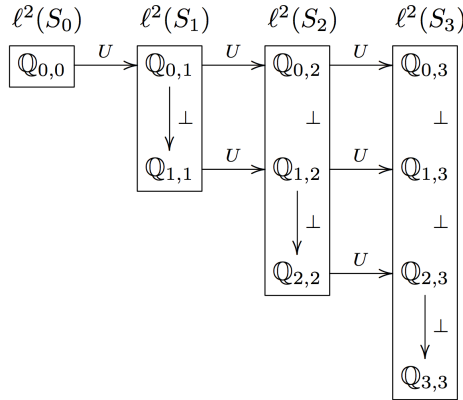
We start by noticing that  $\dim \ell^2(S_n) = \prod_{i=0, \dots, n-1} b_i$ , for  $n \geq 1$  and  $\dim \ell^2(S_0) = 1$ . Therefore, as  $U$  is an isometry,  $U\ell^2(S_n) = \ell^2(S_{n+1})$  if and only if  $b_n = 1$ .

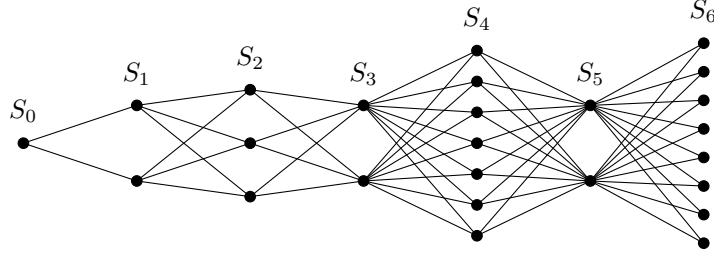
Set  $\mathbb{Q}_{0,0} := \ell^2(S_0)$  and  $\mathbb{Q}_{0,k} := U^k \mathbb{Q}_{0,0}$ , for all  $k \in \mathbb{N}$ . Note that  $\dim \mathbb{Q}_{0,k} = \dim \ell^2(S_0) = 1$ , for all  $k \in \mathbb{N}$ . Moreover, given  $f \in \ell^2(S_k)$ , one has  $f \in \mathbb{Q}_{0,k}$  if and only if  $f$  is constant on  $S_k$ .

We define recursively  $\mathbb{Q}_{n,n+k}$  for  $k, n \in \mathbb{N}$ . Given  $n \in \mathbb{N}$ , suppose that  $\mathbb{Q}_{n,n+k}$  is constructed for all  $k \in \mathbb{N}$ , and set

- $\mathbb{Q}_{n+1,n+1}$  as the orthogonal complement of  $\bigoplus_{i=0, \dots, n} \mathbb{Q}_{i,n+1}$  in  $\ell^2(S_{n+1})$ ,
- $\mathbb{Q}_{n+1,n+k+1} := U^k \mathbb{Q}_{n+1,n+1}$ , for all  $k \in \mathbb{N} \setminus \{0\}$ .

We sum-up the construction in the following diagram:



FIGURE 1. An antitree with spheres  $S_0, \dots, S_6$ .

We point out that  $\dim \mathbb{Q}_{n+1,n+1} = \dim \mathbb{Q}_{n+1,n+k+1}$ , for all  $k \in \mathbb{N}$  and stress that it is 0 if and only if  $b_n = 1$ . Notice that  $U^* \mathbb{Q}_{n,n} = 0$ , for all  $n \in \mathbb{N}$ .

Set finally  $\mathbb{M}_n := \bigoplus_{k \in \mathbb{N}} \mathbb{Q}_{n,n+k}$  and note that  $\ell^2(G) = \bigoplus_{n \in \mathbb{N}} \mathbb{M}_n$ . Moreover, one has that canonically  $\mathbb{M}_n \simeq \ell^2(\mathbb{N}; \mathbb{Q}_{n,n}) \simeq \ell^2(\mathbb{N}) \otimes \mathbb{Q}_{n,n}$ . In this representation, the restriction  $\mathcal{A}_n$  of  $\mathcal{A}$  to the space  $\mathbb{M}_n$  is given by the following tensor product of Jacobi matrices:

$$\mathcal{A}_n \simeq \begin{pmatrix} 0 & \sqrt{b_n} & 0 & 0 & \cdots \\ \sqrt{b_n} & 0 & \sqrt{b_{n+1}} & 0 & \ddots \\ 0 & \sqrt{b_{n+1}} & 0 & \sqrt{b_{n+2}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \otimes \mathbf{1}_{\mathbb{Q}_{n,n}}.$$

Now  $\mathcal{A}$  is given as the direct sum  $\bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$  in  $\bigoplus_{n \in \mathbb{N}} \mathbb{M}_n$ . This decomposition is extracted from [GS] and takes roots in [AF].

In particular, for a binary tree, i.e,  $b_n = 2$  for all  $n \in \mathbb{N}$ ,

$$\mathcal{A}_n \simeq \sqrt{2} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \ddots \\ 0 & 1 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \otimes \mathbf{1}_{\mathbb{Q}_{n,n}}.$$

Hence,  $\mathcal{A}$  is the infinite direct sum of copies of  $\sqrt{2} \mathcal{A}_{\mathbb{N}}$ .

We obtain that

$$\sigma(\mathcal{A}) = [-2\sqrt{2}, 2\sqrt{2}]$$

and  $\mathcal{A}$  has no eigenvalue.

**4.3. Case of an antitree.** We define the class of antitrees. This section is inspired from [BK, GS2]. The *sphere* of radius  $n \in \mathbb{N}$  around a vertex  $v \in \mathcal{V}$  is the set  $S_n(v) := \{w \in \mathcal{V} \mid d_G(v, w) = n\}$ . A graph is an *antitree*, if there exists a vertex  $v \in \mathcal{V}$  such that for all other vertices  $w \in \mathcal{V} \setminus \{v\}$

$$\mathcal{N}(w) = S_{n-1}(v) \cup S_{n+1}(v),$$

where  $n = d_G(v, w) \geq 1$ . The distinguished vertex  $v$  is the *root* of the antitree. Antitrees are bipartite and enjoy *radial symmetry*.



We denote the root by  $v$ , the spheres by  $S_n := S_n(v)$ , and their sizes by  $s_n := |S_n|$ . Further,  $|x| := d(v, x)$  is the distance of  $x \in V$  from the root.

The operator  $P : \ell^2(\mathcal{V}) \rightarrow \ell^2(\mathcal{V})$ , given by

$$Pf(x) := \frac{1}{s_{|x|}} \sum_{y \in S_{|x|}} f(y), \text{ for all } f \in \ell^2(\mathcal{V}) \text{ and } x \in \mathcal{V},$$

averages a function over the spheres. Thereby,  $P = P^2 = P^*$  is the orthogonal projection onto the space of radially symmetric functions in  $\ell^2(\mathcal{V})$ . A function  $f : \mathcal{V} \rightarrow \mathbb{C}$  is radially symmetric, if it is constant on spheres, i.e., for all nodes  $x, y \in \mathcal{V}$  with  $|x| = |y|$ , we have  $f(x) = f(y)$ .

For all radially symmetric  $f$ , we define  $\tilde{f} : \mathbb{N} \rightarrow \mathbb{C}$ ,  $\tilde{f}(|x|) := f(x)$ , for all  $x \in \mathcal{V}$ . Note that

$$P\ell^2(V) = \left\{ f : \mathcal{V} \rightarrow \mathbb{C}, f \text{ radially symmetric}, \sum_{n \in \mathbb{N}} s_n |\tilde{f}(n)|^2 < \infty \right\} \simeq \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}),$$

where  $(s_n)_{n \in \mathbb{N}}$  is now a sequence of weights.

The key observation is that

$$\mathcal{A} = PAP \text{ and } \widetilde{\mathcal{A}P}f(|x|) = s_{|x|-1} \widetilde{P}f(|x|-1) + s_{|x|+1} \widetilde{P}f(|x|+1),$$

for all  $f \in \mathcal{C}_c(V)$ , with the convention  $s_{-1} = 0$ .

Using the unitary transformation

$$U : \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}) \rightarrow \ell^2(\mathbb{N}), \quad U\tilde{f}(n) = \sqrt{s_n} \tilde{f}(n),$$

we see that  $\mathcal{A}$  is unitarily equivalent to the direct sum of 0 on  $(P\ell^2(V))^\perp$  and a Jacobi matrix acting on  $\ell^2(\mathbb{N})$  with 0 on the diagonal and the sequence  $(\sqrt{s_n} \sqrt{s_{n+1}})_{n \in \mathbb{N}}$  on the off-diagonal.

$$\mathcal{A} \simeq 0 \oplus \begin{pmatrix} 0 & \sqrt{s_0} \sqrt{s_1} & 0 & 0 & \cdots \\ \sqrt{s_0} \sqrt{s_1} & 0 & \sqrt{s_1} \sqrt{s_2} & 0 & \ddots \\ 0 & \sqrt{s_1} \sqrt{s_2} & 0 & \sqrt{s_2} \sqrt{s_3} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

In particular, if  $s_n = 2$  for all  $n \in \mathbb{N}$ ,  $\sigma(\mathcal{A}) = [-2, 2]$  and 0 is the only eigenvalue. It is of infinite multiplicity.

## 5. CONTINUOUS FUNCTIONAL CALCULUS

We pursue the analysis of self-adjoint operator and develop the theory of continuous functional calculus.

**5.1. Point and continuous spectrum.** We start with a first decomposition of the spectrum.

**Definition 5.1.** *Let  $H$  be a bounded self-adjoint operator. We set:*

$$\mathcal{H}_p := \mathcal{H}_p(H) := \overline{\{f \in \ker(\lambda - H), \lambda \in \sigma_p(H)\}}$$

the spectral subspace associated to  $\sigma_p(H)$ . We set also:

$$\mathcal{H}_c := \mathcal{H}_c(H) := \mathcal{H}_p^\perp$$

the spectral subspace associated to continuous spectrum of  $H$ .

**Theorem 5.2** (RAGE). *Let  $H$  be self-adjoint in  $\mathcal{H}$  and  $K$  be a compact operator in  $\mathcal{H}$ . Let  $\phi_0 \in \mathcal{H}_c(H)$ . We have:*

$$\frac{1}{T} \int_0^T \|K e^{itH} \phi_0\|^2 dt \rightarrow 0, \quad \text{as } T \rightarrow \infty,$$

where  $e^{itH}\phi$  is the unique solution to the Schrödinger equation:

$$\begin{cases} i(\partial_t \phi)(t) &= (H\phi)(t) \\ \phi(0) &= \phi_0. \end{cases}$$

The RAGE Theorem due to Ruelle [Rue], Amrein and Georgescu [AG] and Enss [E]. A proof may be found in [CFKS] for instance.

**Remark 5.3.** *In the previous examples, by taking  $K = 1_X(Q)$ , where  $X$  is a finite set, we see that if the initial condition is taken in the spectral subspaces associated to the continuous spectrum of  $H$  then it escapes, in average, every compact set.*

**Remark 5.4.** *We refer to C. Rojas-Molina's course for a different presentation. We also mention that she uses this theorem to prove the spectrum is purely point almost surely in the setting of random Schrödinger operators acting on  $\mathbb{Z}^d$ .*

**5.2. Motivation and polynomial case.** The aim is to localise more precisely in the spectrum a vector. For instance, one would like to know around which energy a  $\phi$  is taken in  $\mathcal{H}_c$ . We shall build the continuous functional calculus.

We take for instance  $\mathcal{A}_{\mathbb{Z}}$ . We have that

$$\mathcal{A}_{\mathbb{Z}} = \mathcal{F} 2 \cos(Q) \mathcal{F}^{-1},$$

where  $\mathcal{F}$  is a unitary transform.

Given  $f \in \mathcal{C}(\sigma(\mathcal{A}_{\mathbb{Z}}))$ , we can define the

$$f(\mathcal{A}_{\mathbb{Z}}) := \mathcal{F} f(2 \cos(Q)) \mathcal{F}^{-1}$$

Unfortunately, for a general self-adjoint operator  $H$ , it is complicate to find such a unitary transformation. To overcome this problem we will build directly  $f(H)$  by first considering polynomials and then by proceeding by density.

We aim at defining the continuous functional calculus for bounded self-adjoint operator. We start with polynomials. We define the operator  $P(T) \in \mathcal{L}(\mathcal{H})$  by:

$$P(T) := \sum_{k=0}^n a_k T^k, \quad \text{when } P(X) := \sum_{k=0}^n a_k X^k, \quad \text{with } n \in \mathbb{N} \text{ and } a_i \in \mathbb{C}.$$

Note that, given  $P, Q \in \mathbb{C}[X]$  and  $\lambda, \mu \in \mathbb{C}$ , we have:

$$\begin{aligned} (\lambda P + \mu Q)(T) &= \lambda(P(T)) + \mu(Q(T)) \\ (PQ)(T) &= P(T)Q(T) = Q(T)P(T). \end{aligned}$$

**Proposition 5.5** (Spectral mapping). *Given  $T \in \mathcal{L}(\mathcal{H})$  and  $P \in \mathbb{C}[X]$ , we have:*

$$P(\sigma(T)) = \sigma(P(T))$$

*Proof.* Let  $\lambda \in \mathbb{C}$ . We have  $\lambda$  root of  $P(\lambda) - P$ . There exists  $Q \in \mathbb{C}[X]$  such that  $P(\lambda) - P(X) = (\lambda - X)Q(X)$ , then

$$P(\lambda)\text{Id} - P(T) = (\lambda\text{Id} - T)Q(T) = Q(T)(\lambda\text{Id} - T).$$

If  $P(\lambda) \notin \sigma(P(T))$ , we set  $S := (P(\lambda)\text{Id} - P(T))^{-1}$ . We get:

$$(\lambda\text{Id} - T)Q(T)S = \text{Id} = SQ(T)(\lambda\text{Id} - T).$$

This implies that  $\lambda\text{Id} - T$  is invertible with inverse  $Q(T)S = SQ(T)$ . In particular  $\lambda \notin \sigma(T)$ . By contraposition we have proven that  $P(\sigma(T)) \subset \sigma(P(T))$ .

We turn to the reverse inclusion. It is enough to deal with  $\deg P = n \geq 1$ . Let  $\mu \in \sigma(P(T))$  and  $\lambda_1, \dots, \lambda_n$  roots of  $P - \mu$ . We have:

$$P(X) - \mu = c(X - \lambda_1) \dots (X - \lambda_n),$$

for some  $c \neq 0$ . This gives:

$$P(T) - \mu\text{Id} = c(T - \lambda_1\text{Id}) \dots (T - \lambda_n\text{Id}).$$

Since  $\mu \in \sigma(P(T))$ ,  $P(T) - \mu\text{Id}$  is not invertible, there exist  $i_0 \in \{1, \dots, n\}$  such that  $(T - \lambda_{i_0}\text{Id})$  is not invertible, then  $\lambda_{i_0} \in \sigma(T)$ . Moreover,  $P(\lambda_{i_0}) = \mu$ . Hence,  $\sigma(P(T)) \subset P(\sigma(T))$ . This concludes the proof.  $\square$

Let  $P \in \mathbb{C}[X]$  be given by  $P = \sum_{k=0}^n a_k X^k$ , we set:

$$\overline{P} := \sum_{k=0}^n \overline{a_k} X^k \quad \text{and} \quad |P|^2 := P\overline{P}.$$

We estimate the norm of  $P(T) := \sum_{k=0}^n a_k T^k$ . This step is crucial so as to extend to the case of continuous functions.

**Proposition 5.6.** *Let  $P \in \mathbb{C}[X]$ . Then  $P(T)^* = \overline{P}(T)$  and*

$$\|P(T)\| = \max_{t \in \sigma(T)} |P(t)|.$$

Note that we have a max because  $\sigma(T)$  is compact and  $P$  is continuous.

*Proof.* The fact that  $P(T)^* = \overline{P}(T)$  follows from  $T^* = T$ . Using (2.32), we obtain

$$\|P(T)\|^2 = \|P(T)P(T)^*\| = \|P(T)\overline{P}(T)\| = \||P|^2(T)\|.$$

Note then that  $|P|^2(T)$  is self-adjoint because

$$(5.2.4) \quad \langle x, |P|^2(T)y \rangle = \langle x, P(T)\overline{P}(T)y \rangle = \langle \overline{P}(T)P(T)x, y \rangle = \langle |P|^2(T)x, y \rangle,$$

for all  $x, y \in \mathcal{H}$ . Moreover  $|P|^2(T) \geq 0$  because

$$(5.2.5) \quad \langle x, |P|^2(T)x \rangle = \langle \overline{P}(T)x, \overline{P}(T)x \rangle \geq 0,$$

for all  $x \in \mathcal{H}$ . Finally,

$$\begin{aligned} \|P(T)\|^2 &= \| |P|^2(T) \|, \quad \text{by (5.2.4)} \\ &= \max \sigma(|P|^2(T)), \quad \text{by (5.2.5) and Proposition 2.37} \\ &= \max_{t \in \sigma(T)} |P|^2(t), \quad \text{by Proposition 5.5} \\ &= \left( \max_{t \in \sigma(T)} |P(t)| \right)^2. \end{aligned}$$

which gives the result.  $\square$

**5.3. The continuous case.** We recall the theorem of Stone-Weierstrass, e.g., [RS1, Theorem IV.9].

**Theorem 5.7** (Stone-Weierstrass). *Let  $K$  a Hausdorff compact space. Let  $\mathcal{A}$  be a sub-algebra of  $\mathcal{C}(K; \mathbb{C})$ , endowed with the uniform norm, such that:*

- 1) *If  $f \in \mathcal{A}$  then  $\bar{f} \in \mathcal{A}$ .*
- 2)  *$\mathcal{A}$  separates points, i.e., for all  $x \neq y$  in  $K$ , there exists  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .*
- 3) *The identity belongs to  $\mathcal{A}$ .*

*Then  $\overline{\mathcal{A}} = \mathcal{C}(K; \mathbb{C})$ .*

We deduce the main theorem.

**Theorem 5.8** (Continuous functional calculus). *Let  $T \in \mathcal{L}(\mathcal{H})$  be a self-adjoint operator. There is a unique continuous morphism  $\Phi : \mathcal{C}(\sigma(T)) \rightarrow \mathcal{L}(\mathcal{H})$  (of  $*$ -algebra) satisfying:*

- 1)  $\Phi(P) = P(T)$ , for all  $P \in \mathbb{C}[X]$ ,
- 2)  $\Phi(f + \lambda g) = \Phi(f) + \lambda \Phi(g)$ ,
- 3)  $\Phi(fg) = \Phi(f)\Phi(g)$ ,
- 4)  $\Phi(\bar{f}) = (\Phi(f))^*$ ,

*for all  $f, g \in \mathcal{C}(\sigma(T))$  and  $\lambda \in \mathbb{C}$ . Moreover,  $\Phi$  is an isometry, i.e.,*

$$(5.3.6) \quad \|\Phi(f)\| = \max_{t \in \sigma(T)} |f(t)|, \text{ for all } f \in \mathcal{C}(\sigma(T)).$$

**Remark 5.9.** *In the sequel, we shall denote  $\Phi(f)$  by  $f(T)$ .*

*Proof.* We set

$$\Phi_0 : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}), \Phi_0(f) := f(T)$$

where

$$\mathcal{A} := \{P|_{\sigma(T)}, \text{ with } P \in \mathbb{C}[X]\},$$

endowed with the sup norm.

First note that if  $P$  and  $Q$  are two polynomials with the same restriction to  $\sigma(T)$ . Then,

$$\|P(t) - Q(T)\| = \|(P - Q)(T)\| = \max_{t \in \sigma(T)} |(P - Q)(t)| = 0.$$

This means that  $P(T) = Q(T)$ . Therefore  $\Phi_0$  is well-defined.

Notice that  $\Phi_0$  is an isometry. By Stone-Weierstrass Theorem we see that  $\mathcal{A}$  is dense in  $\mathcal{C}(\sigma(T))$ , for the sup norm. By density, there exists a unique linear map

$$\Phi : \mathcal{C}(\sigma(T)) \rightarrow \mathcal{L}(\mathcal{H})$$

such that  $\Phi|_{\mathcal{A}} = \Phi_0$  and such that  $\|\Phi\|_{\mathcal{L}(\mathcal{C}(\sigma(T)), \mathcal{H})} = \|\Phi_0\|_{\mathcal{L}(\mathcal{A}, \mathcal{H})}$ . Moreover, since  $\Phi_0$  satisfy 2, 3 et 4 and that is an isometry, by density  $\Phi$  also satisfies the points.  $\square$

**Remark 5.10.** *We stress that if  $\lambda \in \rho(T)$ , we obtain:*

$$\|(\lambda \text{Id} - T)^{-1}\| = \frac{1}{d(\lambda, \sigma(T))}.$$

*This equality does not hold true in general for bounded operators but also holds true for normal operators.*

We link the functional calculus with the spectrum.

**Proposition 5.11.** *Let  $H$  be self-adjoint. We have:*

$$\sigma(H) = \{\lambda \in \mathbb{R}, \varphi(H) \neq 0, \text{ for all } \varphi \in \mathcal{C}(\sigma(H); \mathbb{C}) \text{ with } \varphi(\lambda) \neq 0\}$$

*Proof.* Let  $\lambda \in \sigma(H)$ . Take  $\varphi \in \mathcal{C}(\sigma(H); \mathbb{C})$  with  $\varphi(\lambda) \neq 0$ . By (5.3.6), we see that  $\|\varphi(H)\| \geq |\varphi(\lambda)| \neq 0$ . In particular,  $\varphi(H) \neq 0$ . Assume now that  $\lambda \notin \sigma(H)$ . There exists  $\varphi \in \mathcal{C}(\sigma(H); \mathbb{C})$  such that  $\varphi(\lambda) \neq 0$  and such that  $\varphi|_{\sigma(H)} = 0$ . By (5.3.6), we obtain that  $\|\varphi(H)\| = 0$ , i.e.,  $\varphi(H) = 0$ . By contraposition we obtain the reverse inclusion.  $\square$

We pursue the link with the spectrum and extend the spectral mapping theorem.

**Proposition 5.12** (Spectral mapping). *Given  $T \in \mathcal{L}(\mathcal{H})$  self-adjoint and  $f \in \mathcal{C}(\sigma(T); \mathbb{C})$ . Then,*

$$\sigma(f(T)) = f(\sigma(T)).$$

*Proof.* Let  $\lambda \notin f(\sigma(T))$ . We set  $g(t) := (\lambda - f(t))^{-1}$ . We have  $g \in \mathcal{C}(\sigma(T))$ . By functional calculus,

$$g(T)(\lambda \text{Id} - f(T)) = (\lambda \text{Id} - f(T))g(T) = \text{Id}.$$

Then,  $\lambda \notin \sigma(f(T))$ , i.e.,  $\sigma(f(T)) \subset f(\sigma(T))$ .

Set now  $\lambda \in f(\sigma(T))$ . For all  $n \in \mathbb{N}$ , we choose  $g_n \in \mathcal{C}_c(\mathbb{R}; [0, 1])$  being 1 in  $\lambda$  and 0 away from  $[\lambda - 1/n, \lambda + 1/n]$ . By functional calculus,

$$\|(\lambda \text{Id} - f(T))g_n(T)\| = \max_{t \in [\lambda - 1/n, \lambda + 1/n] \cap \sigma(T)} |(\lambda - f(t))g_n(t)| \rightarrow 0,$$

when  $n \rightarrow \infty$ .

Note also that  $\|g_n(T)\| = 1$ . Then, there exists a sequence  $x_n$  with norm 1 such that  $\|g_n(T)x_n\| \geq 1/2$ . We set

$$y_n := \frac{g_n(T)x_n}{\|g_n(T)x_n\|}.$$

We have  $\|y_n\| = 1$  and

$$\|(\lambda \text{Id} - f(T))y_n\| \leq 2\|(\lambda \text{Id} - f(T))g_n(T)\| \cdot \|x_n\| \rightarrow 0.$$

In particular  $\lambda \in \sigma(f(T))$ .  $\square$

The functional calculus is stable by composition.

**Proposition 5.13.** *Let  $T \in \mathcal{L}(\mathcal{H})$  be self-adjoint,  $f \in \mathcal{C}(\sigma(T))$  and  $g \in \mathcal{C}(f(\sigma(T)))$ . Then,*

$$g(f(T)) = (g \circ f)(T).$$

Recall that  $f(\sigma(T)) = \sigma(f(T))$ . Then  $g(f(T))$  has a meaning by applying the functional calculus for  $f(T)$ .

*Proof.* Set

$$\mathcal{A} := \{g \in \mathcal{C}(f(\sigma(T))), g(f(T)) = (g \circ f)(T)\}.$$

Clearly  $\mathcal{A}$  is an algebra and  $\mathcal{A}$  contains the function 1. Moreover, the function  $g$  defined by  $g(x) = x$  is in  $\mathcal{A}$ , because  $g(f(T)) = f(T)$  and  $g \circ f = f$ . Besides, the functions separates points. Take now  $g \in \mathcal{A}$ . We have:

$$\bar{g}(f(T)) = (g(f(T)))^* = ((g \circ f)(T))^* = \bar{g} \circ f(T),$$

the  $\mathcal{A}$  is stable by conjugaison. By Stone-Weirstrass, we get:  $\bar{\mathcal{A}} = \mathcal{C}(f(\sigma(T)))$ .

It remains to show that  $\mathcal{A}$  is closed. Let  $g_n \in \mathcal{C}(f(\sigma(T)))$  that tends to  $g \in \mathcal{C}(f(\sigma(T)))$  for the sup norm. By functional calculus for  $f(T)$ , we see that  $\|g(f(T)) - g_n(f(T))\| \rightarrow 0$ , when  $n \rightarrow \infty$ . Then, by functional calculus for  $T$ , as  $g_n \circ f$  tends uniformly to  $g \circ f$ , we have that  $\|(g \circ f)(T) - (g_n \circ f)(T)\| \rightarrow 0$ , when  $n \rightarrow \infty$ . Then  $g \in \mathcal{A}$  and  $\mathcal{A}$  is closed.  $\square$

**Exercise 5.1.** *Let  $H$  be a self-adjoint operator.*

1) *Prove that*

$$e^{itH} = \sum_{n=0}^{\infty} \frac{(itH)^n}{n!},$$

*where the left hand side is given by functional calculus.*

2) *Prove that  $e^{itH}$  is unitary.*

**Exercise 5.2.** *Let  $H \in \mathcal{L}(\mathcal{H})$  such that  $\langle f, Hf \rangle \geq 0$ , for all  $f \in \mathcal{H}$ .*

- 1) *Prove that  $H$  is self-adjoint. (Hint: Use the polarisation identity)*
- 2) *Prove that  $\sigma(H) \subset [0, \infty[$ .*
- 3) *Prove that there is (a unique)  $T$  self-adjoint with  $\sigma(T) \subset [0, \infty[$ , such that  $T^2 = H$ . It is the square root of  $H$ .*

## 6. OTHER FUNCTIONAL CALCULI

We give now more or less explicit ways to deal with the functional calculus of  $H$  self-adjoint.

**6.1. The Fourier approach.** Let  $f \in L^1(\mathbb{R}; \mathbb{C})$ . Set

$$\hat{f}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} f(t) e^{-it\xi} dt,$$

Assume that  $\hat{f} \in L^1(\mathbb{R}; \mathbb{C})$ . Then we have:

$$f(H) = \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi H} d\xi,$$

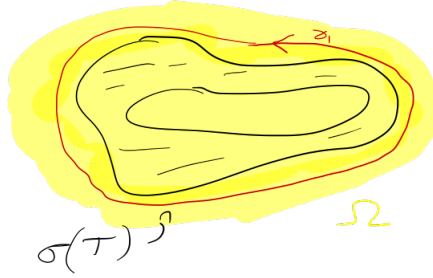
where the integral exists in  $\mathcal{L}(\mathcal{H})$ .

**Exercise 6.1.** Where do we use that  $\hat{f} \in L^1(\mathbb{R}; \mathbb{C})$ ? Prove the equality.

**6.2. The Holomorphic approach.** Let  $f$  be holomorphic in an open neighbourhood  $\Omega$  of  $\sigma(H)$ , where  $H$  is bounded

$$f(H) = \int_{\Gamma} f(z)(H - z)^{-1} dz,$$

where the integral exists in  $\mathcal{L}(\mathcal{H})$  and  $\Gamma$  is a contour with indice 1 that circumvents  $\sigma(T)$ . [Da2, Section 1.5]



**6.3. Helffer-Sjöstrand's formula.** For  $\rho \in \mathbb{R}$ , let  $\mathcal{S}^\rho$  be the class of function  $\varphi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{C})$  such that

$$(6.3.7) \quad \forall k \in \mathbb{N}, \quad C_k(\varphi) := \sup_{t \in \mathbb{R}} \langle t \rangle^{-\rho+k} |\partial_t^k \varphi(t)| < \infty.$$

We also write  $\varphi^{(k)}$  for  $\partial_t^k \varphi$ . Equipped with the semi-norms defined by (6.3.7),  $\mathcal{S}^\rho$  is a Fréchet space. Leibniz' formula implies the continuous embedding:

$$\mathcal{S}^\rho \cdot \mathcal{S}^{\rho'} \subset \mathcal{S}^{\rho+\rho'}.$$

**Lemma 6.1.** Let  $\varphi \in \mathcal{S}^\rho$  with  $\rho \in \mathbb{R}$ . For all  $l \in \mathbb{N}$ , there is a smooth function  $\varphi^{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$ , call an almost analytic extension of  $\varphi$ , such that:

$$\begin{aligned} \varphi^{\mathbb{C}}|_{\mathbb{R}} &= \varphi, & \left| \frac{\partial \varphi^{\mathbb{C}}}{\partial \bar{z}}(z) \right| &\leq c_1 \langle \Re(z) \rangle^{\rho-1-l} |\Im(z)|^l \\ \text{supp } \varphi^{\mathbb{C}} &\subset \{x + iy \mid |y| \leq c_2 \langle x \rangle\}, \\ \varphi^{\mathbb{C}}(x + iy) &= 0, \text{ if } x \notin \text{supp } \varphi. \end{aligned}$$

for constants  $c_1, c_2$  depending on the semi-norms (6.3.7) of  $\varphi$  in  $\mathcal{S}^\rho$ .

Let  $\rho < 0$  and  $\varphi \in \mathcal{S}^\rho$ . The bounded operator  $\varphi(A)$  can be recover by Helffer-Sjöstrand's formula:

$$\varphi(A) = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \varphi^{\mathbb{C}}}{\partial \bar{z}} (z - A)^{-1} dz \wedge d\bar{z},$$

where the integral exists in the norm topology. We refer for instance to [Da] for a self-contained approach.

**Exercise 6.2.** Using  $\|(z - A)^{-1}\| \leq 1/|\Im(z)|$ , show that the integral converges in norm.

## 7. THE DISCRETE AND THE ESSENTIAL SPECTRUM

We already separated the spectrum in two part that suited perfectly with the RAGE Theorem. Here the aim is to analyse the part of the spectrum is stable under compact perturbation.

## 7.1. Definition of the essential spectrum.

**Definition 7.1.** Let  $T \in \mathcal{L}(\mathcal{H})$  be a self-adjoint operator. We set

$$\begin{aligned}\sigma_d(T) &:= \{\lambda \in \mathbb{R}, \lambda \text{ is an isolated eigenvalue of finite multiplicity}\}, \\ \sigma_{\text{ess}}(T) &:= \sigma(T) \setminus \sigma_d(T).\end{aligned}$$

These spectra are called discret and essential, respectively.

**Proposition 7.2.** Let  $T$  be self-adjoint in a Hilbert space  $\mathcal{H}$  of infinite dimension, then  $\sigma_{\text{ess}}(T) \neq \emptyset$ .

*Proof.* Suppose that the spectrum is purely discret. Since it is contained in a compact there is a sub-sequence of eigenvalues that converges to a point of the spectrum. The later is not isolated. Contradiction.  $\square$

## 7.2. Link with the functional calculus.

**Proposition 7.3.** Let  $T$  be a self-adjoint operator acting in  $\mathcal{H}$  and  $\lambda \in \sigma(T)$  isolated.

- 1)  $\lambda \in \sigma_p(T)$ .
- 2) Given  $\varphi \in \mathcal{C}(\sigma(T))$  defined by 1 on  $\lambda$  and 0 elsewhere, we have that  $\varphi(T)$  is an orthogonal projection with range  $\ker(\lambda \text{Id} - T)$ .

*Proof.* First since  $\varphi(\lambda) = 1$ ,  $\varphi(T)$  is a projection. Indeed,

$$\|\varphi^2(T) - \varphi(T)\| = \sup_{t \in \sigma(T)} |\varphi^2(t) - \varphi(t)| = |\varphi^2(\lambda) - \varphi(\lambda)| = 0.$$

Moreover, the projection is orthogonal because  $\varphi$  is with real values and therefore  $\varphi(T)^* = \overline{\varphi}(T) = \varphi(T)$ . Then we show that  $\text{ran}(\varphi(T)) \subset \ker(\lambda \text{Id} - T)$ . We have:

$$\|(\lambda \text{Id} - T)\varphi(T)\| = \sup_{t \in \sigma(T)} |(\lambda - t)\varphi(t)| = 0.$$

Take now  $x \in \ker(\lambda \text{Id} - T)$ . We have:

$$\begin{aligned}(\text{Id} - \varphi(T))x &= \Phi\left(\underbrace{(1 - \varphi(\cdot))(\lambda - \cdot)^{-1}(\lambda - \cdot)}_{\in \mathcal{C}(\sigma(T))}\right)x \\ &= \Phi((1 - \varphi(\cdot))(\lambda - \cdot)^{-1})(\lambda \text{Id} - T)x = 0\end{aligned}$$

Then  $\text{ran}(\varphi(T)) = \ker(\lambda \text{Id} - T)$ . Finally since  $\varphi(T) \neq 0$  by functional calculus and then  $\lambda \in \sigma_p(T)$ .  $\square$

**Proposition 7.4.** Let  $T$  be self-adjoint in  $\mathcal{H}$  and  $\lambda \in \sigma(T)$ . Then,

- 1)  $\lambda \in \sigma_d(T)$ , if and only if there exists  $\varepsilon > 0$  and  $\varphi \in \mathcal{C}(\sigma(T); \mathbb{R})$  such that  $\text{supp}(\varphi) \subset [\lambda - \varepsilon, \lambda + \varepsilon]$  with  $\varphi(\lambda) = 1$  and such that  $\varphi(T)$  is compact.



2)  $\lambda \in \sigma_{\text{ess}}(T)$ , if and only if for all  $\varepsilon > 0$  and for all  $\varphi \in \mathcal{C}(\sigma(T); \mathbb{R})$  such that  $\text{supp}(\varphi) \subset [\lambda - \varepsilon, \lambda + \varepsilon]$  with  $\varphi(\lambda) = 1$ , we have that  $\varphi(T)$  is non-compact.

Note that in both cases that, since  $\lambda \in \sigma(T)$  and that  $\varphi(\lambda) = 1$ , functional calculus ensures that  $\varphi(T) \neq 0$ .

*Proof.* Note that 1) and 2) are equivalent (by taking the negation).

We suppose that there exist  $\varepsilon > 0$  and  $\varphi \in \mathcal{C}(\sigma(T); \mathbb{R})$  such that  $\text{supp}(\varphi) \subset [\lambda - \varepsilon, \lambda + \varepsilon]$  with  $\varphi(\lambda) = 1$  and such that  $\varphi(T)$  is compact.

Suppose that  $\lambda$  is not isolated. There exist a sequence  $\lambda_n \in \sigma(T)$  (note that  $\lambda$  could belong to an interval) such that  $\lambda_n \rightarrow \lambda$ . By spectral mapping, the spectrum of  $\varphi(T)$  is contained in  $\varphi(\lambda_n)$  and  $1 = \varphi(\lambda)$ . By continuity we have  $\varphi(\lambda_n) \rightarrow 1$ . This is a contradiction with the fact that  $\varphi(T)$  is compact (because 0 is the only possible accumulation point). Contradiction.

We have that  $\lambda$  is isolated. Let  $\varphi_0 \in \mathcal{C}(\sigma(T))$  with  $\varphi_0(\lambda) = 1$  and 0 elsewhere.

$$\|\varphi_0(H) - \varphi_0(H)\varphi(H)\| = \max_{t \in \sigma(T)} |\varphi_0(t) - \varphi_0(t)\varphi(t)| = |\varphi_0(\lambda) - \varphi_0(\lambda)\varphi(\lambda)| = 0.$$

Then  $\varphi_0(T) = \varphi_0(T)\varphi(T)$  is compact, because it is a product of a compact operator and a bounded operator. By the previous proposition  $\varphi_0(T)$  is a orthogonal projection with image  $\ker(\lambda \text{Id} - T)$ . Since it is compact we deduce that it is finite (Riesz' Theorem). In particular  $\lambda \in \sigma_d(T)$ .  $\square$

**7.3. Stability and characterisation.** We are now able to prove that the essential spectrum is stable with respect to compact perturbations.

**Theorem 7.5** (Weyl). *Let  $T$  and  $V$  be two self-adjoint operators on  $\mathcal{H}$ . If  $V \in \mathcal{K}(\mathcal{H})$ , i.e., compact, then*

$$\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T + V).$$

*Proof.* We set

$$\mathcal{A} := \{\varphi \in \mathcal{C}(\sigma(T) \cup \sigma(T + V)), \varphi(T) - \varphi(T + V) \in \mathcal{K}(\mathcal{H})\}$$

First  $\mathcal{A}$  is an algebra. 1 is in  $\mathcal{A}$  because  $\text{Id} - \text{Id} = 0$  is compact. Then by taking  $\varphi(t) = t$ , we see that  $\varphi(T) - \varphi(T + V) = -V \in \mathcal{K}(\mathcal{H})$ . This function separates points. Suppose now that  $\varphi \in \mathcal{A}$ , we have:

$$\overline{\varphi}(T) - \overline{\varphi}(T + V) = (\varphi(T))^* - (\varphi(T + V))^* = (\varphi(T) - \varphi(T + V))^* \in \mathcal{K}(\mathcal{H}).$$

Because the adjoint of a compact operator is compact. By Stone-Weierstrass we deduce that  $\overline{\mathcal{A}} = \mathcal{C}(\sigma(T) \cup \sigma(T + V))$ . It remains to show that  $\mathcal{A}$  is closed. Let  $\varphi_n \in \mathcal{A}$  that tends to  $\varphi \in \mathcal{C}(\sigma(T) \cup \sigma(T + V))$  for the uniform norm. We have  $\|\varphi_n(T) - \varphi(T)\| \rightarrow 0$  and  $\|\varphi_n(T + V) - \varphi(T + V)\| \rightarrow 0$  when  $n \rightarrow \infty$ . In particular,

$$\varphi_n(T) - \varphi_n(T + V) \rightarrow \varphi(T) - \varphi(T + V),$$

in norm then  $\varphi(T) - \varphi(T + V) \in \mathcal{K}(\mathcal{H})$ , because  $\mathcal{K}(\mathcal{H})$  is closed.

Finally since  $\varphi(T) - \varphi(T + V)$  is compact for all  $\varphi \in \mathcal{C}(\sigma(T) \cup \sigma(T + V))$  the previous proposition gives  $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T + V)$ .  $\square$

In the spirit of Proposition 2.38, we turn to a characterisation of the essential spectrum in term of sequences.

**Proposition 7.6** (Weyl's criterion). *Let  $T$  be self-adjoint on  $\mathcal{H}$ . Then  $\lambda \in \sigma_{\text{ess}}(T)$  if and only if there exist  $f_n \in \mathcal{H}$  such that :*

$$\|f_n\| = 1, \quad f_n \rightharpoonup 0 \quad \text{et} \quad \|(\lambda \text{Id} - T)f_n\| \rightarrow 0,$$

when  $n \rightarrow \infty$  and where  $\rightharpoonup$  denotes the weak convergence.

*Proof.* Let  $\lambda \in \sigma_{\text{ess}}(T)$ . Suppose first that  $\lambda$  is isolated. We have that  $\lambda$  is an eigenvalue of infinite multiplicity. Take  $(f_n)_n$  to be an orthonormal basis of  $\ker(\lambda \text{Id} - T)$ .

Suppose now that  $\lambda$  is not isolated. There exist  $\lambda_n \in \sigma(T)$ , two by two distinct, such that  $\lambda_n \rightarrow \lambda$ , when  $n \rightarrow \infty$ . Up to a sub-sequence or considering  $-T$ , we can suppose that  $\lambda_n$  is strictly increasing. We then construct  $\varphi_n \in \mathcal{C}(\sigma(T); [0, 1])$  such that  $\varphi_n(\lambda_n) = 1$  and such that  $\text{supp}(\varphi_n) \subset [(2\lambda_n + \lambda_{n-1})/3, (2\lambda_n + \lambda_{n+1})/3]$ . In particular,  $\varphi_n$  has support two by two disjoint and  $\|\varphi_n(T)\| = 1$ . Take now  $x_n \in \mathcal{H}$  such that  $\|\varphi_n(T)x_n\| \geq 1/2$ . We have

$$f_n := \frac{\varphi_n(T)x_n}{\|\varphi_n(T)x_n\|}$$

which is of norm 1. We see that  $f_n$  tends weakly to 0 because for  $n \neq m$

$$\langle f_n, f_m \rangle = \left\langle \frac{x_n}{\|\varphi_n(T)x_n\|}, \overbrace{\frac{\varphi_n(T)\varphi_m(T)x_m}{\|\varphi_m(T)x_m\|}}^{=0} \right\rangle = 0,$$

due to the support of  $\varphi_n$  and by functional calculus. Finally we have:

$$\|(\lambda \text{Id} - T)f_n\| \leq 2\|(\lambda \text{Id} - T)g_n(T)\| \cdot \|x_n\| \rightarrow 0,$$

by functional calculus.  $\square$

**7.4. Examples.** Denote by  $H$  one of the discrete operator that we have considered before. Then take  $V$  be a function with value in  $\mathbb{R}$  such that  $\lim_{|x| \rightarrow \infty} V(x) = 0$ , where  $|x|$  denotes the distance to a fixed point of the space. We have:

$V(Q)$  is a compact operator

and therefore

$$\sigma_{\text{ess}}(H) = \sigma(H) = \sigma_{\text{ess}}(H + V).$$

Here we used only the 1-point compactification by considering this limit. On  $\mathbb{Z}$ , it natural to consider the 2-point compactification, i.e., look at limits in  $\pm\infty$ . The result is formulated as follows:

**Exercise 7.1.** Let  $\mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C})$ . Let  $V : \mathbb{Z} \rightarrow \mathbb{R}$  such that  $c^\pm := \lim_{n \rightarrow \pm\infty} V(n)$  exists and is finite. Using that  $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N}$ , prove that

$$\begin{aligned} \sigma_{\text{ess}}(\mathcal{A}_{\mathbb{Z}} + V(Q)) &= [-2 + c^-, 2 + c^-] \cup [-2 + c^+, 2 + c^+] \\ &= [-2, 2] + \{c_-, c_+\}. \end{aligned}$$

**Exercise 7.2.** Same exercise but use the Weyl's criterion.

On a tree there is a natural compactification that correspond to add one infinity for each geodesic that goes to infinity. This is the hyperbolic compactification. With more work, in [Go] one shows:

**Theorem 7.7.** *Let  $\mathcal{G} := (\mathcal{E}, \mathcal{V})$  be a binary tree. Let  $\mathcal{H} := \ell^2(\mathcal{V}; \mathbb{C})$ . Let  $\hat{\mathcal{V}} := \mathcal{V} \cup \partial\mathcal{V}$  be the hyperbolic compactification of  $\mathcal{V}$ . Suppose that  $V : \mathcal{V} \rightarrow \mathbb{R}$  is bounded and extends continuously to  $\hat{\mathcal{V}}$ . Then we have:*

$$\sigma_{\text{ess}}(\mathcal{A} + V(Q)) = [-2\sqrt{2}, 2\sqrt{2}] + V(\partial\mathcal{V}).$$

## 8. BORELIAN FUNCTIONNAL CALCULUS

**8.1. Spectral measure.** The aim now is to define the spectral measure of an operator. We would like to be able to define  $1_X(H)$ , where  $X$  is a Borelian set. In a second step we will relate some properties of the measure to the dynamical behaviour of the Schrödinger equation.

To go from the polynomial functional calculus to the continuous one, we use the density of polynomial with respect to the sup norm. Here in order to obtain a Borelian fonctionnal calculus, we will also rely on an argument of density but we need to work with a weaker topology. This approach is similar to the difference that one finds between  $C^*$ -algebra and  $W^*$ -algebra.

We will work in three steps :

- 1) Define the spectral measure associated to a vector.
- 2) Construct  $\varphi(H)$  with  $\varphi : \mathbb{R} \rightarrow \mathbb{C}$  being Borelian bounded.
- 3) Check that this construction is compatible with the property of a functional calculus.

Step 1: Let  $H \in \mathcal{L}(\mathcal{H})$  be a self-adjoint operator. Let  $f \in \mathcal{H} \setminus \{0\}$ . By functional calculus, we have that

$$\Phi : \mathcal{C}(\sigma(T); \mathbb{C}) \rightarrow \mathbb{C}, \quad \text{given by} \quad \Phi(\varphi) := \langle f, \varphi(H)f \rangle$$

is continuous and positive (if  $\varphi \geq 0$  then  $\Phi(\varphi) \geq 0$ ).

Therefore by Riesz-Markov's Theorem there is a unique measure  $m_f$  such that

$$\langle f, \varphi(H)f \rangle = \int_{\sigma(H)} \varphi(t) dm_f(t).$$

**Definition 8.1.** *The measure  $m_f$  is called the spectral measure of  $H$  associated to  $f$ .*

**Remark 8.2.** *If  $\|f\| = 1$ , note that  $m_f$  is a probability measure.*

Step 2: Given  $\varphi \in \mathcal{B}(\sigma(H)) = \mathcal{B}(\sigma(H); \mathbb{C})$ , i.e, a borelian bounded function, we set:

$$\langle f, \varphi(H)f \rangle := \int_{\sigma(H)} \varphi(t) dm_f(t).$$

We now explain why  $\varphi(H)$  is a well-defined bounded operator (why does  $\varphi(H)$  is linear? Does it depend on the choice of  $f$ ?).

Given  $\varphi \in \mathcal{C}(\sigma(H))$ . For  $f \in \mathcal{H}$ , we set

$$B_\varphi(f, f) := \langle f, \varphi(H)f \rangle = \int_{\sigma(H)} \varphi(t) dm_f(t)$$

and stress that  $m_f$  is a bounded measure. Indeed,

$$m_f(\sigma(H)) = \int_{\sigma(H)} 1 dm_f(t) = \langle f, 1(H)f \rangle = \|f\|^2,$$

because  $1(H) = \text{Id}$ . (recall the starting point with polynomials).

We also set

$$B_\varphi(f, g) := \langle f, \varphi(H)g \rangle$$

Recalling the polarisation formula

$$B_\varphi(f, g) = \frac{1}{4} \sum_{k=0}^3 i^k B_\varphi(i^k f + g, i^k f + g),$$

we see that there is a complex measure  $m_{f,g}$  such that:

$$B_\varphi(f, g) = \int_{\sigma(H)} \varphi(t) dm_{f,g}(t), \text{ where } m_{f,g} := \frac{1}{4} \sum_{k=0}^3 i^k m_{i^k f + g}.$$

Notice that:

$$m_{\lambda f + g, h} = \bar{\lambda} m_{f, h} + m_{g, h} \quad \text{and} \quad m_{h, \lambda f + g} = \lambda m_{h, f} + m_{h, g}.$$

We now take  $\varphi \in \mathcal{B}(\sigma(H))$ . We extend the definition of  $B_\varphi$  in the following way:

$$B_\varphi(f, g) := \int_{\sigma(H)} \varphi(t) dm_{f,g}(t).$$

By the property of the measure we see that:

$$B_\varphi \text{ is a sesquilinear form.}$$

We now prove that it is continuous. First we note that:

$$(8.1.8) \quad |B_\varphi(f, f)| \leq \|\varphi\|_\infty \int_{\sigma(H)} 1 dm_f(t) = \|\varphi\|_\infty \langle f, 1(H)f \rangle = \|\varphi\|_\infty \|f\|^2.$$

We aim at showing:

$$|B_\varphi(f, g)| \leq \|\varphi\|_\infty \|f\| \cdot \|g\|, \quad \text{for all } f, g \in \mathcal{H}.$$

Set:

$$\alpha := \sup_{\|f\|=1} |B_\varphi(f, f)|.$$

It is enough to show that  $|B_\varphi(f, g)| \leq \alpha$  for all  $f$  et  $g$  such that  $\|f\| = \|g\| = 1$ .

If  $B_\varphi(f, g) = 0$  there is nothing to do. We set

$$\lambda := \frac{\overline{B_\varphi(f, g)}}{|B_\varphi(f, g)|}.$$

Note that  $|\lambda| = 1$ .

By polarisation, we have:

$$\begin{aligned}
|B_\varphi(f, g)| &= B_\varphi(f, \lambda g) = \Re B_\varphi(f, \lambda g) = \Re \left( \frac{1}{4} \sum_{k=0}^3 i^k \underbrace{B_\varphi(i^k f + \lambda g, i^k f + \lambda g)}_{\in \mathbb{R}} \right) \\
&= \frac{1}{4} (B_\varphi(f + \lambda g, f + \lambda g) - B_\varphi(-f + \lambda g, -f + \lambda g)) \\
&\leq \frac{\alpha}{4} (\|f + \lambda g\|^2 + \|-f + \lambda g\|^2) \leq \alpha,
\end{aligned}$$

where we used in the last line that  $\|x\| = \|y\| = |\lambda| = 1$ .

We turn to the existence of  $\varphi(H)$ .

Note that  $f \mapsto B_\varphi(f, g)$  is a continuous anti-linear form. Therefore there exists  $T(g)$  such that

$$B_\varphi(f, g) = \langle f, T(g) \rangle, \quad \text{for all } f \in \mathcal{H}.$$

It is easy to see that  $T(g_1 + \lambda g_2) = T(g_1) + \lambda T(g_2)$ .

Moreover, by Riesz' isomorphism, we get:

$$\|Tg\| = \|f \mapsto B_\varphi(f, g)\| \leq \|\varphi\|_\infty \|g\|.$$

Therefore  $T$  is a linear bounded operator. We denote it by  $\varphi(H)$ .

Using (8.1.8), we infer:

$$(8.1.9) \quad \|\varphi(H)\| \leq \sup_{x \in \sigma(H)} |\varphi(x)|, \quad \text{for all } \varphi \in \mathcal{B}(\sigma(H)).$$

**Step 3:** We now need a result of density. It is a consequence of the monotone class Theorem.

**Definition 8.3.** Let  $\Omega$  be a real and open interval. A family  $\varphi_n : \Omega \rightarrow \mathbb{C}$ , with  $n \in \mathbb{N}$ , is boundedly convergent if:

- 1) There is a finite  $M$  such that  $|\varphi_n(x)| \leq M$  for all  $n \in \mathbb{N}$  and  $x \in \Omega$ .
- 2) If  $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$  exists for all  $x \in \Omega$ .

**Proposition 8.4.** Let  $\Omega$  be a real and open interval. Let  $\mathcal{F}$  be a family of bounded complex functions on  $\Omega$ , such that  $\mathcal{C}_c(\Omega) \subset \mathcal{F}$ . If  $\mathcal{F}$  is stable under bounded convergence of sequences, then the set of bounded Borelian function  $\mathcal{B}(\Omega)$  is included in  $\mathcal{F}$ .

We now obtain:

**Theorem 8.5.** Let  $H$  be self-adjoint operator acting on  $\mathcal{H}$ . There is a unique map  $\hat{\Phi} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$  such that:

- 1)  $\hat{\Phi}(\varphi + \lambda \psi) = \hat{\Phi}(\varphi) + \lambda \hat{\Phi}(\psi)$ ,
- 2)  $(\hat{\Phi}(\varphi))^* = \hat{\Phi}(\overline{\varphi})$ ,
- 3)  $\hat{\Phi}(\varphi \times \psi) = \hat{\Phi}(\varphi) \hat{\Phi}(\psi)$ ,
- 4)  $\hat{\Phi}(x) = H$ ,
- 5) If  $\varphi_n(x) \rightarrow \phi(x)$  for all  $x \in \mathbb{R}$  and if  $\sup_n \|\varphi_n\|_\infty < \infty$  then for all  $f \in \mathcal{H}$ ,  $\hat{\Phi}(\varphi_n)f \rightarrow \hat{\Phi}(\phi)f$ , as  $n \rightarrow \infty$ .

Moreover we have:

- 6)  $\|\hat{\Phi}(H)\| \leq \|\varphi|_{\sigma(H)}\|_\infty$
- 7) If  $Hf = \lambda f$ , then  $\Phi(\varphi)f = \varphi(\lambda)f$ ,
- 8) If  $\varphi \geq 0$  then  $\sigma(\Phi(\varphi)) \geq 0$ .

**Remark 8.6.** As before we denote  $\Phi(\varphi)$  by  $\varphi(H)$ .

**Remark 8.7.** Given a Borel set  $\mathcal{I} \subset \sigma(H)$ , we have that  $E_{\mathcal{I}}(H) := 1_{\mathcal{I}}(H)$  is an orthogonal projector. Moreover,

$$\langle f, E_{\mathcal{I}}(H)f \rangle = \int_{\mathcal{I}} dm_f(t) = m_f(\mathcal{I}).$$

and

$$\langle f, E_{\mathcal{I}}(H)g \rangle = \int_{\mathcal{I}} dm_{f,g}(t) = m_{f,g}(\mathcal{I}).$$

Therefore  $\mathcal{I} \rightarrow E_{\mathcal{I}}(H)$  is a measure with projector values in  $\mathcal{L}(\mathcal{H})$ .

Using for instance the Bochner integral, we can prove that for  $\varphi \in \mathcal{B}(\sigma(H))$

$$\varphi(H) = \int_{\sigma(H)} \varphi(t) dE_t(H).$$

*Proof. Unicity:* Suppose that  $\Phi$  and  $\tilde{\Phi}$  are satisfying the points 1) to 5). Using Theorem 5.8 we see that

$$\mathcal{C}_c(\mathbb{R}) \subset \{\varphi \in \mathcal{B}(\mathbb{R}), \Phi(\varphi) = \tilde{\Phi}(\varphi)\}.$$

Thanks to point 5) and Proposition 8.4, we infer that  $\Phi(\varphi) = \tilde{\Phi}(\varphi)$ , for all  $\varphi \in \mathcal{B}(\mathbb{R})$ .

*Existence:* We set  $\Phi(\varphi) := \varphi(H)$ , where the latter is constructed in the second step.

Take  $\varphi, \psi \in \mathcal{B}(\mathbb{R})$ ,  $f \in \mathcal{H}$  and  $\lambda \in \mathbb{C}$ . We have:

$$\begin{aligned} \langle f, (\varphi + \lambda\psi)(H)f \rangle &= \int_{\sigma(H)} (\varphi + \lambda\psi)(t) dm_f(t) \\ &= \int_{\sigma(H)} \varphi(t) dm_f(t) + \int_{\sigma(H)} \lambda\psi(t) dm_f(t) = \langle f, (\varphi(H) + \lambda\psi(H))f \rangle \end{aligned}$$

In particular by Proposition 2.15, we obtain 1).

Now notice that, since  $m_f$  is

$$\langle f, (\varphi(H))^* f \rangle = \overline{\langle f, \varphi(H)f \rangle} = \overline{\int_{\sigma(H)} \varphi(t) dm_f(t)} = \int_{\sigma(H)} \overline{\varphi(t)} dm_f(t) = \langle f, \overline{\varphi}(H)f \rangle.$$

By Proposition 2.15, we obtain  $(\varphi(H))^* = \overline{\varphi}(H)$ .

We turn to 3). First we set

$$S_0 := \{\varphi \in \mathcal{B}(\mathbb{R}), (\varphi\psi)(H) = \varphi(H)\psi(H), \forall \psi \in \mathcal{C}_c(\mathbb{R})\}.$$

By Theorem 5.8,  $\mathcal{C}_c(\mathbb{R}) \subset S_0$ . We show that  $S_0$  is stable under boundedly convergence. Suppose that the family  $\varphi_n \in \mathcal{C}_c(\mathbb{R})$ , with  $n \in \mathbb{N}$ , is boundedly convergent. We denote its limit by  $\varphi$ . We have that there is  $M > 0$  such that  $|\varphi_n| \leq M$ , for all  $n \in \mathbb{N}$ . Let  $\varphi \in \mathcal{C}_c(\mathbb{R})$ . By Theorem 5.8,

$$\langle f, (\varphi_n\psi)(H)f \rangle = \langle f, \varphi_n(H)\psi(H)f \rangle$$

On one side we have:

$$\begin{aligned}\langle f, \varphi_n(H)\psi(H)f \rangle &= \int_{\sigma(H)} \varphi_n(t) dm_{f, \psi(H)f}(t) \rightarrow \int_{\sigma(H)} \varphi(t) dm_{f, \psi(H)f}(t) \\ &= \langle f, \varphi(H)\psi(H)f \rangle, \text{ as } n \rightarrow \infty,\end{aligned}$$

by dominated convergence. On the other side, note that  $|(\psi\varphi_n)| \leq M\|\psi\|_\infty$ , for all  $n \in \mathbb{N}$ . Hence,

$$\begin{aligned}\langle f, (\varphi_n\psi)(H)f \rangle &= \int_{\sigma(H)} (\varphi_n\psi)(t) dm_{f,f}(t) \rightarrow \int_{\sigma(H)} \varphi\psi(t) dm_{f,f}(t) \\ &= \langle f, (\varphi\psi)(H)f \rangle, \text{ as } n \rightarrow \infty,\end{aligned}$$

by dominated convergence. Therefore,  $\psi \in S_0$ . The set  $S_0$  is stable under boundedly convergence. Using Proposition 8.4, we obtain that  $S_0 = \mathcal{B}(\mathbb{R})$ .

Next, we set:

$$S := \{\psi \in \mathcal{B}(\mathbb{R}), (\varphi\psi)(H) = \varphi(H)\psi(H), \forall \varphi \in \mathcal{B}(\mathbb{R})\}.$$

Since  $S_0 = \mathcal{B}(\mathbb{R})$  we obtain that  $\mathcal{C}_c(\mathbb{R}) \subset S$ . We show that  $S_0$  is stable under boundedly convergence. Suppose that the family  $\psi_n \in \mathcal{C}_c(\mathbb{R})$ , with  $n \in \mathbb{N}$ , is boundedly convergent. We denote its limit by  $\psi$ . We have that there is  $M > 0$  such that  $|\psi_n| \leq M$ , for all  $n \in \mathbb{N}$ . Let  $\varphi \in \mathcal{B}(\mathbb{R})$ . Since  $S_0 = \mathcal{B}(\mathbb{R})$ ,

$$\langle f, (\varphi\psi_n)(H)f \rangle = \langle f, \varphi(H)\psi_n(H)f \rangle$$

On one side we have:

$$\begin{aligned}\langle f, \varphi(H)\psi_n(H)f \rangle &= \langle \varphi(H)^*f, \psi_n(H)f \rangle \\ &= \int_{\sigma(H)} \psi_n(t) dm_{\varphi(H)^*f, f}(t) \rightarrow \int_{\sigma(H)} \psi(t) dm_{\varphi(H)^*f, f}(t) \\ &= \langle \varphi(H)^*f, \psi(H)f \rangle = \langle f, \varphi(H)\psi(H)f \rangle, \text{ as } n \rightarrow \infty,\end{aligned}$$

by dominated convergence. On the other side, note that  $|\varphi\psi_n| \leq M\|\varphi\|_\infty$ , for all  $n \in \mathbb{N}$ . Hence,

$$\begin{aligned}\langle f, (\varphi\psi_n)(H)f \rangle &= \int_{\sigma(H)} (\varphi\psi_n)(t) dm_{f,f}(t) \rightarrow \int_{\sigma(H)} \varphi\psi(t) dm_{f,f}(t) \\ &= \langle f, (\varphi\psi)(H)f \rangle, \text{ as } n \rightarrow \infty,\end{aligned}$$

by dominated convergence. Therefore,  $\psi \in S$ . The set  $S$  is stable under boundedly convergence. Using Proposition 8.4, we obtain that  $S = \mathcal{B}(\mathbb{R})$ . In particular 3) holds true.

The point 4) is clear thanks to Theorem 5.8. We check the point 5). Take  $\varphi_n$  and  $\varphi$  as in 5).

$$\begin{aligned}\|(\varphi_n(H) - \varphi(H))f\|^2 &= \langle f, (\varphi_n(H) - \varphi(H))^*(\varphi_n(H) - \varphi(H))f \rangle \\ &= \langle f, (\overline{\varphi_n}(H) - \overline{\varphi}(H))(\varphi_n(H) - \varphi(H))f \rangle, \text{ by 2)} \\ &= \langle f, (|\varphi_n|^2(H) - (\overline{\varphi_n}\varphi)(H) - (\overline{\varphi}\varphi_n)(H) + |\varphi|^2(H))f \rangle, \text{ by 3)} \\ &= \int_{\sigma(H)} (|\varphi_n|^2(t) - (\overline{\varphi_n}\varphi)(t) - (\overline{\varphi}\varphi_n)(t) + |\varphi|^2(t)) dm_f(t) \\ &= \int_{\sigma(H)} |\varphi_n(t) - \varphi(t)|^2 dm_f(t) \rightarrow 0, \text{ as } n \rightarrow \infty,\end{aligned}$$

by dominated convergence.

The point 6) is given by (8.1.9). The point 8), comes from the fact that if  $\varphi \geq 0$ , in particular,  $\varphi$  is real-valued function. Hence, by 2),  $\varphi(H)$  is self-adjoint. Moreover,

$$\langle f, \varphi(H)f \rangle = \int_{\sigma(H)} \varphi(t) dm_f(t) \geq 0$$

Therefore, Proposition 2.37 ensue that  $\sigma(\varphi(H)) \subset \mathbb{R}^+$ .

We finish with point 7). Suppose that  $Hf = \lambda f$ . Set:

$$T := \{\varphi \in \mathcal{B}(\mathbb{R}), \varphi(H)f = \varphi(\lambda)f\}$$

Since  $H^n f = \lambda^n f$  by induction, we infer that  $\mathbb{C}[X] \subset T$ . By Stone-Weirstrass, applied on all compact sets, we obtain that  $\mathcal{C}_c(\mathbb{R}) \subset T$ . Finally using 5), we see that  $T$  is stable under boundedly convergence. This yields  $T = \mathcal{B}(\mathbb{R})$ .  $\square$

We strengthen Proposition 5.11

**Proposition 8.8.** *Let  $H$  be a self-adjoint operator.*

$$\sigma(H) = \{\lambda \in \mathbb{R}, E_{[\lambda-\varepsilon, \lambda+\varepsilon]}(H) \neq 0, \text{ for all } \varepsilon > 0\}.$$

*Proof.* We recall

$$\sigma(H) = \{\lambda \in \mathbb{R}, \varphi(H) \neq 0, \text{ for all } \varphi \in \mathcal{C}(\sigma(H); \mathbb{C}) \text{ with } \varphi(\lambda) \neq 0\}.$$

Take  $\lambda \in \sigma(H)$  and  $\varepsilon > 0$ . Let  $\varphi \in \mathcal{C}_c([\lambda - \varepsilon, \lambda + \varepsilon]; \mathbb{R})$  such that  $\varphi(\lambda) \neq 0$ . We have  $\|\varphi(H)\| > 0$  by (5.3.6). We have:

$$0 < \|\varphi(H)\| = \|\varphi(H)E_{[\lambda-\varepsilon, \lambda+\varepsilon]}(H)\| \leq \|\varphi(H)\| \cdot \|E_{[\lambda-\varepsilon, \lambda+\varepsilon]}(H)\|.$$

Therefore,  $E_{[\lambda-\varepsilon, \lambda+\varepsilon]}(H) \neq 0$ .

Take now  $\lambda \notin \sigma(H)$ . There is  $\varepsilon > 0$  such that  $[\lambda - \varepsilon, \lambda + \varepsilon] \cap \sigma(H) = \emptyset$ . Therefore by (8.1.9) we see that  $E_{[\lambda-\varepsilon, \lambda+\varepsilon]}(H) = 0$ .  $\square$

**8.2. Nature of the spectral measure.** There is a link between the spectrum and nature of the spectral measure. The easiest one is given in the next exercise.

**Exercise 8.1.**  $E_{\{\lambda\}}(H) \neq 0$  if and only if  $\lambda$  is an eigenvalue of  $H$ . Moreover  $E_{\{\lambda\}}(H)$  is an orthogonal projector with image  $\ker(\lambda - H)$ .

To go further we recall some general facts about measure theory.

**Definition 8.9.** Let  $\mu$  be a Borel sigma-finite measure on  $\mathbb{R}$ .

- 1) We say that  $x \in \mathbb{R}$  is an atom for  $\mu$  if  $\mu(\{x\}) > 0$ .
- 2) We say that  $\mu$  is continuous if  $\mu$  has no atom.
- 3) We say that  $\mu$  is supported by Borel set  $\Sigma$  if  $\mu(\mathbb{R} \setminus \Sigma) = 0$ .
- 4) We say that  $\mu$  is absolutely continuous with respect to the Lebesgue measure if  $\mu(\mathcal{I}) = 0$  when  $\text{Leb}(\mathcal{I}) = 0$ . We denote it by  $\mu \ll \text{Leb}$ .
- 5) We say that  $\mu$  is singular with respect to the measure  $\nu$  when there exists a Borel set  $\Sigma$  such that  $\mu(\mathbb{R} \setminus \Sigma) = 0$  and  $\nu(\Sigma) = 0$ . We denote it by  $\mu \perp \nu$ .

We recall the Theorem of Radon-Nykodim and refer to [Rud, Chapter 6].



**Theorem 8.10** (Radon-Nykodim). *Let  $\mu$  be a Borel sigma-finite measure on  $\mathbb{R}$  which is absolutely continuous with respect to the Lebesgue measure. Then there exists  $f \in L^1_{\text{loc}}(\mathbb{R}, dx)$  such that*

$$\mu(A) = \int_A f(x) dx,$$

for all  $A$  Borel sets.

We now turn to the decomposition of the spectral measure. We recall the Lebesgue decomposition of a measure, e.g., [Rud, Chapter 6].

**Theorem 8.11** (Lebesgue decomposition). *Given  $\mu$  be a Borel sigma-finite measure on  $\mathbb{R}$ . There are measures  $\mu^p$  and  $\mu^c$  which are purely atomic and continuous, respectively, such that:*

$$\mu = \mu^p + \mu^c.$$

We have  $\mu^p \perp \mu^c$ .

Moreover, there are measures  $\mu^{\text{ac}}$  and  $\mu^{\text{sc}}$ , which are continuous with respect to the Lebesgue measure and singular with respect to it, respectively, such that:

$$\mu^c = \mu^{\text{ac}} + \mu^{\text{sc}}.$$

We have  $\mu^{\text{ac}} \perp \mu^{\text{sc}}$ .

We now put into practice this decomposition. Given  $f \in \mathcal{H}$  and  $H$  self-adjoint, we have:

$$\begin{aligned} \|f\|^2 &= \langle f, f \rangle = \int_{\mathbb{R}} dm_f(x) \\ &= \int_{\mathbb{R}} dm_f^p(x) + \int_{\mathbb{R}} dm_f^{\text{ac}}(x) + \int_{\mathbb{R}} dm_f^{\text{sc}}(x) \\ &= \int_{\mathbb{R}} 1_{\Sigma^p}(x) dm_f(x) + \int_{\mathbb{R}} 1_{\Sigma^{\text{ac}}}(x) dm_f(x) + \int_{\mathbb{R}} 1_{\Sigma^{\text{sc}}}(x) dm_f(x) \\ &= \langle f, E_{\Sigma^p}(H)f \rangle + \langle f, E_{\Sigma^{\text{ac}}}(H)f \rangle + \langle f, E_{\Sigma^{\text{sc}}}(H)f \rangle \\ &= \|E_{\Sigma^p}(H)f\|^2 + \|E_{\Sigma^{\text{ac}}}(H)f\|^2 + \|E_{\Sigma^{\text{sc}}}(H)f\|^2, \end{aligned}$$

where  $\Sigma^p$ ,  $\Sigma^{\text{ac}}$ , and  $\Sigma^{\text{sc}}$  are Borel sets that are supporting the discrete, ac, sc part, respectively.

**Danger:** These sets depend a priori on  $f$ .

Using the separability of the space and cyclic vectors, we can prove, e.g., [Ja, Section 4.8]:

**Theorem 8.12.** *Let  $H$  be self-adjoint in  $\mathcal{H}$ , there are closed (Hilbert) subspaces  $\mathcal{H}^p$ ,  $\mathcal{H}^{\text{ac}}$ , and  $\mathcal{H}^{\text{sc}}$  such that*

$$\mathcal{H} = \mathcal{H}^p \oplus \underbrace{\mathcal{H}^{\text{ac}} \oplus \mathcal{H}^{\text{sc}}}_{\mathcal{H}^c}$$

and, denoting by  $m_f$  the spectral measure of  $H$  associated to  $f$ ,

- 1) if  $f \in \mathcal{H}^p$  then  $m_f$  is atomic,
- 2) if  $f \in \mathcal{H}^{\text{ac}}$  then  $m_f$  is absolutely continuous with respect to the Lebesgue measure,
- 3) if  $f \in \mathcal{H}^{\text{sc}}$  then  $m_f$  is singularly continuous with respect to the Lebesgue measure.

We denote by  $P^p$ ,  $P^{ac}$ , and  $P^{sc}$  the respective projection.

Moreover,  $\varphi(H)\mathcal{H}^X \subset \mathcal{H}^X$ , for  $X \in \{p, ac, sc\}$  and  $\varphi \in \mathcal{B}(\mathbb{R})$ .

**Remark 8.13.** Note that

$$P^p = E_{\sigma_p}(H).$$

We now decompose the spectrum. Set  $X \in \{p, ac, sc\}$  and let

$$\sigma^X(H) := \sigma^X(H|_{\mathcal{H}^X}).$$

We have:

$$\sigma(H) = \sigma^p(H) \cup \sigma^{ac}(H) \cup \sigma^{sc}(H).$$

**Be careful:** We do not have in general that the different spectra are two by two disjoint. We could have mixed spectrum. For instance, by taking a direct sum, it is easy to construct an example such that

$$\sigma(H) = \sigma^p(H) = \sigma^{ac}(H) = \sigma^{sc}(H) = [0, 1].$$

**Proposition 8.14.** Given  $f \in \mathcal{H}^{ac}$ . Let  $K$  be a compact operator. Then

$$K e^{-itH} f \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

**Remark 8.15.** Recall that given  $f \in \mathcal{H}^{sc} \subset \mathcal{H}^c$  and  $K$  a compact operator, the RAGE's theorem ensures a priori solely:

$$\frac{1}{T} \int_0^T \|K e^{-itH} f\|^2 dt \rightarrow 0, \quad \text{as } T \rightarrow \infty.$$

**Remark 8.16.** Take  $K = 1_X(Q)$ , where  $X$  is a finite set in the examples of graphs, by denoting by  $H$  the studied operator, we see that for  $f \in \mathcal{H}^{ac}$  we have

$$1_X(Q) e^{-itH} f \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

The particle escapes to infinity.

*Proof.* Let  $f \in \mathcal{H}^{ac}$  and let  $g \in \mathcal{H}$ . We denote by

$$m_{g,f}(\mathcal{I}) := \langle g, E_{\mathcal{I}}(H)f \rangle.$$

This measure is purely absolutely continuous with respect to the Lebesgue measure because, for  $\mathcal{I}$  such that  $\text{Leb}(\mathcal{I}) = 0$ , we have:

$$|m_{g,f}(\mathcal{I})| = |\langle g, E_{\mathcal{I}}(H)f \rangle| \leq \|g\|^2 \cdot \|E_{\mathcal{I}}(H)f\|^2 = 0.$$

By the Riemann-Lebesgue's Theorem, we have that

$$t \mapsto \widehat{m_{g,f}}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} dm_{g,f}(x) \in \mathcal{C}_0(\mathbb{R}),$$

where  $\mathcal{C}_0(\mathbb{R})$  denotes the continuous functions that tend to 0 at infinity. Using functional calculus, we infer

$$\langle g, e^{-itH} f \rangle \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Therefore for  $\{g_j\}_{j=1,\dots,N} \subset \mathcal{H}$ , we get:  $\langle \sum_j g_j, e^{-itH} f \rangle \rightarrow 0$ , as  $t \rightarrow \infty$ . By density of the finite rank operator in the set of compact operator, for  $K \in \mathcal{K}(\mathcal{H})$ , we obtain:

$$K e^{-itH} f \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

This finishes the proof.  $\square$

**Example 8.17.** We now decompose the spectrum of  $\mathcal{A}_{\mathbb{Z}}$  which acts on  $\mathcal{H} := \ell^2(\mathbb{Z})$ .

First note that the different spectra are stable by unitary equivalence. We recall that  $\mathcal{A}_{\mathbb{Z}}$  is unitarily equivalent to

$$\varphi(Q) \text{ in } L^2(-\pi, \pi),$$

where  $\varphi(x) := 2 \cos(x)$ . Note that  $\varphi(Q)L^2(0, \pi) \subset L^2(0, \pi)$  and  $\varphi(Q)L^2(-\pi, 0) \subset L^2(-\pi, 0)$ .

Take  $f$  in  $L^2(0, \pi)$ . Set  $\mathcal{I} \subset (0, \pi)$  such that  $\text{Leb}(\mathcal{I}) = 0$ .

$$\begin{aligned} \|E_{\mathcal{I}}(\varphi(Q))f\|^2 &= \|E_{\varphi^{-1}(\mathcal{I})}(Q)f\|^2 = \int_{\varphi^{-1}(\mathcal{I})} |f(x)|^2 dx \\ &= \int_{\mathcal{I}} \underbrace{|f(\varphi(x))|^2 |\varphi'(x)|}_{\in L^1} dx = 0. \end{aligned}$$

Do the same with  $f$  in  $L^2(-\pi, 0)$ . Therefore, that the spectrum of  $\mathcal{A}_{\mathbb{Z}}$  is purely absolutely continuous with respect to the Lebesgue measure.

**Exercise 8.2.** Let  $\varphi \in C^1([-\pi, \pi]; \mathbb{R})$  such that  $\varphi'(x) = 0$  if and only if  $x \in [-1, 1]$ . Let  $H := \varphi(Q)$  in  $L^2([-\pi, \pi])$ . Show that:

$$\sigma^p(H) = \{\varphi(0)\}, \quad \sigma^{\text{ac}}(H) = \varphi([-\pi, \pi]), \text{ and } \sigma^{\text{sc}}(H) = \emptyset.$$

**8.3. The cantor measure.** Set  $f : [0, 1] \rightarrow [0, 1]$  given by:

$$f(x) := \begin{cases} 3x, & \text{if } x \in \left[0, \frac{1}{3}\right], \\ 0, & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 3x - 2, & \text{if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

For  $n \in \mathbb{N}^*$ , set  $E_{n+1} := f^{-1}(E_n)$ , where  $E_0 := [0, 1]$ . This gives

$$E_1 = [0, 1/3] \cup [2/3, 1],$$

$$E_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9]$$

and so on. We have

$$C := \bigcap_{n \in \mathbb{N}} E_n.$$

This is the triadic Cantor set. Note that  $C$  is compact,  $C \neq \emptyset$ , and  $\text{Leb}(C) = 0$ .

Let  $\alpha \in \mathcal{C}([0, 1])$  be constructed as follows.

$$\alpha(x) := \begin{cases} \frac{1}{2}, & \text{for } x \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ \frac{1}{4}, & \text{for } x \in \left(\frac{1}{9}, \frac{2}{9}\right), \\ \frac{3}{4}, & \text{for } x \in \left(\frac{7}{9}, \frac{8}{9}\right), \\ \text{etc...} \end{cases}$$

and extended by continuity on  $[0, 1]$ , e.g., [DMRV].

The function  $\alpha$  is strictly increasing and its derivative is 0 almost everywhere. The Cantor measure is defined by prescribing

$$(8.3.10) \quad \mu_C(a, b) := \alpha(b) - \alpha(a).$$

and extending it to the Borel sets. We have that  $\mu_C(C) = 1$  and that  $\text{Leb}(C) = 0$ . Note also that  $\mu_C(x) = 0$ , for all  $x \in C$ . Therefore  $\mu_C$  is singular continuous with respect to the Lebesgue measure.

**8.4. Putnam's theorem and a.c. spectrum.** Using the spectral theorem, we establish Stone's formula:

$$\frac{1}{2} \langle f, (E_{[a,b]}(H) + E_{(a,b)}(H))f \rangle = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_a^b \text{Im}(\langle f, (H - \lambda - i\varepsilon)^{-1} f \rangle) dx.$$

where  $f \in \mathcal{H}$ , e.g., [RS1, Theorem VII.13].

**Proposition 8.18.** *Let  $H$  be self-adjoint in Hilbert space  $\mathcal{H}$ . Set  $a < b$ . Suppose that there is  $f \in \mathcal{H}$  such that*

$$c(f) := \sup_{\varepsilon \in (0,1)} \sup_{\lambda \in (a,b)} |\text{Im}(\langle f, (H - \lambda - i\varepsilon)^{-1} f \rangle)| < \infty$$

*Then  $E_{(a,b)}(H)f \in \mathcal{H}^{\text{ac}}$ .*

*Assume that  $\{f, c(f) < \infty\}$  is dense in  $\mathcal{H}$ , then:*

$$\sigma(H)|_{(a,b)} = \sigma^{\text{ac}}(H)|_{(a,b)}, \quad \sigma^{\text{p}}(H)|_{(a,b)} = \sigma^{\text{sc}}(H)|_{(a,b)} = \emptyset.$$

*Proof.* Set  $f \in \mathcal{H}$ . By Stone's formula and the fact that given a set  $J$ ,  $\|E_J(H)f\| \leq \|E_{\overline{J}}(H)f\|$ , we have for  $c < d$

$$0 \leq \langle f, E_{(c,d)}(H)f \rangle \leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_c^d \text{Im}(\langle f, (H - \lambda - i\varepsilon)^{-1} f \rangle) dx.$$

Set  $S := \cup_{i=1}^N (a_i, b_i)$  is open in  $(a, b)$ , where the intervals are taken two by two disjoint. Suppose first that  $N < \infty$ . We have:

$$\begin{aligned} \|E_S(H)f\|^2 &\leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_S \text{Im}(\langle f, (H - \lambda - i\varepsilon)^{-1} f \rangle) dx. \\ &\leq C \sum_i \int_{a_i}^{b_i} dx = C \cdot \text{Leb}(S). \end{aligned}$$

Suppose then that  $N = \infty$ . For  $m \in \mathbb{N}$ , set  $S_m := \cup_{i=1}^m (a_i, b_i)$ .

$$\|E_S(\varphi(Q))f\|^2 = \lim_{m \rightarrow \infty} \|E_{S_m}(\varphi(Q))f\|^2 \leq C \lim_{m \rightarrow \infty} \text{Leb}(S_m) = C \cdot \text{Leb}(S).$$

Take finally  $\mathcal{I} \subset (a, b)$  be such that  $\text{Leb}(\mathcal{I}) = 0$ . Since the Lebesgue measure is outer-regular for all  $k \in \mathbb{N}^*$  there is an open set  $S^{(k)}$  such that  $\mathcal{I} \subset S^{(k)}$  and  $|S^{(k)}| \leq 1/k$ . This implies that  $\|E_{\mathcal{I}}(H)f\| = 0$ . This gives  $E_{(a,b)}(H)f \in \mathcal{H}^{\text{ac}}$ .

Assume that  $\{f, c(f) < \infty\}$ . Since  $\mathcal{H}^{\text{ac}}$  is closed we obtain that  $E_{(a,b)}(H)f \in \mathcal{H}^{\text{ac}}$  for all  $f \in \mathcal{H}$ .  $\square$

We give a basic and central result in the theory of positive commutator. The first stone was set by C.R. Putnam, see [Put].

**Proposition 8.19** (Putnam). *Let  $H$  be a bounded self-adjoint operator acting in a Hilbert space  $\mathcal{H}$ . Suppose that there is a bounded self-adjoint operator  $A$ , such that:*

$$(8.4.11) \quad [H, iA] = C^*C,$$

where  $C$  is a bounded and injective operator. Then,

$$\sup_{\varepsilon > 0} \sup_{\lambda \in \mathbb{R}} |\langle f, \operatorname{Im}(H - \lambda - i\varepsilon)^{-1} f \rangle| \leq 4\|A\| \cdot \|(C^*)^{-1} f\|^2,$$

for all  $f \in \mathcal{D}((C^*)^{-1})$ . In particular, the spectrum of  $H$  is purely absolutely continuous.

**Remark 8.20.** Note that  $(C^*)^{-1}$  is an unbounded operator with dense domain, since  $C$  is injective.

*Proof.* Set  $R(z) := (z - H)^{-1}$ . Then

$$\begin{aligned} \|CR(\lambda \pm i\varepsilon)\|^2 &= \|R(\lambda \mp i\varepsilon)C^*CR(\lambda \pm i\varepsilon)\| \\ &= \|R(\lambda \mp i\varepsilon)[H, iA]R(\lambda \pm i\varepsilon)\| \\ &= \|R(\lambda \mp i\varepsilon)[H - \lambda \mp i\varepsilon, iA]R(\lambda \pm i\varepsilon)\| \\ &\leq \|AR(\lambda \pm i\varepsilon)\| + \|R(\lambda \mp i\varepsilon)A\| + 2\varepsilon\|R(\lambda \mp i\varepsilon)AR(\lambda \pm i\varepsilon)\| \leq 4\|A\|/\varepsilon. \end{aligned}$$

Therefore, we obtain

$$2\|C\operatorname{Im}R(\lambda \pm i\varepsilon)C^*\| = \|2i\varepsilon CR(\lambda + i\varepsilon)R(\lambda - i\varepsilon)C^*\| \leq 8\|A\|.$$

Therefore,

$$\sup_{\varepsilon > 0} \sup_{\lambda \in \mathbb{R}} |\langle f, \Im(H - \lambda - i\varepsilon)^{-1} f \rangle| \leq 4\|A\| \cdot \|(C^*)^{-1} f\|^2.$$

Stone's formula ensures that the measure given by  $\|E_{(\cdot)}(H)f\|^2$  is purely-absolutely continuous for all  $f \in \mathcal{D}((C^*)^{-1})$ . Since the domain is dense in  $\mathcal{H}$  and that  $\mathcal{H}^{\text{ac}}$  is closed, we obtain the result.  $\square$

Here we have proved a stronger result than the absence of singularly continuous spectrum

$$\sup_{\varepsilon > 0} \sup_{\lambda \in \mathbb{R}} |\langle f, \operatorname{Im}(H - \lambda - i\varepsilon)^{-1} f \rangle| \leq 4\|A\| \cdot \|(C^*)^{-1} f\|^2,$$

For the a.c. spectrum it would suffice to have on the right hand side a constant that depends on  $f$ . Here we have an explicit dependency of  $f$  that is uniform in a certain sense.

The bound that we obtain is in fact equivalent to the global propagation estimate:

$$\int_{\mathbb{R}} \|C^* e^{-itH} f\|^2 dt \leq c\|f\|^2,$$

for some  $c > 0$  and all  $f \in \mathcal{H}$ , e.g., [RS4, Theorem XIII.25].

The particle not only escape to infinity but will localise where  $C^*$  is small.

**8.5. On the stability of the a.c. spectrum.** We now aim at perturbation theory. We start with a negative result. Adding something which is too big compare to  $H$  will destroy the a.c. part of  $H$ , e.g., [Ka, Section X.2.1].

**Theorem 8.21.** *Let  $H$  be a self-adjoint operator. There exists a compact and self-adjoint operator  $K$  such that*

$$\sigma^{\text{pp}}(H + K) \cap \sigma^{\text{ess}}(H) = \sigma^{\text{ess}}(H).$$

**Remark 8.22.** *Whereas a compact perturbation is small when we study the stability of the essential spectrum, here we see that the nature of the spectrum, for instance being purely-a.c., is not stable under this class.*

We turn to a positive result, e.g., [RS3, Theorem IX.8].

**Theorem 8.23** (Kato-Rosenblum). *Let  $H$  be a self-adjoint operator. Let  $T$  be self-adjoint and trace class, i.e.,  $T$  compact such that  $\sum_i |\lambda_i(T)| < \infty$ . Then,  $\mathcal{H}^{\text{ac}}(H)$  is unitarily equivalent to  $\mathcal{H}^{\text{ac}}(H + T)$ . In particular,*

$$\sigma^{\text{ac}}(H) = \sigma^{\text{ac}}(H + T).$$

**Remark 8.24.** *Even if  $\mathcal{H}^{\text{ac}}(H) = \mathcal{H}$  the theorem does not guarantee that  $\mathcal{H}^{\text{ac}}(H + T) = \mathcal{H}$ . We could have that  $\mathcal{H}^{\text{sc}}(H + T) \neq \emptyset$ .*

We now prove the remark. Given a self-adjoint operator  $H$  and  $f \in \mathcal{H}$ . Set  $m_f(\cdot) := \langle f, E_{(\cdot)}(H)f \rangle$ . We define the *Borel transform* of  $m_f$  by setting:

$$F_{m_f}(x) := \int_{\mathbb{R}} \frac{dm_f(\xi)}{\xi - x}.$$

The de la Vallée-Poussin's result links the boundary value of  $F_{m_f}$  with the Lebesgue decomposition of  $m_f$ .

**Theorem 8.25** (de la Vallée-Poussin). *Let*

$$A_{m_f} := \{x, \lim_{\varepsilon \rightarrow 0^+} F_{m_f}(x + i\varepsilon) = \infty\}$$

and

$$B_{m_f} := \{x, \lim_{\varepsilon \rightarrow 0^+} F_{m_f}(x + i\varepsilon) \text{ is finite and } \text{Im} F_{m_f}(x + i0^+) > 0\}.$$

Then,  $m_f(\mathbb{R} \setminus (A_{m_f} \cup B_{m_f})) = 0$ ,  $m_f^{\text{ac}}(\mathbb{R} \setminus B_{m_f}) = 0$ ,  $m_f^{\text{s}}(\mathbb{R} \setminus A_{m_f}) = 0$ .

Let  $L^2([0, 1], \text{Leb}|_{[0, 1]} + m_C)$ , where  $m_C$  is defined in (8.3.10). We see that

$$\sigma(Q) = [0, 1], \sigma^{\text{pp}}(Q) = \emptyset, \sigma^{\text{ac}}(Q) = [0, 1], \text{ and } \sigma^{\text{sc}}(Q) = C.$$

For  $\lambda \in \mathbb{R}$ , we set

$$H_\lambda := Q + \lambda P_{\{1\}},$$

where

$$P_{\{1\}} := 1 \langle 1, \cdot \rangle.$$

We have that for  $\lambda \in \mathbb{R} \setminus \{0\}$ ,

$$\sigma^{\text{ess}}(H_\lambda) = [0, 1], \sigma^{\text{ac}}(H_\lambda) = [0, 1], \text{ and } \sigma^{\text{sc}}(H_\lambda) = \emptyset.$$

*Proof.* A direct computation gives:

$$\frac{1}{\pi} \text{Im} F_m(x + i0^+) = \begin{cases} 1, & x \in (0, 1), \\ 1/2, & x \in \{0, 1\}, \\ 0, & x \notin [0, 1]. \end{cases}$$

and for  $x \in (0, 1)$ :

$$\text{Re} F_m(x + i0^+) = \ln \left( \frac{x}{1-x} \right),$$

for  $x \in (0, 1)$ . Since for any measure  $\mu$  we have

$$\operatorname{Im} F_\mu(x_0 + i\varepsilon) \geq \mu(\{y, |x - y| \leq \varepsilon\}),$$

we infer:

$$F_{\mu_C}(x + i0^+) = \begin{cases} +\infty, & x \in C, \\ 0, & x \notin C, \end{cases} \quad \text{since the measure is not supported here}$$

Recall that

$$F_{\mu_\lambda}(z) = \langle 1, (H_\lambda - z)^{-1} 1 \rangle = \int (x - z)^{-1} d\mu_\lambda(x),$$

i.e.,  $\mu_\lambda$  is the spectral measure associated to  $H_\lambda$  and to the vector 1.

We now turn to the study of  $\mu_\lambda$  and focus on  $F_{\mu_\lambda}(z)$ , for all  $z \in \mathbb{C} \setminus \mathbb{R}$ . The resolvent identity gives

$$(H_\lambda - z)^{-1} = (H_0 - z)^{-1} - \lambda(H_\lambda - z)^{-1} P_1 (H_0 - z)^{-1}.$$

This gives:

$$F_{\mu_\lambda}(z) = F_{\mu_0}(z) - \lambda F_{\mu_\lambda}(z) F_{\mu_0}(z).$$

Therefore

$$F_{\mu_\lambda}(z) = \frac{F_{\mu_0}(z)}{1 + \lambda F_{\mu_0}(z)}.$$

This yields

$$\operatorname{Im}(F_{\mu_\lambda}(z)) = \frac{\operatorname{Im}(F_{\mu_0}(z))}{(1 + \lambda \operatorname{Re}(F_{\mu_0}(z)))^2 + \lambda^2 \operatorname{Im}(F_{\mu_0}(z))^2}.$$

The singular part of the spectrum of  $H_\lambda$  is supported by:

$$A_\lambda := \{x, \lim_{\varepsilon \rightarrow 0^+} F_{\mu_\lambda}(x + i\varepsilon) = \infty\}.$$

Given  $\lambda \neq 0$ , we see that  $[0, 1] \cap A_\lambda = \emptyset$ . Therefore there is no singular spectrum for  $H_\lambda$ . The spectrum of  $H_\lambda$  is purely absolutely continuous.  $\square$

## 9. THE MOURRE THEORY

**9.1. Motivation.** It is very complicated to apply the Putnam theorem in practice because of the boundedness of  $A$ .

We sacrifice the boundedness of  $A$  in the Putnam theorem and try to exploit the positivity of a commutator.

We start with  $\varphi(Q) := 2 \cos(Q)$  on  $\mathcal{H} := L^2(-\pi, \pi)$ . For  $f \in \mathcal{C}_c^\infty((-\pi, \pi))$  we set:

$$A_0 f := \frac{1}{2} (i \partial_x \varphi'(Q) + \varphi'(Q) i \partial_x).$$

This operator is essentially self-adjoint and we denote by  $A_0$  its closure.

For  $f \in \mathcal{C}_c^\infty((-\pi, \pi))$ , we have:

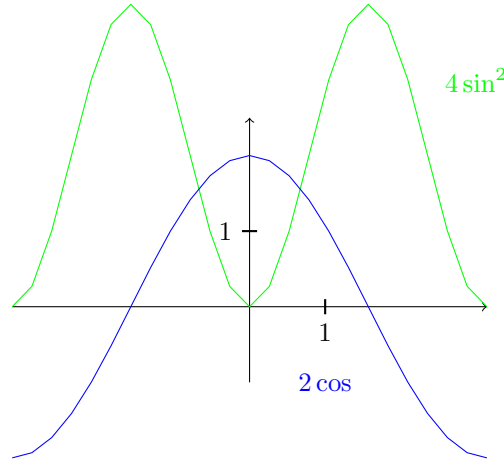
$$\begin{aligned}
 2[\varphi(Q), iA_0]f &= -[\varphi(Q), \partial_x \varphi'(Q) + \varphi'(Q) \partial_x]f \\
 &= (\partial_x \varphi'(Q) + \varphi'(Q) \partial_x) \varphi(Q) f - \varphi(Q) (\partial_x \varphi'(Q) + \varphi'(Q) \partial_x) f \\
 &= \varphi''(Q) \varphi(Q) f + (\varphi'(Q))^2 f + \varphi'(Q) \varphi(Q) f' + (\varphi'(Q))^2 f + \varphi'(Q) \varphi(Q) f' \\
 &\quad - (\varphi''(Q) \varphi(Q) f + \varphi'(Q) \varphi(Q) f' + \varphi'(Q) \varphi(Q) f') \\
 &= 2\varphi^2(Q) f.
 \end{aligned}$$

In other words, using the density of  $\mathcal{C}_c^\infty$  in  $\mathcal{H}$ , we infer:

$$[\varphi(Q), iA_0] = (\varphi'(Q))^2.$$

This gives:

$$[\varphi(Q), iA_0] = 4 \sin^2(Q) = (2 - 2 \cos(Q))(2 + 2 \cos(Q)).$$



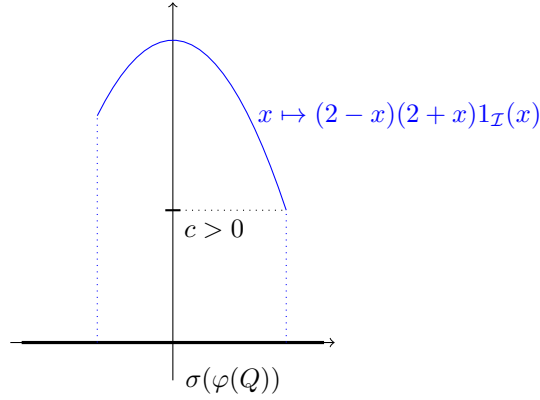
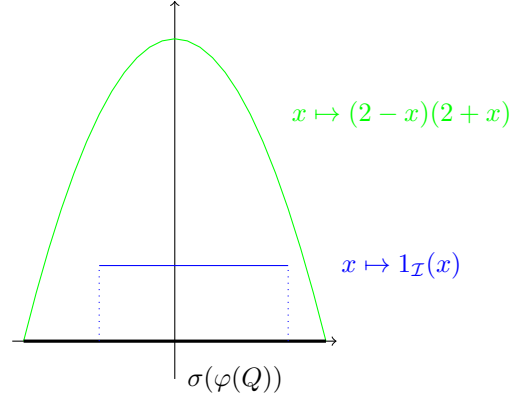
**Remark 9.1.** Note that  $4 \sin^2(x) = 0$  if and only if  $\cos'(x) = 0$ .

The operator  $4 \sin^2(Q)$  is injective and non-negative. Taking apart that  $A_0$  is unbounded, we are in the setting of Putnam's theory. We hope to deduce that  $2 \cos(Q)$  is purely a.c. by this method.

Take  $\mathcal{I}$  be a closed subset included in the interior of  $[-2, 2] = \sigma(\varphi(Q))$ . We have:

$$E_{\mathcal{I}}(\varphi(Q))[\varphi(Q), iA_0]E_{\mathcal{I}}(\varphi(Q)) = E_{\mathcal{I}}(\varphi(Q))(2 - \varphi(Q))(2 + \varphi(Q))E_{\mathcal{I}}(\varphi(Q))$$





There is  $c > 0$ , for all  $f \in \mathcal{H}$ ,

$$\begin{aligned}
 \langle f, E_{\mathcal{I}}(\varphi(Q))[\varphi(Q), \mathbf{i}A_0]E_{\mathcal{I}}(\varphi(Q))f \rangle &= \langle f, E_{\mathcal{I}}(\varphi(Q))(2 - \varphi(Q))(2 + \varphi(Q))E_{\mathcal{I}}(\varphi(Q))f \rangle \\
 &= \int_{\sigma(\varphi(Q))} 1_I(x)(2 - x)(2 + x)1_I(x)dm_f(\varphi(Q))(x) \\
 &\geq c \int_{\sigma(\varphi(Q))} 1_I(x)dm_f(\varphi(Q))(x) \\
 &= c \langle E_{\mathcal{I}}(\varphi(Q))f, E_{\mathcal{I}}(\varphi(Q))f \rangle.
 \end{aligned}$$

In other words we have that there is  $c > 0$  such that

$$E_{\mathcal{I}}(\varphi(Q))[\varphi(Q), \mathbf{i}A_0]E_{\mathcal{I}}(\varphi(Q)) \geq cE_{\mathcal{I}}(\varphi(Q)),$$

holds in the form sense, i.e., when applied to any  $f$  on both side.

We now go back to  $\mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C})$  and will go into perturbation theory.

Recall that the Fourier transform  $\mathcal{F} : \ell^2(\mathbb{Z}) \rightarrow L^2([-\pi, \pi])$  is defined by

$$(\mathcal{F}f)(x) := \frac{1}{\sqrt{2\pi}} \sum_n f(n)e^{-inx}, \text{ for all } f \in \ell^2(\mathbb{Z}) \text{ and } x \in [-\pi, \pi].$$

The adjacency matrix is given by:

$$(\mathcal{A}_{\mathbb{Z}}f)(n) := f(n-1) + f(n+1), \quad \text{for } f \in \mathcal{H}.$$

and

$$\mathcal{A}_{\mathbb{Z}} = \mathcal{F}^{-1} 2 \cos(Q) \mathcal{F}$$

Moreover, for  $f \in \mathcal{C}_c(\mathbb{Z})$ , the set of function with compact support, a direct computation gives:

$$Af := \mathcal{F}^{-1} A_0 \mathcal{F} f = i \left( \frac{1}{2} (U^* + U) + Q(U^* - U) \right) f,$$

where

$$Uf(n) := f(n-1) \quad \text{and} \quad (U^*f)(n) = f(n+1).$$

The operator  $A$  is essentially self-adjoint on  $\mathcal{C}_c(\mathbb{Z})$ . We denote its closure with the same symbol. We refer to Remark 9.4 for the notion of unbounded self-adjoint operator.

Thanks to the previous calculus, we have:

$$[\mathcal{A}_{\mathbb{Z}}, iA] = (2 - \mathcal{A}_{\mathbb{Z}})(2 + \mathcal{A}_{\mathbb{Z}})$$

and, given  $\mathcal{I}$  closed included in the interior of  $[-2, 2]$ , the spectrum of  $\mathcal{A}_{\mathbb{Z}}$ , there is a positive constant  $c > 0$ :

$$E_{\mathcal{I}}(\mathcal{A}_{\mathbb{Z}})[\mathcal{A}_{\mathbb{Z}}, iA]E_{\mathcal{I}}(\mathcal{A}_{\mathbb{Z}}) \geq cE_{\mathcal{I}}(\mathcal{A}_{\mathbb{Z}}),$$

in the form sense, i.e., when applied to  $f \in \mathcal{H}$  on both side.

We now add a perturbation. Let  $V : \mathbb{Z} \rightarrow \mathbb{R}$  be such that

$$\lim_{n \rightarrow \pm\infty} V(n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \pm\infty} n(V(n) - V(n+1)) = 0.$$

In particular, we have :

$$V(Q) \in \mathcal{K}(\mathcal{H}) \quad \text{and} \quad Q(V(Q) - V(Q+1)) \in \mathcal{K}(\mathcal{H}).$$

Take  $f \in \mathcal{C}_c$ . We have:

$$\begin{aligned} [U^*, V(Q)]f(n) &= (U^*V(Q)f)(n) - (V(Q)U^*f)(n) \\ &= (V(Q)f)(n+1) - V(n)f(n+1) = (V(n+1) - V(n))f(n+1) \\ &= ((V(Q+1) - V(Q))U^*f)(n). \end{aligned}$$

We obtain:

$$[U^*, V] = (V(Q+1) - V(Q))U^* \quad \text{and} \quad [U, V] = (V(Q-1) - V(Q))U.$$

This yields:

$$\begin{aligned}
2[V(Q), iA]f &= 2 \left[ V(Q), i \cdot i \left( \frac{1}{2}(U^* + U) + Q(U^* - U) \right) \right] \\
&= [(U^* + U) + Q(U^* - U), V(Q)] f \\
&= [U^*, V]f + [U, V]f + Q[U^*, V]f - Q[U, V]f, \text{ since } [Q, V(Q)] = 0 \\
&= \underbrace{(V(Q+1) - V(Q)) U^* f}_{\text{compact}} + \underbrace{(V(Q-1) - V(Q)) U f}_{\text{compact}} \\
&\quad + \underbrace{Q(V(Q+1) - V(Q)) U^* f}_{\text{compact}} - \underbrace{Q(V(Q-1) - V(Q)) U f}_{\text{compact}}.
\end{aligned}$$

Therefore,

$$[V(Q), iA] \in \mathcal{K}(\mathcal{H}).$$

We plug this information into the previous estimate. We set  $H := \mathcal{A}_{\mathbb{Z}} + V(Q)$

$$\begin{aligned}
[H, iA] &= [\mathcal{A}_{\mathbb{Z}}, iA] + [V(Q), iA] = (2 - \mathcal{A}_{\mathbb{Z}})(2 + \mathcal{A}_{\mathbb{Z}}) + \text{compact} \\
&= (2 - \mathcal{A}_{\mathbb{Z}} - V(Q))(2 + \mathcal{A}_{\mathbb{Z}} + V(Q)) + \text{compact} \\
&= (2 - H)(2 + H) + \text{compact}.
\end{aligned}$$

Recall that, by the Weyl's Theorem,  $\sigma_{\text{ess}}(H) = [-2, 2]$ , therefore by taking  $\mathcal{I}$  being closed in the interior of the essential spectrum of  $H$  we get, there are

$$c := \inf_{x \in \mathcal{I}} (2 - x)(2 + x) > 0$$

and a compact operator  $K$  such that

$$(9.1.12) \quad E_{\mathcal{I}}(H)[H, iA]E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H) + \underbrace{E_{\mathcal{I}}(H)KE_{\mathcal{I}}(H)}_{\text{compact}},$$

in the form sense. This is a Mourre estimate.

**9.2. General Theory.** In the beginning of the eighties, E. Mourre had the brilliant idea to localize (8.4.11) in energy. His theory was developed in [Mou] to show the absolute continuity of the continuous spectrum of 3-body Schrödinger operators and to study their scattering theory. His work was immediately generalized to the  $N$ -body context in [PSS]. In particular, one wanted to show their asymptotic completeness and the Mourre estimate, c.f., (9.1.12), played a crucial role in the proof. Now, Mourre's commutator theory is fundamental tool to develop the stationary scattering theory of general self-adjoint operators. We refer to [ABG] for some historical developments and to [Go2] for a more complete introduction to the subject.

**Definition 9.2.** *Given a bounded operator  $H$  acting in a complex Hilbert space  $\mathcal{H}$  and  $k \in \mathbb{N}$ , one says that  $H \in \mathcal{C}^k(A)$  if  $t \mapsto e^{-itA} H e^{itA} f$  is  $\mathcal{C}^k$  for all  $f \in \mathcal{H}$ .*

**Proposition 9.3.** *Let  $H$  be a bounded operator and  $A$  be a (possibly unbounded) self-adjoint operator. The following assertions are equivalent:*

- 1)  $H \in \mathcal{C}^1(A)$ .

2) There is a constant  $c > 0$  such that

$$(9.2.13) \quad |\langle Hf, Af \rangle - \langle Af, Hf \rangle| \leq c\|f\|^2,$$

for all  $f \in \mathcal{D}(A)$ .

**Remark 9.4.** Be careful, in practice  $A$  is a unbounded self-adjoint operator. We did not develop its spectral theory in this course. One should keep in mind that the spectrum of  $A$  is real and that the theory of Borel functional calculus is also valid in this context. To prove it, one replaces the statements about polynomial with rational functions that are bounded on  $\mathbb{R}$ .

Note that, by density of  $\mathcal{D}(A)$ , (9.2.13) defines a bounded operator that we denote by  $[H, A]_\circ$ , or simply  $[H, A]$  when no confusion can arise.

We can prove that the derivative of  $t \mapsto e^{-itA} H e^{itA} f$  for  $t = 0$  is equal to  $[H, A]_\circ$ .

One can show that  $H\mathcal{D}(A) \subset \mathcal{D}(A)$  and therefore  $[H, A]$  has a meaning, on the operator sense, on  $\mathcal{D}(A)$  is equal to  $[H, A]_\circ|_{\mathcal{D}(A)} = (HA - AH)|_{\mathcal{D}(A)}$ .

**Remark 9.5.** In our example,  $H \in \mathcal{C}^1(A)$ .

**Proposition 9.6** (“Virial Theorem”). Let  $H \in \mathcal{C}^1(A)$  with  $H$  bounded and self-adjoint and  $A$  self-adjoint.

1) If the following Mourre estimate holds true

$$E_{\mathcal{I}}(H)[H, iA]_\circ E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H) + K,$$

where  $K \in \mathcal{K}(\mathcal{H})$  and  $c > 0$ , then  $H$  has a finite number of eigenvalue in  $\mathcal{I}$ , counted with multiplicity.

2) If the following strict Mourre estimate holds true

$$E_{\mathcal{I}}(H)[H, iA]_\circ E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H),$$

where  $c > 0$ , then  $H$  has no eigenvalue in  $\mathcal{I}$ .

**Remark 9.7.** The operator  $A$  is the conjugate operator associated to  $H$ .

**Remark 9.8.** In our example, the eigenvalues of  $H$  that do not belong to  $\{-2, 2\}$  are of finite multiplicity and can only accumulate to  $\{-2, 2\}$ .

*Proof.* Let  $f$  be an eigenfunction of  $H$  associated to  $\lambda \in \mathcal{I}$ . To motivate the approach we start with a wrong proof.

$$\begin{aligned} \langle f, [H, iA]_\circ f \rangle &= \langle f, [H - \lambda, iA]_\circ f \rangle \\ &= i \langle \underbrace{(H - \lambda)f}_{=0}, \underbrace{Af}_{f \in \mathcal{D}(A)?} \rangle - i \langle \underbrace{Af}_{f \in \mathcal{D}(A)?}, \underbrace{(H - \lambda)f}_{=0} \rangle = 0? \end{aligned}$$

As in general  $f \notin \mathcal{D}(A)$ , we change slightly the approach. Set for  $\tau \neq 0$ ,

$$A_\tau := \frac{1}{i\tau}(e^{iA\tau} - \text{Id})$$

Note that for  $g \in \mathcal{D}(A)$ ,

$$\lim_{\tau \rightarrow 0} A_\tau g = Ag.$$

Moreover, we have for all  $g \in \mathcal{H}$

$$[A, H]_{\circ} g = \lim_{\tau \rightarrow 0} \frac{1}{i\tau} (e^{i\tau A} H e^{-i\tau A} - H) g = \lim_{\tau \rightarrow 0} \frac{1}{i\tau} [e^{i\tau A}, H] e^{-i\tau A} g = \lim_{\tau \rightarrow 0} [A_{\tau}, H] g.$$

We can compute in a legal way.

$$\begin{aligned} \langle f, [H, iA]_{\circ} f \rangle &= \lim_{\tau \rightarrow 0} \langle f, [H, iA_{\tau}] f \rangle \\ &= \lim_{\tau \rightarrow 0} \langle f, [H - \lambda, iA_{\tau}]_{\circ} f \rangle \\ &= \lim_{\tau \rightarrow 0} i \underbrace{\langle (H - \lambda) f, \underbrace{A_{\tau} f}_{A_{\tau} f \in \mathcal{H}} \rangle}_{=0} - i \underbrace{\langle \underbrace{A_{\tau} f}_{A_{\tau} f \in \mathcal{H}}, (H - \lambda) f \rangle}_{=0} = 0. \end{aligned}$$

We turn to the point 2. We apply the strict Mourre estimate to  $f$ , where  $Hf = \lambda f$  and  $\lambda \in \mathcal{I}$ . Note first that

$$f = E_{\{\lambda\}}(H)f = E_{\mathcal{I}}(H)f.$$

Therefore, we get

$$\begin{aligned} \|f\|^2 &= \|E_{\mathcal{I}}(H)f\|^2 \leq \frac{1}{c} \langle f, E_{\mathcal{I}}(H)[H, iA]_{\circ} E_{\mathcal{I}}(H)f \rangle \\ &= \frac{1}{c} \langle E_{\mathcal{I}}(H)f, [H, iA]_{\circ} E_{\mathcal{I}}(H)f \rangle \\ &= \frac{1}{c} \langle f, [H, iA]_{\circ} f \rangle = 0. \end{aligned}$$

Therefore  $H$  has no eigenvalue in  $\mathcal{I}$ .

We now prove the point 1. Suppose that, for all  $n \in \mathbb{N}$ , there are  $\lambda_n \in \mathcal{I}$  and  $f_n \in \mathcal{H}$  such that  $\|f_n\| = 1$  and  $Hf_n = \lambda_n f_n$ .

We apply the Mourre estimate to  $f_n$ . We get:

$$\begin{aligned} 0 &= \langle f_n, E_{\mathcal{I}}(H)[H, iA]_{\circ} E_{\mathcal{I}}(H)f_n \rangle \geq c \langle E_{\mathcal{I}}(H)f_n, E_{\mathcal{I}}(H)f_n \rangle + \langle E_{\mathcal{I}}(H)f_n, K E_{\mathcal{I}}(H)f_n \rangle \\ &\geq c \underbrace{\langle f_n, f_n \rangle}_{=1} + \langle f_n, K f_n \rangle \\ &\geq c - \|K f_n\|^2 \end{aligned}$$

Or  $f_n \rightharpoonup 0$  (weak convergence) and  $K$  compact, therefore  $\|K f_n\| \rightarrow 0$ . With  $n$  large enough we obtain a contradiction with the fact that  $c > 0$ .  $\square$

**Proposition 9.9.** *Let  $H \in \mathcal{C}^1(A)$  with  $H$  bounded and self-adjoint and  $A$  self-adjoint. Assume that the following Mourre estimate holds true*

$$E_{\mathcal{I}}(H)[H, iA]_{\circ} E_{\mathcal{I}}(H) \geq c E_{\mathcal{I}}(H) + K,$$

where  $K \in \mathcal{K}(\mathcal{H})$ .

If  $H$  has no eigenvalue in  $\mathcal{I}$ , then for all  $\lambda$  in the interior of  $\mathcal{I}$  there is  $\mathcal{J} := [\lambda - \varepsilon, \lambda + \varepsilon]$ , with  $\varepsilon > 0$  small enough, such that

$$E_{\mathcal{J}}(H)[H, iA]_{\circ} E_{\mathcal{J}}(H) \geq \frac{c}{2} E_{\mathcal{J}}(H),$$

holds true.

*Proof.* Set  $\mathcal{I}_n := [\lambda - 1/n, \lambda + 1/n]$ . Since there is no eigenvalue in  $\mathcal{I}$ , we have that for all  $f \in \mathcal{H}$  that

$$\|E_{\mathcal{I}_n}(H)f\|^2 = \int_{\mathcal{I}_n} dm_f(x) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

by dominated convergence.

Since  $K$  is compact, we have that  $\|KE_{\mathcal{I}_n}(H)\| \rightarrow 0$ , as  $n \rightarrow \infty$ . Therefore, for  $n$  large enough, we obtain that  $\|KE_{\mathcal{I}_n}(H)\| \leq c\|E_{\mathcal{I}_n}(H)\|/2$ . Therefore we obtain:

$$E_{\mathcal{I}_n}(H)[H, \mathfrak{i}A]_{\circ} E_{\mathcal{I}_n}(H) \geq \frac{c}{2} E_{\mathcal{I}_n}(H).$$

We conclude by setting  $\mathcal{J} := \mathcal{I}_n$ . □

Assume that  $H \in \mathcal{C}^1(A)$  and

$$E_{\mathcal{I}}(H)[H, \mathfrak{i}A]_{\circ} E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H).$$

We will deduce some dynamical properties.

Given  $f \in \mathcal{H}$  and  $f_t := e^{-itH}f$  its evolution at time  $t \in \mathbb{R}$  under the dynamic generated by the Hamiltonian  $H$ , one looks at the Heisenberg picture:

$$(9.2.14) \quad \mathcal{H}_f(t) := \langle f_t, Af_t \rangle.$$

As  $A$  is an unbounded self-adjoint operator, we take  $f := \varphi(H)g$ , with  $g \in \mathcal{D}(A)$  and  $\varphi \in \mathcal{C}_c^\infty(\mathcal{I})$ . We can prove that  $\mathcal{H}_f$  is well-defined as  $e^{-itH}\varphi(H)$  stabilises the domain of  $A$ . This implies also that  $\mathcal{H}_f \in \mathcal{C}^1(\mathbb{R})$ .

**Remark 9.10.** Note that  $E_{\mathcal{I}}(H)f = E_{\mathcal{I}}(H)\varphi(H)g = \varphi(H)g = f$ .

Since  $H \in \mathcal{C}^1(A)$ , the commutator  $[H, \mathfrak{i}A]_{\circ}$  is a bounded operator. We denote by  $C$  its norm.

$$\mathcal{H}'_f(t) = \langle f_t, [H, \mathfrak{i}A]_{\circ} f_t \rangle = \langle f_t, E_{\mathcal{I}}(H)[H, \mathfrak{i}A]_{\circ} E_{\mathcal{I}}(H)f_t \rangle.$$

We now use the Mourre estimate above  $\mathcal{I}$  and since  $e^{itH}$  is unitary, one gets:

$$c\|f\|^2 \leq \mathcal{H}'_f(t) \leq C\|f\|^2.$$

Now integrate the previous inequality and obtain

$$ct\|f\|^2 \leq \mathcal{H}_f(t) - \mathcal{H}_f(0) \leq Ct\|f\|^2, \quad \text{for } t \geq 0$$

The transport of the particle is therefore ballistic with respect to  $A$ , we have some transport in the direction given by  $A$ . Purely absolutely continuous spectrum is therefore expected.

**Theorem 9.11.** Suppose that  $H$  is a bounded and self-adjoint operator and that  $A$  is self-adjoint. Assume that  $H \in \mathcal{C}^2(A)$  and that

$$E_{\mathcal{I}}(H)[H, \mathfrak{i}A]_{\circ} E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H),$$

holds true for some non-empty and closed interval  $\mathcal{I}$ . Then:

- 1) The spectrum of  $H$  restricted to  $\mathcal{I}$  is purely absolutely continuous.

- 2) Given  $\mathcal{J}$  a closed interval included in the interior of  $\mathcal{I}$ , for all  $s > 1/2$  there is a constant  $c > 0$ , such that the following limiting absorption principle holds true:

$$\sup_{\lambda \in \mathcal{J}} \sup_{\varepsilon > 0} |\langle f, (H - \lambda - i\varepsilon)^{-1} f \rangle| \leq c \|\langle A \rangle^s f\|^2,$$

where  $\langle x \rangle := \sqrt{1 + x^2}$ .

- 3) There is  $c > 0$  such that for all  $f \in \mathcal{H}$ ,

$$\int_{\mathbb{R}} \|\langle A \rangle^{-s} e^{-itH} E_{\mathcal{J}}(H) f\|^2 dt \leq c \|f\|^2.$$

The original proof given in [Mou], see also [PSS], relies on the use of a differential inequality. Their hypothesis was slightly stronger than the one given by the class  $\mathcal{C}^2(A)$ . The latter was introduced by V. Georgescu and its collaborators. They greatly improved this hypothesis. We refer to [ABG] for historical developments. Under the class  $\mathcal{C}^2(A)$ , one can provide a different proof which relies on commutator expansions, see [GJ, Gé] and also [Go2] for a sketch of proof and a review for recent applications.

**9.3. Final example.** A direct application of the previous theorem gives:

**Theorem 9.12.** Suppose that  $H := \mathcal{A}_{\mathbb{Z}} + V(Q)$ , where

$$\lim_{n \rightarrow \pm\infty} V(n) = 0, \quad \lim_{n \rightarrow \pm\infty} n(V(n) - V(n+1)) = 0, \quad \text{and} \quad \sup_n n^2 |V(n) - V(n+1)| < \infty$$

Then:

- 1) The essential spectrum of  $H$  is  $\sigma_{\text{ess}}(H) = [-2, 2]$ .
- 2) The eigenvalues of  $H$  that do not belong to  $\{-2, 2\}$  are of finite multiplicity and can only accumulate to  $\{-2, 2\}$ .
- 3)  $\sigma^{\text{sc}}(H) = \emptyset$ .
- 4) Given  $\mathcal{J}$  a closed interval included in the interior of  $\mathcal{I}$ , for all  $s > 1/2$  there is a constant  $c > 0$ , such that the following limiting absorption principle holds true:

$$\sup_{\lambda \in \mathcal{J}} \sup_{\varepsilon > 0} |\langle f, (H - \lambda - i\varepsilon)^{-1} f \rangle| \leq c \|\langle Q \rangle^s f\|^2,$$

for all  $f \in \mathcal{D}(\langle Q \rangle^s)$ .

- 5) There is  $c > 0$  such that for all  $f \in \mathcal{H}$ ,

$$\int_{\mathbb{R}} \|\langle A \rangle^{-s} e^{-itH} E_{\mathcal{J}}(H) f\|^2 dt \leq c \|f\|^2.$$

With more technology, we can prove that

- 1) Under the hypothesis that there is  $\varepsilon > 0$  such that

$$\lim_{n \rightarrow \pm\infty} V(n) = 0, \quad \lim_{n \rightarrow \pm\infty} n^{1+\varepsilon} (V(n) - V(n+1)) = 0,$$

the conclusions of the Theorem remain true, e.g., [BoSa].

- 2) Under the hypothesis that  $n \mapsto V(n+k) - V(n) \in \ell^1(\mathbb{Z})$  holds true for some  $k \in \mathbb{Z}$ , we have that

$$\sigma^{\text{sc}}(H) = \emptyset$$

and that there is no eigenvalue in  $(-2, 2)$ . We refer to [GH] for historical developments and application to the Dirac case.

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