Basic fluid mechanics for civil engineers

Maxime Nicolas
maxime.nicolas@univ-amu.fr

POLYTECH MARSEILLE
Aix Marseille Université
Département génie civil

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Course 1 outline

PREAMBLE

Course syllabus

Schedule:

- 10 lectures
- 10 workshops

Assessment and exam:

activity	percentage
homework	20%
	bonus +1 if written in english
final exam (Dec. 5th)	80%
flash quiz	+1 point on the final grade

Online

This course is available on ENT/AmeTice :

Sciences & technologies ▸ Polytech ▸ Génie civil ▸
[16] - S5 - JGC51B - Mécanique des fluides (Maxime Nicolas)

with

- slides
- workshops texts
- equation forms

Working advices

- personal work is essential
- read your notes before the next class and before the workshop
- be curious
- work for you (not for the grade)

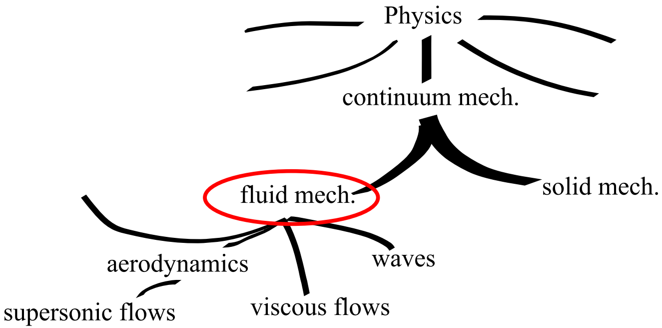
Course outline

1. Introduction and basic concepts — vector calculus
2. Statics — hydrostatic pressure, Archimede's principle
3. Kinematics — Euler and Langrage description, mass conservation
4. Balance equations — mass and momentum cons. equation
5. Flows classification and Bernoulli — Venturi effect
6. The Navier-Stokes equation — Poiseuille and Couette flows
7. The Stokes equation — Flow Sedimentation
8. Non newtonian fluids — Concrete flows
9. Flow in porous media — Darcy
10. Surface tension effects — Capillarity

INTRODUCTION AND BASIC CONCEPTS

Description of a fluid

What is fluid mechanics?



What is fluid mechanics?

Fluid mechanics is the mechanical science for gazes or liquids, at rest or flowing.

Large set of applications :

- blood flow
- atmosphere flows, oceanic flows, lava flows
- pipe flow (water, oil, vapor)
- flight (birds, planes)
- pumping
- dams, harbours
- ...

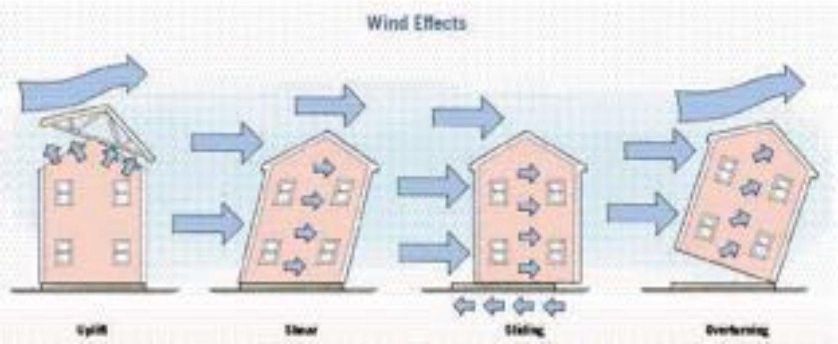
Large atmospheric phenomena

Ouragan Katrina, 29 août 2005

FM for civil engineering: dams

Hoover dam, 1935

FM for civil engineering: wind effects on structures



from `timberframehome.wordpress.com`

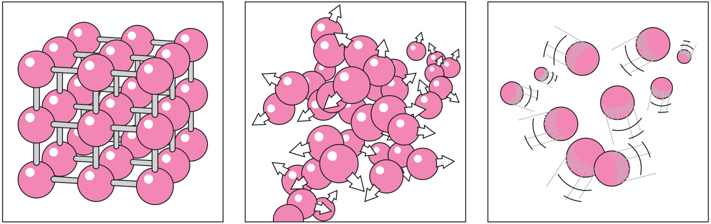
FM for civil engineering: harbor structures

from www.marseille-port.fr

FM for civil engineering: concrete flows

from http://www.chantiersdefrance.fr

What is a fluid?



(a) (b) (c)

Main concepts

- density
- stresses and pressure
- viscosity
- superficial tension

density

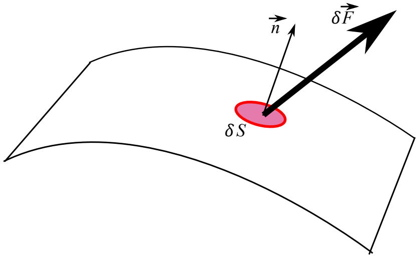
density = weight per unit volume
unit : $\mathrm{kg \cdot m^{-3}}$

fluid	density in $\mathrm{kg \cdot m^{-3}}$
air	1.29
water	1 000
concrete	2 500
molten iron	$\approx$ 7 000

Notice: density decreases with temperature increase

Stress

Elementary force $\delta\vec{F}$ applying on an elementary surface δS.



Ratio is

$$\vec{\sigma} = \frac{\delta\vec{F}}{\delta S}$$

the stress vector.
Standard unit : Pa (pascal). $1\ \text{Pa} = 1\ \text{N}{\cdot}\text{m}^{-2} = 1\ \text{kg}{\cdot}\text{m}^{-1}{\cdot}\text{s}^{-2}$

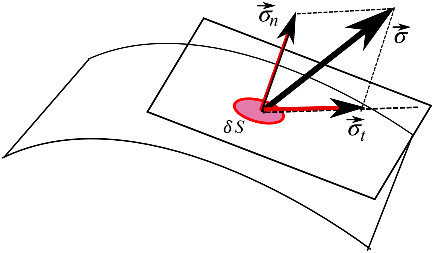
Stress

The surface element δS is oriented by a unit vector $\vec{n}$.
$\vec{n}$ is normal (perpendicular) to the tangential plane.

$$\vec{\sigma} = \vec{\sigma_n} + \vec{\sigma_t}$$

with

$$\vec{\sigma_n} = (\vec{\sigma} \cdot \vec{n})\vec{n} \qquad \vec{\sigma_t} = \vec{\sigma} - \vec{\sigma_n} = (\vec{\sigma} \cdot \vec{t})\vec{t}$$



stress vector = normal stress ($\perp$) + shear stress (//)

Pressure

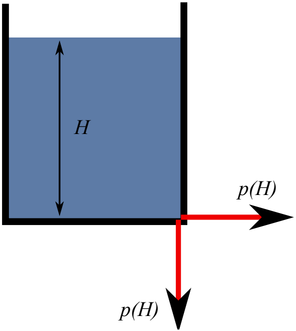
The pressure is a normal stress.

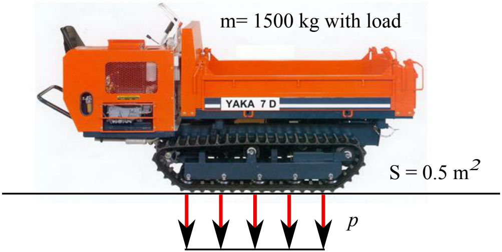
Notation : p
S.I. unit : pascal (Pa) $1\ \mathrm{Pa} = 1\ \mathrm{N{\cdot}m^{-2}} = 1\ \mathrm{kg{\cdot}m^{-1}{\cdot}s^{-2}}$

basic interpretation: normal force applied on a surface

The pressure in a fluid is an isotropic stress: its intensity does not depend on the direction.

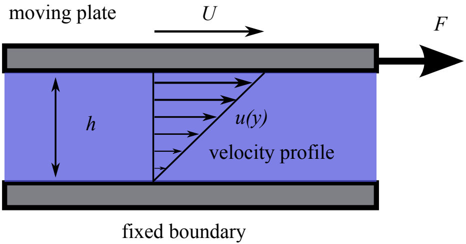
Pressure examples





viscosity

A macroscopic view on viscosity :



Tangential (shear) stress: $\sigma_t = \tau = \frac{F}{A}$
Shear rate: $\dot{\gamma} = \frac{U}{h}$

For a newtonian fluid :

$$\tau = \eta \, \dot{\gamma}$$

η is the dynamic viscosity of the fluid

viscosity

Standard unit: $[\eta]$=Pa·s

1 Pa·s=1 kg·m^{-1}· s^{-1}

fluid	η (Pa·s)
air	$1.8\,10^{-5}$
water	10^{-3}
blood	$6\,10^{-3}$
honey	10
fresh concrete	5–25 ⚠non-newtonian fluid

Also useful : kinematic viscosity

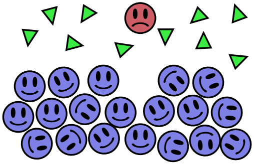
$$\nu = \frac{\eta}{\rho}$$

with $[\nu]$=m^2·s^{-1}

superficial tension

The superficial tension applies only at the interface between 2 different fluids (e.g. water and air).

The molecules of a fluid like to be surrounded by some molecules of the same kind.

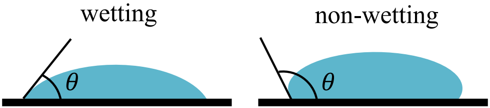


A drop of liquid on a solid surface does not flatten completely under gravity:

superficial tension and wettability

- symbol: γ
- unit: $[\gamma]$=N·m^{-1}
- order of magnitude: 0.02 to 0.075 N·m^{-1}
- most common: $\gamma_{water/air}$ = 0.073 N·m^{-1}

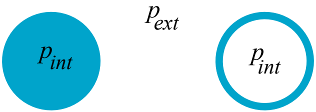
When the fluid molecules are preferring the contact with a solid surface rather than the surrounding air, it is said that the fluid is wetting the solid.



drops and bubbles

When the water/air interface is curved, the surface tension is balanced with a pressure difference, according to Laplace's law:

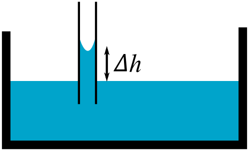
$$\Delta p = p_{int} - p_{ext} = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$



$$\Delta p_{drop} = \frac{2\gamma}{R} \qquad \Delta p_{bubble} = \frac{4\gamma}{R}$$

capillary rise

The capillary rise is a very common phenomena (rise of water in sils, rocks or concrete), and can be illustrated with a single tube:



wetting → curvature → pressure difference → rise

$$\Delta h = \frac{4\gamma \cos\theta}{\rho g d}$$

INTRODUCTION AND BASIC CONCEPTS

Maths for fluid mechanics

Maths for fluid mechanics

- scalar, vector, tensor
- scalar fields $f(x,y,z)$
- vector fields $\overrightarrow{A}(x,y,z)$
- differential operators : gradient, divergence, curl, laplacian
- partial differential equations

scalars and scalar field

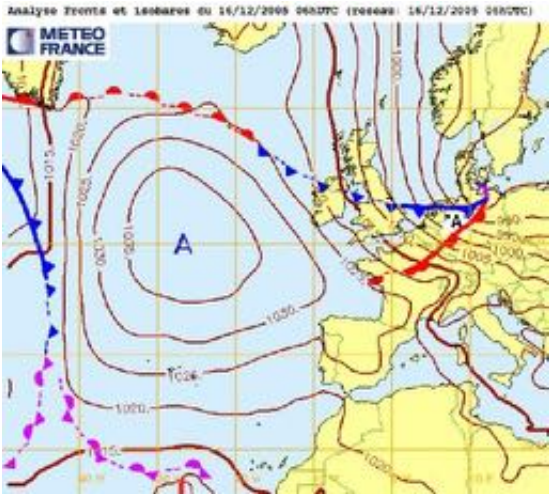
A scalar is a one-value object. mass, volume, density, temperature...

A scalar field is a multi-variable scalar function $p(x,y,z) = p(\overrightarrow{r})$

Without time, stationary scalar field $p(\overrightarrow{r})$

With time, unstationary scalar field $p(\overrightarrow{r},t)$

Scalar field mapping



Vectors

A vector is a multi-value object. Useful to represent forces, velocities, accelerations.
In 3 dimensions,

$$\overrightarrow{A} = (A_x, A_y, A_z) = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

example of the gravity acceleration:

$$\overrightarrow{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

Vector field

A vector field is a set of scalar functions, each function is a component of a vector.

$$\overrightarrow{A}(x,y,z) = \begin{pmatrix} A_x(x,y,z) \\ A_y(x,y,z) \\ A_z(x,y,z) \end{pmatrix}$$

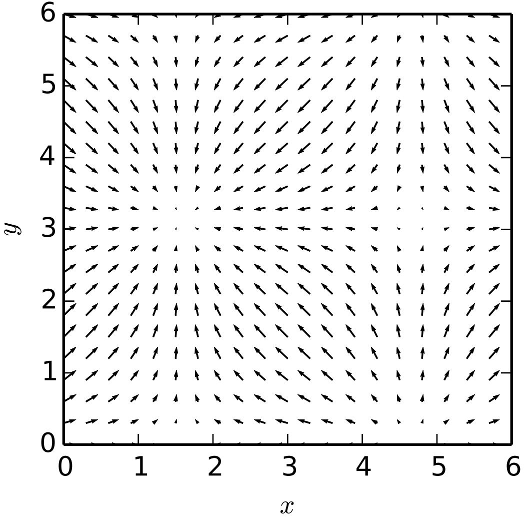
and for an unstationary vector field

$$\overrightarrow{A}(x,y,z,t) = \begin{pmatrix} A_x(x,y,z,t) \\ A_y(x,y,z,t) \\ A_z(x,y,z,t) \end{pmatrix}$$

The value of the vector has to be computed at each space point and for each time.

Vector field

Plot of $\overrightarrow{A} = (\cos x, \sin y, 0)$



Vector field

Review of vector and differential calculus

derivative definition for a single variable function:

$$\frac{d}{dt}f(t) = \frac{f(t+\delta t) - f(t)}{\delta t}, \quad \text{as} \quad \delta t \to 0$$

but many useful functions in fluid mechanics are multi-variables functions (pressure, velocity).
Partial derivative:

$$\frac{\partial f(x,y,z,t)}{\partial y} = \frac{f(x,y+\delta y,z,t) - f(x,y,z,t)}{\delta y}, \quad \text{as} \quad \delta y \to 0$$

Important implication :

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Example: compute $\frac{\partial^2}{\partial x \partial y}(x^2 y + 1)$

Integration of a partial derivative

Let's define

$$\frac{\partial f(x,y,z)}{\partial y} = k(x,y,z)$$

Integrating along a single coordinate (here y) gives

$$f(x,y,z) = \int k(x,y,z)dy + C(x,z)$$

The integration constant C does not depend on the integration coordinate.

Example: $k = \frac{\partial f}{\partial y} = xy^2$, please find $f(x,y)$

A very useful differential operator

Let's define for (x,y,z) coordinates

$$\overrightarrow{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad \text{nabla or del}$$

⚠ it is not a true vector, but we will often treat it as a vector

gradient

The gradient operator applies to a scalar function:

$$\overrightarrow{grad} f = \overrightarrow{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

$$\text{scalar} \xrightarrow{\overrightarrow{grad}} \text{vector}$$

Consequence: the gradient of a scalar field is a vector field.

Example: compute $\overrightarrow{\nabla}(x^2yz + 2)$

divergence

The divergence of a vector field is a scalar field:

$$\vec{\nabla} \cdot \vec{A} = \mathrm{div}\,\vec{A} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{vector} \xrightarrow{div} \text{scalar}$$

Example 1: compute $\vec{\nabla} \cdot \vec{A}$ with $\vec{A} = (x,y,z)$
Example 2: compute $\vec{\nabla} \cdot \vec{A}$ with $\vec{A} = (y,z,x)$

Why $\vec{\nabla}$ is not a true vector?

Let's compare $\vec{\nabla} \cdot \vec{A}$ and $\vec{A} \cdot \vec{\nabla}$

$\vec{\nabla} \cdot \vec{A}$ (the divergence) of $\vec{A}$ is a scalar
$\vec{A} \cdot \vec{\nabla}$ is an scalar differential operator:

$$\vec{A} \cdot \vec{\nabla} = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}$$

Obviously $\vec{\nabla} \cdot \vec{A} \neq \vec{A} \cdot \vec{\nabla}$

curl

The curl of a vector field is

$$\overrightarrow{\nabla} \times \overrightarrow{A} = \overrightarrow{\nabla} \wedge \overrightarrow{A} = \overrightarrow{curl}\,\overrightarrow{A} = \overrightarrow{rot}\,\overrightarrow{A} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\overrightarrow{\nabla} \times \overrightarrow{A} = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix}$$

$$\text{vector} \xrightarrow{\overrightarrow{curl}} \text{vector}$$

curl

Alternate method:

$$\overrightarrow{\nabla} \times \overrightarrow{A} = \det \begin{pmatrix} \overrightarrow{e}_x & \overrightarrow{e}_y & \overrightarrow{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{pmatrix}$$

Example: calculate $\overrightarrow{\nabla} \times \overrightarrow{A}$ for

$$\overrightarrow{A} = \begin{pmatrix} x^2 - y^2 \\ y^2 - z^2 \\ z^2 - x^2 \end{pmatrix}$$

laplacian

The laplacian is the divergence of the gradient:

$$\Delta f = \vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$$

and for a (x,y,z) coordinate,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{scalar} \xrightarrow{\Delta} \text{scalar}$$

But a laplacian can also apply to a vector:

$$\Delta \vec{A} = \begin{pmatrix} \Delta A_x \\ \Delta A_y \\ \Delta A_z \end{pmatrix}$$

$$\text{vector} \xrightarrow{\Delta} \text{vector}$$

Useful formulae

The curl of a gradient is always zero :

$$\vec{\nabla} \times \vec{\nabla} f = 0$$

Prove it!

The divergence of a curl is always zero:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

The double curl:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A}$$

Memo

$$\begin{aligned} \text{scalar} &\xrightarrow{\overrightarrow{grad}} \text{vector} \\ \text{vector} &\xrightarrow{div} \text{scalar} \\ \text{vector} &\xrightarrow{\overrightarrow{curl}} \text{vector} \\ \text{scalar} &\xrightarrow{\Delta} \text{scalar} \\ \text{vector} &\xrightarrow{\Delta} \text{vector} \end{aligned}$$

Other coordinate systems

The cartesian (x,y,z) is not always the best.

Flow in a pipe: $\overrightarrow{v}(\overrightarrow{r},t)$ and $p(\overrightarrow{r},t)$

$$\overrightarrow{v}(r,\theta,z,t), \quad p(r,\theta,z,t)$$

In this course, only the cartesian and cylindrical coordinate systems will be used.

Differential operators in cylindrical coordinates

$$\overrightarrow{\nabla} f = \begin{pmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial f}{\partial \theta} \\ \dfrac{\partial f}{\partial z} \end{pmatrix}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{\nabla} \times \overrightarrow{A} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial A_z}{\partial \theta} - \dfrac{\partial A_\theta}{\partial z} \\ \dfrac{\partial A_z}{\partial r} - \dfrac{\partial A_r}{\partial z} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{1}{r}\dfrac{\partial A_r}{\partial \theta} \end{pmatrix}$$

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Basic fluid mechanics for civil engineers
Lecture 2

Maxime Nicolas
maxime.nicolas@univ-amu.fr

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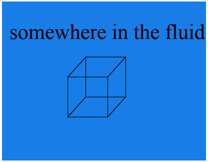
Lecture 2 outline

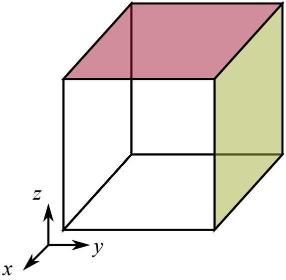
Force balance for a fluid at rest

Cube at equilibrium

Hypothesis: homogeneous fluid at rest under gravity.

Imagine a cube of virtual fluid immersed in the same fluid :





$$\overrightarrow{W} + \overrightarrow{F}_p = 0$$

Continuous approach

weight of an infinitesimal volume of fluid δV of mass m:

$$\overrightarrow{W} = \delta m \overrightarrow{g} = \iiint_{\delta V} \rho \overrightarrow{g} \, dV$$

pressure forces acting on surface δS, boundary of V:

$$\overrightarrow{F}_p = - \iint_{\delta S} p(M) dS \overrightarrow{n}$$

at equilibrium, $\overrightarrow{W} + \overrightarrow{F}_p = 0$, written as

$$\iiint_{\delta V} \rho \overrightarrow{g} \, dV - \iint_{\delta S} p(M) dS \overrightarrow{n} = 0$$

Useful theorem

The gradient theorem

$$\iint_S f\, dS \overrightarrow{n} = \iiint_V \overrightarrow{\nabla} f\, dV$$

Thus

$$\iiint_{\delta V} \rho \overrightarrow{g}\, dV - \iiint_{\delta V} \overrightarrow{\nabla} p(M) dV = 0$$

and

$$\iiint_{\delta V} \left(\rho \overrightarrow{g}\, dV - \overrightarrow{\nabla} p(M) dV \right) = 0$$

finally

$$\boxed{\overrightarrow{\nabla} p - \rho \overrightarrow{g} = 0}$$

integration

for $\overrightarrow{g} = (0,0-g)$ and $p = p(z)$,

$$-\rho g - \frac{dp}{dz} = 0$$

which gives

$$p(z) = p_0 - \rho g z$$

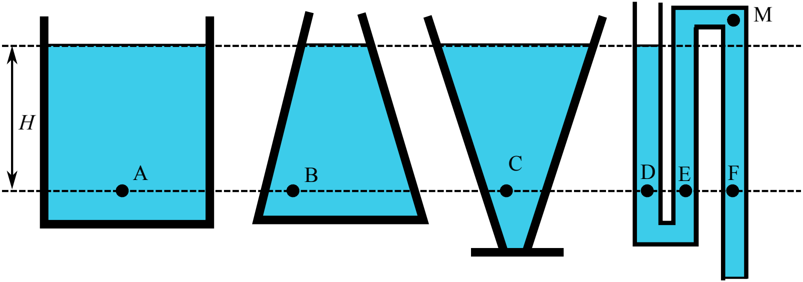
with p_0 the reference pressure at $z = 0$.

if $z = 0$ is the free water/air surface, then $p_0 = p_{atm}$, and the relative pressure is

$$p_{rel} = p - p_{atm} = -\rho g z$$

The hydrostatics « paradox »

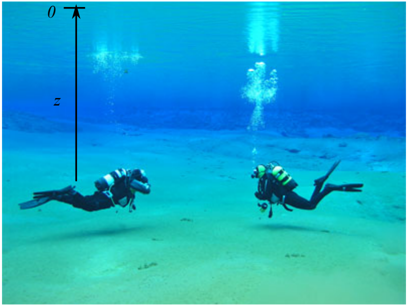
Pressure does not depend on the volume.



$$p_A = p_B = p_C = p_D = p_E = p_F$$

what do you think of pressure at M ?

numerical example

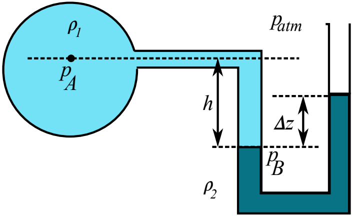


for $z = -10$ m,

$$p_{rel} = p - p_{atm} = \rho g z = 10^3 \times 10 \times 10 = 10^5 \quad \mathrm{Pa}$$

absolute pressure is $\approx 2\,10^5$ Pa (twice the atmospheric pressure)

pressure measurements: the manometer



Calculate p_A in the tank.

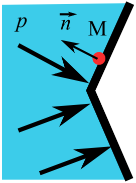
Pressure forces on surfaces

Pressure force on a arbitrary surface

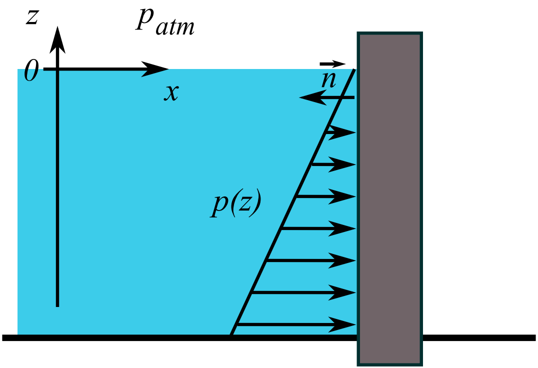
The total pressure force acting on a surface S in contact with a fluid is

$$\overrightarrow{F}_p = -\iint_S p(M)\,\overrightarrow{n}\,dS$$

⚠ remember $\overrightarrow{n}$ is an outgoing unit vector



Pressure force on a vertical wall



H: height of the wetted wall, $L=$ width of the wetted wall

pressure center

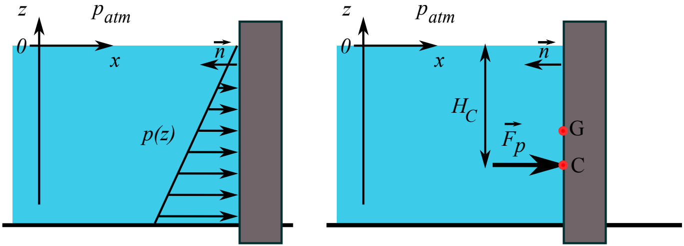
definition: the pressure center C is defined by

$$\overrightarrow{OC} \times \overrightarrow{F}_p = -\iint_S \overrightarrow{OM} \times (p\,\overrightarrow{n})dS, \quad M,P \in S$$

applying $\overrightarrow{F}_p$ on P does not induce rotation of the surface.

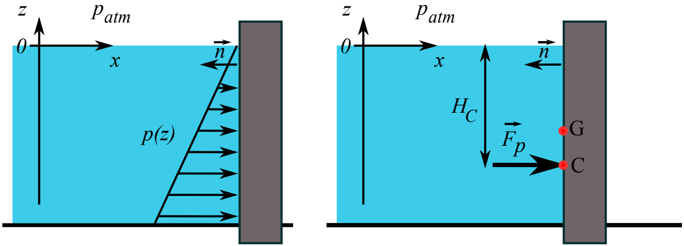
$\overrightarrow{OC} \times \overrightarrow{F}_p$ and $\overrightarrow{OM} \times (p\,\overrightarrow{n})$ are both torques.

Pressure center on a vertical wall



H: height of the wetted wall, $L=$ width of the wetted wall

Pressure center on a vertical wall



H: height of the wetted wall, $L=$ width of the wetted wall

pressure center located at 2/3 of the depth

$$h = \frac{2}{3}H$$

pressure center and barycenter

the pressure center is always below the gravity center (barycenter). It can be proved that

$$H_C = H_G + \frac{I}{H_G S}$$

with

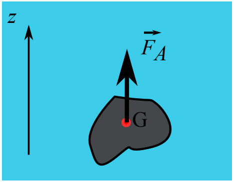
- H_C: depth of the pressure center
- H_G: depth of the gravity center
- S: wetted surface
- I: moment of inertia

see Workshop #2

Archimedes' principle

The buoyancy principle

In Syracuse (now Sicily), in -250 (est.), Archimedes writes:
A body immersed in a fluid experiences a buoyant vertical force upwards.
This force is equal to the weight of the displaced fluid.



This force applies at the buoyancy center: barycenter of the immersed volume.

Modern formulation of the principle

the pressure force acting on the surface S of a fully immersed body is

$$\overrightarrow{F}_p = -\iint_S p\,\overrightarrow{n}\,dS$$

from the gradient theorem,

$$\overrightarrow{F}_p = -\iiint_V \overrightarrow{\nabla} p\,dV$$

and combining with the hydrostatics law $\overrightarrow{\nabla} p = \rho\overrightarrow{g}$, we have

$$\overrightarrow{F}_p = -\iiint_V \rho\overrightarrow{g}\,dV = -m_f\,\overrightarrow{g} = \overrightarrow{F}_A$$

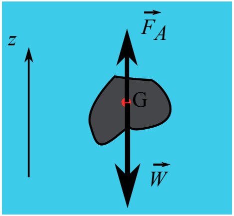
a density difference

Writing ρ_s the solid density of the body, its weight is

$$\overrightarrow{W} = \iiint_V \rho_s \overrightarrow{g}\, dV$$

and the weight + the pressure force is

$$\overrightarrow{R} = \overrightarrow{W} + \overrightarrow{F}_A = (\rho_s - \rho) V \overrightarrow{g}$$



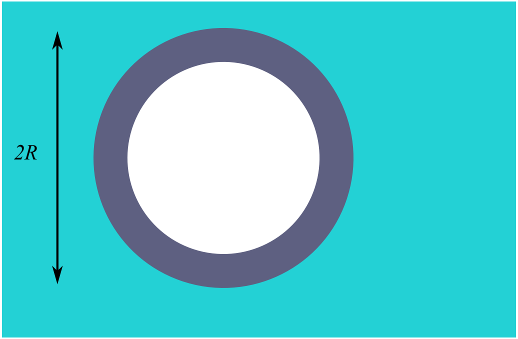
this $\overrightarrow{R}$ force may be positive or negative (the sign of the density difference $\rho_s - \rho$).

pressure center of an immersed body

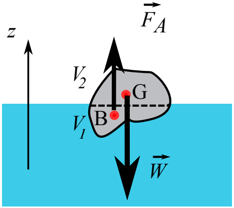
the buoyancy center B of the fully immersed body is the barycenter G.

Example: how to avoid buoyancy

Consider a hollow sphere made of steel, outer radius R and wall width w. Find the width w for which the sphere does not sink nor float.



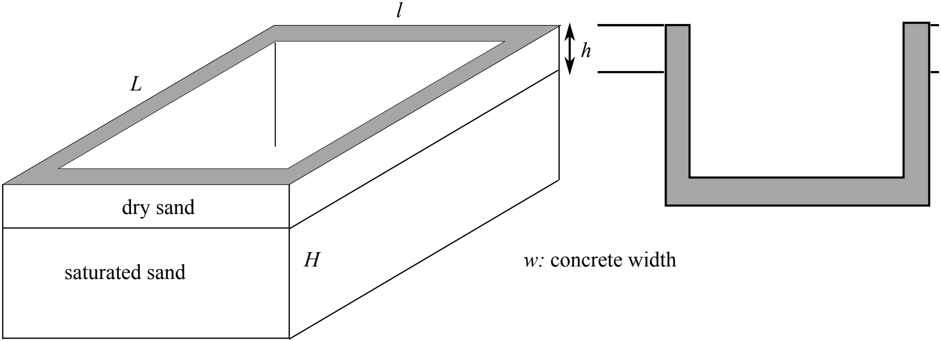
Buoyancy of a partially immersed body



$$\vec{F}_A = -\iiint_{V_1} \rho_1 \vec{g}\, dV - \iiint_{V_2} \rho_2 \vec{g}\, dV = -(\rho_1 V_1 + \rho_2 V_2)\vec{g}$$

⚠ the buoyancy center B is the barycenter of the immersed volume V_1 and is in general different from G.

Example: stability of a diaphragm wall



Find h for which the structure starts to uplift.
Use $H = 8$ m, $L = 30$ m, $l = 20$ m, $w = 0.6$ m,
$\rho = 1000$ kg·m^{-3}, $\rho_s = 2500$ kg·m^{-3}

Basic fluid mechanics for civil engineers
Lecture 3

Maxime Nicolas
maxime.nicolas@univ-amu.fr

POLYTECH MARSEILLE
Aix*Marseille Université
Département génie civil

september–december 2016

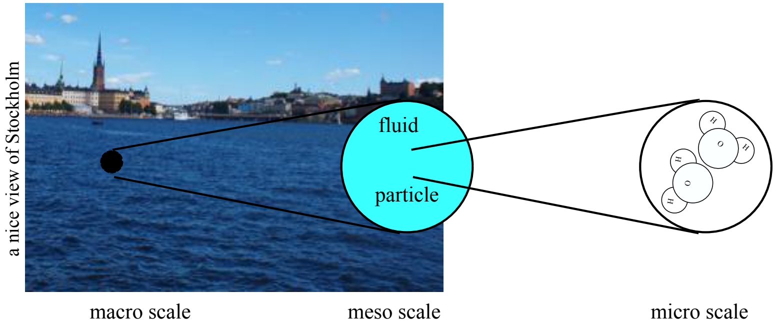
Lecture 3 outline

1. Eulerian and Lagrangian descriptions
2. Mass conservation

Eulerian and Lagrangian descriptions

Fluid particle

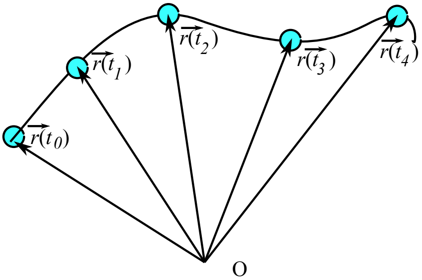
A fluid particle is a mesoscopic scale containing a very large number of fluid molecules, but much smaller than the macroscopic flow scale.



The travel of a fluid particle

Lagrange's description of the path of a fluid particle:

$$\vec{r} = \vec{r}(\vec{r}_0, t)$$



BUT TOO MANY FLUID PARTICLES TO FOLLOW

except for diluted gas, sprays.

The travel of a fluid particle

Eulerian description: the motion of the fluid is determined by a velocity field

$$\overrightarrow{u} = \overrightarrow{u}(\overrightarrow{r}, t)$$

with

$$\overrightarrow{u} = \frac{d\overrightarrow{r}}{dt}$$

Integration of $\overrightarrow{u}$ gives $\overrightarrow{r}$ (if needed)

Steady flow

A steady flow is such $\vec{u}(\vec{r})$ only: no time dependence.

⚠steady $\neq$ static !

photo A. Duchesne, MSC lab, Paris

unsteady flow when $\vec{u}$ is time-dependent: $\vec{u}(\vec{r},t)$

Flow example of the day

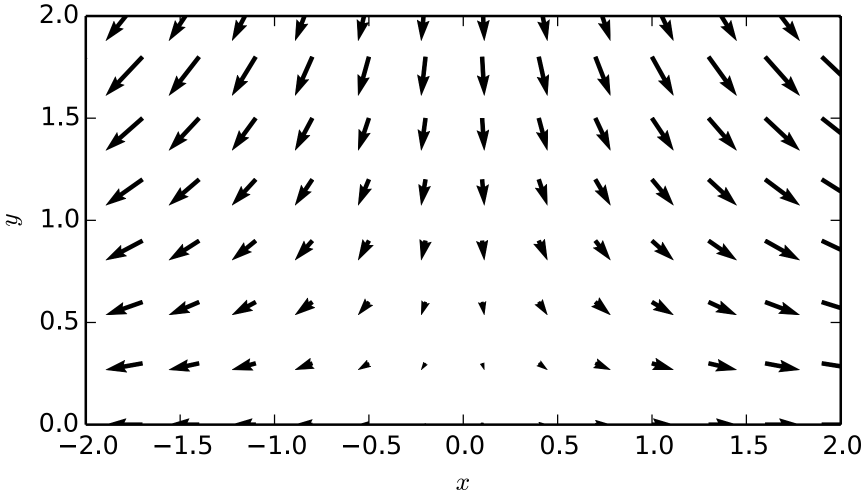
Consider the 2D steady flow

$$\vec{u} = \frac{U_0}{L}\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix}$$

where U_0 is a characteristic velocity, and L a characteristic lengths (both space and time constants)

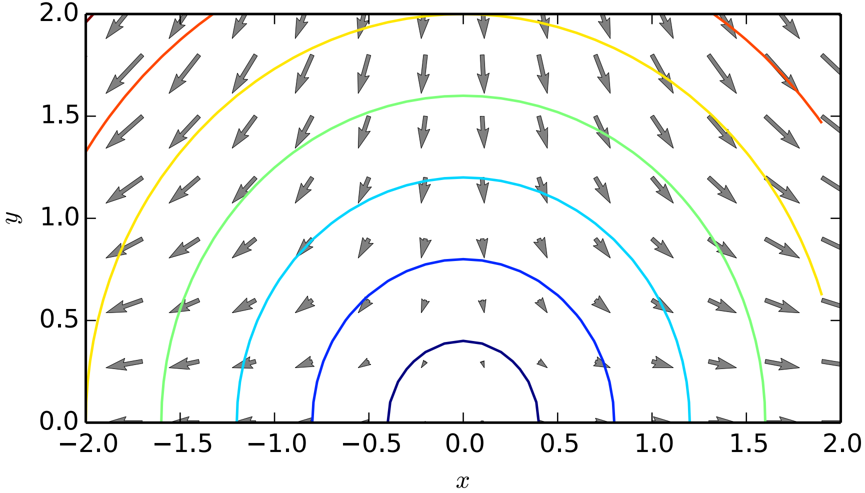
Flow example of the day

$\overrightarrow{u}$ vector field (python code available on Ametice)



Flow example of the day

iso-velocity lines ($\| \overrightarrow{u} \|$ =constant)



Acceleration

from t to $t+\delta t$, the particle moves from $\overrightarrow{r}$ to a new position $\overrightarrow{r}+\delta\overrightarrow{r}$ and has a new velocity $\overrightarrow{u}+\delta\overrightarrow{u}$

$$\overrightarrow{r}+\delta\overrightarrow{r}=(x+\delta x, y+\delta y, z+\delta z), \quad \overrightarrow{u}+\delta\overrightarrow{u}=(u_x+\delta u_x, u_y+\delta u_y, u_z+\delta u_z)$$

The acceleration (change of velocity) has two origins :

- variation of velocity at the same location
- variation of velocity by a change of location

Acceleration

since each velocity component is a 4 variables function

$$u_x = u_x(x,y,z,t)$$

its total derivative is

$$\delta u_x = \frac{\partial u_x}{\partial x}\delta x + \frac{\partial u_x}{\partial y}\delta y + \frac{\partial u_x}{\partial z}\delta z + \frac{\partial u_x}{\partial t}\delta t + \dots$$

the same for u_y and u_z:

$$\delta u_y = \frac{\partial u_y}{\partial x}\delta x + \frac{\partial u_y}{\partial y}\delta y + \frac{\partial u_y}{\partial z}\delta z + \frac{\partial u_y}{\partial t}\delta t + \dots$$

$$\delta u_z = \frac{\partial u_z}{\partial x}\delta x + \frac{\partial u_z}{\partial y}\delta y + \frac{\partial u_z}{\partial z}\delta z + \frac{\partial u_z}{\partial t}\delta t + \dots$$

Acceleration

alternate writing:

$$\delta u_x = \delta x \frac{\partial u_x}{\partial x} + \delta y \frac{\partial u_x}{\partial y} + \delta z \frac{\partial u_x}{\partial z} + \delta t \frac{\partial u_x}{\partial t}$$

and the x-acceleration is

$$\frac{\delta u_x}{\delta t} = u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} + \frac{\partial u_x}{\partial t}$$

or

$$\frac{\delta u_x}{\delta t} = \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right) u_x$$

particular derivative

the particular derivative is an operator with two terms:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \overrightarrow{u} \cdot \overrightarrow{\nabla}$$

and the particular acceleration is

$$\frac{D\overrightarrow{u}}{Dt} = \frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u} \cdot \overrightarrow{\nabla})\overrightarrow{u}$$

$$\overrightarrow{u} \cdot \overrightarrow{\nabla} = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}$$

particular derivative for a steady flow

in the case of a steady flow $\overrightarrow{u}(\overrightarrow{r})$, the particular derivative reduces to

$$\frac{D\overrightarrow{u}}{Dt} = (\overrightarrow{u} \cdot \overrightarrow{\nabla})\overrightarrow{u}$$

Plane flow : $\overrightarrow{u} = (u_x(y,z),0,0)$, then

$$(\overrightarrow{u} \cdot \overrightarrow{\nabla})\overrightarrow{u} = 0$$

Flow example of the day

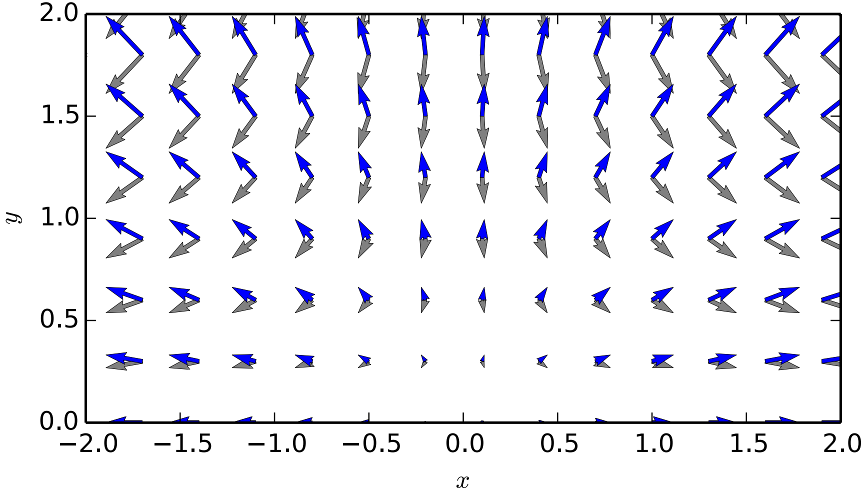
$$\overrightarrow{u} = \frac{U_0}{L}\left(\begin{array}{c} x \\ -y \\ 0 \end{array}\right)$$

the acceleration is :

$$\overrightarrow{a} =$$

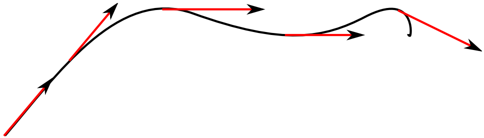
Flow example of the day

acceleration vector field (in blue)



Streamlines

Def: In every point of the flow field, the tangent to a streamline is given by the velocity vector $\overrightarrow{u}$.



a streamline is not the path of a single fluid particle

with $\overrightarrow{dl}$ a curve element, and $\overrightarrow{u}$ the fluid velocity, $\overrightarrow{dl}$ and $\overrightarrow{u}$ must be colinear :

$$\overrightarrow{dl} \times \overrightarrow{u} = 0$$

so

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

Flow example of the day

$$\vec{u} = \frac{U_0}{L}\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix}$$

the streamline equation is

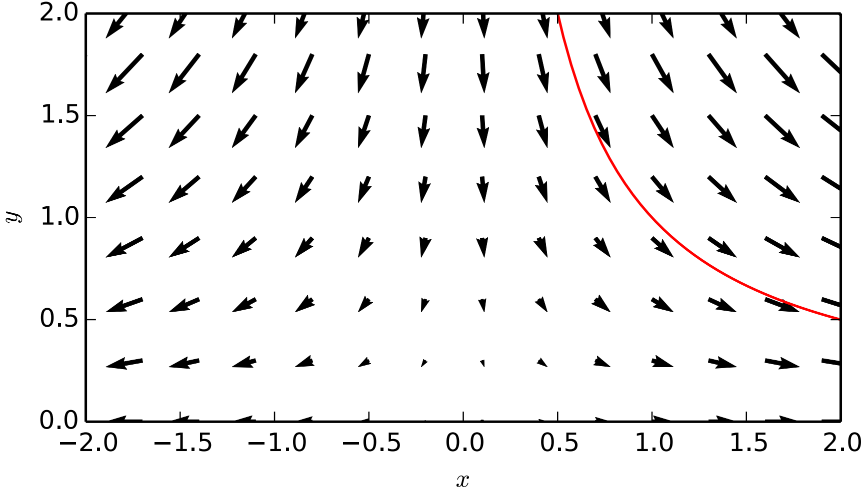
$$\frac{dx}{u_x} = \frac{dy}{u_y}$$

so

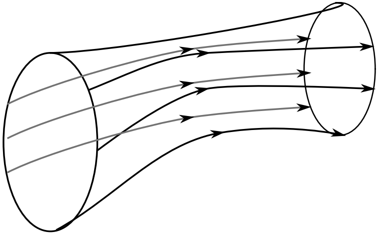
$$\frac{dx}{x} = -\frac{dy}{y}$$

Flow example of the day

streamline $y = C/x$ for $C = 1$:



Stream tubes



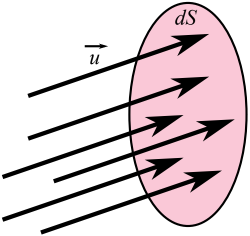
a stream tube

Flow rate

The volume flow rate:

$$dQ = \overrightarrow{u} \cdot \overrightarrow{n}\, dS$$

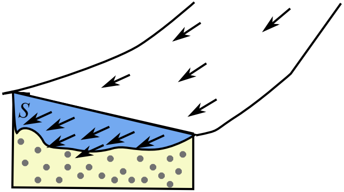
this is the volume of fluid crossing dS during a unit time.



Integration over a surface give the flow rate

$$Q = \iint_S dQ = \iint_S \overrightarrow{u} \cdot \overrightarrow{n}\, dS \quad \text{in m}^3 \cdot \text{s}^{-1}$$

Example of a river flow rate



Rhône river in Valence, data from rdbrmc

syringe

flowrate is

$$Q = U_1 S_1 = U_2 S_2$$

since $S_2 \ll S_1$, $U_2 \gg U_1$ and the fluid has a large kinetic energy !

Mass conservation

Mass conservation equation

the mass variation in a reference volume is due to the flow (in/out) through the surface of this volume:

$$\frac{\partial}{\partial t} \iiint_V \rho dV = - \iint_S \rho \vec{u} \cdot \vec{n} \, dS$$

using Ostrogradski,

$$\iiint_V \frac{\partial \rho}{\partial t} dV = - \iiint_V \vec{\nabla} \cdot (\rho \vec{u}) dV$$

then

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) \right] dV = 0$$

Mass conservation

local mass conservation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

since

$$\vec{\nabla} \cdot (\rho \vec{u}) = \vec{u} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{u},$$

the mass conservation equation is

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{u} = 0$$

or

$$\boxed{\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{u} = 0}$$

Mass conservation for a incompressible flow

⚠ all fluids are compressible

$$\chi_{air} = 6.610^{-5}\ \mathrm{Pa}^{-1}, \quad \chi_{water} = 4.610^{-10}\ \mathrm{Pa}^{-1}$$

but the flow may be incompressible = no significant variation of ρ during the flow.

A flow is seen as incompressible when

- the characteristic velocity of the flow is much lower than the sound velocity: $V \ll c_{sound}$.

$$c_{air} = 340\ \mathrm{m}\cdot\mathrm{s}^{-1} \quad c_{water} = 1\,500\ \mathrm{m}\cdot\mathrm{s}^{-1}$$

- the relative pressure is $\ll$ than absolute pressure (10^5)

Mass conservation for a incompressible flow

if ρ is a constant during the flow,

$$\frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \overrightarrow{\nabla} \rho = 0,$$

and the mass conservation equation reduces to

$$\boxed{\overrightarrow{\nabla} \cdot \overrightarrow{u} = 0}$$

Flow example of the day

Mass conservation for

$$\vec{u} = \frac{U_0}{L}\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix}$$

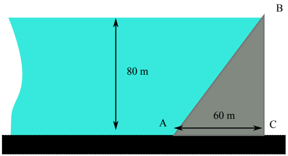
$$\vec{\nabla} \cdot \vec{u} = \frac{U_0}{L}\,(1-1) = 0$$

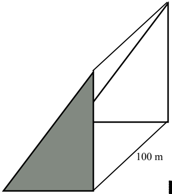
WS2 preparation

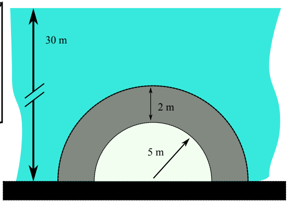
WS2 preparation

Three hydrostatics problems:

- pressure force on a dam
- uplift of an empty swimming pool
- tunnel







Basic fluid mechanics for civil engineers
Lecture 4

Maxime Nicolas
maxime.nicolas@univ-amu.fr

Département génie civil

september–december 2016

Lecture 4 outline: conservation equations

Introduction

AXIOMS CONCERNING LAWS OF MOTION, in Principia Mathematica (1687)

Mutationem motus proportionalem esse vi motrici impressae, & fieri secundum lineam rectam qua vis illa imprimitur.

Introduction

Newton's second law from *Principia Mathematica* (1687)

> *The rate of change of the momentum of a body is directly proportional to the net force acting on it, and the direction of the change in momentum takes place in the direction of the net force.*

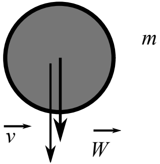
Modern formulation:

$$F = \frac{d}{dt}(mv)$$

where F is a force, and mv a momentum.

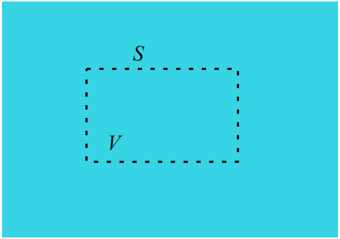
Introduction

Newton's second law for a rigid body:



$$\frac{d}{dt}(m\vec{v}) = \vec{W}$$

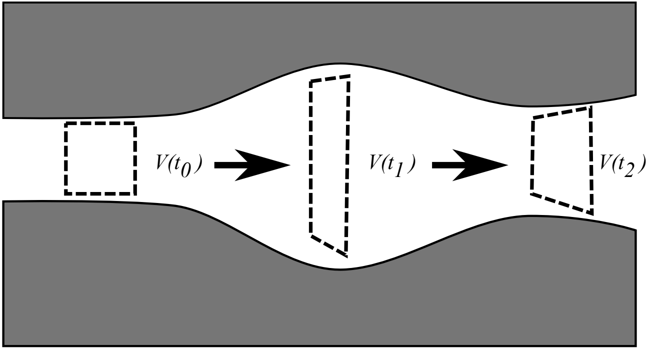
How to transpose this law to a fluid particle (infinitesimal volume)?



$$\frac{d}{dt}\iiint_V \rho\,\vec{u}\,dV = \vec{F}_v + \vec{F}_s$$

A general transport law

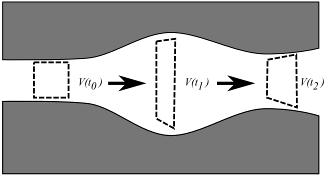
control volume



the mass inside the control volume is constant

$f(\vec{r},t)$ is a scalar function transported by the flow-field $\vec{u}$.

control volume



We aim to calculate the variation of f during the transport:

$$\frac{d}{dt}\iiint_{V(t)} f(\overrightarrow{r},t)dV$$

difficulty: the integration volume evolves with time.

A general transport law for a scalar

The variation of f in V has two terms:

- the local variation of f (at a fixed location)
- the flux of f through S, the surface of V

$$\frac{d}{dt}\iiint_V f\,dV = \iiint_V \frac{\partial f}{\partial t}\,dV + \iint_S f\,\vec{u}\cdot\vec{n}\,dS$$

using the divergence theorem,

$$\frac{d}{dt}\iiint_V f\,dV = \iiint_V \left(\frac{\partial f}{\partial t} + \vec{\nabla}\cdot[f\,\vec{u}\,]\right) dV$$

known as Reynolds's theorem.

Vector transport law

With a vector $\overrightarrow{A} = (A_x, A_y, A_z)$ transported by the flow-field $\overrightarrow{u}$, each component (scalar) follows

$$\frac{d}{dt}\iiint_V A_i \, dV = \iiint_V \left(\frac{\partial A_i}{\partial t} + \overrightarrow{\nabla} \cdot [A_i \overrightarrow{u}] \right) dV$$

It follows that

$$\frac{d}{dt}\iiint_V \overrightarrow{A} \, dV = \iiint_V \frac{\partial \overrightarrow{A}}{\partial t} dV + \iint_S \overrightarrow{A} (\overrightarrow{u} \cdot \overrightarrow{n}) \, dS$$

Mass conservation equation

Mass conservation as a transport law

Taking $f = \rho$, we write

$$\frac{d}{dt}\iiint_V \rho\, dV = \iiint_V \left(\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla}\cdot[\rho\overrightarrow{u}]\right) dV$$

The **mass conservation** is

$$\frac{d}{dt}\iiint_V \rho\, dV = 0$$

this implies

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla}\cdot[\rho\overrightarrow{u}] = 0$$

or

$$\frac{D\rho}{Dt} + \rho\overrightarrow{\nabla}\cdot\overrightarrow{u} = 0$$

Reminder

Mass conservation equation from lecture 3:

$$\frac{D\rho}{Dt} + \rho \overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$$

For an steady incompressible flow:

$$\overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$$

Momentum conservation equation

Transport of $\rho \overrightarrow{u}$

With the momentum density $\overrightarrow{A} = \rho \overrightarrow{u}$

$$\frac{d}{dt} \iiint_V \rho \overrightarrow{u} \, dV = \iiint_V \frac{\partial}{\partial t}(\rho \overrightarrow{u}) \, dV + \iint_S \rho \overrightarrow{u} \, (\overrightarrow{u} \cdot \overrightarrow{n}) \, dS$$

We know that

$$\frac{\partial}{\partial t}(\rho \overrightarrow{u}) = \rho \frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{u} \frac{\partial \rho}{\partial t}$$

and the divergence theorem gives

$$\iint_S \rho \overrightarrow{u} \, (\overrightarrow{u} \cdot \overrightarrow{n}) \, dS = \iiint_V \overrightarrow{\nabla} \cdot (\rho \overrightarrow{u} \otimes \overrightarrow{u}) dV$$

$\boxed{\text{STOP}}$ NEW CONCEPT!

$\overrightarrow{u} \otimes \overrightarrow{u}$ is a rank 2 tensor (= a matrix)

The symbol $\otimes$ means a tensor product

Maths: tensor product

For two 3-components vectors $\overrightarrow{u}$ and $\overrightarrow{v}$:

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \otimes \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} u_x v_x & u_x v_y & u_x v_z \\ u_y v_x & u_y v_y & u_y v_z \\ u_z v_x & u_z v_y & u_z v_z \end{pmatrix}$$

and for our need

$$\overrightarrow{u} \otimes \overrightarrow{u} = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_y u_x & u_y^2 & u_y u_z \\ u_z u_x & u_z u_y & u_z^2 \end{pmatrix}$$

Note that $\overrightarrow{u} \otimes \overrightarrow{u}$ is symmetric.

Maths: divergence of a tensor

Let **A** be a rank 2 tensor. Its divergence is

$$\overrightarrow{\nabla} \cdot \mathbf{A} = \begin{pmatrix} \overrightarrow{\nabla} \cdot \overrightarrow{A}_x \\ \overrightarrow{\nabla} \cdot \overrightarrow{A}_y \\ \overrightarrow{\nabla} \cdot \overrightarrow{A}_z \end{pmatrix}$$

with $\overrightarrow{A}_x = (Axx, Axy, Axz)$ the x-line of **A**.

$$\text{2-tensor (matrix)} \xrightarrow{div} \text{1-tensor (vector)}$$

Maths: divergence of $\vec{u} \otimes \vec{u}$

Write on the board:

$$\vec{\nabla} \cdot (\vec{u} \otimes \vec{u}) = \ldots$$

$$\vec{\nabla} \cdot (\vec{u} \otimes \vec{u}) = \vec{u}(\vec{\nabla} \cdot \vec{u}) + (\vec{u} \cdot \vec{\nabla})\vec{u}$$

and therefore

$$\vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u}) = \vec{u}(\vec{\nabla} \cdot [\rho \vec{u}]) + \rho(\vec{u} \cdot \vec{\nabla})\vec{u}$$

back to the $\rho\vec{u}$ transport law

$$\frac{d}{dt}\iiint_V \rho\vec{u}\,dV = \iiint_V \frac{\partial}{\partial t}(\rho\vec{u})\,dV + \iint_S \rho\vec{u}\,(\vec{u}\cdot\vec{n})\,dS$$

$$\begin{aligned}\frac{d}{dt}\iiint_V \rho\vec{u}\,dV &= \iiint_V \left[\left(\rho\frac{\partial\vec{u}}{\partial t} + \vec{u}\frac{\partial\rho}{\partial t}\right)\right. \\ &\qquad \left. + \vec{u}\,(\vec{\nabla}\cdot[\rho\vec{u}]) + \rho(\vec{u}\cdot\vec{\nabla})\vec{u}\right] dV \\ &= \iiint_V \left[\vec{u}\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u})\right)\right. \\ &\qquad \left. +\rho\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right)\right] dV\end{aligned}$$

back to the $\rho\,\overrightarrow{u}$ transport law

Since

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho\,\overrightarrow{u}) = 0$$

and

$$\frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u} \cdot \overrightarrow{\nabla})\,\overrightarrow{u} = \frac{D\,\overrightarrow{u}}{Dt},$$

then

$$\boxed{\frac{d}{dt} \iiint_V \rho\,\overrightarrow{u}\; dV = \iiint_V \rho \frac{D\,\overrightarrow{u}}{Dt}\; dV}$$

Newton's law for a fluid

Newton's law

What does Newton says: The variation of momentum is balanced by the sum of forces applying on the volume V bounded by a surface S.

$$\frac{d}{dt}\iiint_V \rho\,\overrightarrow{u}\,dV = \iiint_V \rho\frac{D\overrightarrow{u}}{Dt}\,dV = \sum_i \overrightarrow{F}_i$$

Two kinds of forces:

- volume forces
- surface forces

volume force

the weight is the only volume force for a dielectric and non-magnetic fluid.

$$\overrightarrow{F}_v = \overrightarrow{W} = \iiint_V \rho \overrightarrow{g}\, dV$$

surface forces

We write the total surface forces as the sum of local forces applying on the surface S:

$$\overrightarrow{F}_s = \iint_S \overrightarrow{T}(M, \overrightarrow{n}) dS$$

where $\overrightarrow{T}(M, \overrightarrow{n})$ is a stress vector for all $M \in S$, for a unit vector $\overrightarrow{n}$ on each element of S.

$$\overrightarrow{T} = \boldsymbol{\sigma} \overrightarrow{n}$$

$\boxed{\text{STOP}}$ 2-tensor × vector (see next slide)

and using (again) the divergence theorem

$$\iint_S \overrightarrow{T}(M, \overrightarrow{n}) dS = \iint_S \boldsymbol{\sigma} \overrightarrow{n} dS = \iiint_V \overrightarrow{\nabla} \cdot \boldsymbol{\sigma} dV$$

Maths: product of a 2-tensor with a vector

We need to calculate

$$\boldsymbol{\sigma}\,\vec{n}$$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z \\ \sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z \\ \sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_{zz} n_z \end{pmatrix}$$

which is a vector

Newton's law

$$\iiint_V \rho \frac{D\vec{u}}{Dt}\, dV = \vec{W} + \vec{F}_s$$

$$\iiint_V \rho \frac{D\vec{u}}{Dt}\, dV = \iiint_V \rho \vec{g}\, dV + \iint_S \vec{T}(M, \vec{n}) dS$$

$$\iiint_V \rho \frac{D\vec{u}}{Dt}\, dV = \iiint_V \rho \vec{g}\, dV + \iiint_V \vec{\nabla} \cdot \boldsymbol{\sigma}\, dV$$

and locally,

$$\boxed{\rho \frac{D\vec{u}}{Dt} = \rho \vec{g} + \vec{\nabla} \cdot \boldsymbol{\sigma}}$$

For each component $(i = x, y, z)$,

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

Lecture abstract

Mass conservation equation:

$$\frac{D\rho}{Dt} + \rho \overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$$

Momentum conservation equation:

$$\rho \frac{D\overrightarrow{u}}{Dt} = \rho \overrightarrow{g} + \overrightarrow{\nabla} \cdot \boldsymbol{\sigma}$$

remember that

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \overrightarrow{u} \cdot \overrightarrow{\nabla}$$

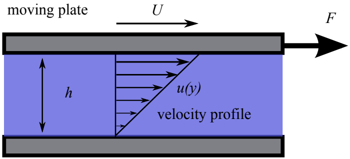
The fluid stress tensor

The fluid stress tensor gathers all the information about surface forces:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{D}$$

pressure and shear

$\mathbf{I}$ is the unity tensor
the tensor $\mathbf{D}$ has the information about the **rheology** of the fluid.

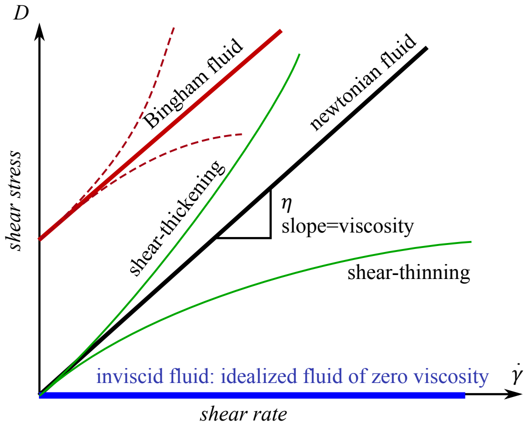


$D = \frac{F}{S}$, a function of $\dot{\gamma} = \frac{U}{h}$

Basic rheology

shear stress D, as a function of the shear rate $\dot{\gamma}$

$$D = f(\dot{\gamma})$$



Basic fluid mechanics for civil engineers
Lecture 5

Maxime Nicolas
maxime.nicolas@univ-amu.fr

POLYTECH MARSEILLE
Aix*Marseille Université
Département génie civil

september–december 2016

Lecture 5 outline: Inviscid flows

1. Flash test
2. Flows classification
3. Bernoulli's theorem

Flash test

Flash test rules

- 5 questions
- 15 min to answer
- work for yourself
- NO CHEATING PLEASE!

Flash test: 5 questions

1. what is the Archimede's force of a $1\,\mathrm{m}^3$ sphere of concrete under water?
2. calculate the relative pressure at a 2 meters depth under fresh concrete
3. calculate $\overrightarrow{\nabla} f$ with $f = (y - z)/x^2$
4. $$\overrightarrow{u} = \frac{U_0}{L} \begin{pmatrix} 2x \\ 2y \\ -4z \end{pmatrix}$$

   is this an incompressible flow?
5. calculate $D\overrightarrow{u}/Dt$ with

   $$\overrightarrow{u} = \frac{gt}{L} \begin{pmatrix} -x \\ y \end{pmatrix}$$

Last lecture abstract

Mass conservation equation:

$$\frac{D\rho}{Dt} + \rho \overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$$

Momentum conservation equation:

$$\rho \frac{D\overrightarrow{u}}{Dt} = \rho \overrightarrow{g} + \overrightarrow{\nabla} \cdot \boldsymbol{\sigma}$$

remember that

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \overrightarrow{u} \cdot \overrightarrow{\nabla}$$

The fluid stress tensor

The fluid stress tensor gathers all the information about surface forces:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{D}$$

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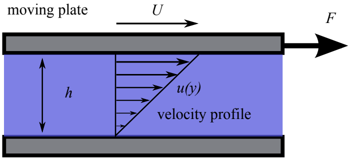
I is the unity tensor
the tensor **D** has the information about the **rheology** of the fluid.



$$D = \frac{F}{S}, \text{ a function of } \dot{\gamma} = \frac{U}{h}$$

introducing a useful dimensionless number

Momentum conservation equation:

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \vec{\nabla})\vec{u} = \rho \vec{g} - \vec{\nabla} p + \vec{\nabla} \cdot \mathbf{D}$$

Inertia term:

$$\|\rho(\vec{u} \cdot \vec{\nabla})\vec{u}\| \quad \propto \quad \rho\left(U\frac{1}{L}\right)U \quad = \quad \rho\frac{U^2}{L}$$

rheology (viscous) term:

$$\|\vec{\nabla} \cdot \mathbf{D}\| \quad \propto \quad \eta\frac{U}{L^2}$$

To compare :

$$\frac{\|\rho(\vec{u} \cdot \vec{\nabla})\vec{u}\|}{\|\vec{\nabla} \cdot \mathbf{D}\|} = \frac{\rho U L}{\eta}$$

The Reynolds number

this dimensionless number is named the Reynolds number (symbol: Re) [1883 by Osborne Reynolds]

$$Re = \frac{\rho U L}{\eta}, \qquad 0 < Re < \infty$$

with

- ρ fluid density
- U characteristic velocity
- L characteristic length
- η fluid dynamic viscosity

Re is used to **classify** the different possible flows

Flows classification: $Re \ll 1$

Flow is dominated by viscous (stress) effects (low velocity or small size flow or large viscosity → Lecture 7).

Flows classification: $Re \gg 1$

Flow is dominated by inertia effects (high velocity or large size or low viscosity).

Wind over a small-scale house in a wind tunnel. Photo from Leibniz Institut

`www.atb-postdam.de`

The fluid stress tensor for an inviscid flow

Assume $Re \gg 1$

In this particular case (no $\mathbf{D}$), the stress tensor reduces to

$$\boldsymbol{\sigma} = -p\mathbf{I}$$

and

$$\overrightarrow{\nabla} \cdot \boldsymbol{\sigma} = -\overrightarrow{\nabla} p$$

only pressure gradient remains

Conservation eq. for inviscid flows

under the assumptions of inviscid, steady an incompressible flow, the conservation equations are

$$\overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$$

and

$$\rho(\overrightarrow{u} \cdot \overrightarrow{\nabla})\overrightarrow{u} = \rho\overrightarrow{g} - \overrightarrow{\nabla} p$$

The famous Bernoulli's theorem

rewriting the momentum cons. eq.

With a little chunk of maths, we write

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{\nabla}\left(\frac{u^2}{2}\right) + (\vec{\nabla}\times\vec{u})\times\vec{u}$$

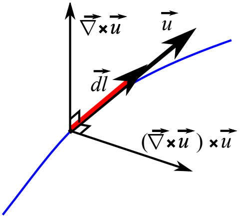
so that the momentum conservation equation is now

$$\begin{aligned}
\vec{\nabla}\left(\frac{u^2}{2}\right) + (\vec{\nabla}\times\vec{u})\times\vec{u} &= \vec{g} - \frac{1}{\rho}\vec{\nabla}p \\
&= -\vec{\nabla}(gz) - \frac{1}{\rho}\vec{\nabla}p
\end{aligned}$$

along a streamline

Following a streamline, with $\overrightarrow{dl}$ a small oriented line element of the streamline

$$\vec{\nabla}\left(\frac{u^2}{2}\right)\cdot\overrightarrow{dl} + [(\vec{\nabla}\times\vec{u})\times\vec{u}]\cdot\overrightarrow{dl} \quad = \quad \vec{g}\cdot\overrightarrow{dl} - \frac{1}{\rho}\left(\vec{\nabla}p\right)\cdot\overrightarrow{dl}$$



Since $\vec{u}//\overrightarrow{dl}$, $\quad [(\vec{\nabla}\times\vec{u})\times\vec{u}]\cdot\overrightarrow{dl} = 0$

$$\vec{\nabla}\left(\frac{u^2}{2}\right)\cdot\overrightarrow{dl} \quad = \quad \vec{g}\cdot\overrightarrow{dl} - \frac{1}{\rho}\left(\vec{\nabla}p\right)\cdot\overrightarrow{dl}$$

along a streamline

$$\overrightarrow{\nabla}\left(\frac{u^2}{2} + gz + \frac{p}{\rho}\right) \cdot \overrightarrow{dl} = 0$$

meaning that

$$\frac{u^2}{2} + gz + \frac{p}{\rho} = C'$$

or

$$\boxed{\rho\frac{u^2}{2} + \rho gz + p = C}$$

this was first proved by Daniel Bernoulli in 1738.

Bernoulli's theorem

Under the assumptions of

- inviscid flow
- steady flow
- incompressible flow

the quantity

$$p + \rho\frac{u^2}{2} + \rho g z = C$$

is constant along a streamline.

the constant C is named the force potential (*charge* in french). Unit: Pa

Understanding Bernoulli

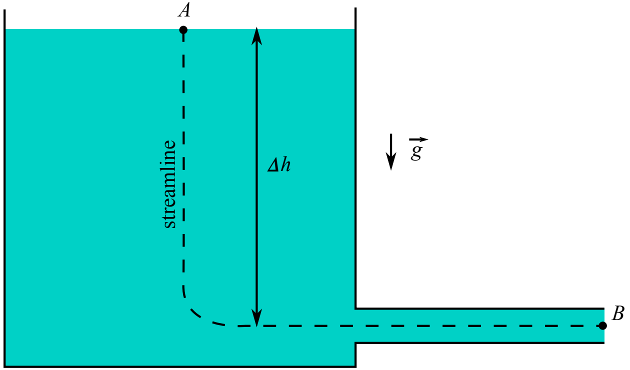
Along a streamline, the energy density (energy per unit volume) is conserved.
Multiplying by a volume V transported along the streamline:

$$\frac{1}{2}\rho V u^2 + pV + \rho V g z = CV$$

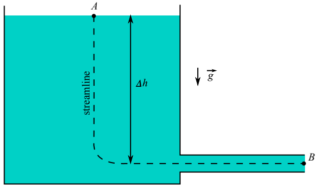
or

$$\frac{1}{2} m u^2 + pV + mgz = CV$$

Example: emptying a water tank



Example: emptying a water tank



output velocity:

$$u_B = \sqrt{2g\Delta h}$$

known as the Toricelli's formula (1608–1647)

Basic fluid mechanics for civil engineers
Lecture 6

Maxime Nicolas
maxime.nicolas@univ-amu.fr

Département génie civil

september–december 2016

Lecture 6 outline: the Navier-Stokes equation

1. The Navier-Stokes equation
2. Known solutions of steady NS
3. CFD

Flash test stats

Bonus stats:

Bonus	number
0.0	1
0.2	8
0.4	11
0.6	19
0.8	13
1.0	2

Mean bonus is 0.55

Last lecture abstract

Mass conservation equation:

$$\frac{D\rho}{Dt} + \rho \overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$$

Momentum conservation equation:

$$\rho \frac{D\overrightarrow{u}}{Dt} = \rho \overrightarrow{g} + \overrightarrow{\nabla} \cdot \boldsymbol{\sigma}$$

with

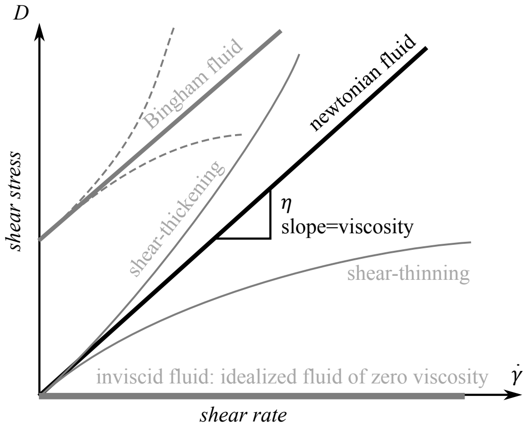
$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{D}$$

The Navier-Stokes equation

Newtonian fluids

The Navier-Stokes equation is the momentum conservation equation for 3D newtonian fluids:

linearity between shear stress and shear rate



tensor D

for an incompressible flow

$$\mathbf{D} = 2\eta\mathbf{E}$$

with

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

example :

$$E_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Note: the **E** tensor will be presented extensively in Elasticity class (6th semester).

the divergence of **D**

We write **D**

$$\mathbf{D} = \eta \begin{pmatrix} \frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} & \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z} \end{pmatrix}$$

and

$$(\overrightarrow{\nabla} \cdot \mathbf{D})_x = \eta(\ldots)$$

which is

$$(\overrightarrow{\nabla} \cdot \mathbf{D})_x = \eta \Delta u_x$$

and finally the divergence of **D** is

$$\overrightarrow{\nabla} \cdot \mathbf{D} = \eta \Delta \overrightarrow{u}$$

The Navier-Stokes equation

Now we write the Navier-Stokes equation for the incompressible flow of a newtonian fluid:

$$\boxed{\rho \frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla} p + \eta\Delta\vec{u}}$$

or

$$\rho \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = \rho\vec{g} - \vec{\nabla} p + \eta\Delta\vec{u}$$

What is needed to solve NS?

NS is a set of 3 partial differential equations (PDEs) coupled with the mass conservation equation.

As any differential equation, the complete solving needs:

- boundary conditions (BC) for velocity and/or stress
- boundary conditions for pressure
- initial conditions for $\overrightarrow{u}$ and p (unsteady flows only)

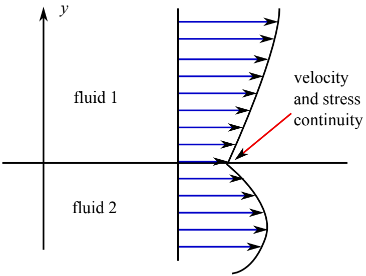
Velocity and stress continuity

The velocity must be continuous at an interface:

$$u\big|_{y=0+} = u\big|_{y=0-}$$

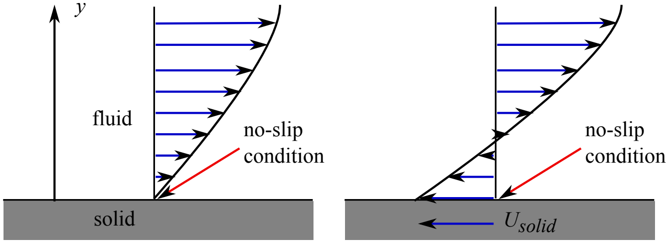
The tangential stress must be continuous at an interface:

$$\eta_1 \left.\frac{\partial u}{\partial y}\right|_{y=0+} = \eta_2 \left.\frac{\partial u}{\partial y}\right|_{y=0-}$$



Example of BC

Example: $\overrightarrow{u} = \overrightarrow{U}_{solid}$ at a solid non-deformable surface.



If the solid is at rest (left), then $\overrightarrow{u} = 0$ at the interface.

Known solutions of steady NS

Steady NS analytical solutions

Main classification:

	plane	cylindrical
Boundary-driven	Example 1	Example 4
Pressure-driven	Example 2	Example 3
Boundary-driven + pressure-driven	WS6	

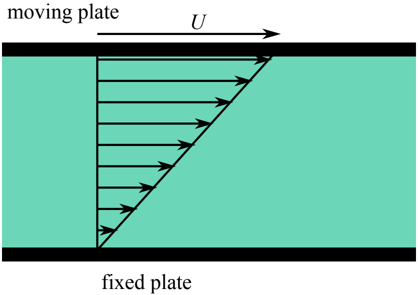
Example 1: plane boundary-driven flow

Flow: $\overrightarrow{u} = u_x(y)$, no pressure gradient
Boundary conditions: $\overrightarrow{u} = 0$ at $y = 0$, $\overrightarrow{u} = U$ at $y = h$

Plane boundary-driven flow

Solution:

$$u_x(y) = U\frac{y}{h}, \quad p = constant$$



Pure shear flow: Couette flow (from Maurice Couette 1858–1943)

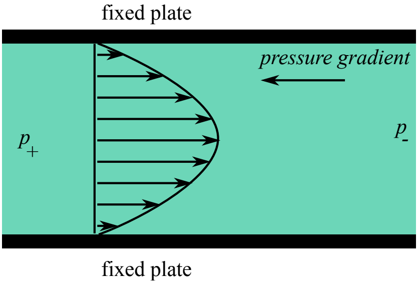
Example 2: plane pressure-driven flow

Flow: $\overrightarrow{u} = u_x(y)$, constant pressure gradient along x: $\partial p/\partial x = K$
Boundary conditions: $\overrightarrow{u} = 0$ at $y = 0$ and at $y = h$

Plane pressure-driven flow

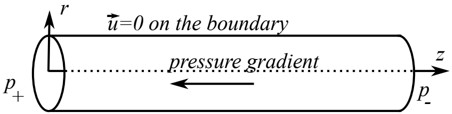
Solution:

$$u_x(y) = \frac{1}{2\eta}\frac{dp}{dx}(y-h)y$$



Example3: cylindrical pressure-driven flow

Flow: $\overrightarrow{u} = u_z(r)$, constant pressure gradient $dp/dx = K$
Boundary conditions: $\overrightarrow{u} = 0$ at $r = R$



Cylindrical pressure-driven flow

We need to write:

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \begin{pmatrix} u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \\ u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \\ u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

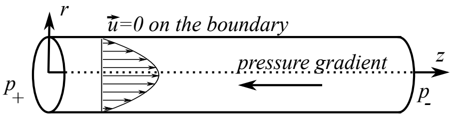
$$\vec{\nabla} p = \begin{pmatrix} \frac{\partial p}{\partial r} \\ \frac{1}{r}\frac{\partial p}{\partial \theta} \\ \frac{\partial p}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial p}{\partial z} \end{pmatrix}$$

$$\Delta \vec{u} = \begin{pmatrix} \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \\ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \\ \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\partial u_z}{\partial r} \right] \end{pmatrix}$$

Cylindrical pressure-driven flow

Solution:

$$u_z(r) = -\frac{1}{4\eta}\frac{dp}{dz}(R^2 - r^2)$$



parabolic Poiseuille flow (Jean-Léonard-Marie Poiseuille, 1797–1869)

Cylindrical pressure-driven flow

$$u_z(r) = -\frac{1}{4\eta}\frac{dp}{dz}(R^2 - r^2)$$

flow-rate through the pipe:

$$q = \iint_s u dS = 2\pi \int_0^R u_z(r) r \, dr = -\frac{\pi}{8\eta}\frac{dp}{dz}R^4$$

mean velocity:

$$\bar{u} = \frac{q}{\pi R^2} = -\frac{1}{8\eta}\frac{dp}{dz}R^2$$

stresses on the wall

Because of the non-slip condition on the wall, the fluid exerts a stress on the wall. The local shear stress at $r = R$ is

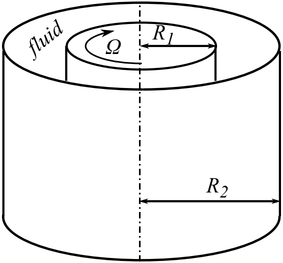
$$\sigma = \left(\eta \frac{\partial u_z}{\partial r}\right)_{r=R}$$

Then the total viscous force on a pipe of length L is

$$F_v = \int_0^L dz 2\pi R \sigma dz = 2\pi LR \times \frac{R}{2}\frac{dp}{dz} = \pi R^2 L \frac{dp}{dz}$$

Example 4: Cylindrical BC flow

Axisymmetric Couette flow between 2 coaxial cylinders:



Example 4: Cylindrical BC flow

velocity field:

$$\vec{u} = (u_r, u_\theta, u_z)$$

Mass conservation eq.:

$$\vec{\nabla} \cdot \vec{u} = 0 = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\vec{u} = (0, u_\theta(r), 0)$$

Example 4: Cylindrical BC flow

NS:

$$
\begin{aligned}
0 &= -\frac{\partial p}{\partial r} \\
0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \eta \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}\right] \\
\rho g &= -\frac{\partial p}{\partial z}
\end{aligned}
$$

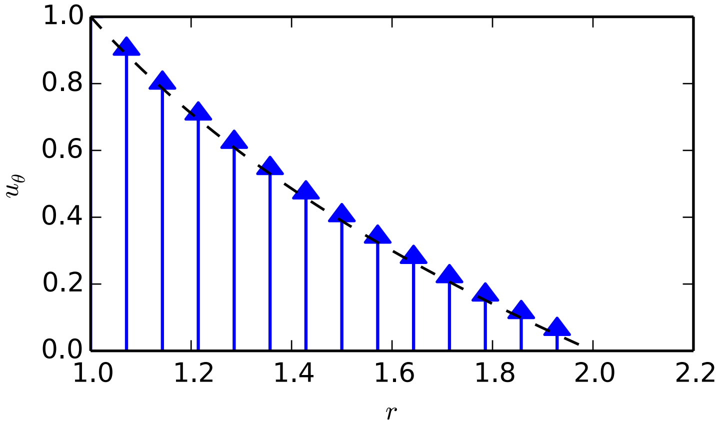
with BC:

$$u_\theta(R_1) = \Omega R_1 \quad \text{and} \quad u_\theta(R_2) = 0$$

Example 4: Cylindrical BC flow

solution:

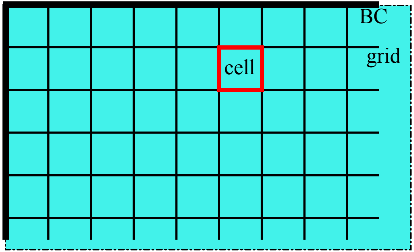
$$u_\theta(r) = \frac{\Omega R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2 - r^2}{r}\right)$$



CFD

beyond the analytical solutions

when no analytical solution is available, Computational Fluid Dynamics (CFD) helps a lot!



- fluid domain is discretized on grid
- NS is solved on each grid cell
- continuity of velocity, stress and pressure must be checked
- BC and IC

CFD basic principle

Equations are discretized on a grid.

Example for a 1D-domain (hydrostatics)

$$0 = -\rho g - \frac{dp}{dz}$$

$$\frac{dp}{dz} \approx \frac{p(z+dz) - p(z)}{dz} = \frac{p_{i+1} - p_i}{dz}$$

$$p_{i+1} = p_i - \rho g dz$$

BC : $p = p_{atm}$ at $z = 0$

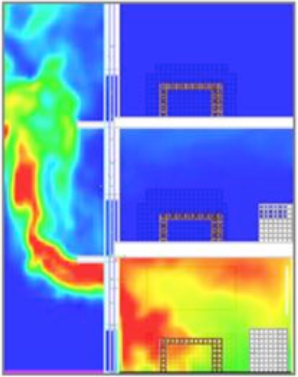
$$p(-dz) = p_{atm} + \rho g dz$$

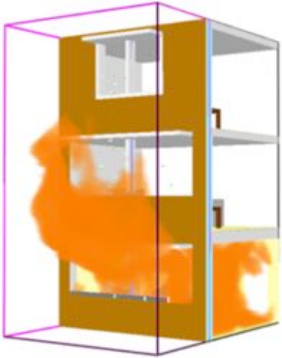
CFD softwares

- home-made codes
- open-source codes
- commercial softwares
    - Autodesk CFD
    - ComSol
    - Fluent
    - StarCCM+
    - ...

Example of CFD result

Fire simulation





Basic fluid mechanics for civil engineers
Lecture 7

Maxime Nicolas
maxime.nicolas@univ-amu.fr

Département génie civil

september–december 2016

Lecture 7 outline: the Stokes equation

The Stokes equation

From Navier-Stokes to the Stokes equation

Navier-Stokes:

$$\rho \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = \rho \vec{g} - \vec{\nabla} p + \eta \Delta \vec{u}$$

Reynolds number:

$$Re = \frac{\| \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} \|}{\| \vec{\nabla} \cdot \mathbf{D} \|} = \frac{\| \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} \|}{\| \eta \Delta \vec{u} \|} = \frac{\rho U L}{\eta}$$

Hypothesis:

- very low Reynolds numbers $Re \to 0$
- steady flow: $\partial \vec{u} / \partial t = 0$

Within this frame , the NS equation reduces to

$$0 = \rho \vec{g} - \vec{\nabla} p + \eta \Delta \vec{u}$$

From NS to the Stokes equation

Writing the pressure as

$$p = p' - \rho g z$$

gives

$$\boxed{\vec{\nabla} p' = \eta \Delta \vec{u}}$$

named the Stokes equation.

George Gabriel Stokes (1819–1903), English physicist and mathematician

Properties of the Stokes equation

Properties of the Stokes equation

The stokes equation $\overrightarrow{\nabla} p' = \eta \Delta \overrightarrow{u}$ has 4 interesting properties:

1. Unicity of the solution
2. Linearity
3. Reversibility
4. Minimum of energy dissipation

Unicity of the solution

Assume that the BC are known (either at infinite or at finite distance from an interface).

If $(\overrightarrow{u_1}, p_1')$ is a solution and $(\overrightarrow{u_2}, p_2')$ is another solution, it can be proved that

$$(\overrightarrow{u_1}, p_1') = (\overrightarrow{u_2}, p_2')$$

meaning that the solution is unique.

Linearity

Suppose two solutions of the Stokes equation:

- $(\overrightarrow{u_1}, p'_1)$ for BC1
- $(\overrightarrow{u_2}, p'_2)$ for BC2

The flow

$$(\lambda_1 \overrightarrow{u_1} + \lambda_2 \overrightarrow{u_2} \ , \ \lambda_1 p'_1 + \lambda_2 p'_2)$$

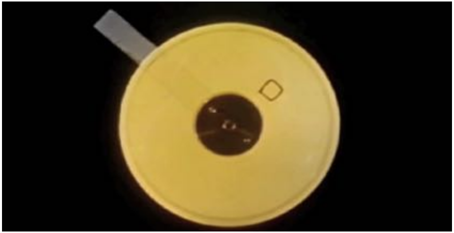
is also a solution for the boundary conditions BC1+BC2

Reversibility

No time variable in the Stokes eq.

- the flow is instantaneous: no delay between the driving BC or driving force and the flow
- the flow is reversible

Experimental evidence of the reversibility:



A movie featuring G.I. Taylor illustrating low-Reynolds number flows
www.youtube.com/watch?v=QcBpDVzBPMk

Reynolds numbers for swimming

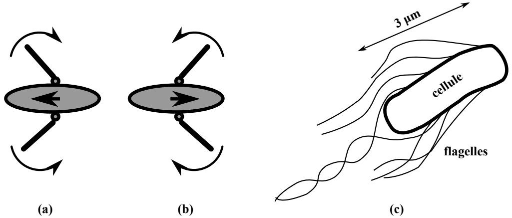
Let's calculate a few *Re* numbers:

animal	speed (m/s)	size (m)	η (Pa.s)	*Re*
mako shark	14	4	10^{-3}	10^7
human	2.5	2	10^{-3}	10^6
goldfish	1.5	0.05	10^{-3}	10^4
E. Coli	4×10^{-5}	3×10^{-6}	10^{-3}	10^{-4}
sperm	5×10^{-5}	6×10^{-5}	50	10^{-8}

Consequence of the reversibility

How do small animals or living cells swim?

The simple swimming motion :



Swimming at low *Re*

flagella ad cilia → helicoidal motion (like a corkscrew)

Minimum of energy dissipation

The loss of energy is due to the viscous forces of the flow.

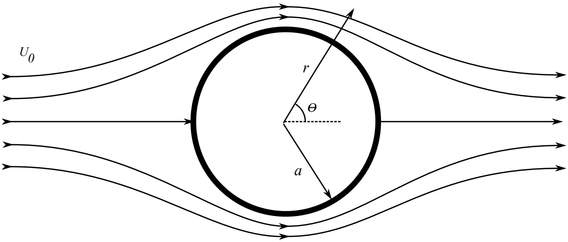
It can be proved that the solution of the Stokes equation (for a given set of BC) is the flow which minimizes the loss of energy.

Drag force on a sphere

Drag force on a sphere

In 1853, G. Stokes derived the exact expression for the drag force of a sphere moving at velocity U_0 in a viscous fluid at rest (or the drag force of a steady sphere in flow of velocity U_0 far from the sphere).

$$\boxed{\overrightarrow{F}_{Stokes} = -6\pi\eta a \overrightarrow{U}_0}$$



Problem formulation

Solve

$$\vec{\nabla} p' = \eta \Delta \vec{u}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

in spherical coordinates

with BC :

- $\vec{u} = 0$ at the sphere surface $r = a$
- $\vec{u} = \vec{U}_0$ far from the sphere $(r \to \infty)$

then calculate the drag force as

$$F_{Stokes} = F_{pressure} + F_{shear}$$

A few steps to the solution (1/6)

1] Flow symmetry:

$$\vec{u} = \begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \\ 0 \end{pmatrix}, \quad p' = p'(r,\theta)$$

2] Mass conservation eq. (incompressible flow)

$$\vec{\nabla} \cdot \vec{u} = 0$$

leads to a stream function ψ such as

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}$$

A few steps to the solution (2/6)

3] The Stokes equation may be rewritten as

$$-\eta \overrightarrow{\nabla} \times (\overrightarrow{\nabla} \times \overrightarrow{u}) = \overrightarrow{\nabla} p'$$

which gives

$$\frac{\partial p}{\partial r} = \frac{\eta}{r^2 \sin\theta} \frac{\partial}{\partial \theta}(Z\psi), \quad \frac{\partial p}{\partial \theta} = -\frac{\eta}{r \sin\theta} \frac{\partial}{\partial r}(Z\psi)$$

with an operator

$$Z \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right)$$

A few steps to the solution (3/6)

4] Writing

$$\psi = f(r) \sin\theta$$

leads to a 4th order differential equation

$$r^4 \frac{\partial^4 f}{\partial r^4} - 4r^2 \frac{\partial^2}{\partial r^2} + 8r \frac{\partial f}{\partial r} - 8f = 0$$

With the test solution $f = r^k$, the characteristic polynome is

$$k(k-1)(k-2)(k-3) - 4k(k-1) + 8k - 8 = 0$$

or

$$(k-1)(k-2)(k+1)(k-4) = 0$$

So that

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4$$

A few steps to the solution (4/6)

5] Using the 4 BC, we find the stream function

$$\psi = U_0 \left(\frac{a^3}{4r} - \frac{3ar}{4} + \frac{r^2}{2} \right) \sin^2\theta$$

Finally, the velocity field is

$$u_r = U_0 \left(\frac{a^3}{2r^3} - \frac{3a}{2r} + 1 \right) \cos\theta, \quad u_\theta = U_0 \left(\frac{a^3}{4r^3} - \frac{3a}{4r} - 1 \right) \sin\theta$$

and the pressure is

$$p' = -\frac{3a\eta U_0}{2r^2} \cos\theta$$

A few steps to the solution (5/6)

6] the shear stress of the fluid on the sphere is

$$\tau_{r\theta} = -\eta\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

and the shear force

$$F_{shear} = 2\pi a^2 \int_0^\pi \tau_{r\theta}\sin^2\theta d\theta = 4\pi a\eta U_0$$

7] The pressure force is

$$F_{pressure} = 2\pi a^2 \int_0^\pi p\sin\theta\cos\theta d\theta = 2\pi a\eta U_0$$

A few steps to the solution (6/6)

And the total drag force on the sphere is

$$F_{drag} = F_{shear} + F_{pressure} = 4\pi a\eta U_0 + 2\pi a\eta U_0 = 6\pi a\eta U_0$$

Obviously the drag force is opposed to the motion:

$$\boxed{\overrightarrow{F}_{Stokes} = -6\pi\eta a\overrightarrow{U}_0}$$

and this is the (long) way to prove the Stokes drag force !

Sedimentation

What is sedimentation

Motion of solid particles under gravity in a fluid.

Sedimentation occurs in

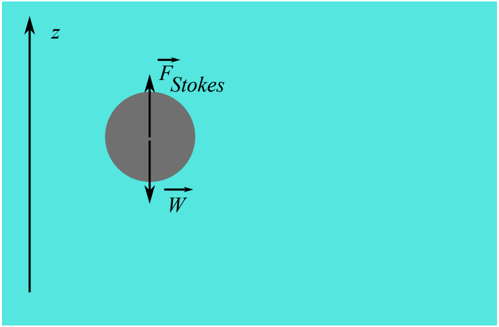
- geophysical flows
- transport then settling of particles in rivers
- industrial mixtures
- building materials (concrete)

Simple sedimentation of a sphere

Consider a sphere of density ρ_p immersed in a fluid $(\eta,\ \rho)$. We suppose

- the particle is at rest at $t = 0$
- $\rho_p > \rho$
- $Re \ll 1$

We aim to compute the sphere motion. . .



Sedimentation

Motion equation:

$$m\frac{d^2z}{dt^2} = -\frac{4}{3}\pi a^3(\rho_p - \rho)g - 6\pi\eta a\frac{dz}{dt}$$

or

$$\frac{4}{3}\pi a^3\rho_p\frac{dU}{dt} = -\frac{4}{3}\pi a^3(\rho_p - \rho)g - 6\pi\eta aU$$

or

$$\frac{dU}{dt} = -\left(\frac{\rho_p - \rho}{\rho_p}\right)g - \frac{9}{2}\frac{\eta}{a^2\rho_p}U$$

Sedimentation

A dimensional analysis gives

$$\left[\frac{\eta}{a^2\rho_p}\right] = \frac{\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-1}}{\mathcal{L}^2\mathcal{M}\mathcal{L}^{-3}} = \mathcal{T}^{-1}$$

so we can define a Stokes time

$$T = \frac{2}{9}\frac{a^2\rho_p}{\eta}$$

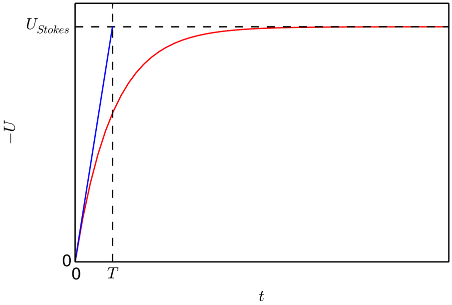
and the motion equation is

$$\frac{dU}{dt} = -\left(\frac{\rho_p - \rho}{\rho_p}\right)g - \frac{U}{T}$$

Sedimentation

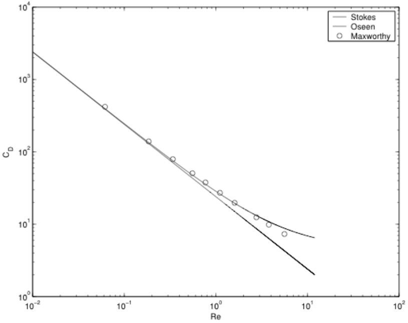
This last equation solution is

$$U = -U_{Stokes}(1 - \mathrm{e}^{-t/T}), \quad U_{Stokes} = \frac{2}{9}\frac{(\rho_p - \rho)a^2 g}{\eta}$$



Validity of the Stokes equation

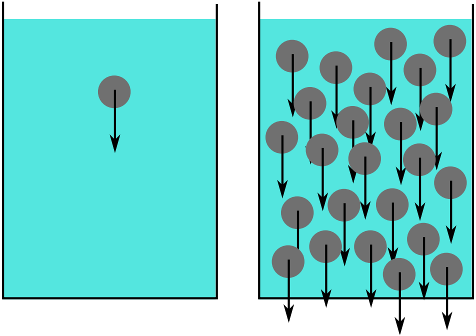
Remember that we made the hypothesis that $Re \ll 1$. Comparison with experiments ($C_D = F_{drag}/(0.5\pi a^2 \rho U^2)$):



(Maxworthy, 1964, experiments with a saphire sphere)

The Stokes drag force should not be applied for $Re \geqslant 1$.

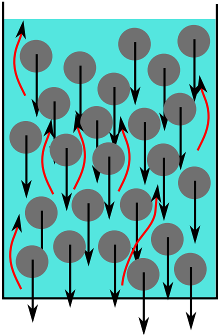
Sedimentation in a finite volume



Sedimentation in a finite volume

Non-permeable boundaries of the tank induce a back-flow, hindering the settling of the particles.
An important parameter is the volume fraction

$$\phi = \frac{volume\ \ occupied\ \ by\ \ the\ \ particles}{total\ \ volume}, \quad 0 < \phi < 1$$



Sedimentation in a finite volume

On average, the settling velocity of the particles is

$$U_s = U_{Stokes} F(\phi)$$

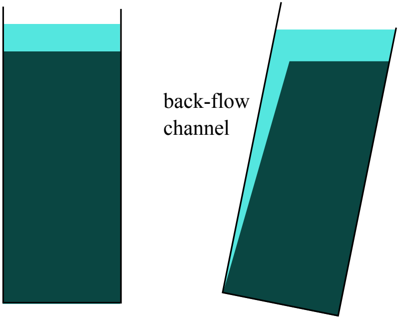
with an empirical hindering function (Richardson & Zaki, 1954):

$$F(\phi) = (1 - \phi)^n$$

The exponent n (close to 4.5) may decrease with increasing Reynolds number.

The Boycott effect

When the vessel is inclined (even slightly), the settling velocity of the particles is enhanced.



Basic fluid mechanics for civil engineers
Lecture 8

Maxime Nicolas
maxime.nicolas@univ-amu.fr

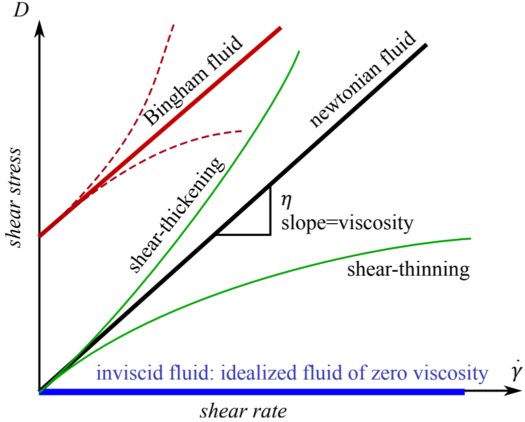
POLYTECH MARSEILLE
Aix Marseille Université
Département génie civil

september–december 2016

Lecture 8 outline: introduction to non newtonian fluid mechanics

The rheology zoo

Stress-strain relation



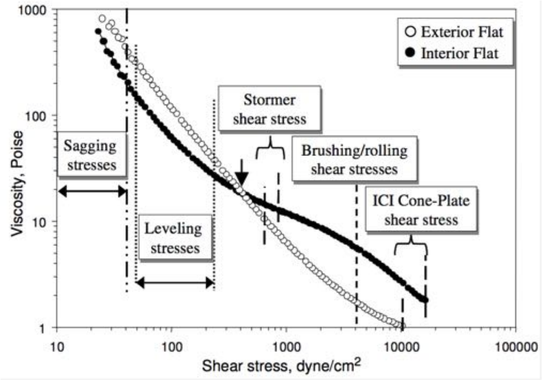
Power-law fluids

$$\tau = k\dot{\gamma}^n$$

Depending on the exponent n, the behavior is

- $n < 1$ Shear-thinning fluids (shampoo, paint)
- $n = 1$ newtonian fluids (air, water, honey)
- $n > 1$ Shear-tickenning fluids

Shear-thinning fluids = fluides rhéofluidifiants



from R. R. Eley, *Rheology Reviews* 2005, pp 173 - 240

For paints $n \approx 0.5$, with $k \approx 10^3$ Pa·s^2.

Shear-tickenning fluids = fluides rhéoépaississants

Easy kitchen experiment:

- 50 % corn starch (Maizena)
- 50 % water

Mix and play !

Yield stress fluids = fluides à seuils

In general

$$\tau = \tau_0 + F(\dot{\gamma})$$

means that a minimal stress must be applied to trigger the motion.

The simplest yield stress model is the **Bingham** model:

$$\tau = \tau_0 + \eta_{app}\dot{\gamma}$$

with an apparent viscosity η_{app} and a yield stress τ_0.

Generalized Stokes equation

$$\rho \frac{D\overrightarrow{u}}{Dt} = \rho \overrightarrow{g} + \overrightarrow{\nabla} \cdot \boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{D}$$

or

$$\rho \frac{D\overrightarrow{u}}{Dt} = -\overrightarrow{\nabla} p' + \overrightarrow{\nabla} \cdot \mathbf{D}$$

For any steady and parallel flow, one can write a balance between the pressure gradient and the shear stress

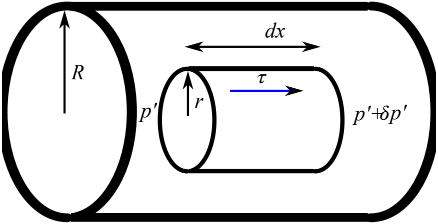
$$\overrightarrow{\nabla} p' = \overrightarrow{\nabla} \cdot \mathbf{D}$$

The Rabinovitch-Mooney formula

Flow in a pipe

For a cylindrical pipe, the force balance for a cylindrical element of fluid is

$$\delta p' \pi r^2 = -2\pi r dx \tau$$



$$\tau = -\frac{r}{2}\frac{\delta p'}{\delta x} = -K\frac{r}{2}$$

For $r = R$, $\tau(R) = \tau_w$, so that

$$\frac{\tau}{\tau_w} = \frac{r}{R}$$

Flow in a pipe

The flow-rate

$$Q = \int_0^R 2\pi r u_z(r) dr$$

can be expressed as

$$Q = \pi \left[r^2 u_z(r) \right]_0^R - \pi \int_0^R r^2 \frac{du_z}{dr} dr$$

The first term is zero, then, using

$$\dot{\gamma} = -\frac{du_z}{dr}, \quad r = R\frac{\tau}{\tau_w}$$

$$Q = \pi \int_0^R \left(R\frac{\tau}{\tau_w} \right)^2 \dot{\gamma} d \left(R\frac{\tau}{\tau_w} \right)$$

The Rabinovitch-Mooney formula

Finally,

$$Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau^2 \dot{\gamma}(\tau)\, d\tau$$

known as the Rabinovitch-Mooney formula, valid for any rheology.

- newtonian: $\dot{\gamma} = \frac{\tau}{\eta}$
- power-law fluid: $\dot{\gamma} = \left(\frac{\tau}{K}\right)^{1/n}$
- Bingham fluid: $\dot{\gamma} = \frac{\tau - \tau_0}{\eta_{app}}$
- Herschel-Bulkley : $\dot{\gamma} = \left(\frac{\tau - \tau_0}{K}\right)^{1/n}$
- many other models ...

The Rabinovitch-Mooney formula

Let's check the RM formula for a newtonian fluid

$$Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau^2 \dot{\gamma}(\tau)\, d\tau$$

$$\tau_w = -K \frac{R}{2}$$

$$\dot{\gamma}(\tau) = \frac{\tau}{\eta}$$

We find

$$Q = -\frac{\pi}{8\eta} K R^4$$

as in Lecture #6 (p. 23)

Flow of Bingham fluids

Flow of Bingham fluids in a pipe

Bingham rheology

$$\tau = \tau_0 - \eta_{app} \frac{du_z}{dr}$$

We assume there exists a radius r_0 which separates a shear zone ($\tau > \tau_0$) and a non-shear zone ($\tau < \tau_0$) with

$$\tau(r_0) = \tau_0$$

$$-K \frac{r}{2} = \tau_0 - \eta_{app} \frac{du_z}{dr}$$

easily integrated to get $u_z(r)$

Flow of Bingham fluids in a pipe

BC:

- at the wall: $u_z(R) = 0$
- at $r = r_0 = -\frac{2\tau_0}{K}$: $\tau = \tau_0$

After a few lines, we find

$$u_z(r) = \frac{1}{\eta_{app}}\left[\tau_0(r - R) + \frac{K}{4}(r^2 - R^2)\right], \quad r > r_0$$

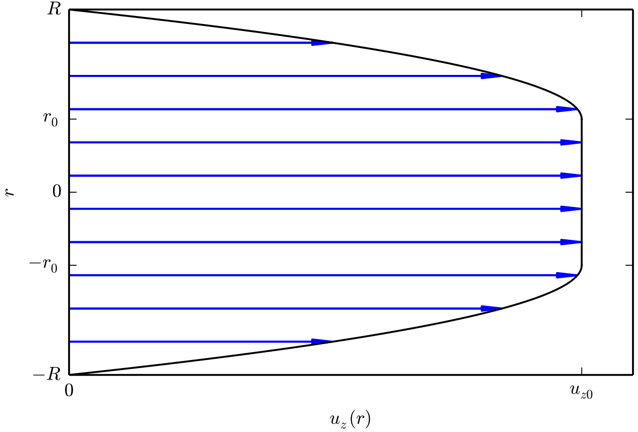
$$u_z(r) = u_{z0} = \frac{1}{\eta_{app}}\left[\tau_0(r_0 - R) + \frac{K}{4}(r_0^2 - R^2)\right], \quad r < r_0$$

or

$$u_{z0} = -\frac{1}{\eta_{app}}\left[\frac{\tau_0^2}{K} + \tau_0 R + \frac{K}{4}R^2\right]$$

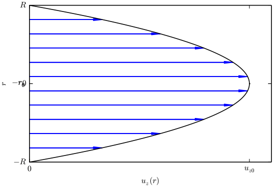
View of the flow field

A characteristic flow field is

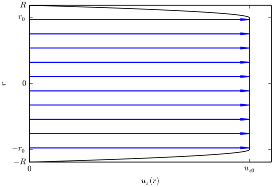


Limit cases

In the limit of $\tau_0 \to 0$, the Poiseuille flow is found.



With a high yield stress $\tau_0 \to K\frac{R}{2}$



The flow is called a plug-flow: no-shear except at the wall.

Flow rate

Using the RM formula

$$Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau^2 \dot{\gamma}(\tau)\, d\tau$$

$$\tau_w = -K\frac{R}{2}$$

with a Bingham rheology

$$\dot{\gamma}(\tau) = \frac{1}{\eta_{app}}(\tau - \tau_0)$$

We find

$$Q = -\frac{\pi R^3}{\eta_{app}}\left(\frac{\tau_0}{3} + \frac{KR}{8}\right)$$

Practical cases

Concrete pump

Concrete pump: tech.spec.

Pump specifications:

Technical data

Model		BSA 1005 D	BSA 1005 E
Material number		102310.000	102311.000
Output	m³/h	52	48
Delivery pressure	bar	70	
Delivery cylinder	Ø mm	180	
Delivery cyl. stroke	mm	1000	

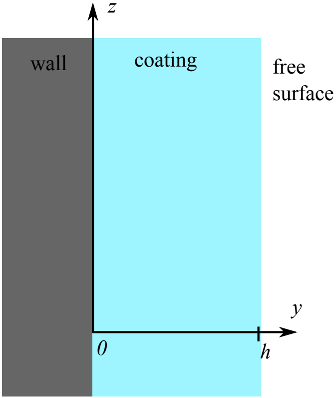
Concrete rheology:

$\tau_0 = 200$ Pa, $\eta_{app} = 400$ Pa·s

What is the maximum length of the pipe?

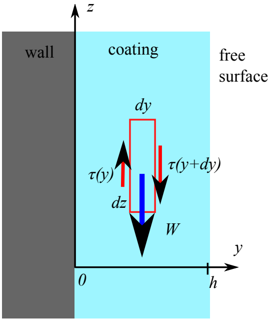
Vertical coating

A vertical fluid coating is applied on a vertical wall:



- fluid of rheology $\tau(\dot{\gamma})$
- no stress at the free surface $\tau(y = h) = 0$
- no-slip condition at the wall: $u_z(y = 0) = 0$

Vertical coating



Force balance on a small fluid element:

$$(dz\,dx)\left[-\tau(y+dy)+\tau(y)\right]-dy(dz\,dx)\rho g=0$$

$$\frac{d\tau}{dy}=-\rho g$$

Vertical coating

Integration with stress BC:

$$\tau(y) = \rho g(h - y)$$

Maximum stress at the wall: $\tau_{max} = \rho gh$ Critical thickness : $h_0 = \tau_0/\rho g$

Bingham rheology:

- If $\tau_{max} < \tau_0$ (or $h < h_0$), the fluid is at rest (no flow)
- If $\tau_{max} > \tau_0$ (or $h > h_0$), the fluid flows downwards.

Vertical coating

Bingham rheology:

$$\tau = \tau_0 + \eta \frac{du_z}{dy} = \rho g (h - y)$$

integrates to

$$u_z(y) = \frac{y}{\eta} \left[\rho g \left(h - \frac{y}{2} \right) - \tau_0 \right]$$

with a BC $u_z(0) = 0$

Maximum velocity u_{zM} is reached at

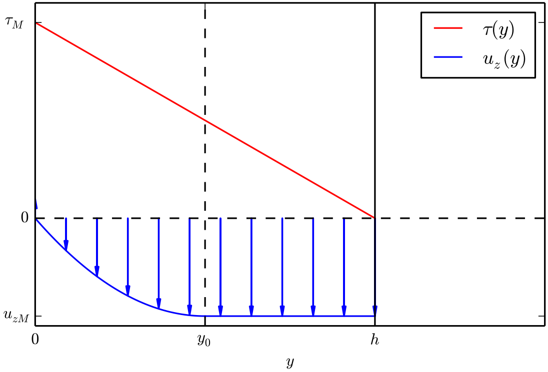
$$y_0 = h - \frac{\tau_0}{\rho g} = h - h_0$$

and is

$$u_{zM} = \frac{1}{2} \frac{\rho g}{\eta} (h - h_0)^2$$

Vertical coating

Stress and velocity field:

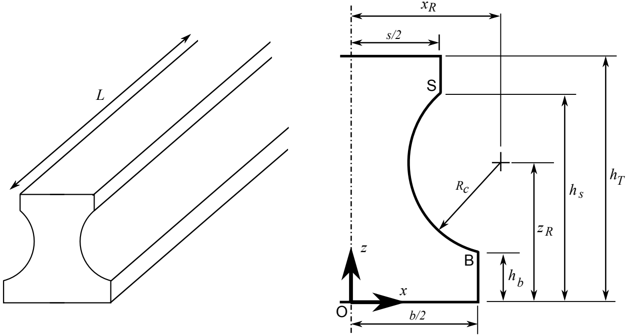


Homework 2016

The context: slipforming of a road barrier

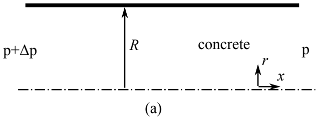
slipforming = coffrage glissant
Recent tools to produce elongated concrete structures on-site.

HW2016: Part 1: hydrostatics



1. Pressure profile on the form
2. Total pressure force $\mathbf{F}_p/L$
3. pressure center

HW2016: Part 2: Bingham flow in a pipe



1. Flow-rate Q_1 without lubrication
2. Flow-rate Q_2 with lubrication
3. Effect of the oil layer thickness

Advices

- do not loose time finding the solution on the internet
- try to work for yourself to learn something and improve your skills
- do not detail all the calculations
- if you introduce assumptions or hypothesis, write them clearly

Due December 9th during the final exam

Basic fluid mechanics for civil engineers
Lecture 9

Maxime Nicolas
maxime.nicolas@univ-amu.fr

POLYTECH MARSEILLE
Aix Marseille Université
Département génie civil

september–december 2016

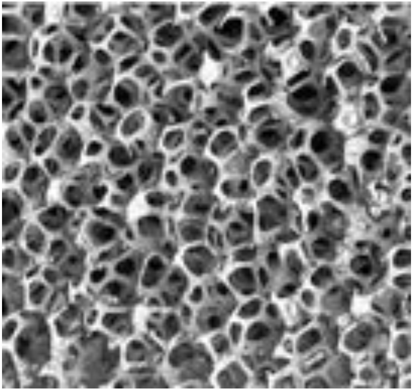
Lecture 9 outline: Flow in porous media

1. Flow in porous media
   - Darcy's law
   - Measuring the permeability
2. Flow through an earth dam

Flow in porous media

Porous material

A porous material has a complex but continuous pore space.



porous media: geometric description

Each point of the volume is occupied by

- solid phase
- fluid phase

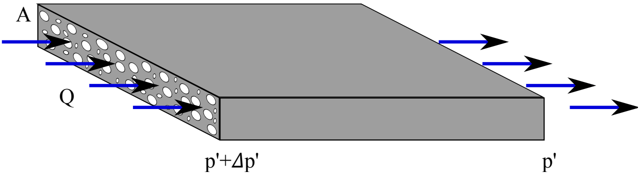
Solid volume fraction:

$$\phi = \frac{\textit{volume occupied by the solid}}{\textit{total volume}}$$

Porosity:

$$\varepsilon = 1 - \phi = \frac{\textit{volume occupied by the fluid}}{\textit{total volume}}$$

Darcy's law



flow-rate as a function of pressure gradient: Darcy's law

$$Q = -\frac{k}{\eta} A \frac{\Delta p'}{L}$$

with k the intrinsic permeability.
or

$$\bar{u} = \frac{Q}{A} = -\frac{k}{\eta}\frac{\Delta p'}{L}$$

by Henri Darcy (1803–1858).

Permeability

Dimension and unit:

$$[k] = \frac{[\bar{u}][\eta][L]}{[\Delta p']} = \mathcal{L}^2$$

the S.I. unit of k is m^2.

A practical unit is the darcy:

$$1 \text{ darcy} = 1 \ d = (1 \ \mu\text{m})^2 = 10^{-12} \ \text{m}^2$$

Permeability of soils and rocks

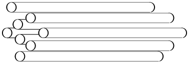
order of magnitude for common soil materials:

material	permeability (darcy)
gravel, pebble bed	10^5
highly fracturated rock	10^5
sand and gravel mixture	10^2
oil reservoir rock	10 to 10^{-1}
fine sand, silt	10^{-3}
sandstone	10^{-3}
granite	10^{-6}

Modeling the permeability

model of porous media: network of parallel tubes (radius a, length L).
flow rate for a single tube (Poiseuille flow, see Lecture # 6) :

$$\delta Q = -\frac{\pi}{8\eta}\frac{\Delta p'}{L}a^4$$



With n the cross-section density of tubes (number of tubes per unit surface), the porosity is

$$\varepsilon = n\pi a^2$$

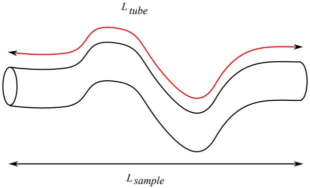
and the permeability is

$$k = \varepsilon\frac{a^2}{8}$$

Modeling the permeability

With a tortuous tube model, we introduce the tortuosity factor τ:

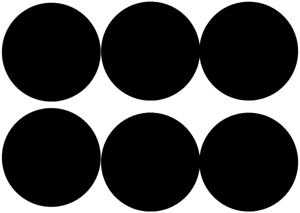
$$L_{tube} = \tau \times L_{sample}$$



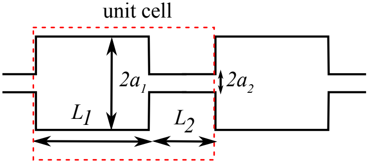
$$k = \frac{\varepsilon}{\tau}\frac{a^2}{8}$$

Modeling the permeability

Model of a porous media made of grains:



With a network of tubes with changing radius:



Flow through a heterogeneous porous media

1. Parallel permeabilities

$$\frac{Q_1}{A_1} = -\frac{k_1}{\eta}\frac{\Delta p}{L}, \quad \frac{Q_2}{A_2} = -\frac{k_2}{\eta}\frac{\Delta p}{L}$$

Total flow rate:

$$Q = Q_1 + Q_2 = -(k_1 A_1 + k_2 A_2)\frac{\Delta p}{\eta L}$$

Effective permeability:

$$k_{parallel} = \left(\frac{A_1}{A_1 + A_2}\right) k_1 + \left(\frac{A_2}{A_1 + A_2}\right) k_2$$

Flow through a heterogeneous porous media

2. Serial permeabilities

$$\frac{Q}{A} = -\frac{k_1}{\eta}\frac{\Delta p_1}{L_1}, \quad \frac{Q}{A} = -\frac{k_2}{\eta}\frac{\Delta p_2}{L_2}$$

Total pressure drop:

$$\Delta p = \Delta p_1 + \Delta p_2 = -\frac{Q}{A}\eta\frac{L_1 + L_2}{k_{serial}}$$

Effective permeability:

$$k_{serial} = \frac{L_1 + L_2}{\frac{L_1}{k_1} + \frac{L_2}{k_2}}$$

Flow through a heterogeneous porous media

If $k_1 \gg k_2$,

$$k_{parallel} \approx \frac{A_1}{A_1 + A_2} k_1$$

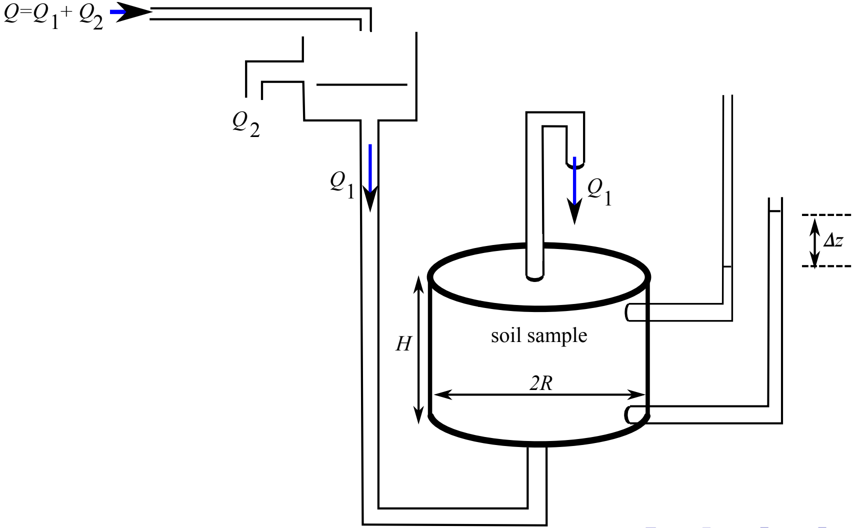
the flow is governed by the larger permeability

$$k_{serial} \approx \frac{L_1 + L_2}{L_2} k_2$$

the flow is governed by the smaller permeability

Constant pressure permeameter

A simple design to measure the permeability of a soil sample:



Constant pressure permeameter

Darcy:

$$Q_1 = \pi R^2 \frac{k}{\eta} \frac{\Delta p}{H}$$

Pressure drop:

$$\Delta p = \rho g \Delta z$$

Intrinsic permeability:

$$k = \frac{\eta}{\rho g} \frac{Q_1}{\pi R^2} \frac{H}{\Delta z}$$

or hydraulic permeability

$$K = \frac{\rho g}{\eta} k$$

Hydraulic permeability

$$K = \frac{\rho g}{\eta} k$$

Dimension:

$$[K] = \frac{[\rho][g][k]}{[\eta]} = \mathcal{L} \cdot \mathcal{T}^{-1}$$

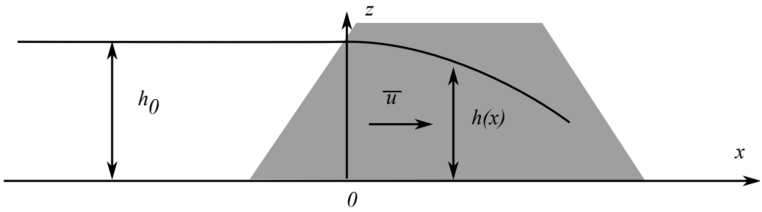
Unit: K in m·s^{-1} (as a velocity)

Flow through an earth dam

Earth dam are used to protect from floods

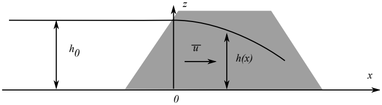
Flow through an earth dam

Flow through an earth dam



$$\bar{u} = -\frac{k}{\eta}\frac{dp}{dx}, \quad \frac{dp}{dx} = \rho g \frac{dh}{dx}$$

flow through an earth dam



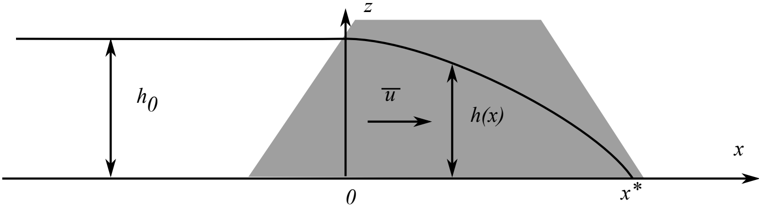
flow rate (per unit of dam length):

$$\frac{Q}{L} = \int_0^{h(x)} \bar{u}\, dz = -\frac{k\rho g}{2\eta} \frac{d[h^2]}{dx}$$

the free surface of water in the dam is

$$h(x) = \sqrt{h_0^2 - \frac{2\eta}{k\rho g} \frac{Q}{L} x}$$

flow through an earth dam



The minimum width x^* of the dam to avoid leakage is thus

$$x^* = \frac{k\rho g}{2\eta}\frac{L}{Q}h_0^2$$

Basic fluid mechanics for civil engineers
Lecture 10

Maxime Nicolas
maxime.nicolas@univ-amu.fr

POLYTECH MARSEILLE
Aix*Marseille Université
Département génie civil

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Lecture 10 outline

1. Capillary effects
2. Course summary
3. Open discussion

Capillary effects

Rise of water in a capillary tube

Observing a simple experiment: vertical tubes in a tank of liquid

Jurin

The rise of the liquid in the tube follows a law established by Jurin:

$$\Delta h = \frac{2\gamma\cos\theta}{R\rho g}$$

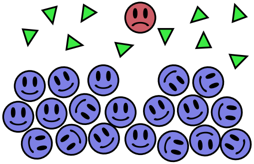
where

- γ is the interfacial tension between liquid and air
- θ is the wetting angle between liquid and tube material
- R is the tube radius

superficial tension

The superficial tension applies only at the interface between 2 different fluids (e.g. water and air).

The molecules of a fluid like to be surrounded by some molecules of the same kind.



A drop of liquid on a solid surface does not flatten completely under gravity:

Superficial tension

For water, the interfacial tension with air is

$$\gamma_{water/air} = 73 \text{ mN}\cdot\text{m}$$

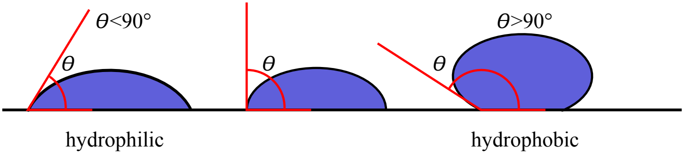
The Laplace pressure scales as γ/d where d is a characteristic length. Comparing with hydrostatic pressure $p = \rho g d$ leads to

$$d = \sqrt{\frac{\gamma}{\rho g}}$$

For water $d \approx 2.7$ mm.

Contact angle

A puddle of water on a solid substrate is either flat or round. The contact angle represents the hydrophilic/hydrophobic nature of the surface.



Walk on water with surface tension

How to float on water

Despite $\rho_{steel} > \rho_{water}$, the paper clip floats!

Hydrophobic natural surfaces

Water drops on a lotus leaf

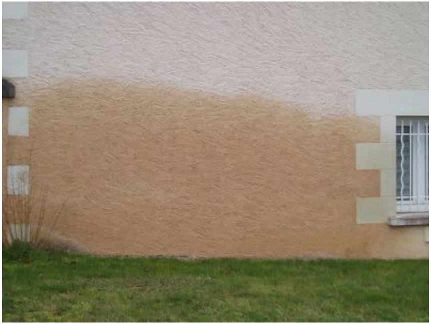
Hydrophobic artificial surfaces

Hydrophobic glass

Capillary rise in porous materials

The capillary rise occurs naturally in

- sugar cube with coffee (or any other liquid)
- soils: from saturated zone to dry zone
- concrete: rise from ill-drained foundation



Course summary

Problem solving method

Before attempting to solve any problem, a few questions have to be addressed:

1. geometry and symmetry
2. steady or not steady
3. dominant forces (inertia or viscous force)
4. relevance of hydrostatics
5. rheology of the liquid
6. boundary conditions
7. initial conditions (for unsteady flows only)

General equations

A minimal set[1] of general equations is
mass conservation eq.:

$$\overrightarrow{\nabla} \cdot \overrightarrow{u} = 0, \quad \text{incompressible flow}$$

momentum conservation eq.:

$$\rho \frac{D\overrightarrow{u}}{Dt} = \rho \overrightarrow{g} + \overrightarrow{\nabla} \cdot \boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{D}$$

The tensor **D** expresses the rheology of the fluid

1. without temperature or reactive effects

Navier-Stokes equation

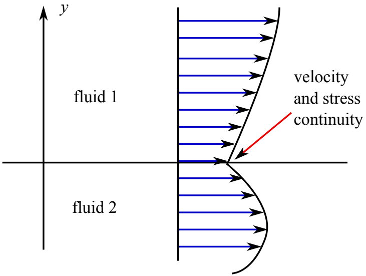
For a newtonian fluid of viscosity η, the needed equations are

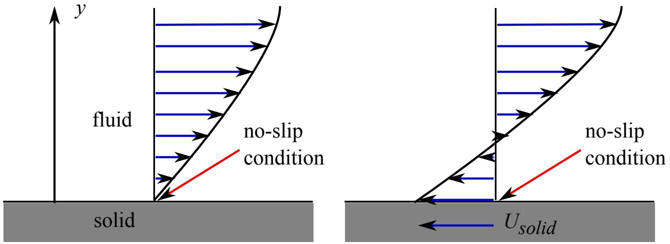
$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{u} &= 0 \\
\rho \frac{D\overrightarrow{u}}{Dt} &= \rho \overrightarrow{g} - \overrightarrow{\nabla} p + \eta \Delta \overrightarrow{u}
\end{aligned}$$

with only a few analytical solutions for small *Re*:

- BC driven flow: Couette flows
- pressure-driven flow: Poiseuille flow

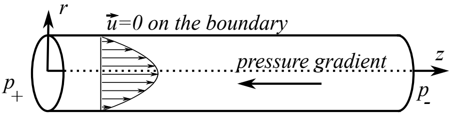
Velocity and stress continuity





Cylindrical pressure-driven flow

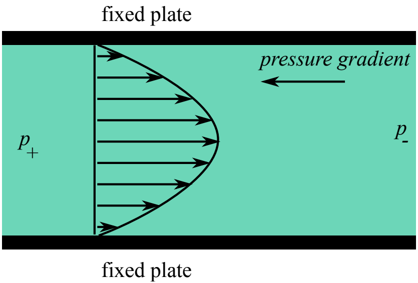
Poiseuille flow:



$$u_z(r) = -\frac{1}{4\eta}\frac{dp}{dz}(R^2 - r^2)$$

$$Q = -\frac{\pi}{8\eta}\frac{dp}{dz}R^4$$

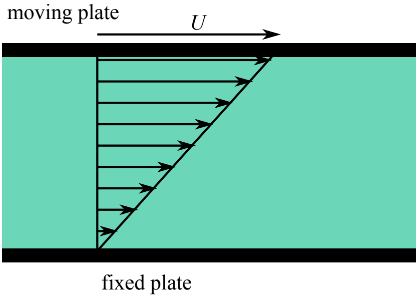
Plane pressure-driven flow



$$u_x(y) = \frac{1}{2\eta}\frac{dp}{dx}(y-h)y$$

Plane boundary-driven flow

Couette flow:



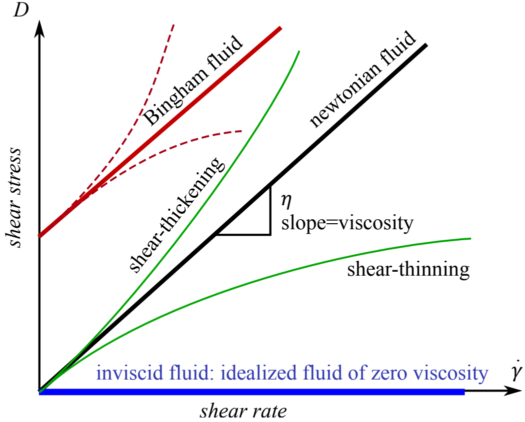
$$u_x(y) = U\frac{y}{h}, \quad p = constant$$

$Re \gg 1$ and steady flows

Bernoulli's equation: along a streamline,

$$\frac{1}{2}\rho v^2 + \rho g z + p = C$$

Stress-strain relation



The Rabinovitch-Mooney formula

Finally,

$$Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau^2 \dot{\gamma}(\tau)\, d\tau$$

known as the Rabinovitch-Mooney formula, valid for any rheology.

- newtonian: $\dot{\gamma} = \frac{\tau}{\eta}$
- power-law fluid: $\dot{\gamma} = \left(\frac{\tau}{K}\right)^{1/n}$
- Bingham fluid: $\dot{\gamma} = \frac{\tau - \tau_0}{\eta}$

Open discussion

Is there any muddy points?