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Waves & vibrations

Maxime Nicolas

► **To cite this version:**

| Maxime Nicolas. Waves & vibrations. Engineering school. France. 2016. cel-01440543

HAL Id: cel-01440543

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Submitted on 25 Jan 2017

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Waves & vibrations

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Département génie civil

november 2016 – january 2017

Course 1 outline

- 1 Preamble
 - Course schedule
 - Online
 - Course outline
- 2 Introduction
- 3 The harmonic oscillator

PREAMBLE

Course syllabus

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Bâtiment Fermi, bureau 212

Schedule:

- 5 lectures
- 5 workshops for civil eng. students
- 3 workshops for mech. eng. students

Final exam: January 25th, 2017

Online

This course is available on ENT/AmeTice :

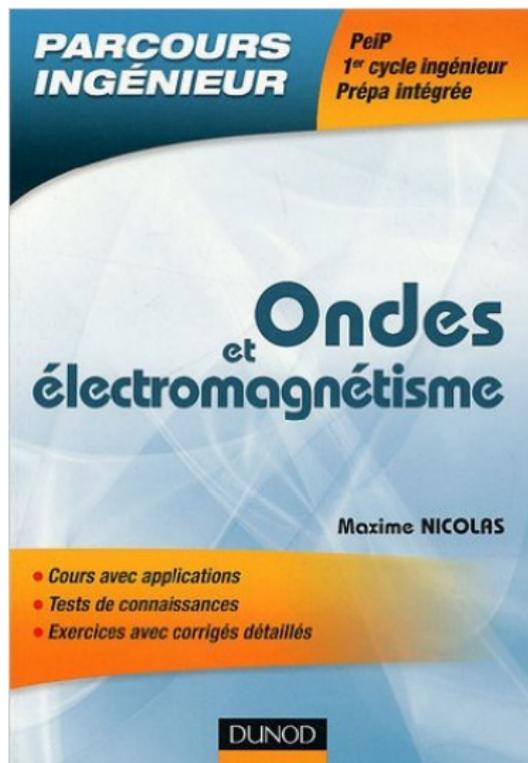
Sciences & technologies ▶ Polytech ▶ Cours communs ▶
[16] - S5 - JGC52D + JME51C - Ondes et vibrations (Maxime Nicolas)

with

- lecture slides
- workshops texts
- past exams

A book

This course is included in a book (paper and pdf versions available):



Course outline

- 1 Introduction & harmonic oscillator
- 2 The wave equation and its solutions
- 3 1D transverse and longitudinal waves
- 4 2D waves: vibration of plates
- 5 beam vibration

Introduction

Where to find waves and vibrations?

Short answer: everywhere :

- waves in fluids → acoustics → sound
- waves in fluids → ripples, waves and tsunamis
- waves in solids → compression waves
- vibrations of structures → planes, cars,

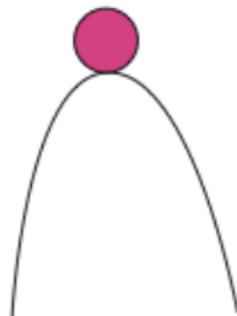
and many more

- electromagnetic waves → light, radio, X-rays, γ -rays
- chemical oscillators
- population dynamics
- health issues due to vibrations

A few definitions

- **wave**: propagation of an oscillation or a vibration
- **vibration**: motion around an equilibrium state
- **oscillation**: motion of a body around an equilibrium point

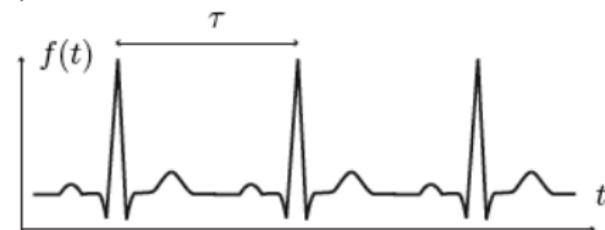
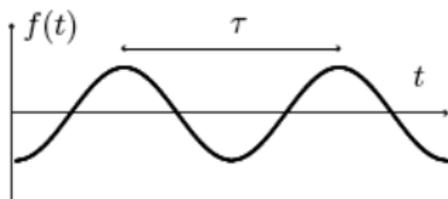
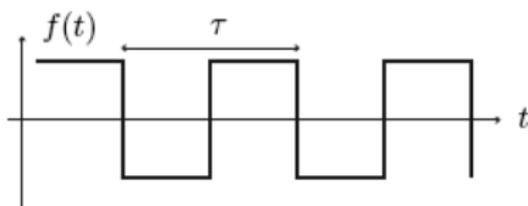
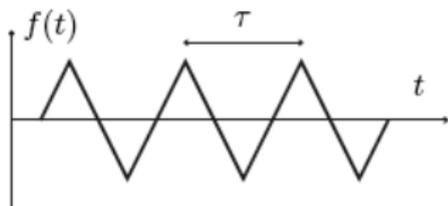
Equilibrium



t -periodical functions

$$F(t + \tau) = F(t), \quad \forall t$$

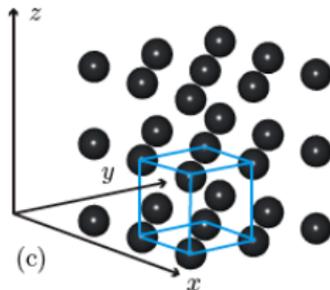
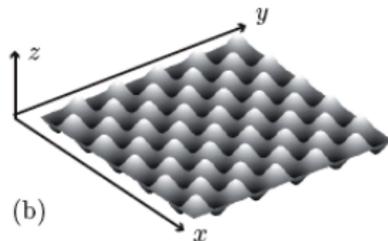
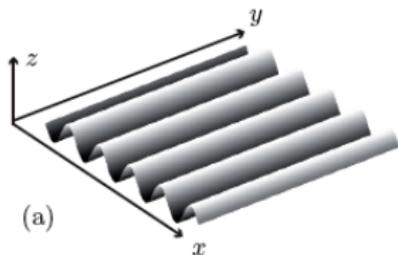
$$f = \tau^{-1}, \quad \omega = 2\pi f = 2\pi/\tau$$



x-periodical functions

$$F(x + \lambda) = F(x), \quad \forall x$$

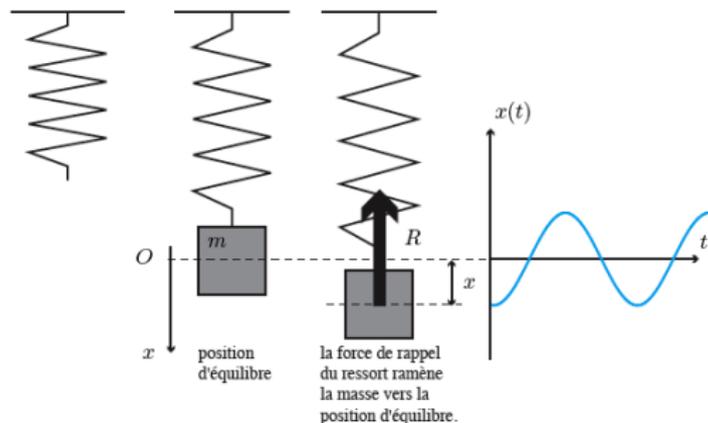
$$k = 2\pi/\lambda$$



The harmonic oscillator

Example 1

The mass-spring system:



$$m \frac{d^2x}{dt^2} + kx = 0$$

Examples comparison

spring-mass

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

$$\omega_0 = \sqrt{k/m}$$

linear equation for x

pendulum

$$mL \frac{d^2 \theta}{dt^2} + mg \sin \theta = 0$$

$$\frac{d^2 \theta}{dt^2} + \omega_0^2 \sin \theta = 0$$

$$\omega_0 = \sqrt{g/L}$$

non linear equation for θ

Linearization

For any (continuous and derivable) function F near x_0 :

$$F(x)_{x \approx x_0} = F(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{d^n f}{dx^n} \right) (x_0) (x - x_0)^n$$

for $F(\theta) = \sin \theta$, $x \approx 0$:

$$F(\theta) = \theta - \frac{\theta^3}{6} + \dots$$

Examples comparison

spring-mass

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

$$\omega_0 = \sqrt{k/m}$$

linear equation for x

pendulum

$$mL \frac{d^2 \theta}{dt^2} + mg \sin \theta = 0$$

$$\frac{d^2 \theta}{dt^2} + \omega_0^2 \theta = 0$$

$$\omega_0 = \sqrt{g/L}$$

linear equation for $\theta \ll 1$

The harmonic oscillator equation

The equation for a physical quantity $A(t)$ (x or θ) is

$$\frac{d^2A}{dt^2} + \omega_0^2 A = 0$$

this is a 2nd order differential equation.

on the blackboard

The harmonic oscillator equation

$$\frac{d^2 A}{dt^2} + \omega_0^2 A = 0$$

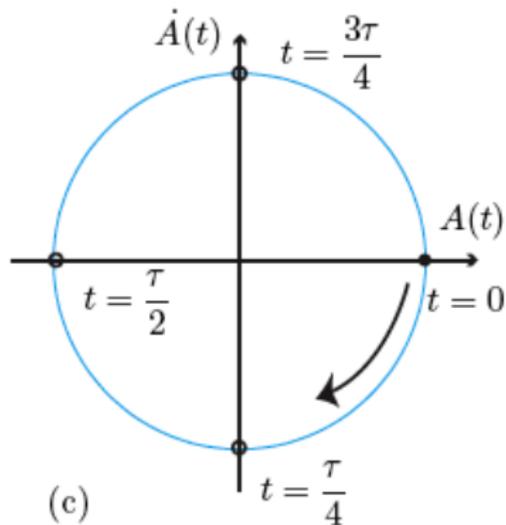
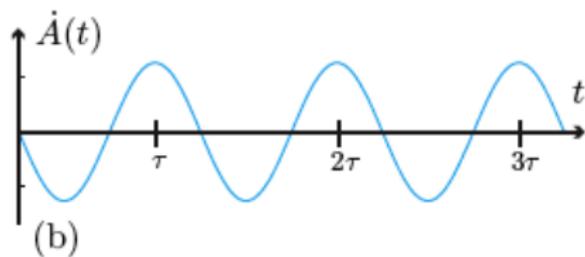
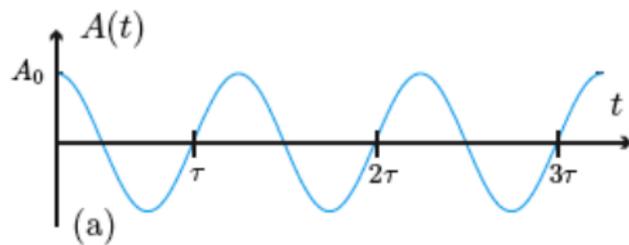
General solution:

$$A(t) = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} = B_1 \cos(\omega_0 t) + B_2 \sin(\omega_0 t)$$

With the initial conditions (A_0, \dot{A}_0)

$$A(t) = A_0 \cos(\omega_0 t) + \frac{\dot{A}_0}{\omega_0} \sin(\omega_0 t)$$

View of the solution



Energy

This equation describes a conservative system (no loss of energy):

$$\frac{d^2 A}{dt^2} + \omega_0^2 A = 0$$

Back to the mass-spring example:

$$m \frac{d^2 x}{dt^2} + kx = 0$$

...

Harmonic oscillator with damping

Introducing a fluid damping force (prop. to velocity):

$$\frac{d^2A}{dt^2} + \gamma \frac{dA}{dt} + \omega_0^2 A = 0$$

Solution with damping

Testing function

$$A = e^{rt}$$

Characteristic equation:

$$r^2 + \gamma r + \omega_0^2 = 0$$

with 2 solutions

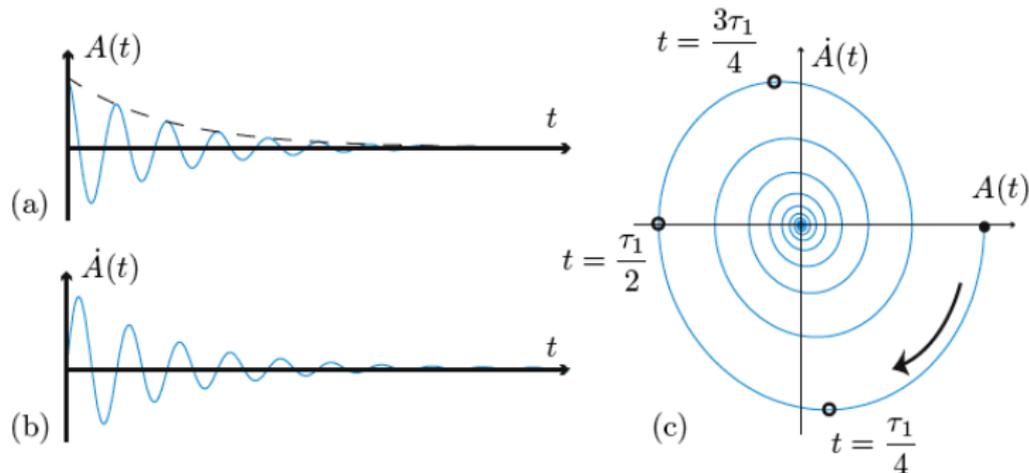
$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{2} \sqrt{\gamma^2 - 4\omega_0^2} = -\frac{\gamma}{2} \pm \alpha$$

and

$$A(t) = e^{-\gamma t/2} (A_1 e^{\alpha t} + A_2 e^{-\alpha t})$$

Weak damping solution

$$\gamma^2 - 4\omega_0^2 < 0, \quad \alpha = i\omega_1, \quad \omega_1 = \frac{1}{2}\sqrt{4\omega_0^2 - \gamma^2}$$



$$A(t) = e^{-\gamma t/2} \left[A_0 \cos(\omega_1 t) + \frac{1}{\omega_1} \left(\frac{\gamma A_0}{2} + \dot{A}_0 \right) \sin(\omega_1 t) \right]$$

Energy loss with damping

$$\frac{1}{2}\dot{A}^2 + \frac{1}{2}\omega_0^2 A^2 = -\gamma \int \dot{A}^2 dt$$

$$E(t) = -\gamma \int \dot{A}^2 dt$$

$$\frac{dE}{dt} = -\gamma \dot{A}^2$$

Oscillator with energy input

$$\frac{d^2 A}{dt^2} + \gamma \frac{dA}{dt} + \omega_0^2 A = A_F \cos(\omega t)$$

with A_F the forcing amplitude, and ω the forcing angular frequency.

proposed long time solution:

$$A(t) = A_1 \cos(\omega t + \varphi)$$

Now find A_1 and $\varphi \dots$

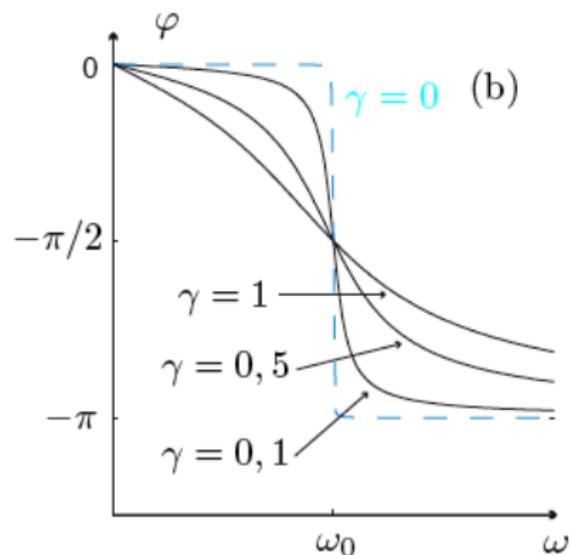
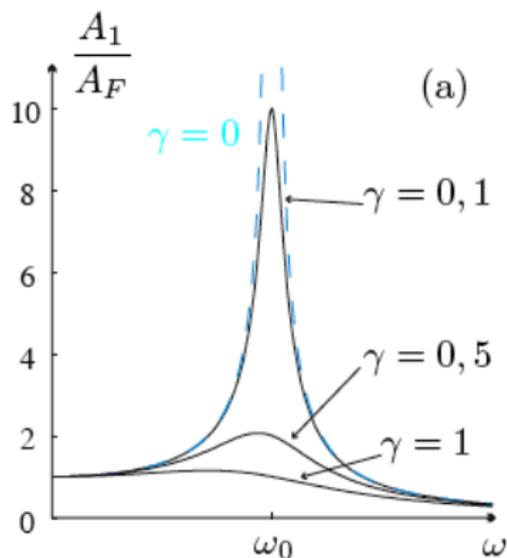
Solution

$$A(t) = A_1 \cos(\omega t + \varphi)$$

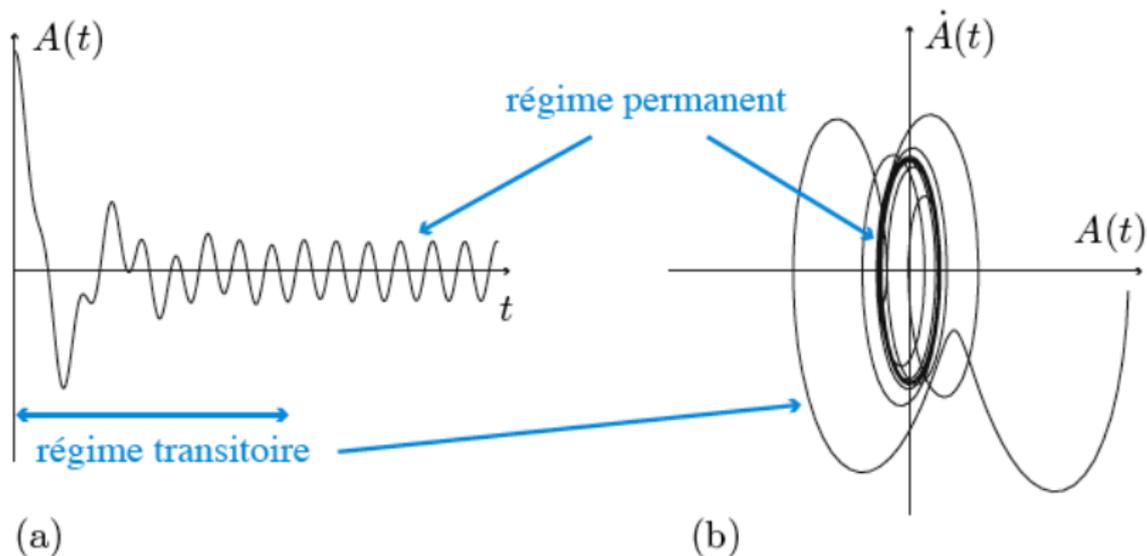
$$\frac{A_1}{A_F} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}$$

$$\varphi = \arctan\left(\frac{-\omega\gamma}{\omega_0^2 - \omega^2}\right)$$

View



View



Waves & vibrations

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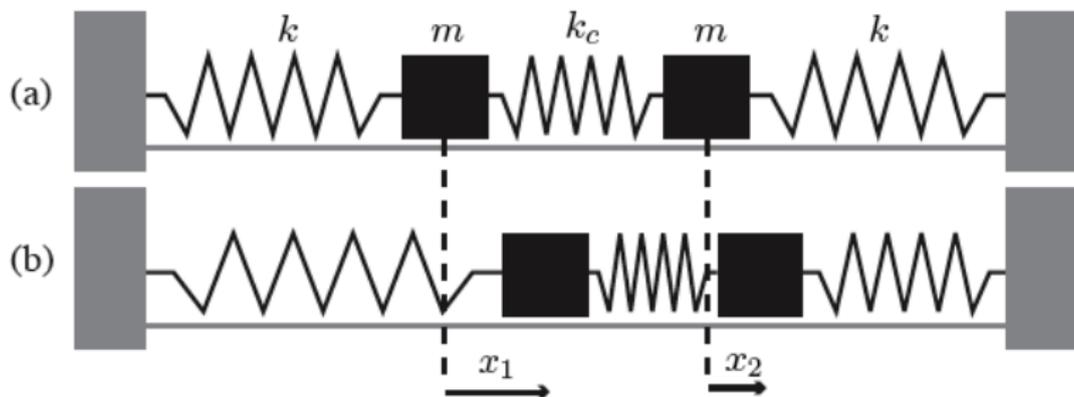
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Course 2 outline

- 1 Coupled oscillators
- 2 1D infinite chain of oscillators
- 3 The simple wave equation

Coupled oscillators

Coupled oscillators



2 identical oscillators + coupling spring

Coupled equations

Reminder: x_i variables are perturbations out of equilibrium.

$$m\ddot{x}_1(t) = -kx_1 - k_c(x_1 - x_2)$$

$$m\ddot{x}_2(t) = -kx_2 - k_c(x_2 - x_1)$$

or

$$\ddot{x}_1(t) = -\omega_0^2 x_1 - \omega_c^2 (x_1 - x_2)$$

$$\ddot{x}_2(t) = -\omega_0^2 x_2 - \omega_c^2 (x_2 - x_1)$$

with oscillating solutions:

$$x_1 = X_1 \exp(i\omega t), \quad x_2 = X_2 \exp(i\omega t)$$

Coupled equations as a linear problem

$$\begin{pmatrix} \omega_0^2 - \omega^2 + \omega_c^2 & -\omega_c^2 \\ -\omega_c^2 & \omega_0^2 - \omega^2 + \omega_c^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0$$

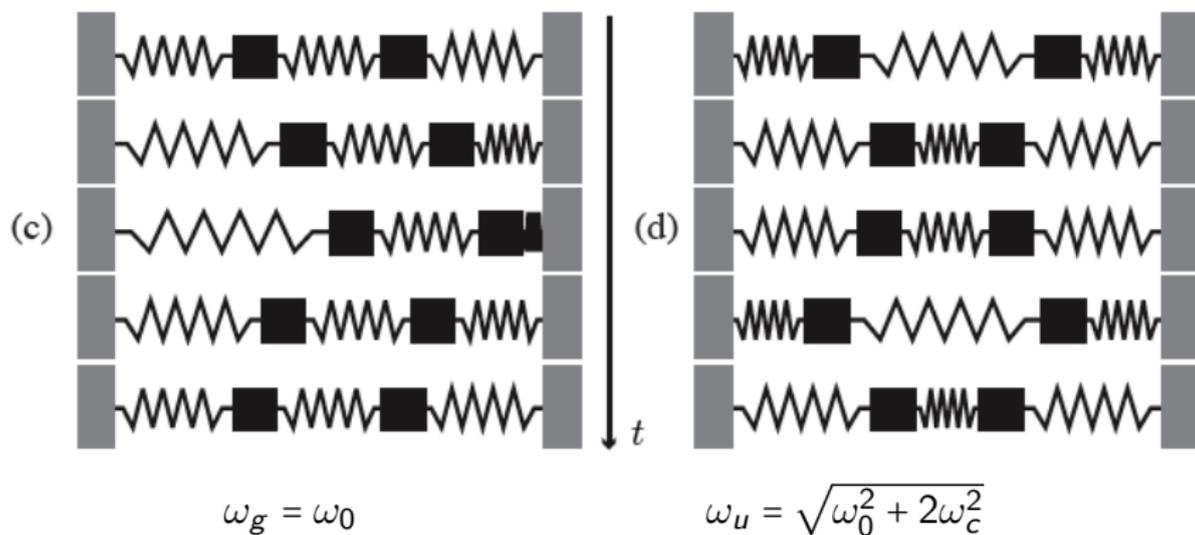
- trivial solution: $X_1 = X_2 = 0$, static equilibrium
- non trivial solution $X_1 \neq 0$, $X_2 \neq 0$

$$\det(M) = 0$$

Two natural frequencies:

$$\begin{aligned} \omega_g &= \omega_0 \\ \omega_u &= \sqrt{\omega_0^2 + 2\omega_c^2} \end{aligned}$$

View of the solutions



Complete solution

The complete solution of the coupled oscillators problem is

$$x_1(t) = A_g \cos(\omega_g t + \varphi_g) + A_u \cos(\omega_u t + \varphi_u)$$

$$x_2(t) = A_g \cos(\omega_g t + \varphi_g) - A_u \cos(\omega_u t + \varphi_u)$$

The 4 constants A_g , A_u , φ_g and φ_u are to be determined by 4 initial conditions:

- $x_1(t=0)$ and $x_2(t=0)$
- $\dot{x}_1(t=0)$ and $\dot{x}_2(t=0)$

N coupled oscillators

For a set of N coupled oscillators, we can write a set of N equations

$$\ddot{x}_i = -\omega_0^2 x_i - \sum_{j \neq i}^N \omega_c^2 (x_i - x_j)$$

with harmonic solutions

$$x_i = X_i \exp(i\omega t)$$

The problem writes

$$M \vec{X} = 0$$

with M a $N \times N$ matrix and $\vec{X} = (X_1, \dots, X_N)$ an amplitude vector.

N coupled oscillators

- trivial solution: $\vec{X} = 0$, static equilibrium
- non trivial solution $X_i \neq 0$

$$\det(M) = 0$$

which leads to N natural frequencies $\omega_1, \dots, \omega_N$ and the complete solution is a linear combination of these N individual solutions:

$$x_i = \sum_j A_j \cos(\omega_j t + \varphi_j)$$

the $2N$ constants A_j and φ_j are to be determined with $2N$ initial conditions.

Partial conclusion

- $N = 2$ easy
- $N = 3$ less easy but possible
- $N \geq 4$ computer help is needed

1D infinite chain of oscillators

$N \rightarrow \infty$

For a very large number of oscillators, the previous method is too expensive!



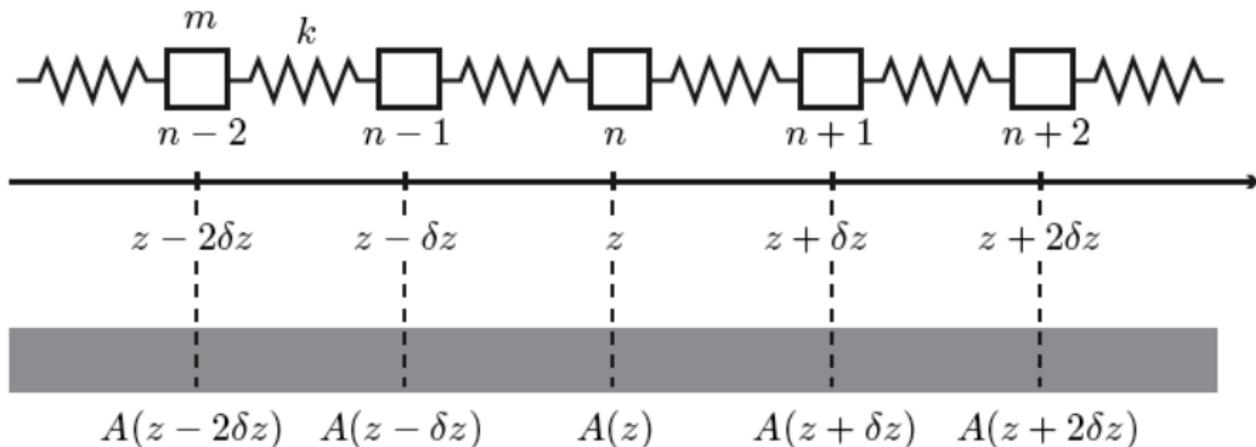
N equations

$$m \frac{d^2 A_n}{dt^2} = -k(A_n - A_{n+1}) - k(A_n - A_{n-1})$$

or

$$\begin{aligned} \frac{d^2 A_n}{dt^2} &= -\omega_0^2(A_n - A_{n+1}) - \omega_0^2(A_n - A_{n-1}) \\ &= \omega_0^2(A_{n+1} - 2A_n + A_{n-1}) \end{aligned}$$

Continuous description



$$A_n(t) = A(z, t)$$

Continuous description

Two neighboring oscillator have a very close behavior

$$A_{n+1} = A(z + \delta z) = A(z) + \delta z \left(\frac{\partial A}{\partial z} \right) + \frac{(\delta z)^2}{2} \left(\frac{\partial^2 A}{\partial z^2} \right) + \dots$$

$$A_{n-1} = A(z - \delta z) = A(z) - \delta z \left(\frac{\partial A}{\partial z} \right) + \frac{(\delta z)^2}{2} \left(\frac{\partial^2 A}{\partial z^2} \right) + \dots$$

summing these 2 equations gives

$$A_{n+1} + A_{n-1} = 2A(z) + (\delta z)^2 \left(\frac{\partial^2 A}{\partial z^2} \right)$$

then

$$A_{n+1} - 2A_n + A_{n-1} = (\delta z)^2 \left(\frac{\partial^2 A}{\partial z^2} \right)$$

Motion equation

Back to the motion equation

$$\frac{d^2 A_n}{dt^2} = \omega_0^2 (A_{n+1} - 2A_n + A_{n-1})$$

with the continuous approach

$$\frac{d^2 A_n}{dt^2} \longrightarrow \frac{\partial^2 A}{\partial t^2}$$

$$A_{n+1} - 2A_n + A_{n-1} = (\delta z)^2 \left(\frac{\partial^2 A}{\partial z^2} \right)$$

and the motion equation is now

$$\frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0, \quad c = \omega_0 \delta z$$

The simple wave equation

Wave equation

It is a 1D wave equation

$$\frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0$$

with a constant velocity c .

If a 3D propagation is needed, the wave equation writes

$$\Delta A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0$$

with the laplacian operator

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The wave equation general solution

The wave eq. can be written as (remember $a^2 - b^2 = (a - b)(a + b)$):

$$\left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t}\right) \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) A = 0$$

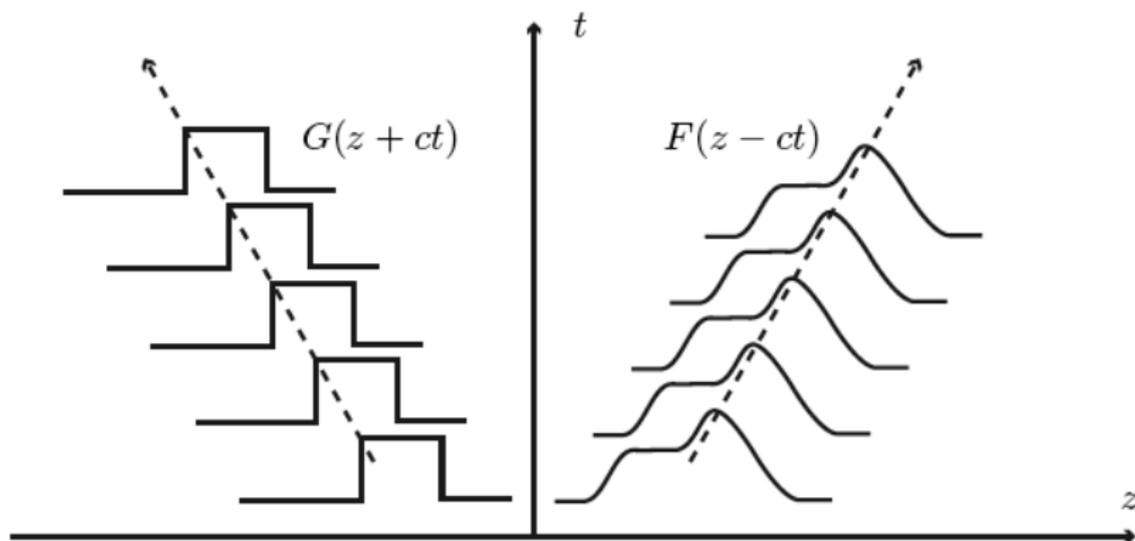
The solution of the wave eq. is the sum of two functions:

$$A(z, t) = F(z - ct) + G(z + ct)$$

F and G are arbitrary functions.

Proof on the board ...

Example



Exercise

Let F a function such that

- $F = 1$ if $-1 < z < 1$
- $F = 0$ elsewhere

With $c = 3$, draw F for

- $t=0$
- $t=1$
- $t=2$

The complete wave solution

The complete solution needs:

- the initial condition $A(z,0)$
- the boundary conditions

but since the WE is linear, its solutions can be written as a sum of elementary solutions

Waves & vibrations

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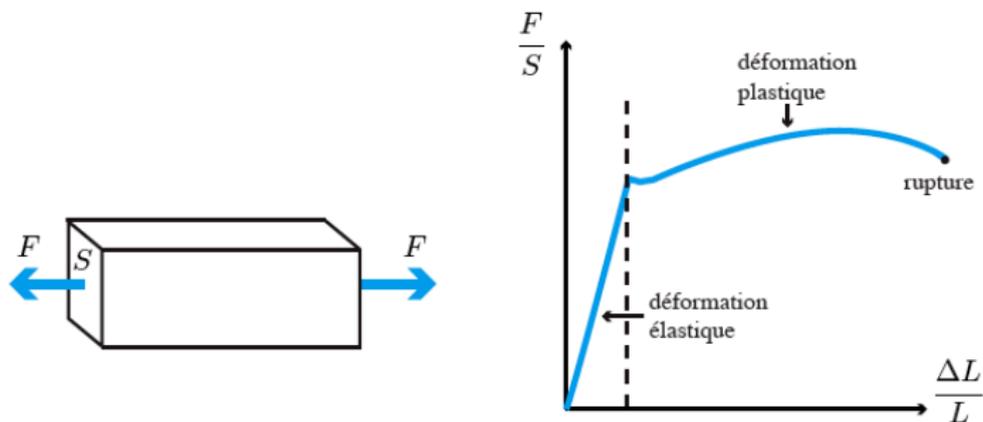
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Lecture 3 outline

- 1 Compression waves
- 2 Vibration of a tensioned string
 - Static equilibrium of a string
 - Vibration of a string

Compression waves

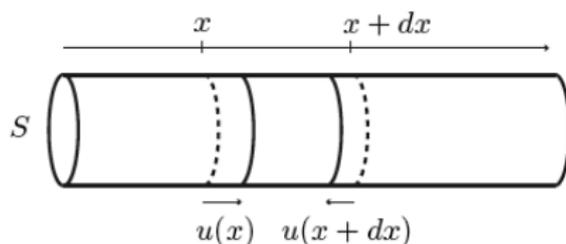
Linear elasticity



Hooke's law:

$$\frac{\Delta L}{L} = \frac{1}{E} \frac{F}{S} = \frac{\sigma}{E}$$

1D compression waves



For $u(x)$ the displacement of a section located at x :

$$\frac{u(x) - u(x + dx)}{dx} = \frac{F}{SE}$$

hence:

$$F(x) = -SE \frac{\partial u}{\partial x}$$

1D compression waves

Newton's equation for the elementary mass dm

$$dm \frac{\partial^2 u}{\partial t^2} = F(x) - F(x + dx)$$

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = ES \left[\left(\frac{\partial u}{\partial x} \right)_{x+dx} - \left(\frac{\partial u}{\partial x} \right)_x \right]$$

which writes as a wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad c = \sqrt{\frac{E}{\rho}}$$

Compression waves velocities

Examples and order of magnitude of compression velocity:

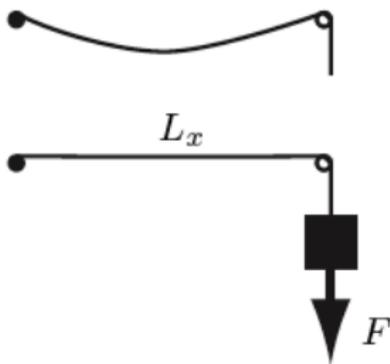
Matériau	ρ (kg.m ⁻³)	E (GPa)	c (m/s)
acier	7 900	200	5 000
cuivre	8 920	124	3 700
béton	2 500	2,7	1 000
plexiglass	1 200	2,4	1 400
verre	2 500	70	5 300

Vibration of a tensioned string

Static equilibrium of a string

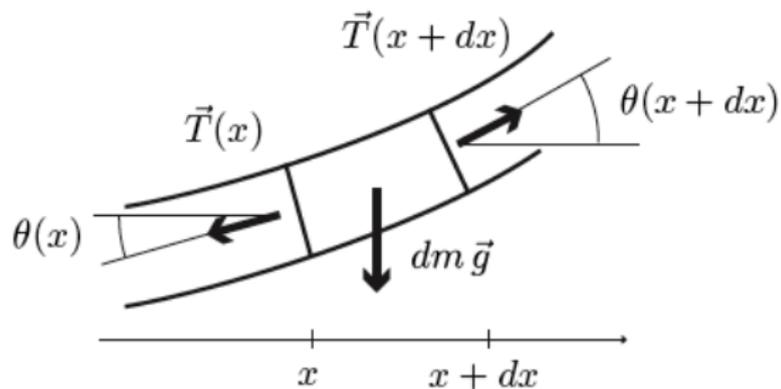
Modeling

- 1D system: $L_x \gg L_y$, $L_x \gg L_z$, $L_y \approx L_z$
- without rigidity (for the moment, see lecture # 5)
- tension force:



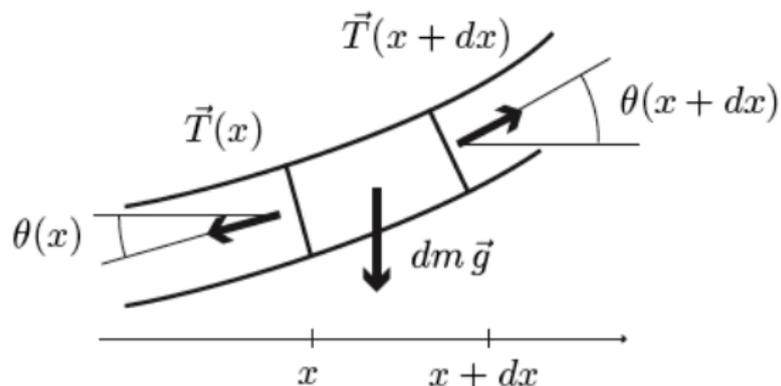
$$F = T_0 = \sigma L_y L_z$$

Shape of a string at equilibrium



$$dm \vec{g} + \vec{T}(x) + \vec{T}(x + dx) = 0$$

Shape of a string at equilibrium



Projection on the x - and y -axis

$$\begin{aligned}
 -T(x) \cos \theta(x) + T(x+dx) \cos \theta(x+dx) &= 0 \\
 -dm g - T(x) \sin \theta(x) + T(x+dx) \sin \theta(x+dx) &= 0
 \end{aligned}$$

Shape of a string at equilibrium

The x-axis equation means

$$T(x) \cos \theta(x) = T_0 = \text{constant}$$

Combining with the y-axis eq. leads

$$-dm g + T_0 [\tan \theta(x + dx) - \tan \theta(x)] = 0$$

or

$$-\frac{mg}{LT_0} + \frac{d}{dx} \tan \theta = -\frac{mg}{LT_0} + \frac{d\theta}{dx} \frac{d \tan \theta}{d\theta} = 0$$

Writing

$$\tan \theta = \frac{dy}{dx}$$

gives

$$\frac{d^2 y}{dx^2} = \left(\frac{mg}{LT_0} \right) \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

Shape of a string at equilibrium

Using a variable $u = dy/dx$, one can finally find

$$y(x) = L_c \left[\cosh\left(\frac{x}{L_c}\right) - \cosh\left(\frac{d}{2L_c}\right) \right], \quad L_c = \frac{LT_0}{mg}$$

the shape of the string of length L , mass m , attached between two fixed points with a d distance.

Maximum bending at the center of the string:

$$y_m = L_c \left[\cosh\left(\frac{d}{2L_c}\right) - 1 \right]$$

Tension of a string at equilibrium

The tensile force is

$$T(x) = \frac{T_0}{\cos \theta} = T_0 \cosh \left(\frac{x}{L_c} \right)$$

and the tensile force variation along the rope is

$$\frac{\Delta T}{T_0} = \frac{y_m}{L} \frac{mg}{T_0}$$

This variation is thus negligible when the string is under tension ($y_m \ll L$)

Vibration of a string

Motion equation

The local motion equation is

$$dm \frac{\partial^2 y}{\partial t^2} = -T(x) \sin \theta(x) + T(x + dx) \sin \theta(x + dx) - dm g$$

For a string under strong tension, the weight is negligible and the tension is constant:

$$\frac{m}{L} dx \frac{\partial^2 y}{\partial t^2} = T_0 [\sin \theta(x + dx) - \sin \theta(x)]$$

Assuming only a weak deviation from the equilibrium ($y = 0$ and $dy/dx = 0$)

$$\sin \theta \approx \tan \theta \approx \frac{\partial y}{\partial x}$$

we obtain

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = 0, \quad c = \sqrt{\frac{T_0}{\rho_L}}, \quad \rho_L = \frac{m}{L}$$

Boundary conditions

The string is attached in two points:

- $y(x = 0) = 0$
- $y(x = L) = 0$

Meaning that the wave propagation is **confined** between 0 and L . This gives a **steady wave**

$$y(x, t) = A(x) \cos(\omega t + \varphi)$$

introducing an amplitude function $A(x)$.

Amplitude equation

$$\frac{d^2 A}{dx^2} + \frac{\omega^2}{c^2} A = 0$$

Its general solution is

$$A = A_1 \cos(kx) + A_2 \sin(kx)$$

and the constants A_1 and A_2 are to be determined with BC

$$\longrightarrow A_1 = 0, \quad A_2 \neq 0 \quad \text{and} \quad \sin(kL) = 0$$

Vibration modes

The vibration is thus possible for a discrete set of k and ω values:

$$k = n \frac{\pi}{L}, \quad \text{and} \quad \omega = n \frac{\pi}{L} c$$

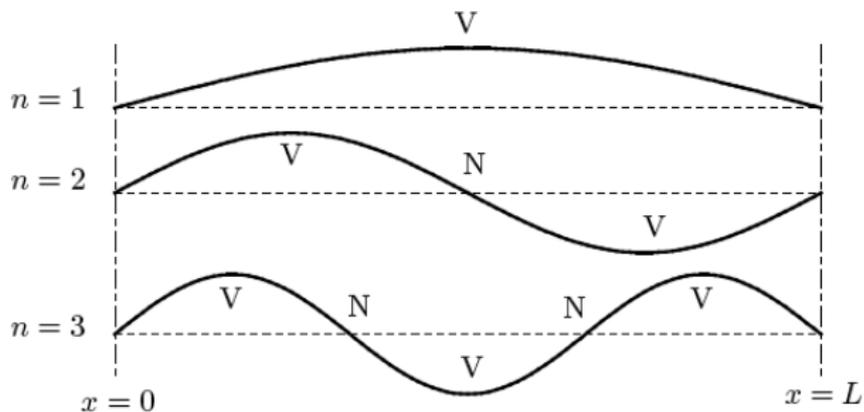
and finally the vibration may be represented as

$$y(x,t) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{nc\pi}{L} t + \varphi_n\right)$$

with an integer $n = 1, 2, 3, \dots$

The C_n and φ_n constants are to be determined by the initial conditions (IC).

Vibration modes



A few definitions

- A nodal point (node):

$$A = 0, 0 < x < L, \forall t$$

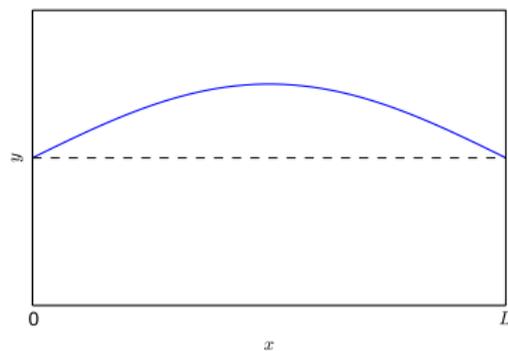
- antinodal points (antinodes):

$$\frac{dA}{dx} = 0, 0 < x < L, \forall t$$

A few properties

- The mode with the lowest frequency is the **fundamental mode**
- mode n has n antinodes and $n - 1$ nodes
- for a 1D system, nodes are points (0D objects)
- for a ND system, nodes are objects of $(N-1)$ D dimension

Movies



View movies:

- mode 1
- mode 3
- modes 1+3+5

Waves & vibrations

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Lecture 4 outline

- 1 Static of a beam
- 2 Vibration of a beam

Static of a beam

Beam characteristics

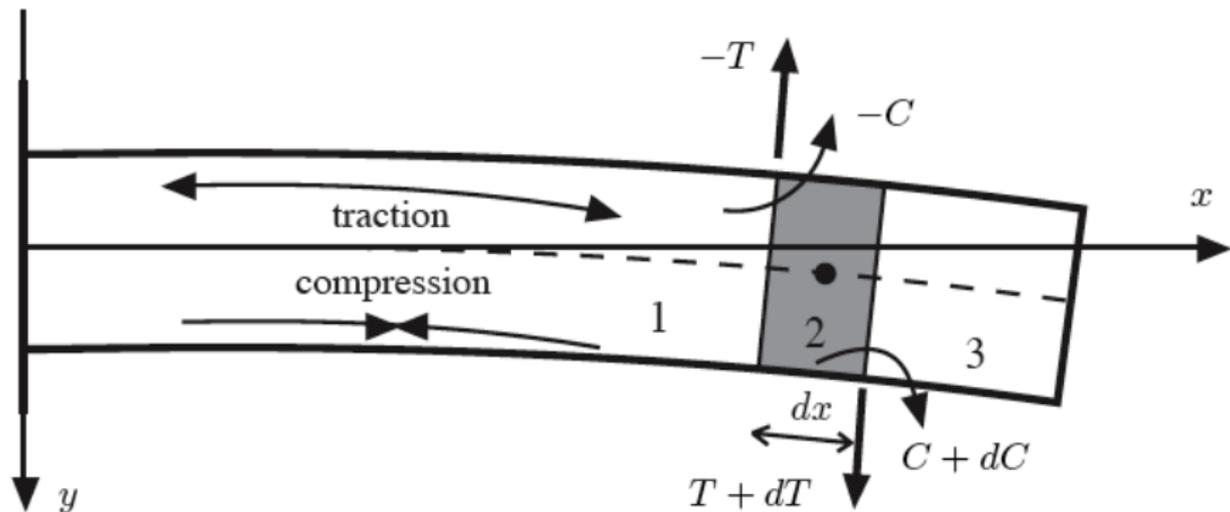
- geometry:

$$L_x > L_y \approx L_z$$

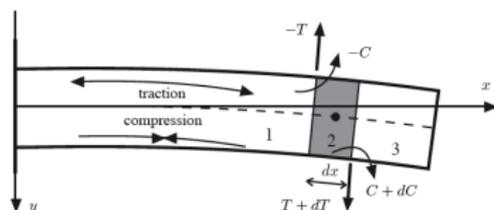
- cross-section S
- quadratic moment of inertia I
- material:

$$E, \rho$$

Beam setup



Force balance

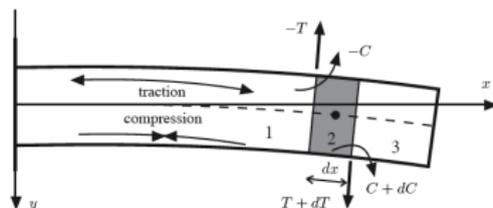


$$\vec{F}_{1 \rightarrow 2} + \vec{F}_2 + \vec{F}_{3 \rightarrow 2} = 0$$

$$-T + S dx \rho g + (T + dT) = 0$$

$$\frac{dT}{dx} = -S \rho g \quad (1)$$

Torque balance



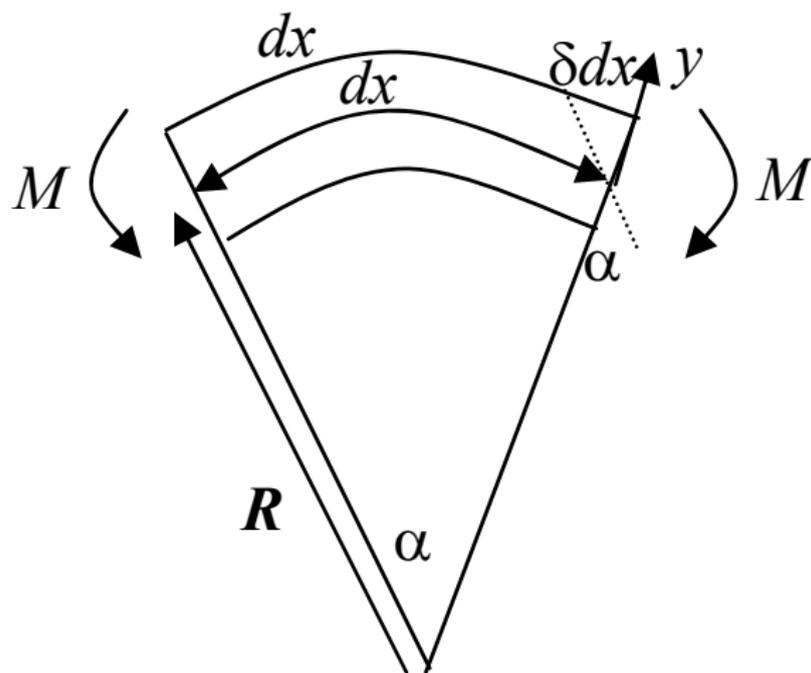
$$-C + (C + dC) + T \frac{dx}{2} + (T + dT) \frac{dx}{2} = 0$$

$$\frac{dC}{dx} = -T$$

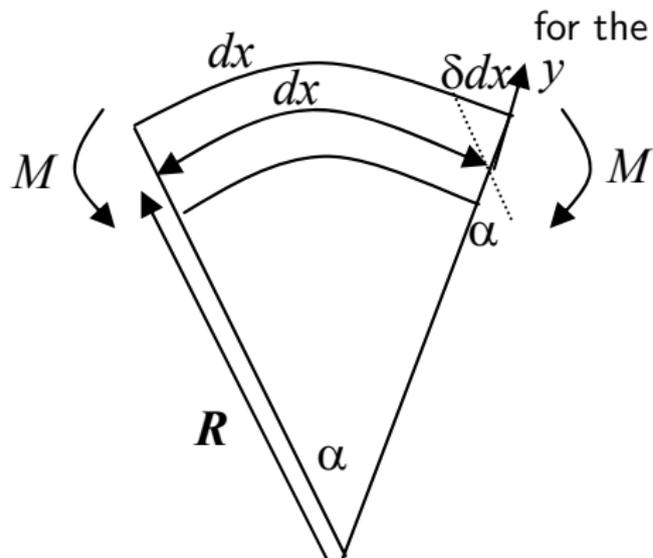
using (1)

$$\frac{d^2 C}{dx^2} = S \rho g \quad (2)$$

Elasticity of the beam



Elasticity of the beam



$$\frac{\delta dx}{dx} = \frac{1}{E} \frac{dF}{dS}$$

$$\frac{\delta dx}{y} = -\frac{dx}{R}$$

$$\frac{\delta dx}{dx} = -\frac{y}{R}$$

$$dF = -\frac{y}{R} E dS$$

Torque

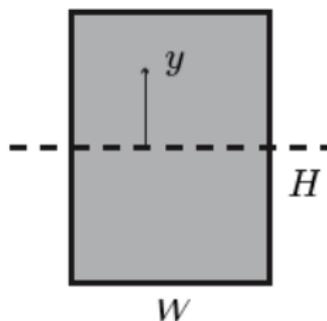
$$\begin{aligned}
 -C &= \iint_S y \, dF = -\frac{E}{R} \iint_S y^2 \, dS \\
 C &= \frac{E}{R} \iint_S y^2 \, dS = \frac{E}{R} I
 \end{aligned} \tag{3}$$

The curvature radius is defined as

$$\frac{1}{R} = \frac{\frac{d^2 y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2 y}{dx^2} \tag{4}$$

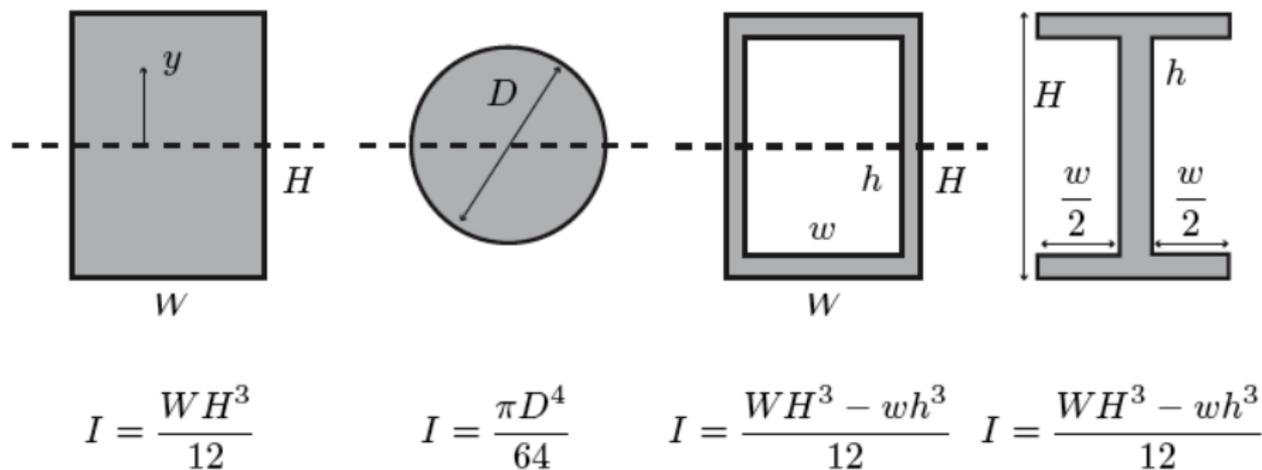
Quadratic moment of inertia

Calculate I for a rectangular cross-section:



$$I = \frac{WH^3}{12}$$

Quadratic moment of inertia



Static shape of a beam

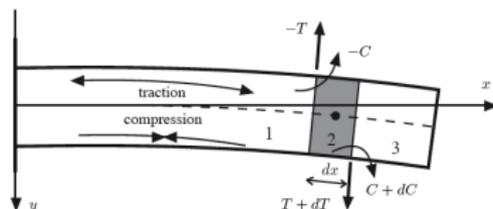
Combining equations (1)-(4) gives

$$\frac{d^4 y}{dx^4} = \frac{S \rho g}{IE} = a$$

Easily integrated as a polynomial function with 4 integration constants.

4 BC needed !

Static shape of a beam



BC:

- $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$
- $T = 0$ and $C = 0$ at $x = L$

Static shape of a beam

$$y(x) = \frac{a}{2}x^2 \left(\frac{x^2}{12} - \frac{Lx}{3} + \frac{L^2}{2} \right)$$

The maximum deviation is at $x = L$

$$y_{max} = \frac{aL^4}{8}$$

Vibration of a beam

Vibration equation

Out of static equilibrium, the motion equation is

$$\frac{d^2 C}{dx^2} = \rho S \left(g - \frac{\partial^2 y}{\partial t^2} \right)$$

Other geometric equations remain unchanged

$$\frac{1}{R} = \frac{d^2 y}{dx^2}, \quad C = \frac{IE}{R}$$

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho S}{IE} \left(\frac{\partial^2 y}{\partial t^2} - g \right) = 0$$

Hypothesis

We assume that

$$g \ll \left| \frac{\partial^2 y}{\partial t^2} \right|$$

so that the vibration equation is

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{r^2 c^2} \frac{\partial^2 y}{\partial t^2} = 0$$

with

$$c = \sqrt{\frac{E}{\rho}}, \quad r = \sqrt{\frac{I}{S}}$$

Gyration radius

The gyration radius r is defined by

$$I = \frac{\pi(2r)^4}{64}$$

or

$$r = \left(\frac{4I}{\pi}\right)^{1/4}$$

Amplitude equation

We seek solution written as

$$y(x,t) = A(x)e^{i\omega t}$$

leading to an amplitude equation

$$\frac{d^4 A}{dx^4} - \frac{\omega^2}{c^2 r^2} A = 0$$

The amplitude $A(x)$ is

$$A(x) = B_1 \cosh(\alpha x) + B_2 \sinh(\alpha x) + B_3 \cos(\alpha x) + B_4 \sin(\alpha x)$$

with

$$\alpha = \sqrt{\omega/cr}$$

the wave number.

BC

The B_i are determined through the BC :

- $A = 0$ at $x = 0$
- $dA/dx = 0$ at $x = 0$
- $d^2A/dx^2 = 0$ at $x = L$
- $d^3A/dx^3 = 0$ at $x = L$

...

BC

$$B_1 = -B_3 \quad B_2 = -B_4$$

$$\begin{pmatrix} \cosh(\alpha L) + \cos(\alpha L) & \sinh(\alpha L) + \sin(\alpha L) \\ \sinh(\alpha L) - \sin(\alpha L) & \cosh(\alpha L) + \cos(\alpha L) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0$$

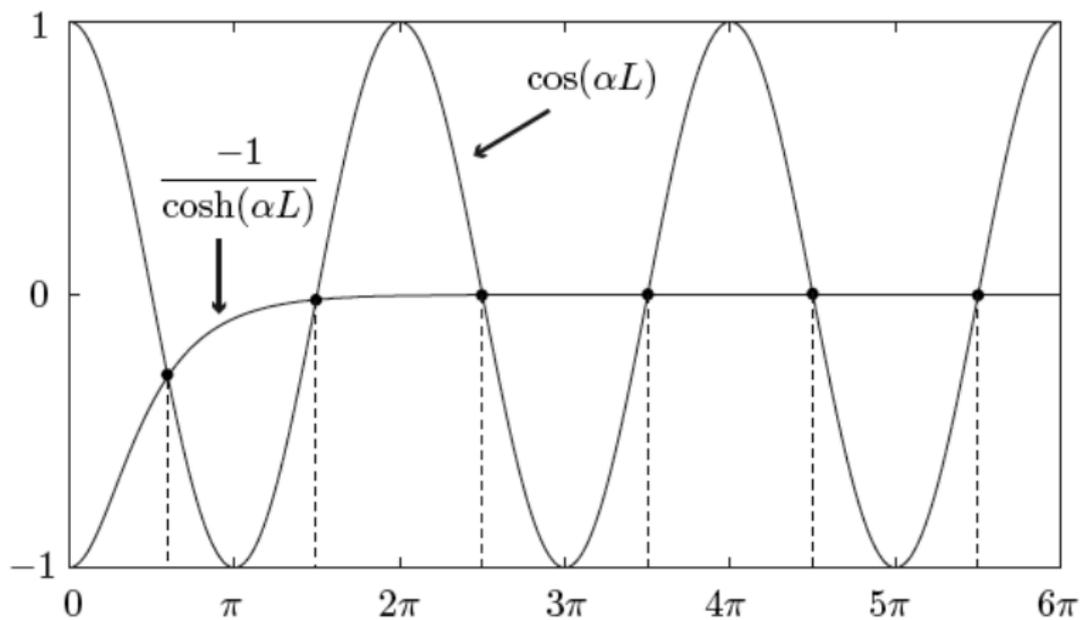
The equation for α is

$$\cosh(\alpha L) \cos(\alpha L) + 1 = 0$$

or

$$\cos(\alpha L) = -\frac{1}{\cosh(\alpha L)}$$

Graphical solution



Vibration modes for the beam

Fundamental mode:

$$\alpha_0 \approx 1.2 \frac{\pi}{2L}, \quad \omega_0 = \frac{1.44\pi^2}{4L^2} \sqrt{\frac{IE}{\rho S}}$$

Other (higher) modes:

$$\alpha_{n \geq 1} = (2n + 1) \frac{\pi}{2L}, \quad \omega_{n \geq 1} = \frac{(2n + 1)^2 \pi^2}{4L^2} \sqrt{\frac{IE}{\rho S}}$$

Back to the hypothesis

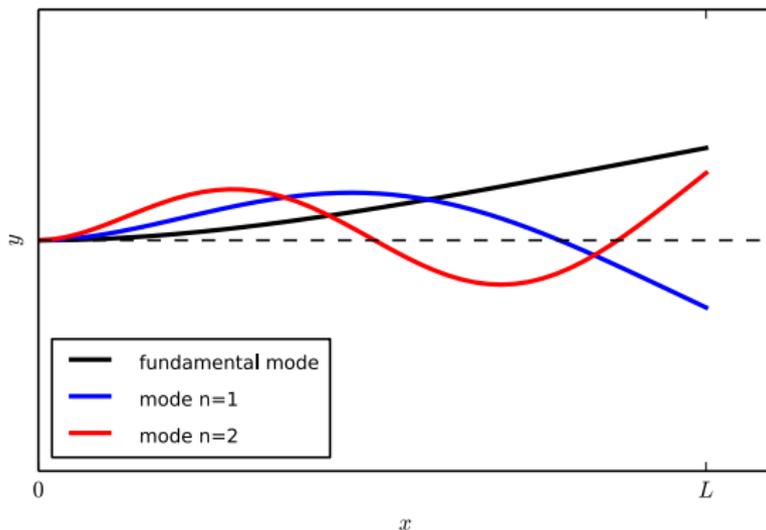
$$\left| \frac{\partial^2 y}{\partial t^2} \right| = \omega A^2 \approx y_m \omega^2$$

$$\frac{g}{\left| \frac{\partial^2 y}{\partial t^2} \right|} = \frac{g}{y_m \omega^2} = \frac{128}{(2n+1)^4 \pi^4}$$

n	0	1	2	3
$g/y_m \omega^2$	0.64	0.02	2×10^{-3}	5×10^{-4}

Concluding remarks

- The shape of the beam in the fundamental mode ($n = 0$) is very close to the shape of the static beam under gravity.
- The shape of the beam of mode $n \geq 1$ has n nodes



Waves & vibrations

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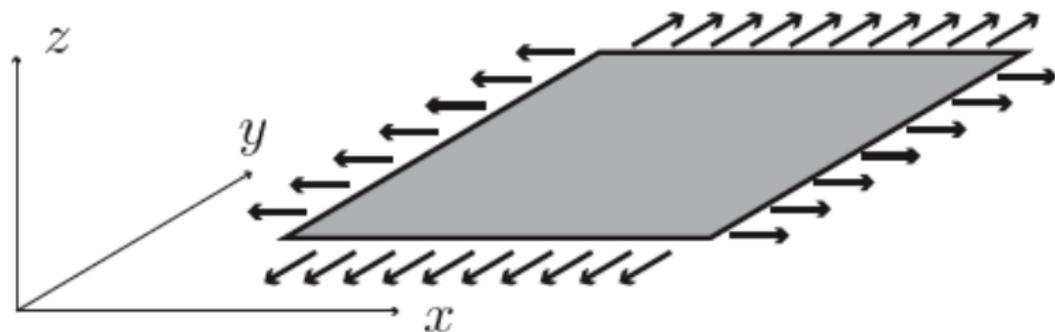
Lecture 5 outline

- 1 Equation of vibration of membranes
- 2 Solving method
- 3 Vibration of rectangular membranes
- 4 Vibration of a square membrane
- 5 Circular membranes
- 6 Vibration of a plate

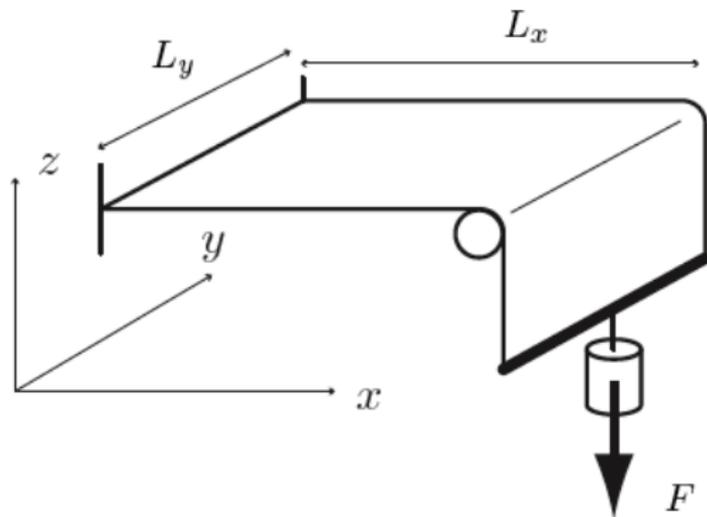
Equation of vibration of membranes

Membrane setup

- 2D system: $L_x \approx L_y$, $L_x \gg L_z$, $L_y \gg L_z$
- without rigidity



Tension of a membrane



$$F = \sigma L_y L_z = T_1 L_y$$

T_1 is a tension per unit length (in $\text{N}\cdot\text{m}^{-1}$)

Evidence of the internal tension of a membrane



The force needed to close the fracture is

$$F = T_1 \times L_{fracture}$$

Motion equation

Hypothesis:

- gravity force is negligible compared to tension force
- tension is uniform and isotropic

We derive the motion equation from the dynamic balance of a small element of the membrane:

$$dm \frac{d^2 \vec{r}}{dt^2} = \int_C \vec{T}_1 dC$$

along the z-axis:

$$\begin{aligned} dm \frac{d^2 z}{dt^2} &= \left(\oint_C \vec{T}_1 dC \right)_z \\ &= T_1 \oint \cos \alpha dC \\ &= T_1 \oint \left(\frac{dz}{dn} \right) dC \end{aligned}$$

Motion equation

Using the Green's theorem:

$$\oint \left(\frac{dz}{dn} \right) dC = \iint_S \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) z dS = S \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)$$

we obtain

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) - \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} = 0$$

which is a 2D wave equation with a velocity

$$c = \sqrt{\frac{T_1}{\rho_S}}$$

with $\rho_S = M/S$ the surface density of the membrane.

About wave velocities

- compression waves (see lecture #3):

$$c_{comp} = \sqrt{\frac{E}{\rho}}$$

- transversal waves of a string:

$$c_{string} = \sqrt{\frac{T_0}{\rho L}}$$

- transversal waves of a membrane:

$$c_{membrane} = \sqrt{\frac{T_1}{\rho S}}$$

About wave velocities

- compression waves (see lecture #3):

$$c_{comp} = \sqrt{\frac{E}{\rho}}$$

- transversal waves of a string (see lecture #3):
with $T_0 = \sigma L_y L_z$ and $\rho_L = M/L_x = \rho L_y L_z$

$$c_{string} = \sqrt{\frac{\sigma}{\rho}}$$

- transversal waves of a membrane:
with $T_1 = \sigma L_z$ and $\rho_S = M/(L_x L_y) = \rho L_z$

$$c_{membrane} = \sqrt{\frac{\sigma}{\rho}}$$

Solving method

What we want to know

From the vibration equation

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) - \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} = 0$$

we want to know:

- the natural frequencies of the membrane
- the shape of the deformed membrane under vibration
- nodes and anti-nodes

Method of the solution

We look for the $z(x,y,t)$ solution of

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) - \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} = 0$$

Separating space and time variables gives

$$z(x,y,t) = A(x,y) \cos(\omega t)$$

and the motion equation writes as an **amplitude equation**

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) A(x,y) + \frac{\omega^2}{c^2} A(x,y) = 0$$

Method of the solution

This amplitude equation ($\Delta_2 A + k^2 A = 0$) can be (*a priori*) solved explicitly with the knowledge of

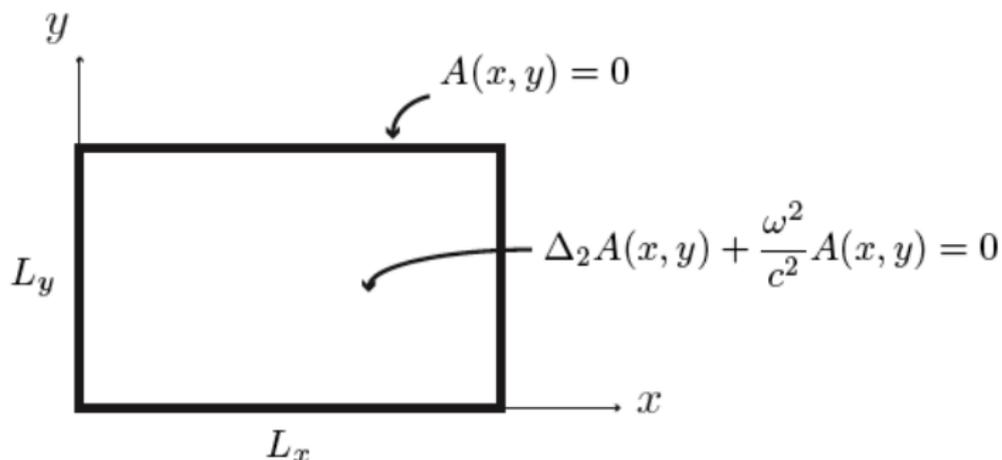
- the boundary conditions (BC)
- the initial conditions (IC)

There is no analytical solution of the membrane vibration for an arbitrary shape! We know solutions for simple shapes:

- a rectangular membrane (or square)
- a circular membrane

Vibration of rectangular membranes

A rectangular frame



amplitude solution (extension from string amplitudes solution):

$$A(x, y) = A_{mn} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right)$$

Vibration freq. of a rectangular membrane

Combining

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) A(x,y) + \frac{\omega^2}{c^2} A(x,y) = 0$$

with

$$A(x,y) = A_{mn} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right)$$

gives

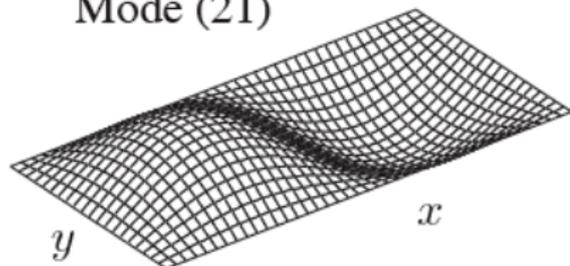
$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2} \sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}$$

The fundamental frequency is for $m = 1$ and $n = 1$ and is

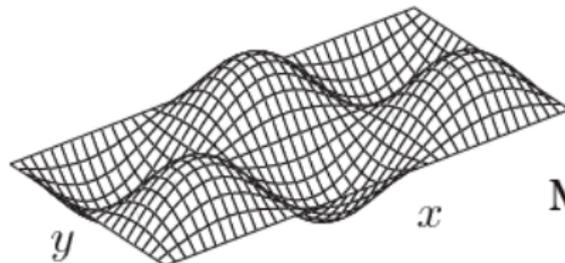
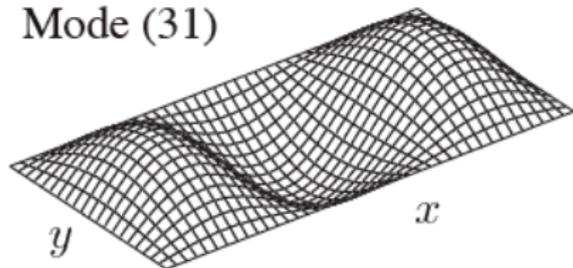
$$f_{11} = \frac{c}{2} \sqrt{\left(\frac{1}{L_x}\right)^2 + \left(\frac{1}{L_y}\right)^2}$$

Viewing a few modes

Mode (21)



Mode (31)



Mode (32)

Complete solution

The complete solution of the linear vibration equation for a rectangular membrane is

$$z(x,y,t) = \sum_{(m,n)} A_{mn} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right) \cos(\omega_{mn}t + \varphi_{mn})$$

with

$$\omega_{mn} = c\pi \sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}, \quad c = \sqrt{\frac{\sigma}{\rho}}$$

and A_{mn} , φ_{mn} determined through IC.

Nodes and anti-nodes

- A nodal line (or curve) is the set of points where

$$A(x,y) = 0, \quad 0 < x < L_x, \quad 0 < y < L_y$$

- A mode (m,n) has $m + n - 2$ nodal lines
- anti-nodes are points where the amplitude is extremal.

Vibration of a square membrane

Square membrane

Taking $L_x = L_y$ describes a square membrane, with natural frequencies

$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2L_x} \sqrt{m^2 + n^2}$$

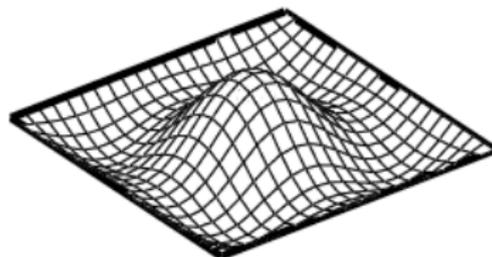
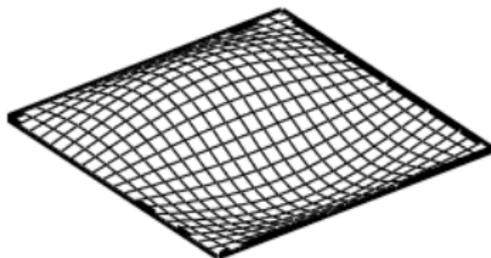
important remark:

$$f_{mn} = f_{nm}$$

for example $f_{21} = f_{12}$.

DEF: degenerated frequency: when two (or more) modes have the same frequency.

Square membrane



Circular membranes

Circular membranes

The amplitude equation

$$\Delta_2 A + k^2 A = 0$$

written in polar (r, θ) coordinates is

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} + \frac{\omega^2}{c^2} = 0$$

with BC

$$A(r = R, \theta) = 0$$

Separating r and θ , the amplitude is

$$A(r, \theta) = \sum_n A_n(r) \cos(n\theta + \varphi_n)$$

Circular membranes

Step to solution: variable change

$$x = \frac{\omega r}{c}$$

the amplitude equation writes now as a **Bessel equation**

$$\frac{d^2 A_n}{dx^2} + \frac{1}{x} \frac{dA_n}{dx} + \left(1 - \frac{n^2}{x^2}\right) A_n = 0$$

Bessel equations and Bessel functions

$$A_n(x) = \alpha_n J_n(x) + \beta_n Y_n(x)$$

with

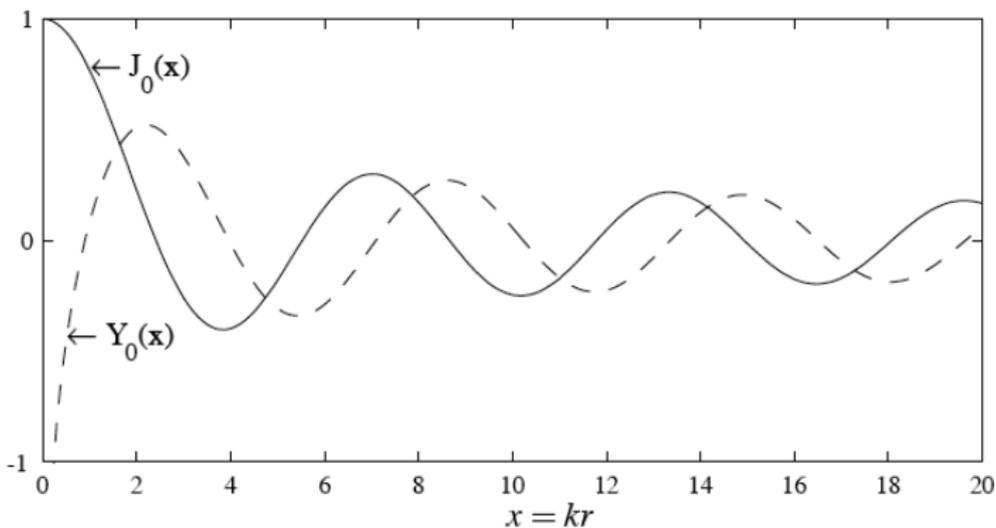
$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-x^2/4)^m}{m! \Gamma(n+m+1)}$$

$$Y_n(x) = \frac{J_n(x) \cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$$

$$\Gamma(k) = \int_0^{\infty} e^{-t} t^{k-1} dt$$

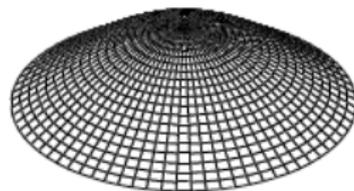
Bessel functions

for $n = 0$, the J_0 and Y_0 Bessel functions are

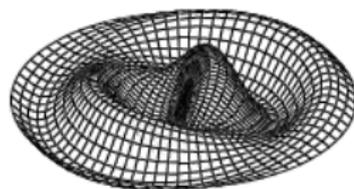


Circular vibration modes

$$z(r, \theta, t) = \cos(\omega t) \sum_n \alpha_n J_n(kr) \cos(n\theta + \varphi_n)$$



mode (01)



mode (13)

Vibration of a plate

Vibration of a rigid membrane

Combining membrane + beam gives

$$\frac{Eh^2}{12\rho(1-\nu^2)}\Delta^2 z + \frac{\partial^2 z}{\partial t^2} = 0.$$

where h is the thickness of the plate