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## Waves & vibrations

Maxime Nicolas

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# Waves & vibrations

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Département génie civil

november 2016 – january 2017

# Course 1 outline

- 1 Preamble
  - Course schedule
  - Online
  - Course outline
- 2 Introduction
- 3 The harmonic oscillator

# PREAMBLE

# Course syllabus

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Bâtiment Fermi, bureau 212

## Schedule:

- 5 lectures
- 5 workshops for civil eng. students
- 3 workshops for mech. eng. students

Final exam: January 25th, 2017

# Online

This course is available on ENT/AmeTice :

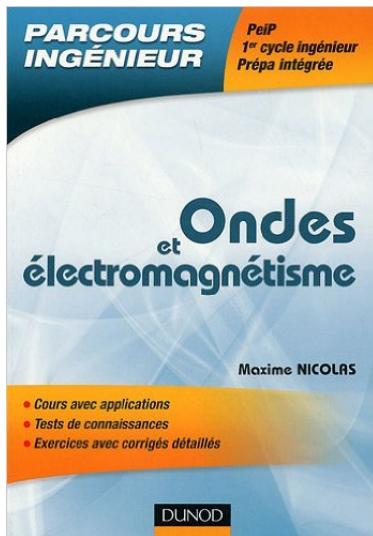
Sciences & technologies ▶ Polytech ▶ Cours communs ▶  
[16] - S5 - JGC52D + JME51C - Ondes et vibrations (Maxime Nicolas)

with

- lecture slides
- workshops texts
- past exams

## A book

This course is included in a book (paper and pdf versions available):



# Course outline

- 1 Introduction & harmonic oscillator
- 2 The wave equation and its solutions
- 3 1D transverse and longitudinal waves
- 4 2D waves: vibration of plates
- 5 beam vibration

# Introduction

# Where to find waves and vibrations?

Short answer: everywhere :

- waves in fluids → acoustics → sound
- waves in fluids → ripples, waves and tsunamis
- waves in solids → compression waves
- vibrations of structures → planes, cars,

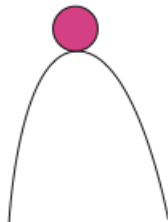
and many more

- electromagnetic waves → light, radio, X-rays,  $\gamma$ -rays
- chemical oscillators
- population dynamics
- health issues due to vibrations

# A few definitions

- **wave**: propagation of an oscillation or a vibration
- **vibration**: motion around an equilibrium state
- **oscillation**: motion of a body around an equilibrium point

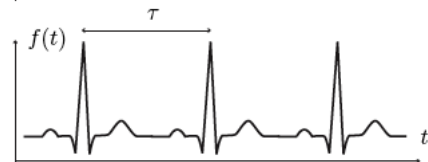
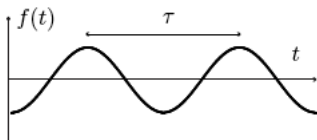
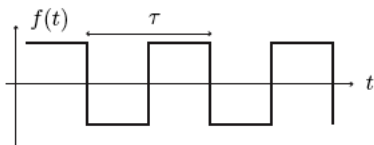
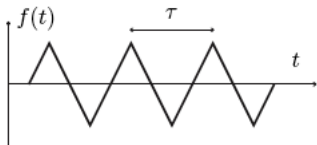
# Equilibrium



# $t$ -periodical functions

$$F(t + \tau) = F(t), \quad \forall t$$

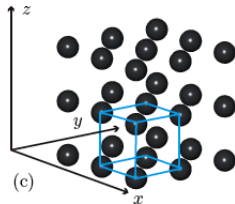
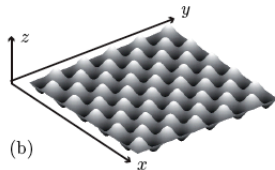
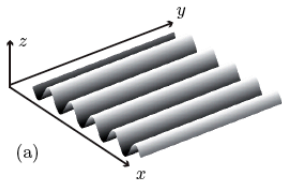
$$f = \tau^{-1}, \quad \omega = 2\pi f = 2\pi/\tau$$



# x-periodical functions

$$F(x + \lambda) = F(x), \quad \forall x$$

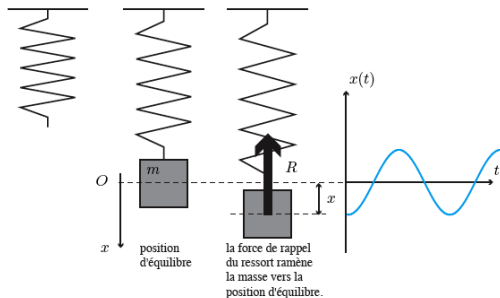
$$k = 2\pi/\lambda$$



# The harmonic oscillator

# Example 1

The mass-spring system:



$$m \frac{d^2 x}{dt^2} + kx = 0$$



## Examples comparison

spring-mass

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

$$\omega_0 = \sqrt{k/m}$$

linear equation for  $x$ 

pendulum

$$mL \frac{d^2 \theta}{dt^2} + mg \sin \theta = 0$$

$$\frac{d^2 \theta}{dt^2} + \omega_0^2 \sin \theta = 0$$

$$\omega_0 = \sqrt{g/L}$$

non linear equation for  $\theta$

# Linearization

For any (continuous and derivable) function  $F$  near  $x_0$ :

$$F(x)_{x \approx x_0} = F(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d^n f}{dx^n} \right) (x_0) (x - x_0)^n$$

for  $F(\theta) = \sin \theta$ ,  $x \approx 0$ :

$$F(\theta) = \theta - \frac{\theta^3}{6} + \dots$$

## Examples comparison

spring-mass

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

$$\omega_0 = \sqrt{k/m}$$

linear equation for  $x$ 

pendulum

$$mL \frac{d^2 \theta}{dt^2} + mg \sin \theta = 0$$

$$\frac{d^2 \theta}{dt^2} + \omega_0^2 \theta = 0$$

$$\omega_0 = \sqrt{g/L}$$

linear equation for  $\theta \ll 1$

# The harmonic oscillator equation

The equation for a physical quantity  $A(t)$  ( $x$  or  $\theta$ ) is

$$\frac{d^2A}{dt^2} + \omega_0^2 A = 0$$

this is a 2nd order differential equation.

on the blackboard

# The harmonic oscillator equation

$$\frac{d^2 A}{dt^2} + \omega_0^2 A = 0$$

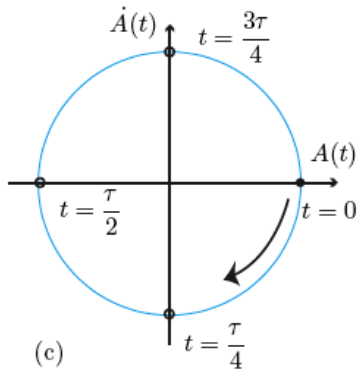
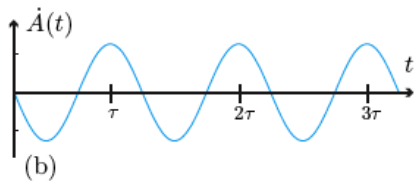
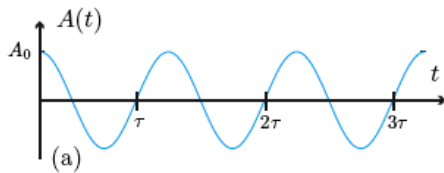
General solution:

$$A(t) = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} = B_1 \cos(\omega_0 t) + B_2 \sin(\omega_0 t)$$

With the initial conditions  $(A_0, \dot{A}_0)$

$$A(t) = A_0 \cos(\omega_0 t) + \frac{\dot{A}_0}{\omega_0} \sin(\omega_0 t)$$

## View of the solution



# Energy

This equation describes a conservative system (no loss of energy):

$$\frac{d^2 A}{dt^2} + \omega_0^2 A = 0$$

Back to the mass-spring example:

$$m \frac{d^2 x}{dt^2} + kx = 0$$

...

# Harmonic oscillator with damping

Introducing a fluid damping force (prop. to velocity):

$$\frac{d^2 A}{dt^2} + \gamma \frac{dA}{dt} + \omega_0^2 A = 0$$

# Solution with damping

Testing function

$$A = e^{rt}$$

Characteristic equation:

$$r^2 + \gamma r + \omega_0^2 = 0$$

with 2 solutions

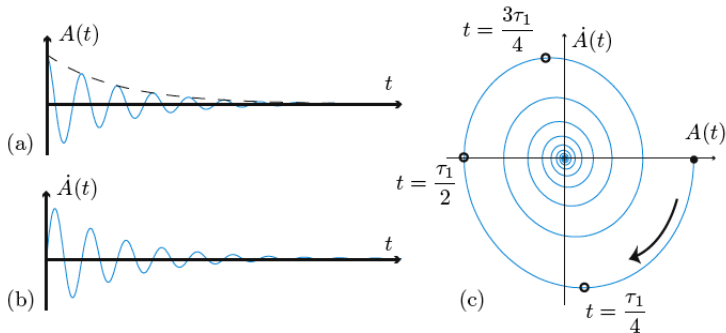
$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{2} \sqrt{\gamma^2 - 4\omega_0^2} = -\frac{\gamma}{2} \pm \alpha$$

and

$$A(t) = e^{-\gamma t/2} (A_1 e^{\alpha t} + A_2 e^{-\alpha t})$$

## Weak damping solution

$$\gamma^2 - 4\omega_0^2 < 0, \quad \alpha = i\omega_1, \quad \omega_1 = \frac{1}{2}\sqrt{4\omega_0^2 - \gamma^2}$$



$$A(t) = e^{-\gamma t/2} \left[ A_0 \cos(\omega_1 t) + \frac{1}{\omega_1} \left( \frac{\gamma A_0}{2} + \dot{A}_0 \right) \sin(\omega_1 t) \right]$$

# Energy loss with damping

$$\frac{1}{2}\dot{A}^2 + \frac{1}{2}\omega_0^2 A^2 = -\gamma \int \dot{A}^2 dt$$

$$E(t) = -\gamma \int \dot{A}^2 dt$$

$$\frac{dE}{dt} = -\gamma \dot{A}^2$$

# Oscillator with energy input

$$\frac{d^2 A}{dt^2} + \gamma \frac{dA}{dt} + \omega_0^2 A = A_F \cos(\omega t)$$

with  $A_F$  the forcing amplitude, and  $\omega$  the forcing angular frequency.

proposed long time solution:

$$A(t) = A_1 \cos(\omega t + \varphi)$$

Now find  $A_1$  and  $\varphi \dots$

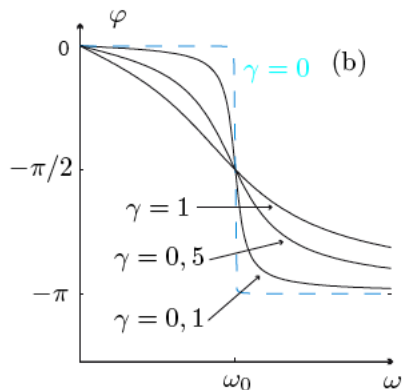
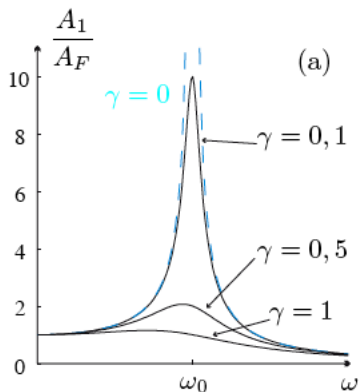
# Solution

$$A(t) = A_1 \cos(\omega t + \varphi)$$

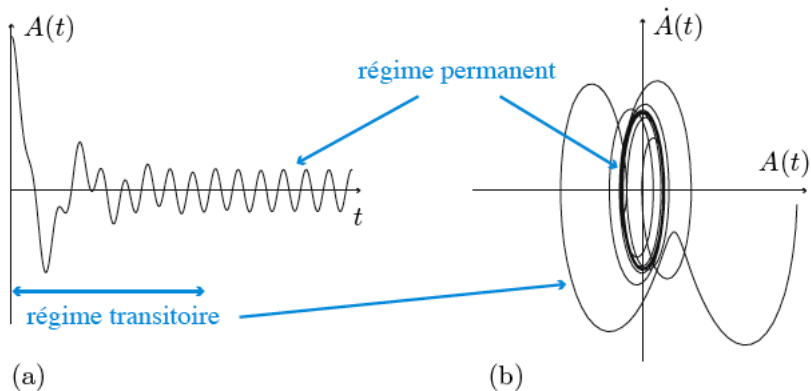
$$\frac{A_1}{A_F} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}$$

$$\varphi = \arctan\left(\frac{-\omega\gamma}{\omega_0^2 - \omega^2}\right)$$

## View



## View



# Waves & vibrations

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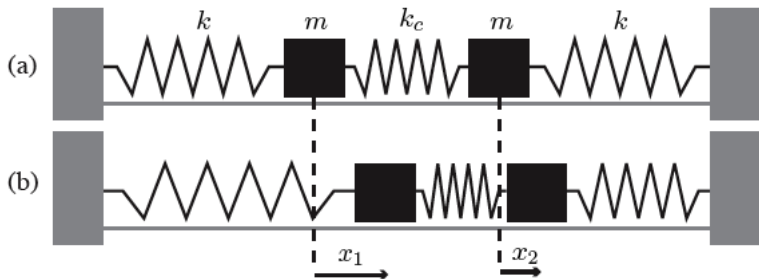
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# Course 2 outline

- 1 Coupled oscillators
- 2 1D infinite chain of oscillators
- 3 The simple wave equation

# Coupled oscillators

# Coupled oscillators



2 identical oscillators + coupling spring

# Coupled equations

Reminder:  $x_i$  variables are perturbations out of equilibrium.

$$m\ddot{x}_1(t) = -kx_1 - k_c(x_1 - x_2)$$

$$m\ddot{x}_2(t) = -kx_2 - k_c(x_2 - x_1)$$

or

$$\ddot{x}_1(t) = -\omega_0^2 x_1 - \omega_c^2 (x_1 - x_2)$$

$$\ddot{x}_2(t) = -\omega_0^2 x_2 - \omega_c^2 (x_2 - x_1)$$

with oscillating solutions:

$$x_1 = X_1 \exp(i\omega t), \quad x_2 = X_2 \exp(i\omega t)$$

## Coupled equations as a linear problem

$$\begin{pmatrix} \omega_0^2 - \omega^2 + \omega_c^2 & -\omega_c^2 \\ -\omega_c^2 & \omega_0^2 - \omega^2 + \omega_c^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0$$

- trivial solution:  $X_1 = X_2 = 0$ , static equilibrium
- non trivial solution  $X_1 \neq 0$ ,  $X_2 \neq 0$

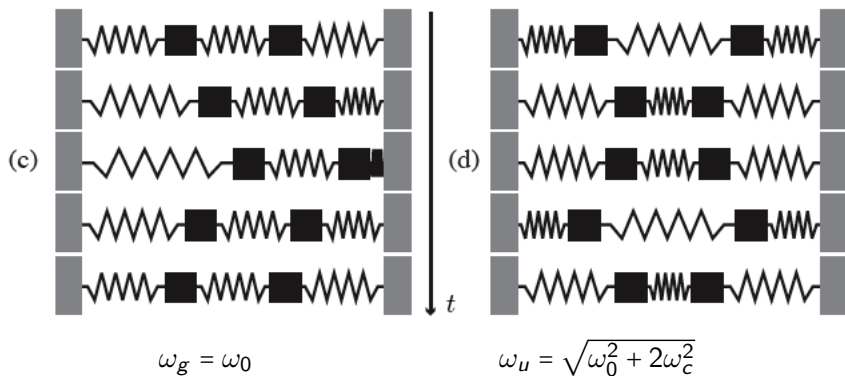
$$\det(M) = 0$$

Two natural frequencies:

$$\omega_g = \omega_0$$

$$\omega_u = \sqrt{\omega_0^2 + 2\omega_c^2}$$

## View of the solutions



# Complete solution

The complete solution of the coupled oscillators problem is

$$x_1(t) = A_g \cos(\omega_g t + \varphi_g) + A_u \cos(\omega_u t + \varphi_u)$$

$$x_2(t) = A_g \cos(\omega_g t + \varphi_g) - A_u \cos(\omega_u t + \varphi_u)$$

The 4 constants  $A_g$ ,  $A_u$ ,  $\varphi_g$  and  $\varphi_u$  are to be determined by 4 initial conditions:

- $x_1(t=0)$  and  $x_2(t=0)$
- $\dot{x}_1(t=0)$  and  $\dot{x}_2(t=0)$

# $N$ coupled oscillators

For a set of  $N$  coupled oscillators, we can write a set of  $N$  equations

$$\ddot{x}_i = -\omega_0^2 x_i - \sum_{j \neq i}^N \omega_c^2 (x_i - x_j)$$

with harmonic solutions

$$x_i = X_i \exp(i\omega t)$$

The problem writes

$$M \vec{X} = 0$$

with  $M$  a  $N \times N$  matrix and  $\vec{X} = (X_1, \dots, X_N)$  an amplitude vector.

# $N$ coupled oscillators

- trivial solution:  $\vec{X} = 0$ , static equilibrium
- non trivial solution  $X_i \neq 0$

$$\det(M) = 0$$

which leads to  $N$  natural frequencies  $\omega_1, \dots, \omega_N$  and the complete solution is a linear combination of these  $N$  individual solutions:

$$x_i = \sum_j A_j \cos(\omega_j t + \varphi_j)$$

the  $2N$  constants  $A_j$  and  $\varphi_j$  are to be determined with  $2N$  initial conditions.

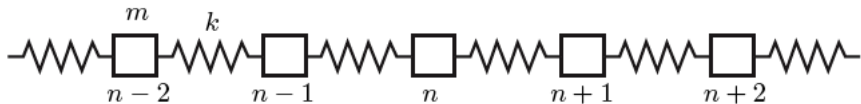
# Partial conclusion

- $N = 2$  easy
- $N = 3$  less easy but possible
- $N \geq 4$  computer help is needed

# 1D infinite chain of oscillators

$N \rightarrow \infty$ 

For a very large number of oscillators, the previous method is too expensive!



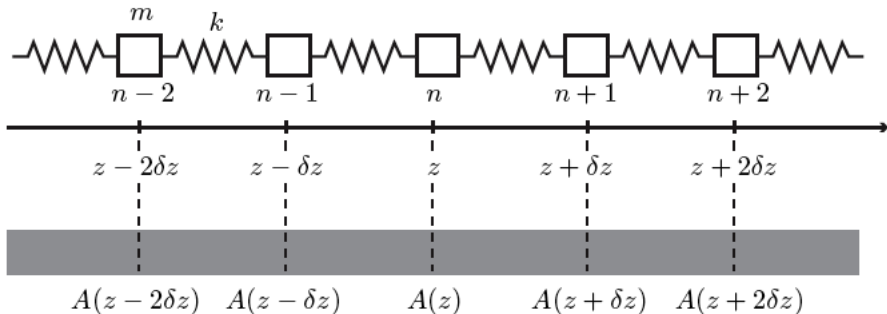
$N$  equations

$$m \frac{d^2 A_n}{dt^2} = -k(A_n - A_{n+1}) - k(A_n - A_{n-1})$$

or

$$\begin{aligned} \frac{d^2 A_n}{dt^2} &= -\omega_0^2(A_n - A_{n+1}) - \omega_0^2(A_n - A_{n-1}) \\ &= \omega_0^2(A_{n+1} - 2A_n + A_{n-1}) \end{aligned}$$

# Continuous description



$$A_n(t) = A(z, t)$$

## Continuous description

Two neighboring oscillator have a very close behavior

$$A_{n+1} = A(z + \delta z) = A(z) + \delta z \left( \frac{\partial A}{\partial z} \right) + \frac{(\delta z)^2}{2} \left( \frac{\partial^2 A}{\partial z^2} \right) + \dots$$

$$A_{n-1} = A(z - \delta z) = A(z) - \delta z \left( \frac{\partial A}{\partial z} \right) + \frac{(\delta z)^2}{2} \left( \frac{\partial^2 A}{\partial z^2} \right) + \dots$$

summing these 2 equations gives

$$A_{n+1} + A_{n-1} = 2A(z) + (\delta z)^2 \left( \frac{\partial^2 A}{\partial z^2} \right)$$

then

$$A_{n+1} - 2A_n + A_{n-1} = (\delta z)^2 \left( \frac{\partial^2 A}{\partial z^2} \right)$$

## Motion equation

Back to the motion equation

$$\frac{d^2 A_n}{dt^2} = \omega_0^2 (A_{n+1} - 2A_n + A_{n-1})$$

with the continuous approach

$$\frac{d^2 A_n}{dt^2} \longrightarrow \frac{\partial^2 A}{\partial t^2}$$

$$A_{n+1} - 2A_n + A_{n-1} = (\delta z)^2 \left( \frac{\partial^2 A}{\partial z^2} \right)$$

and the motion equation is now

$$\frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0, \quad c = \omega_0 \delta z$$

# The simple wave equation

# Wave equation

It is a 1D wave equation

$$\frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0$$

with a constant velocity  $c$ .

If a 3D propagation is needed, the wave equation writes

$$\Delta A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0$$

with the laplacian operator

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

# The wave equation general solution

The wave eq. can be written as (remember  $a^2 - b^2 = (a - b)(a + b)$ ):

$$\left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t}\right) \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) A = 0$$

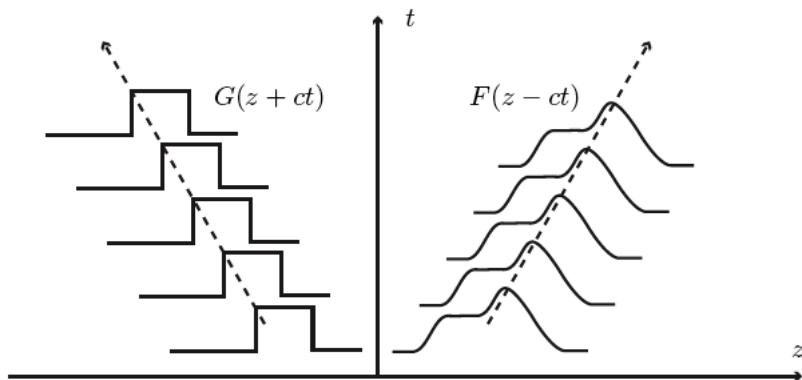
The solution of the wave eq. is the sum of two functions:

$$A(z, t) = F(z - ct) + G(z + ct)$$

$F$  and  $G$  are arbitrary functions.

Proof on the board ...

## Example



# Exercise

Let  $F$  a function such that

- $F = 1$  if  $-1 < z < 1$
- $F = 0$  elsewhere

With  $c = 3$ , draw  $F$  for

- $t=0$
- $t=1$
- $t=2$

# The complete wave solution

The complete solution needs:

- the initial condition  $A(z,0)$
- the boundary conditions

but since the WE is linear, its solutions can be written as a sum of elementary solutions

# Waves & vibrations

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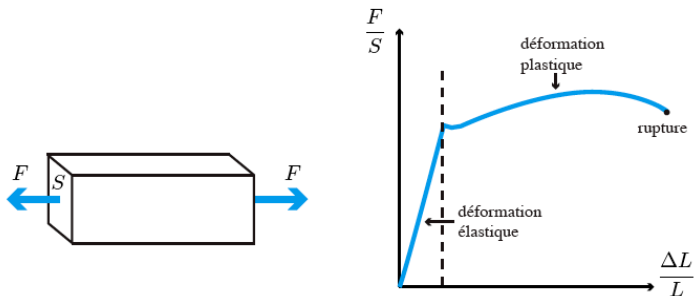
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# Lecture 3 outline

- 1 Compression waves
- 2 Vibration of a tensioned string
  - Static equilibrium of a string
  - Vibration of a string

# Compression waves

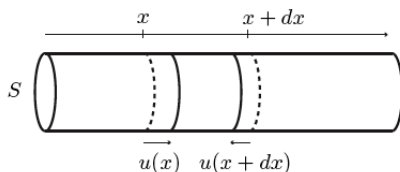
# Linear elasticity



Hooke's law:

$$\frac{\Delta L}{L} = \frac{1}{E} \frac{F}{S} = \frac{\sigma}{E}$$

# 1D compression waves



For  $u(x)$  the displacement of a section located at  $x$ :

$$\frac{u(x) - u(x + dx)}{dx} = \frac{F}{SE}$$

hence:

$$F(x) = -SE \frac{\partial u}{\partial x}$$

# 1D compression waves

Newton's equation for the elementary mass  $dm$

$$dm \frac{\partial^2 u}{\partial t^2} = F(x) - F(x + dx)$$

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = ES \left[ \left( \frac{\partial u}{\partial x} \right)_{x+dx} - \left( \frac{\partial u}{\partial x} \right)_x \right]$$

which writes as a wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad c = \sqrt{\frac{E}{\rho}}$$

# Compression waves velocities

Examples and order of magnitude of compression velocity:

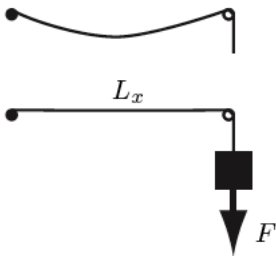
<b>Matériau</b>	$\rho$ (kg.m <sup>-3</sup> )	$E$ (GPa)	$c$ (m/s)
acier	7 900	200	5 000
civre	8 920	124	3 700
béton	2 500	2,7	1 000
plexiglass	1 200	2,4	1 400
verre	2 500	70	5 300

# Vibration of a tensioned string

# Static equilibrium of a string

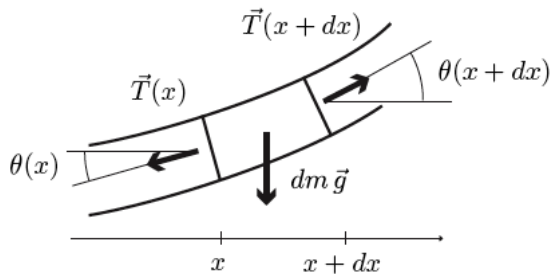
# Modeling

- 1D system:  $L_x \gg L_y$ ,  $L_x \gg L_z$ ,  $L_y \approx L_z$
- without rigidity (for the moment, see lecture # 5)
- tension force:



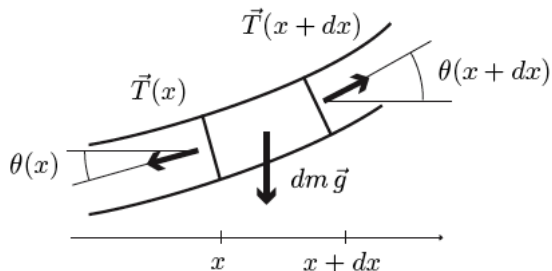
$$F = T_0 = \sigma L_y L_z$$

# Shape of a string at equilibrium



$$dm \vec{g} + \vec{T}(x) + \vec{T}(x + dx) = 0$$

# Shape of a string at equilibrium



Projection on the  $x$ - and  $y$ -axis

$$-T(x) \cos \theta(x) + T(x+dx) \cos \theta(x+dx) = 0$$

$$-dm g - T(x) \sin \theta(x) + T(x+dx) \sin \theta(x+dx) = 0$$

## Shape of a string at equilibrium

The x-axis equation means

$$T(x) \cos \theta(x) = T_0 = \text{constant}$$

Combining with the y-axis eq. leads

$$-dm g + T_0 [\tan \theta(x + dx) - \tan \theta(x)] = 0$$

or

$$-\frac{mg}{LT_0} + \frac{d}{dx} \tan \theta = -\frac{mg}{LT_0} + \frac{d\theta}{dx} \frac{d \tan \theta}{d\theta} = 0$$

Writing

$$\tan \theta = \frac{dy}{dx}$$

gives

$$\frac{d^2 y}{dx^2} = \left( \frac{mg}{LT_0} \right) \sqrt{1 + \left( \frac{dy}{dx} \right)^2}$$

## Shape of a string at equilibrium

Using a variable  $u = dy/dx$ , one can finally find

$$y(x) = L_c \left[ \cosh\left(\frac{x}{L_c}\right) - \cosh\left(\frac{d}{2L_c}\right) \right], \quad L_c = \frac{LT_0}{mg}$$

the shape of the string of length  $L$ , mass  $m$ , attached between two fixed points with a  $d$  distance.

Maximum bending at the center of the string:

$$y_m = L_c \left[ \cosh\left(\frac{d}{2L_c}\right) - 1 \right]$$

# Tension of a string at equilibrium

The tensile force is

$$T(x) = \frac{T_0}{\cos \theta} = T_0 \cosh \left( \frac{x}{L_c} \right)$$

and the tensile force variation along the rope is

$$\frac{\Delta T}{T_0} = \frac{y_m}{L} \frac{mg}{T_0}$$

This variation is thus negligible when the string is under tension ( $y_m \ll L$ )

# Vibration of a string

## Motion equation

The local motion equation is

$$dm \frac{\partial^2 y}{\partial t^2} = -T(x) \sin \theta(x) + T(x + dx) \sin \theta(x + dx) - dm g$$

For a string under strong tension, the weight is negligible and the tension is constant:

$$\frac{m}{L} dx \frac{\partial^2 y}{\partial t^2} = T_0 [\sin \theta(x + dx) - \sin \theta(x)]$$

Assuming only a weak deviation from the equilibrium ( $y = 0$  and  $dy/dx = 0$ )

$$\sin \theta \approx \tan \theta \approx \frac{\partial y}{\partial x}$$

we obtain

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = 0, \quad c = \sqrt{\frac{T_0}{\rho_L}}, \quad \rho_L = \frac{m}{L}$$

## Boundary conditions

The string is attached in two points:

- $y(x = 0) = 0$
- $y(x = L) = 0$

Meaning that the wave propagation is **confined** between 0 and  $L$ . This gives a **steady wave**

$$y(x, t) = A(x) \cos(\omega t + \varphi)$$

introducing an amplitude function  $A(x)$ .

# Amplitude equation

$$\frac{d^2 A}{dx^2} + \frac{\omega^2}{c^2} A = 0$$

Its general solution is

$$A = A_1 \cos(kx) + A_2 \sin(kx)$$

and the constants  $A_1$  and  $A_2$  are to be determined with BC

$$\longrightarrow A_1 = 0, \quad A_2 \neq 0 \quad \text{and} \quad \sin(kL) = 0$$

## Vibration modes

The vibration is thus possible for a discrete set of  $k$  and  $\omega$  values:

$$k = n \frac{\pi}{L}, \quad \text{and} \quad \omega = n \frac{\pi}{L} c$$

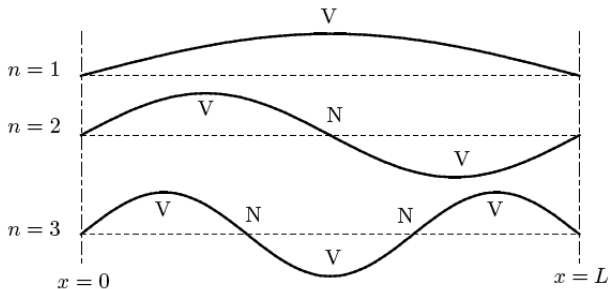
and finally the vibration may be represented as

$$y(x,t) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{nc\pi}{L} t + \varphi_n\right)$$

with an integer  $n = 1, 2, 3, \dots$

The  $C_n$  and  $\varphi_n$  constants are to be determined by the initial conditions (IC).

# Vibration modes



## A few definitions

- A nodal point (node):

$$A = 0, 0 < x < L, \forall t$$

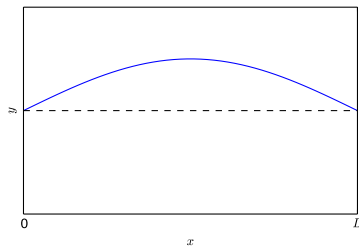
- antinodal points (antinodes):

$$\frac{dA}{dx} = 0, 0 < x < L, \forall t$$

# A few properties

- The mode with the lowest frequency is the **fundamental mode**
- mode  $n$  has  $n$  antinodes and  $n - 1$  nodes
- for a 1D system, nodes are points (0D objects)
- for a ND system, nodes are objects of  $(N-1)$ D dimension

# Movies



View movies:

- mode 1
- mode 3
- modes 1+3+5

# Waves & vibrations

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november 2016 – january 2017

# Lecture 4 outline

- 1 Static of a beam
- 2 Vibration of a beam

# Static of a beam

# Beam characteristics

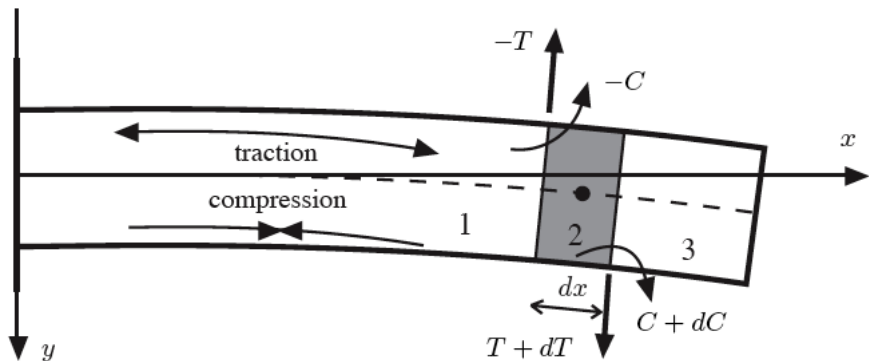
- geometry:

$$L_x > L_y \approx L_z$$

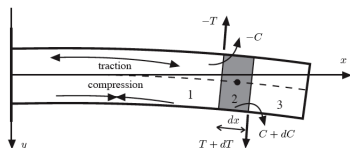
- cross-section  $S$
- quadratic moment of inertia  $I$
- material:

$$E, \rho$$

## Beam setup



## Force balance

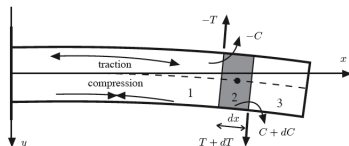


$$\vec{F}_{1 \rightarrow 2} + \vec{F}_2 + \vec{F}_{3 \rightarrow 2} = 0$$

$$-T + S dx \rho g + (T + dT) = 0$$

$$\frac{dT}{dx} = -S \rho g \quad (1)$$

# Torque balance



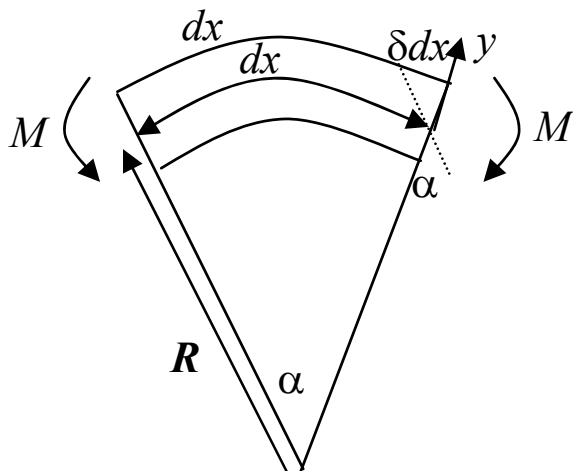
$$-C + (C + dC) + T \frac{dx}{2} + (T + dT) \frac{dx}{2} = 0$$

$$\frac{dC}{dx} = -T$$

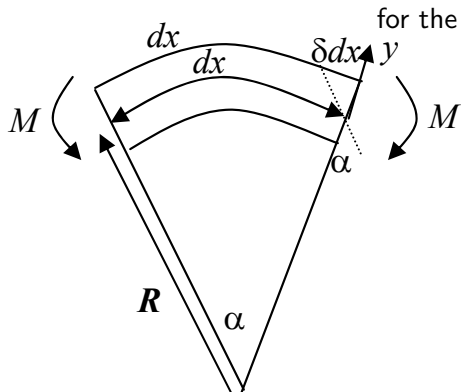
using (1)

$$\frac{d^2 C}{dx^2} = S \rho g \quad (2)$$

## Elasticity of the beam



## Elasticity of the beam



$$\frac{\delta dx}{dx} = \frac{1}{E} \frac{dF}{dS}$$

$$\frac{\delta dx}{y} = -\frac{dx}{R}$$

$$\frac{\delta dx}{dx} = -\frac{y}{R}$$

$$dF = -\frac{y}{R} E dS$$

# Torque

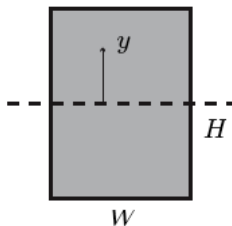
$$\begin{aligned}
 -C &= \iint_S y \, dF = -\frac{E}{R} \iint_S y^2 \, dS \\
 C &= \frac{E}{R} \iint_S y^2 \, dS = \frac{E}{R} I
 \end{aligned} \tag{3}$$

The curvature radius is defined as

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2} \tag{4}$$

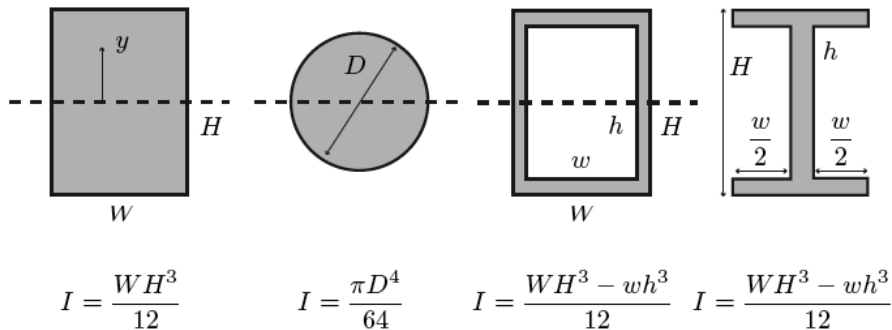
# Quadratic moment of inertia

Calculate  $I$  for a rectangular cross-section:



$$I = \frac{WH^3}{12}$$

# Quadratic moment of inertia



# Static shape of a beam

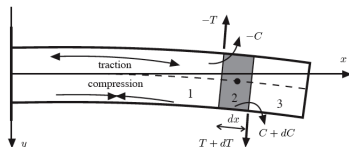
Combining equations (1)-(4) gives

$$\frac{d^4 y}{dx^4} = \frac{S \rho g}{IE} = a$$

Easily integrated as a polynomial function with 4 integration constants.

4 BC needed !

# Static shape of a beam



BC:

- $y = 0$  and  $\frac{dy}{dx} = 0$  at  $x = 0$
- $T = 0$  and  $C = 0$  at  $x = L$

# Static shape of a beam

$$y(x) = \frac{a}{2}x^2 \left( \frac{x^2}{12} - \frac{Lx}{3} + \frac{L^2}{2} \right)$$

The maximum deviation is at  $x = L$

$$y_{max} = \frac{aL^4}{8}$$

# Vibration of a beam

## Vibration equation

Out of static equilibrium, the motion equation is

$$\frac{d^2 C}{dx^2} = \rho S \left( g - \frac{\partial^2 y}{\partial t^2} \right)$$

Other geometric equations remain unchanged

$$\frac{1}{R} = \frac{d^2 y}{dx^2}, \quad C = \frac{IE}{R}$$

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho S}{IE} \left( \frac{\partial^2 y}{\partial t^2} - g \right) = 0$$

# Hypothesis

We assume that

$$g \ll \left| \frac{\partial^2 y}{\partial t^2} \right|$$

so that the vibration equation is

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{r^2 c^2} \frac{\partial^2 y}{\partial t^2} = 0$$

with

$$c = \sqrt{\frac{E}{\rho}}, \quad r = \sqrt{\frac{I}{S}}$$

# Gyration radius

The gyration radius  $r$  is defined by

$$I = \frac{\pi(2r)^4}{64}$$

or

$$r = \left(\frac{4I}{\pi}\right)^{1/4}$$

# Amplitude equation

We seek solution written as

$$y(x,t) = A(x)e^{i\omega t}$$

leading to an amplitude equation

$$\frac{d^4 A}{dx^4} - \frac{\omega^2}{c^2 r^2} A = 0$$

The amplitude  $A(x)$  is

$$A(x) = B_1 \cosh(\alpha x) + B_2 \sinh(\alpha x) + B_3 \cos(\alpha x) + B_4 \sin(\alpha x)$$

with

$$\alpha = \sqrt{\omega/cr}$$

the wave number.

## BC

The  $B_i$  are determined through the BC :

- $A = 0$  at  $x = 0$
- $dA/dx = 0$  at  $x = 0$
- $d^2A/dx^2 = 0$  at  $x = L$
- $d^3A/dx^3 = 0$  at  $x = L$

...

## BC

$$B_1 = -B_3 \quad B_2 = -B_4$$

$$\begin{pmatrix} \cosh(\alpha L) + \cos(\alpha L) & \sinh(\alpha L) + \sin(\alpha L) \\ \sinh(\alpha L) - \sin(\alpha L) & \cosh(\alpha L) + \cos(\alpha L) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0$$

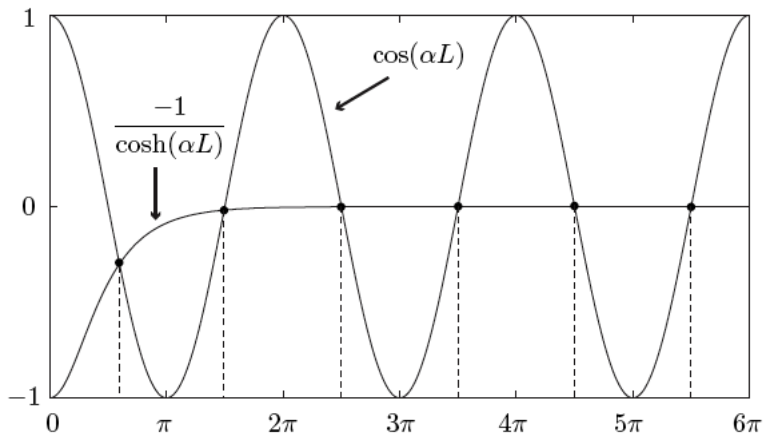
The equation for  $\alpha$  is

$$\cosh(\alpha L) \cos(\alpha L) + 1 = 0$$

or

$$\cos(\alpha L) = -\frac{1}{\cosh(\alpha L)}$$

## Graphical solution



# Vibration modes for the beam

Fundamental mode:

$$\alpha_0 \approx 1.2 \frac{\pi}{2L}, \quad \omega_0 = \frac{1.44\pi^2}{4L^2} \sqrt{\frac{IE}{\rho S}}$$

Other (higher) modes:

$$\alpha_{n \geq 1} = (2n + 1) \frac{\pi}{2L}, \quad \omega_{n \geq 1} = \frac{(2n + 1)^2 \pi^2}{4L^2} \sqrt{\frac{IE}{\rho S}}$$

# Back to the hypothesis

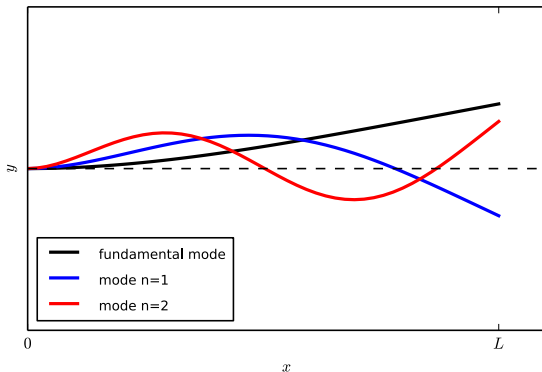
$$\left| \frac{\partial^2 y}{\partial t^2} \right| = \omega A^2 \approx y_m \omega^2$$

$$\frac{g}{\left| \frac{\partial^2 y}{\partial t^2} \right|} = \frac{g}{y_m \omega^2} = \frac{128}{(2n+1)^4 \pi^4}$$

$n$	0	1	2	3
$g/y_m \omega^2$	0.64	0.02	$2 \times 10^{-3}$	$5 \times 10^{-4}$

## Concluding remarks

- The shape of the beam in the fundamental mode ( $n = 0$ ) is very close to the shape of the static beam under gravity.
- The shape of the beam of mode  $n \geq 1$  has  $n$  nodes



# Waves & vibrations

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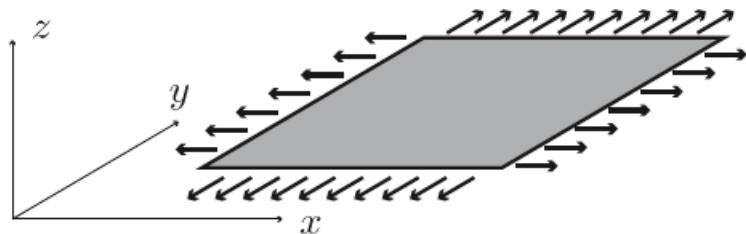
# Lecture 5 outline

- 1 Equation of vibration of membranes
- 2 Solving method
- 3 Vibration of rectangular membranes
- 4 Vibration of a square membrane
- 5 Circular membranes
- 6 Vibration of a plate

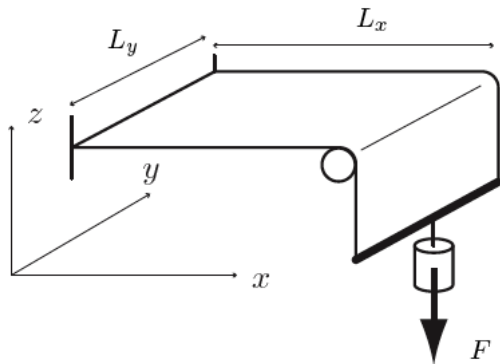
# Equation of vibration of membranes

# Membrane setup

- 2D system:  $L_x \approx L_y$ ,  $L_x \gg L_z$ ,  $L_y \gg L_z$
- without rigidity



# Tension of a membrane



$$F = \sigma L_y L_z = T_1 L_y$$

$T_1$  is a tension per unit length (in  $\text{N}\cdot\text{m}^{-1}$ )

# Evidence of the internal tension of a membrane



The force needed to close the fracture is

$$F = T_1 \times L_{fracture}$$

## Motion equation

Hypothesis:

- gravity force is negligible compared to tension force
- tension is uniform and isotropic

We derive the motion equation from the dynamic balance of a small element of the membrane:

$$dm \frac{d^2 \vec{r}}{dt^2} = \int_C \vec{T}_1 dC$$

along the z-axis:

$$\begin{aligned} dm \frac{d^2 z}{dt^2} &= \left( \oint_C \vec{T}_1 dC \right)_z \\ &= T_1 \oint \cos \alpha dC \\ &= T_1 \oint \left( \frac{dz}{dn} \right) dC \end{aligned}$$

## Motion equation

Using the Green's theorem:

$$\oint \left( \frac{dz}{dn} \right) dC = \iint_S \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) z dS = S \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)$$

we obtain

$$\left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) - \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} = 0$$

which is a 2D wave equation with a velocity

$$c = \sqrt{\frac{T_1}{\rho_S}}$$

with  $\rho_S = M/S$  the surface density of the membrane.

## About wave velocities

- compression waves (see lecture #3):

$$c_{comp} = \sqrt{\frac{E}{\rho}}$$

- transversal waves of a string:

$$c_{string} = \sqrt{\frac{T_0}{\rho L}}$$

- transversal waves of a membrane:

$$c_{membrane} = \sqrt{\frac{T_1}{\rho S}}$$

## About wave velocities

- compression waves (see lecture #3):

$$c_{comp} = \sqrt{\frac{E}{\rho}}$$

- transversal waves of a string (see lecture #3):  
with  $T_0 = \sigma L_y L_z$  and  $\rho_L = M/L_x = \rho L_y L_z$

$$c_{string} = \sqrt{\frac{\sigma}{\rho}}$$

- transversal waves of a membrane:  
with  $T_1 = \sigma L_z$  and  $\rho_S = M/(L_x L_y) = \rho L_z$

$$c_{membrane} = \sqrt{\frac{\sigma}{\rho}}$$

# Solving method

# What we want to know

From the vibration equation

$$\left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) - \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} = 0$$

we want to know:

- the natural frequencies of the membrane
- the shape of the deformed membrane under vibration
- nodes and anti-nodes

## Method of the solution

We look for the  $z(x,y,t)$  solution of

$$\left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) - \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} = 0$$

Separating space and time variables gives

$$z(x,y,t) = A(x,y) \cos(\omega t)$$

and the motion equation writes as an **amplitude equation**

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) A(x,y) + \frac{\omega^2}{c^2} A(x,y) = 0$$

## Method of the solution

This amplitude equation ( $\Delta_2 A + k^2 A = 0$ ) can be (*a priori*) solved explicitly with the knowledge of

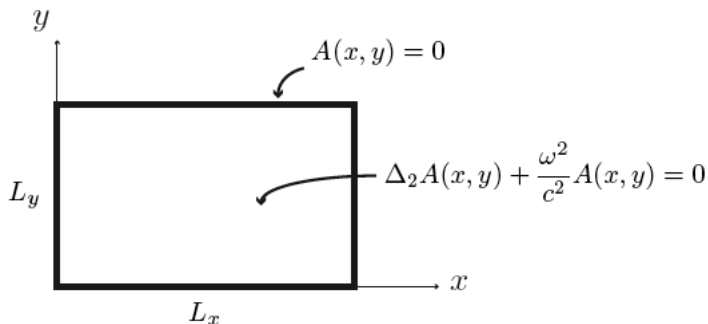
- the boundary conditions (BC)
- the initial conditions (IC)

There is no analytical solution of the membrane vibration for an arbitrary shape! We know solutions for simple shapes:

- a rectangular membrane (or square)
- a circular membrane

# Vibration of rectangular membranes

# A rectangular frame



amplitude solution (extension from string amplitudes solution):

$$A(x, y) = A_{mn} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right)$$

# Vibration freq. of a rectangular membrane

Combining

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) A(x,y) + \frac{\omega^2}{c^2} A(x,y) = 0$$

with

$$A(x,y) = A_{mn} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right)$$

gives

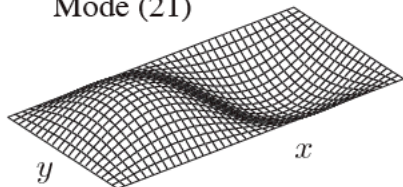
$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2} \sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}$$

The fundamental frequency is for  $m = 1$  and  $n = 1$  and is

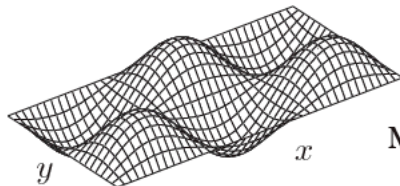
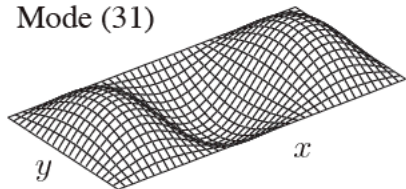
$$f_{11} = \frac{c}{2} \sqrt{\left(\frac{1}{L_x}\right)^2 + \left(\frac{1}{L_y}\right)^2}$$

## Viewing a few modes

Mode (21)



Mode (31)



Mode (32)

## Complete solution

The complete solution of the linear vibration equation for a rectangular membrane is

$$z(x,y,t) = \sum_{(m,n)} A_{mn} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right) \cos(\omega_{mn}t + \varphi_{mn})$$

with

$$\omega_{mn} = c\pi \sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}, \quad c = \sqrt{\frac{\sigma}{\rho}}$$

and  $A_{mn}$ ,  $\varphi_{mn}$  determined through IC.

# Nodes and anti-nodes

- A nodal line (or curve) is the set of points where

$$A(x,y) = 0, \quad 0 < x < L_x, \quad 0 < y < L_y$$

- A mode  $(m,n)$  has  $m + n - 2$  nodal lines
- anti-nodes are points where the amplitude is extremal.

# Vibration of a square membrane

# Square membrane

Taking  $L_x = L_y$  describes a square membrane, with natural frequencies

$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2L_x} \sqrt{m^2 + n^2}$$

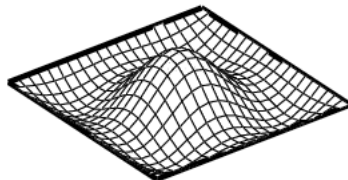
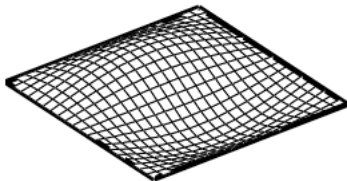
important remark:

$$f_{mn} = f_{nm}$$

for example  $f_{21} = f_{12}$ .

DEF: degenerated frequency: when two (or more) modes have the same frequency.

# Square membrane



# Circular membranes

# Circular membranes

The amplitude equation

$$\Delta_2 A + k^2 A = 0$$

written in polar  $(r, \theta)$  coordinates is

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} + \frac{\omega^2}{c^2} = 0$$

with BC

$$A(r = R, \theta) = 0$$

Separating  $r$  and  $\theta$ , the amplitude is

$$A(r, \theta) = \sum_n A_n(r) \cos(n\theta + \varphi_n)$$

# Circular membranes

Step to solution: variable change

$$x = \frac{\omega r}{c}$$

the amplitude equation writes now as a **Bessel equation**

$$\frac{d^2 A_n}{dx^2} + \frac{1}{x} \frac{dA_n}{dx} + \left(1 - \frac{n^2}{x^2}\right) A_n = 0$$

# Bessel equations and Bessel functions

$$A_n(x) = \alpha_n J_n(x) + \beta_n Y_n(x)$$

with

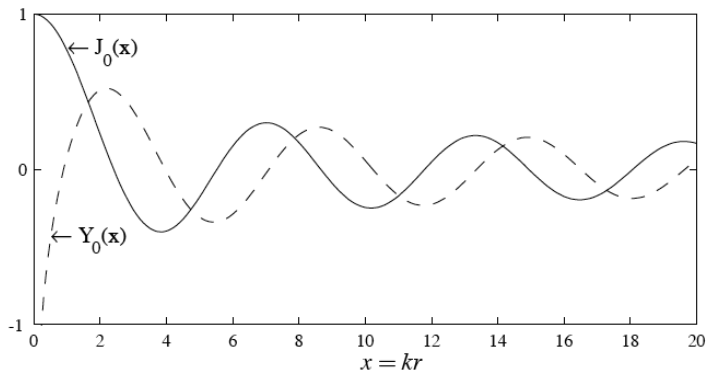
$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-x^2/4)^m}{m! \Gamma(n+m+1)}$$

$$Y_n(x) = \frac{J_n(x) \cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$$

$$\Gamma(k) = \int_0^{\infty} e^{-t} t^{k-1} dt$$

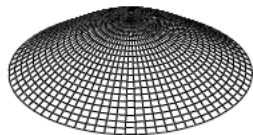
# Bessel functions

for  $n = 0$ , the  $J_0$  and  $Y_0$  Bessel functions are

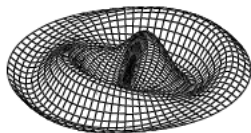


## Circular vibration modes

$$z(r, \theta, t) = \cos(\omega t) \sum_n \alpha_n J_n(kr) \cos(n\theta + \varphi_n)$$



mode (01)



mode (13)

# Vibration of a plate

# Vibration of a rigid membrane

Combining membrane + beam gives

$$\frac{Eh^2}{12\rho(1-\nu^2)}\Delta^2 z + \frac{\partial^2 z}{\partial t^2} = 0.$$

where  $h$  is the thickness of the plate