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► **To cite this version:**

Florence Forbes. Introduction to statistical methods in signal and image processing. Doctoral. Peyresq, France. 2016. cel-01423624

HAL Id: cel-01423624

<https://hal.science/cel-01423624v1>

Submitted on 30 Dec 2016

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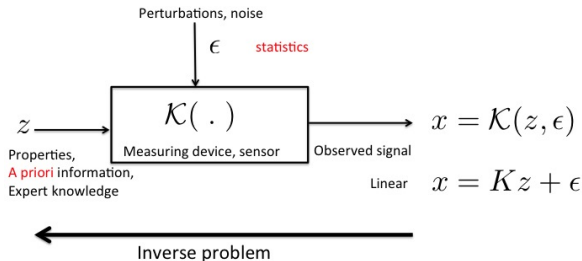
Introduction to statistical methods in signal and image processing

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A methodological framework for inverse problems



Linear Models: convolution (image restoration), projection (tomography), mixtures (source separation), Laplace and Fourier transform (NMR, MRI)

Inversion: instability, non-unicity or existence of the solution
→ **Ill-posed problem**

Regularization: add constraints/hypothesis on the seek solution

- ▶ Bayesian inference: $p(z|x) \propto p(x|z)p(z)$
- ▶ Penalized criterion minimization: $F(z) = L(z, x) + \beta R(z)$

Overview: Part 1- Introduction to Bayesian tools

- ▶ Introduction
- ▶ Statistical inference
 - ▶ Learning and decision
 - ▶ Maximum likelihood
- ▶ Bayesian set up
 - ▶ prior, posterior, etc.
- ▶ Bayesian inference strategies
 - ▶ Point estimators
 - ▶ Fully Bayesian treatment
- ▶ Prior distributions
 - ▶ Conjugate priors and exponential family
 - ▶ Noninformative and Jeffreys' priors
- ▶ Tractability of posteriors

Overview: Part 2- Probabilistic graphical models

- ▶ Directed graphs: Bayesian networks
- ▶ Conditional independence and Markov properties
- ▶ Undirected graphs: Markov random fields
- ▶ Inference and learning
- ▶ Illustration: image segmentation

Introduction

Illustration: Audio-Visual Scene Analysis



- ▶ Estimate the number of audio-visual objects
- ▶ Localize and track every object
- ▶ Determine auditory activity and visibility

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Observed Data

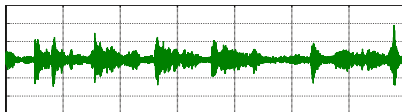
Right camera image:



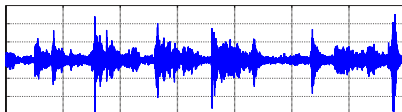
Left camera image:



Left microphone signal:



Right microphone signal:

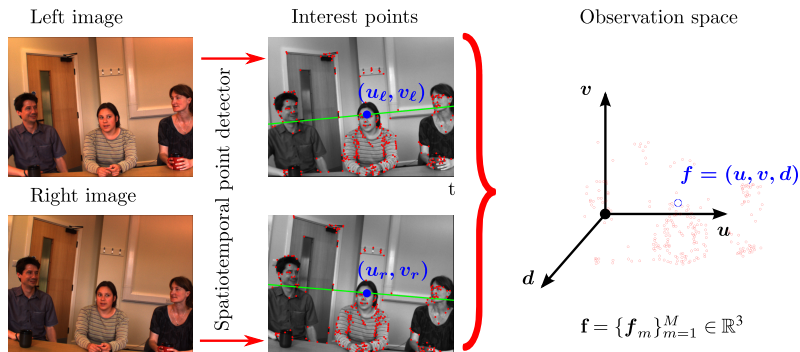


Visual Features Extraction

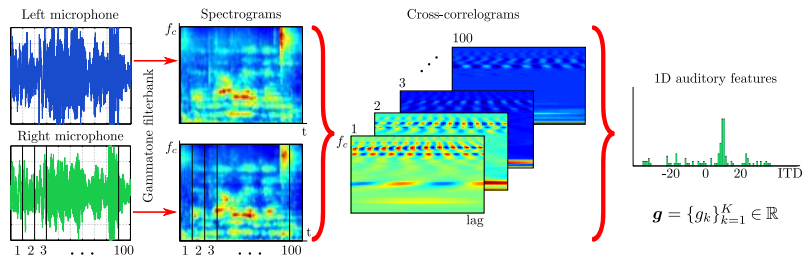
An image pair produces a set of visual observations

$$\mathbf{f} = \{\mathbf{f}_m\}_{m=1}^M \in \mathbb{R}^3;$$

$\mathbf{f} = (u, v, d)$: u, v - image coordinates, d - disparity



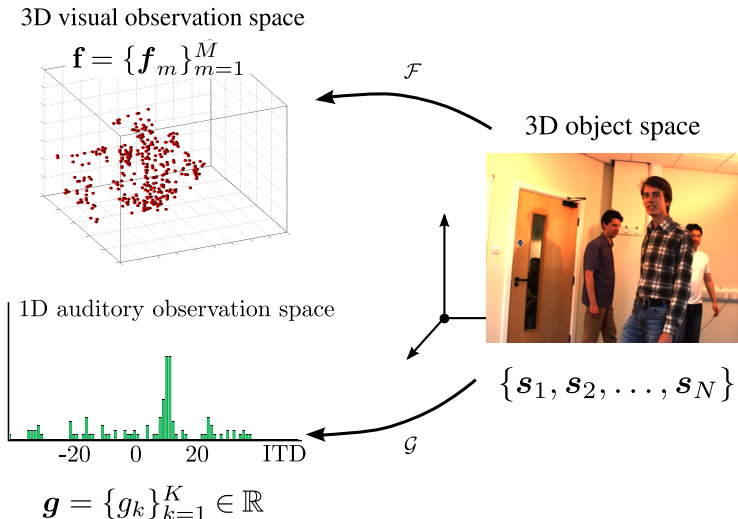
Auditory Features Extraction



ITD = interaural time difference

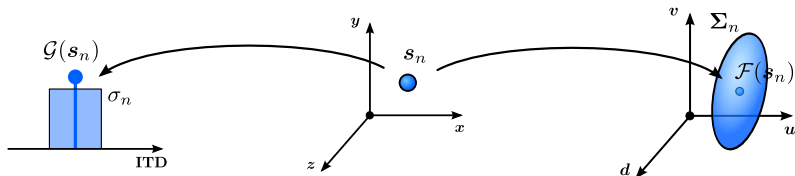
An ITD detection algorithm [H. Christensen, 2007] produces for a 10ms interval of audio signals one auditory observation $g_k \in \mathbb{R}$

Audio-Visual Generative Model



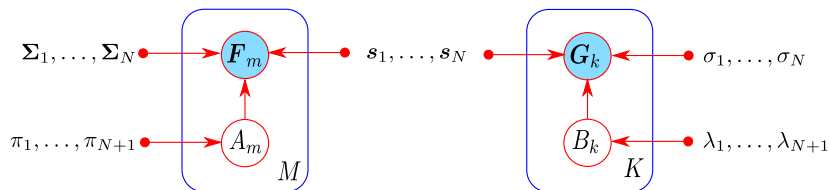
Why statistical modelling in Audio-Visual Scene Analysis ?

- ▶ **Observations are strongly affected by noise:** detector errors, occlusions, reverberations, ambient sounds, can be accounted for with **some probability distributions**.



- ▶ $P(\mathbf{f}_m | A_m = n; \mathbf{s}_n) = \mathcal{N}(\mathbf{f}_m; \mathcal{F}(\mathbf{s}_n), \Sigma_n)$;
 - ▶ $P(g_k | B_k = n; \mathbf{s}_n) = \mathcal{N}(g_k; \mathcal{G}(\mathbf{s}_n), \Gamma_n)$;
- ▶ **Dynamically changing environment:**
can be accounted for with some **prior knowledge**, eg. on motion cues, trajectories are continuous, smooth, etc...

Statistical Model formulation

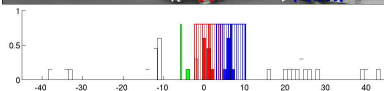
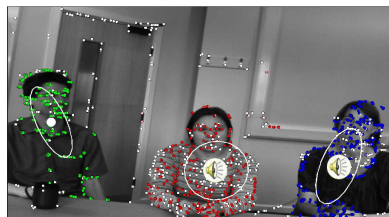
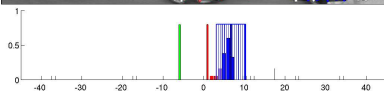
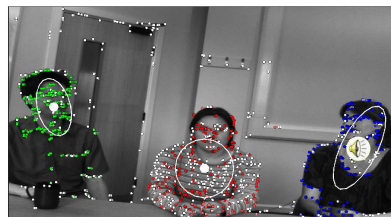


- ▶ $s = \{s_1, \dots, s_n, \dots, s_N\}$ are tying parameters
- ▶ Simultaneous clustering in auditory and visual observation spaces
- ▶ Model parameters: Determine N and s_1, \dots, s_N

$$\theta = \{s_1, \dots, s_N, \Sigma_1, \dots, \Sigma_N, \Gamma_1, \dots, \Gamma_N, \pi_1, \dots, \pi_{N+1}, \lambda_1, \dots, \lambda_{N+1}\}$$

Real Data Results

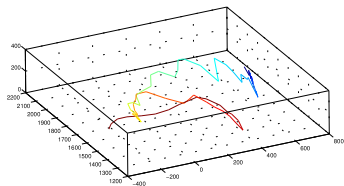
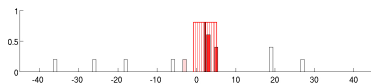
Meeting scenario



- ▶ Estimated speaker locations and their auditory activity for a quasi-stationary scene
- ▶ Error rates for auditory activity detection: 'missed target' = 0.16, 'false alarm' = 0.14
- ▶ Localization error: within 5cm

Real Data Results

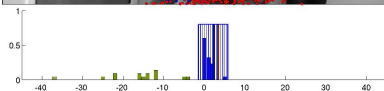
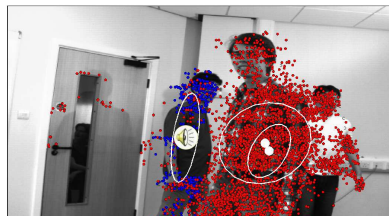
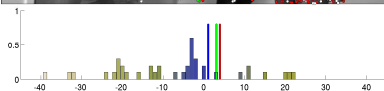
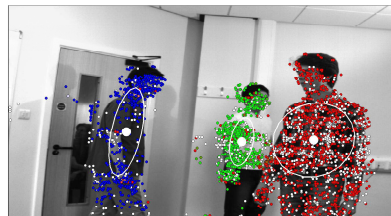
Simple tracking scenario



- ▶ Simple dynamic scene - results on the previous frame are used to initialize the model for the next frame
- ▶ Error rates for auditory activity detection: 'missed target' = 0.13, 'false alarm' = 0.43
- ▶ Localization error: within 10cm

Real Data Results

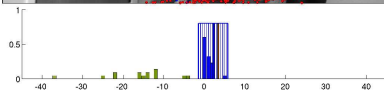
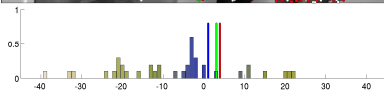
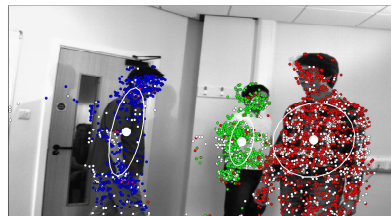
Cocktail party scenario



- ▶ Complex dynamic scene - may fail!
- ▶ Explicit dynamic model is required!

Real Data Results

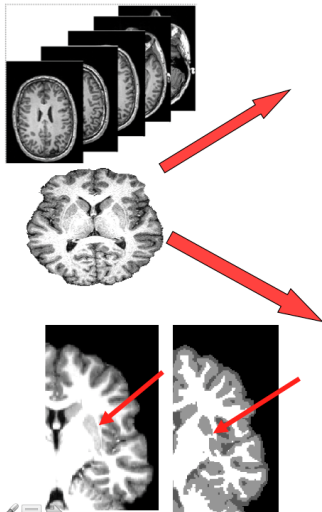
Cocktail party scenario



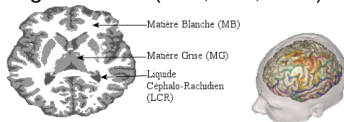
- ▶ Complex dynamic scene - may fail!
- ▶ **Explicit dynamic model is required!**

Illustration: MR Brain scan segmentation

Assign each voxel to a class (label) (among K classes)

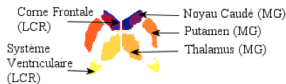


Tissue segmentation (WM, GM, CSF)



→ Cortex 3D reconstruction

Structure segmentation

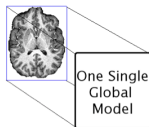
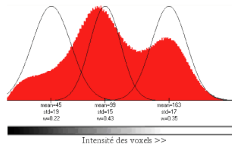


→ Useful for :

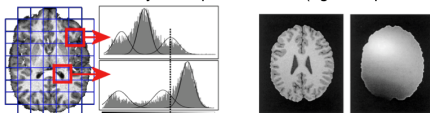
- Distinguishing Cortex GM from Nuclei GM
- volumetric studies
- ...

Tissue segmentation

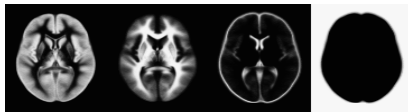
- **Global** estimation of **Gaussian** intensity models for CSF, WM, GM.



- Intensity inhomogeneities are modeled by a unique bias field (eg. Multiplicative) **to be estimated**



Constrain with an atlas



Statistical inference

Statistical inference

From a given set of observation $x = (x_1, \dots, x_N)$, learn a model that best describes the data

- ▶ **Probabilistic parametric model:**

$x = (x_1, \dots, x_N)$ generated from a probability distribution $f(x|\theta)$

$$x = (x_1, \dots, x_N) \sim f(x|\theta)$$

associated **likelihood**: $l(\theta|x) = f(x|\theta)$ viewed as a function of θ

- ▶ **Learning:** estimating θ

e.g. that maximizes $l(\theta|x)$ (**Maximum likelihood inference**)

Decision

Once a model is learned, decide about:

- ▶ The occurrence of an "event",
- ▶ Classify,
- ▶ Or find the value of a variable, etc.

Example 1: Linear model

Assume $x = Kz + \epsilon$

z = clean signal, $z \sim f(z|\theta)$

ϵ = noise, $\epsilon \sim f(\epsilon|\phi)$

x = noisy observed signal

Goal: obtain an estimate for z (\hat{z})

Decision

Example 2: Classification

e.g. 2 groups of objects (people)

$$\theta_1 \longrightarrow f(x|\theta_1) \longrightarrow x \in g_1$$

$$\theta_2 \longrightarrow f(x|\theta_2) \longrightarrow x \in g_2$$

Training data: observations in g_1 and in $g_2 \longrightarrow \hat{\theta}_1, \hat{\theta}_2$

Goal: given x^{new} , decide to which group it belongs

(i.e. compute $p(g|x^{new}, \hat{\theta}_1, \hat{\theta}_2)$)

Maximum likelihood estimation

- ▶ We observe N realizations x_1, \dots, x_N of a variable X
- ▶ Decide on a parametric model for X : $f(x|\theta)$
- ▶ Estimate θ by maximizing $l(\theta|x)$ or $\log l(\theta|x)$

Example 1: Linear Gaussian model $z = Kx + \epsilon$ and $\epsilon \sim \mathcal{N}(\mu_\epsilon, \Sigma_\epsilon)$

$$\log f(z|\theta) = \log \mathcal{N}(Kx + \mu_\epsilon, \Sigma_\epsilon) \propto -(z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)$$

$$\hat{x}_{ML} = \arg \min_x (z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)$$

Normal equations: $(K^T \Sigma_\epsilon^{-1} K) \hat{x}_{ML} = K^T \Sigma_\epsilon^{-1} (z - \mu_\epsilon)$

- ▶ **Least squares:** $\mu_\epsilon = 0$ and $\Sigma_\epsilon = \sigma^2 Id$

$$\implies \hat{x}_{ML} = \arg \min_x \|z - Kx\|_2^2 = \hat{x}_{LS}$$

- ▶ **Weighted least squares:** $\mu_\epsilon = 0$ and $\Sigma_\epsilon = \text{Diag}(\sigma_1^2, \dots, \sigma_N^2)$

$$\implies \hat{x}_{ML} = \arg \min_x \sum_n \frac{(z_n - [Kx]_n)^2}{\sigma_n^2} = \hat{x}_{WLS}$$

Maximum likelihood estimation

Example 2: Man-Woman classification problem

5 subjects in each class were asked if they like football and statistics

	Women g_1					Men g_2				
football	1	1	0	0	0	1	1	1	1	1
statistics	1	0	1	0	1	0	1	0	0	1

ratio	Positive answers	Negative answers	Positive answers	Negative answers
football	$2/5=0.4$	$3/5=0.6$	$5/5=1$	$0/5=0$
statistics	$3/5=0.6$	$2/5=0.4$	$2/5=0.4$	$3/5=0.6$

Observations and notation

- ▶ $N = 10$ responses $x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix} \in \{0, 1\}^2$ (2 questions)

$$\mathbf{x} = \{x_1, \dots, x_N\}$$

- ▶ $N = 10$ group assignments $g_n \in \{Woman, Man\} (= \{1, 2\})$

$$\mathbf{g} = \{g_1, \dots, g_N\}$$

$$\mathbf{x}^{g^1} = \{x_n, g_n = 1\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{x}^{g^2} = \{x_n, g_n = 2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Model

- ▶ Independence: $f(\mathbf{g}) = \prod_{n=1}^N f(g_n)$
- ▶ $f(g_n = \text{woman}) = f(g_n = \text{man}) = 0.5$
- ▶ Conditional independence: $f(\mathbf{x}|\mathbf{g}) = \prod_{n=1}^N f(x_n|g_n)$
- ▶ Independence of the two questions: $f(x_n|g_n) = f(x_{1n}|g_n) f(x_{2n}|g_n)$
- ▶ $\forall n = 1 \dots N, i = \{1, 2\}, g = \{1, 2\}$, Independent Bernoulli distributions ($\theta_i^g \in [0, 1]$):



$$f(x_{in}|g_n = g) = \begin{cases} \theta_i^g & \text{if } x_{in} = 1 \\ 1 - \theta_i^g & \text{if } x_{in} = 0 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently $f(x_{in}|g_n = g) = (\theta_i^g)^{x_{in}} (1 - \theta_i^g)^{1-x_{in}}$

$(\theta_i^g = 0.5 \rightarrow \text{the coin is not biased})$

Likelihood for each group

- ▶ Learning task: Estimate (θ_1^g, θ_2^g) given \mathbf{x}^g ($g = Woman, Man$)
- ▶ Likelihood function:

$$f(\mathbf{x}^g | \theta^g) = \prod_{n, g_n=g} f(x_{1n} | g) f(x_{2n} | g)$$

- ▶ Log-likelihood:

$$\log f(\mathbf{x}^g | \theta^g) = \sum_{n, g_n=g} \sum_{i=1,2} x_{in} \log \theta_i^g + (1 - x_{in}) \log(1 - \theta_i^g)$$

- ▶ Maximization: $\theta_i^g = \frac{\sum_{n, g_n=g} x_{in}}{N_g}$ (mean, frequencies of positive answers)

$$\theta^1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \theta^2 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}$$

Decision: Naive Bayes classifier

Sum-rule: $P(B) = P(B, A) + P(B, A^c)$

Product-rule: $P(A, B) = P(B|A)P(A)$

It follows Bayes' theorem: $P(A|B) = P(B|A)P(A)/P(B)$

with normalization $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$

Goal: Classify a person with $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ie. $g = ??$

▶ Bayes' rule : $f(g|x) = \frac{f(x|g)f(g)}{f(x)} = \frac{f(x|g)f(g)}{\sum_{g'} f(g')f(x|g')}$

▶ Assuming $f(\text{woman}) = f(\text{man}) = 0.5$,

$$f(\text{woman}|x) = \frac{0.4 \times 0.6}{0.4 \times 0.6 + 1 \times 0.4} = 0.375$$

$$f(\text{man}|x) = \frac{1 \times 0.4}{0.4 \times 0.6 + 1 \times 0.4} = 0.625 = 1 - 0.375$$

Decision: Naive Bayes classifier

Goal: classify a person with $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$f(\text{woman}|x) = \frac{0.6 \times 0.6}{0.6 \times 0.6 + 0 \times 0.4} = 1$$

$$f(\text{man}|x) = \frac{0 \times 0.4}{0.6 \times 0.6 + 0 \times 0.4} = 0$$

- ▶ Conclusion: if you don't like football and like statistics, you are almost surely a woman
- ▶ **Overfitting effect** to the small training set
- ▶ Priors over the parameters can avoid overfitting \implies **Bayesian framework**

Bayesian set up

Bayesian concepts

- ▶ **Uncertainty on the parameters θ** of a model modeled through a probability distribution on θ , called **prior distribution**
The prior **encoded the information available a priori, before observing x**

- ▶ Inference based on the distribution of θ conditional on x , **$f(\theta|x)$, called posterior distribution**

Impact

- ▶ From unknown parameters to **random**
- ▶ **Actualization** of the information on θ by extracting the information on θ contained in the observations x
- ▶ Allows **incorporation of imperfect information** in the decision process
- ▶ Unique mathematical way to **condition upon the observations** (conditional perspective)
- ▶ **Penalization** factor

Three basic quantities in Bayesian inference

- ▶ Prior distribution $f(\theta)$
- ▶ likelihood $f(x|\theta)$
- ▶ Posterior distribution $f(\theta|x)$

Forward generative model:

$$f(\theta) \longrightarrow \theta \longrightarrow f(x|\theta) \longrightarrow x$$

→ involves the **prior** and the **likelihood**

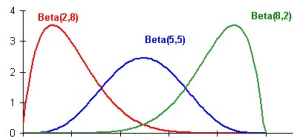
Inference is an inversion problem:

$$x \longrightarrow f(\theta|x) \longrightarrow \hat{\theta}$$

→ involves the **posterior** distribution

Classification example

Assume a **Beta prior** over the Bernoulli parameters: $\theta_i^g \in [0, 1]$



$$f(\theta_i^g) = \text{Beta}(\alpha, \beta) = \mathcal{B}(\alpha, \beta)^{-1} \theta_i^{g\alpha-1} (1 - \theta_i^g)^{\beta-1}$$

$$E[\theta_i^g] = \alpha/(\alpha+\beta), \text{ Mode}[\theta_i^g] = (\alpha-1)/(\alpha+\beta-2)$$

Compute the posterior distribution of θ_i^g :

$$f(\theta_i^g | \mathbf{x}^g) = \frac{f(\mathbf{x}^g | \theta_i^g) f(\theta_i^g)}{f(\mathbf{x}^g)} \propto f(\mathbf{x}^g | \theta_i^g) f(\theta_i^g)$$

$$\log f(\theta_i^g | \mathbf{x}^g) = \text{cst} + (A_i^g - 1) \log \theta_i^g + (B_i^g - 1) \log(1 - \theta_i^g)$$

with $A_i^g = \sum_{n, g_n=g} x_{in} + \alpha$ and $B_i^g = \sum_{n, g_n=g} (1 - x_{in}) + \beta$

so that **posterior distribution**:

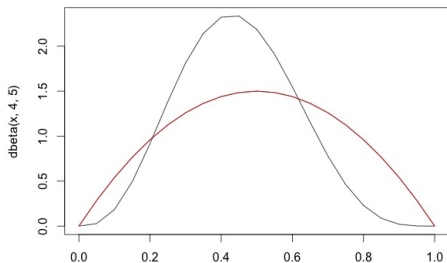
$$\begin{aligned} f(\theta_i^g | \mathbf{x}^g) &\propto \theta_i^{g A_i^g - 1} (1 - \theta_i^g)^{B_i^g - 1} \\ &= \text{Beta}(A_i^g, B_i^g) \end{aligned}$$

Classification example

	Women g_1					Men g_2				
football	1	1	0	0	0	1	1	1	1	1
statistics	1	0	1	0	1	0	1	0	0	1

Example: $\alpha = \beta = 2 \implies A_1^1 = \alpha + 2 = 4, \quad B_1^1 = \beta + 3 = 5$

$\downarrow \mathcal{B}(4, 5)$



$\nwarrow \mathcal{B}(2, 2)$

Bayesian inference strategies

Point estimators

Goal: provide an estimation of θ

The two most common Bayesian estimators are:

- ▶ Maximum a posteriori (MAP) estimator

$$\begin{aligned}\hat{\theta}_{MAP} &= \arg \max_{\theta} f(\theta|x) \\ &= \arg \max_{\theta} f(x|\theta)f(\theta) \\ &= \arg \max_{\theta} \log f(x|\theta) + \log f(\theta)\end{aligned}$$

Note: if $f(\theta) = \text{constant}$ then $\hat{\theta}_{MAP} = \hat{\theta}_{MLE}$

- ▶ Posterior Mean Estimator

$$\hat{\theta}_{PM} = E_{\theta}[\theta|x] = \int \theta f(\theta|x) d\theta$$

Note: $f(\theta|x)$ requires the normalizing term $f(x) = \int f(x|\theta)f(\theta)d\theta$.

$\hat{\theta}_{MAP}$ usually easier to obtain, it involves optimization rather than integration

Posterior mean and Bayesian MSE

The Bayesian Mean Square Error (MSE) is

$$E_{\theta, X}[\|\hat{\theta} - \theta\|_2^2] = \int \int \|\hat{\theta}(x) - \theta\|_2^2 f(\theta, x) d\theta dx$$

Minimum Mean Square Error (MMSE) estimator:

Definition: $\hat{\theta}_{MMSE} = \arg \min_{\hat{\theta}} E_{\theta, X}[\|\hat{\theta} - \theta\|_2^2]$

Solution: $\hat{\theta}_{MMSE} = E_{\theta}[\theta|X] = \hat{\theta}_{PM}$

since $E_{\theta, X}[\|\hat{\theta} - \theta\|_2^2] = E_X[E_{\theta}[\|\hat{\theta} - \theta\|_2^2|X]]$

and $E_{\theta}[\|\hat{\theta} - \theta\|_2^2|X]$ is minimum when $\hat{\theta} = E_{\theta}[\theta|X]$

MAP and 0-1 Loss

The MSE quadratic cost (loss) can be replaced by a 0-1 cost

$$E_{\theta, X}[1 - \delta_{\theta}(\hat{\theta})]$$

where $1 - \delta_{\theta}(\hat{\theta}) = 0$ if $\hat{\theta} = \theta$ (no loss) and 1 otherwise (max loss)

$$\min E_{\theta, X}[1 - \delta_{\theta}(\hat{\theta})] = \max E_X[E_{\theta}[\delta_{\theta}(\hat{\theta})|X]]$$

and $E_{\theta}[\delta_{\theta}(\hat{\theta})|X] = p(\theta = \hat{\theta}|X)$ which is max at the MAP

Linear Minimum MSE

Assume $E[\theta] = E[X] = 0$ and consider an estimator of the form $\hat{\theta} = A^T X$

Goal: find matrix A that minimizes the Bayesian MSE

$$\begin{aligned}MSE(A) &= E_{\theta, X}[\|A^T X - \theta\|_2^2] \\&= E_{\theta, X}[\text{trace}((A^T X - \theta)(A^T X - \theta)^T)] \\&= \text{trace}(E_{\theta, X}[(A^T X - \theta)(A^T X - \theta)^T]) \\&= \text{trace}(E[\theta\theta^T] - A^T E[X\theta^T] - E[\theta X^T]A + A^T E[XX^T]A) \\&= \text{trace}(\Sigma_{\theta} - A^T \Sigma_{x\theta} - \Sigma_{\theta x} A + A^T \Sigma_x A)\end{aligned}$$

$$\frac{\partial}{\partial A} MSE(A) = -2\Sigma_{x\theta} + 2\Sigma_x A = 0$$

$$\hat{A} = \Sigma_x^{-1} \Sigma_{x\theta} \quad \text{Wiener-Hopf equation}$$

$$\hat{\theta}_{LMMSE} = \Sigma_{\theta x} \Sigma_x^{-1} X \quad \text{Wiener filter}$$

$$(\hat{\theta}_{LMMSE} = \Sigma_{\theta x} \Sigma_x^{-1} (X - \mu_x) + \mu_{\theta} \quad \text{in the non centered case})$$

Linear Minimum MSE: Linear model example

Assume $x = K\theta + \epsilon$ where $\theta \sim \mathcal{N}(0, \sigma_\theta^2 I)$ and $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 I)$ are independent

Then $x \sim \mathcal{N}(0, \sigma_\theta^2 K K^T + \sigma_\epsilon^2 I)$ and

$$\begin{aligned}\Sigma_x &= \sigma_\theta^2 K K^T + \sigma_\epsilon^2 I \\ \Sigma_{\theta x} &= E[X\theta^T] = E[K\theta\theta^T + \epsilon\theta^T] = K\Sigma_\theta = \sigma_\theta^2 K\end{aligned}$$

$$\hat{\theta}_{LMMSE} = \sigma_\theta^2 K^T (\sigma_\theta^2 K K^T + \sigma_\epsilon^2 I)^{-1} X = K^T (K K^T + \frac{\sigma_\epsilon^2}{\sigma_\theta^2} I)^{-1} X$$

Note: when SNR increases, $\frac{\sigma_\epsilon^2}{\sigma_\theta^2} \rightarrow 0$, $\hat{\theta}_{LMMSE} \rightarrow \hat{\theta}_{MLE} = (K K^T)^{-1} K^T X$

Classification example

MMSE estimator: Since $f(\theta_i^g | \mathbf{x}^g)$ is a Beta distribution

$$E[\theta_i^g | \mathbf{x}^g] = \frac{\sum_{n, g_n=g} x_{in} + \alpha}{\alpha + \beta + N_g}$$

With $\alpha = \beta = 2$ (mode and mean at 0.5), we get

$$\theta^1 = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix} \text{ and } \theta^2 = \begin{bmatrix} 7/9 \\ 4/9 \end{bmatrix}$$

Then for $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it comes $f(\text{man}|x) = 8/33 = 0.242$

MAP estimator: using the mode of the posterior we get instead :

$$\theta^1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} \text{ and } \theta^2 = \begin{bmatrix} 6/7 \\ 3/7 \end{bmatrix}$$

and for $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it comes $f(\text{man}|x) = 3/19 = 0.158$

Predictive distributions

Use the full posterior rather than a point estimate.

Other distributions of interest are:

Prior predictive (marginal):

Before we observe the data, what do we expect the distribution of observations to be?

$$f(x) = \int f(x|\theta)f(\theta) d\theta$$

- ▶ What we would predict for x given no data
- ▶ Useful for assessing whether choice of prior distribution does capture prior beliefs.

Predictive distributions

Posterior predictive

What is the predictive distribution of a new observation x^{new} given the current data \mathbf{x} ?

$$\begin{aligned} f(x^{new}|\mathbf{x}) &= \int f(x^{new}, \theta|\mathbf{x}) d\theta \\ &= \int f(x^{new}|\theta)f(\theta|\mathbf{x})d\theta \end{aligned}$$

Use the assumption that x^{new} is independent of \mathbf{x} given θ .

Classification example

In each group, the posterior predictive is:

$$f(x^{new} | \mathbf{x}^g) = \int f(x^{new} | \theta^g) f(\theta^g | \mathbf{x}^g) d\theta^g = f(x_1^{new} | \mathbf{x}_1^g) f(x_2^{new} | \mathbf{x}_2^g)$$

$$\begin{aligned} f(x_i^{new} | \mathbf{x}_i^g) &= \int f(x_i^{new} | \theta_i^g) f(\theta_i^g | \mathbf{x}_i^g) d\theta_i^g \\ &= \frac{\mathcal{B}(x_i^{new} + A_i^g, 1 - x_i^{new} + B_i^g)}{\mathcal{B}(A_i^g, B_i^g)} \end{aligned}$$

Then using Bayes' rule:

$$\begin{aligned} f(g^{new} | x^{new}, \mathbf{x}, \mathbf{g}) &\propto f(x^{new} | \mathbf{x}^{g^{new}}) f(g^{new}) \\ &\propto f(x_1^{new} | \mathbf{x}_1^{g^{new}}) f(x_2^{new} | \mathbf{x}_2^{g^{new}}) \times 0.5 \end{aligned}$$

For $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it comes $f(man|x) = 8/33 = 0.242$

Classification example: all results for $f(\text{man}|x = [0, 1]^T)$

Estimator	$f(\text{man} x)$
Maximum likelihood	0
Bayesian MMSE	0.242
Bayesian MAP	0.158
Fully Bayesian	0.242

Prior distributions

From prior information to prior distributions

- ▶ All **computations depends on the prior** choice
- ▶ The prior is a tool summarizing available information as well as uncertainty related with this information
- ▶ The prior distribution is the **key to Bayesian inference** but the available prior information is usually not precise enough to lead to an exact determination

Different strategies are possible:

- ▶ Conjugate priors
- ▶ Noninformative priors
- ▶ Jeffreys prior
- ▶ Hierarchical modelling, etc.

Conjugate priors: a starting point

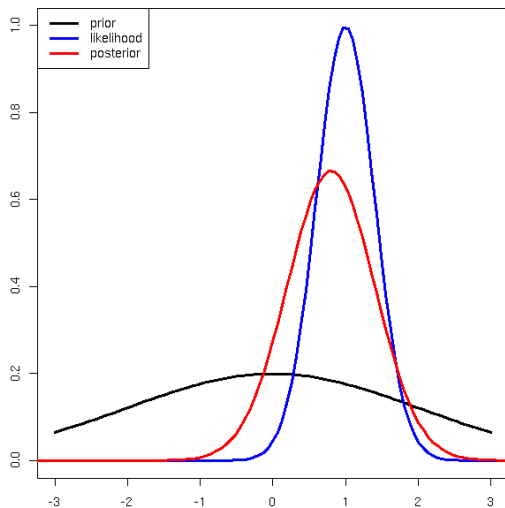
Specific parametric family with convenient analytical properties

Definition: *A family \mathcal{F} of probability distributions on θ is conjugate for a likelihood function $f(x|\theta)$ if, for every $\pi \in \mathcal{F}$, the posterior distribution $f(\theta|x) \propto f(x|\theta)\pi(\theta)$ also belongs to \mathcal{F} .*

Main interest is when \mathcal{F} is parametric: computing the posterior distribution reduces then to an updating of the corresponding parameters of the prior.

- ▶ The prior "structure" on θ is propagated to the posterior (actualisation)
- ▶ **Tractability and simplicity**
- ▶ First approximations to adequate priors

Conjugate priors: Gaussian case



Exponential families

Conjugate priors are usually associated with exponential families of distributions.

Definition: C, h are positive functions, R, T are functions in \mathbb{R}^k

The family of distributions

$$f(x|\theta) = C(\theta)h(x) \exp(R(\theta)T(x))$$

is called an exponential family of dimension k .

When

$$f(x|\theta) = C(\theta)h(x) \exp(\theta x) = \mathbf{h}(\mathbf{x}) \exp(\theta \mathbf{x} - \Psi(\theta))$$

the family is said to be natural.

Exponential families

Interesting analytical properties :

- ▶ Sufficient statistics of constant dimension exist
- ▶ Include common distributions (normal, binomial, Poisson, Wishart, etc.)
- ▶ Availability of the moments:

$$E_X[X|\theta] = \nabla \Psi(\theta), \text{ cov}(X_i, X_j) = \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j}(\theta).$$

- ▶ Allow for conjugate priors

Conjugate distributions for exponential families

If $f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta))$ then

$$f(\theta|\mu, \lambda) = K(\mu, \lambda) \exp(\theta\mu - \lambda\Psi(\theta))$$

where $K(\mu, \lambda)$ is the normalizing constant, **is conjugate** for $f(x|\theta)$.

The posterior is then $f(\theta|\mu + x, \lambda + 1)$.

It follows an "automatic" way to derive prior from $f(x|\theta)$ BUT μ, λ have still to be specified.

Linearity of the posterior mean

$f(x|\theta)$ in the natural exponential family: $f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta))$

$$E_X[X] = m(\theta) = \nabla \Psi(\theta)$$

$f(\theta)$ has a conjugate prior: $f(\theta) \propto \exp(\mu x - \lambda \Psi(\theta))$

$$E_\theta[m(\theta)] = \int m(\theta) f(\theta) d\theta = \frac{\mu}{\lambda}$$

If x_1, \dots, x_N i.i.d $f(x|\theta)$ then

$$f(\theta|x_1, \dots, x_N) \propto f(\theta|x_1) f(x_2|\theta) \dots f(x_N|\theta) = f(\theta|\mu + \sum_{n=1}^N x_n, \lambda + N)$$

$$E_\theta[m(\theta)|x_1, \dots, x_n] = \frac{\mu + \sum_{n=1}^N x_n}{\lambda + N}$$

Common conjugate priors

$f(x \theta)$	$f(\theta)$	$f(\theta x)$
Normal $\mathcal{N}(\theta, \sigma^2)$	Normal $\mathcal{N}(\mu, \tau^2)$	Normal $\mathcal{N}(\frac{\sigma^2\mu + \tau^2x}{\sigma^2 + \tau^2}, (\frac{1}{\sigma^2} + \frac{1}{\tau^2})^{-1})$
Poisson $\mathcal{P}(\theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	Gamma $\mathcal{G}(\alpha + x, \beta + 1)$
Gamma $\mathcal{G}(\nu, \theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	Gamma $\mathcal{G}(\alpha + \nu, \beta + x)$
Binomial $\text{Bin}(n, \theta)$	Beta $B(\alpha, \beta)$	Beta $B(\alpha + x, \beta + n - x)$
Multinomial $\mathcal{M}(\theta_1, \dots, \theta_K)$	Dirichlet $\mathcal{D}(\alpha_1, \dots, \alpha_K)$	Dirichlet $\mathcal{D}(\alpha_1 + x_1, \dots, \alpha_K + x_K)$
Normal $\mathcal{N}(\mu, \frac{1}{\theta})$	Gamma $\mathcal{G}(\alpha, \beta)$	Gamma $\mathcal{G}(\alpha + 1/2, \beta + (x - \mu)^2/2)$

Non informative priors

How to encode absence of prior knowledge?

Is there such a thing as a default prior when prior information is missing?

In the absence of prior information, prior distributions solely derived from the sample distribution $f(x|\theta)$

Uniform priors (Laplace's priors)

Equiprobability of elementary events: the same likelihood to each value of θ

$$\theta \in \{\theta_1, \dots, \theta_p\} \longrightarrow f(\theta_i) = \frac{1}{p}$$

Extensions to continuous spaces:

$$f(\theta) \propto 1 \quad (= \text{constant})$$

Examples:

Location parameters: $f(x|\theta) = f(x - \theta) \longrightarrow f(\theta) \propto 1$

Scale parameters: $f(x|\theta) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right) \longrightarrow f(\theta) \propto \frac{1}{\theta} \quad (f(\log \theta) \propto 1)$

Some drawbacks

Lack of invariance through reparameterization: $\theta \longrightarrow \eta = g(\theta)$

$$f(\theta) \propto 1 \longrightarrow f(\eta) \propto \left| \frac{dg^{-1}(\eta)}{d\eta} \right| \neq \text{constant} \quad (\text{Jacobian formula})$$

Information is not missing anymore !!

May generate improper posterior:

$$x \sim \mathcal{N}(\theta, \sigma^2) \quad \text{with} \quad f(\theta, \sigma^2) \propto 1$$

Then

$$f(\theta, \sigma^2 | x) \propto f(x | \theta) \propto \sigma^{-1} \exp\left(-\frac{(x - \theta)^2}{2\sigma^2}\right)$$
$$\implies f(\sigma^2 | x) \propto 1 \quad \text{is improper, paradoxes occur}$$

\implies Invariant priors

\implies Jeffreys' priors as an alternative

The Jeffreys' priors

Based on Fisher information

Univariate case:

$$I(\theta) = E_X \left[\left(\frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \right] = -E_X \left[\frac{\partial^2 \log f(X|\theta)}{\partial \theta^2} \right]$$

Multivariate case:

$$I(\theta)_{ij} = -E_X \left[\frac{\partial^2 \log f(X|\theta)}{\partial \theta_i \partial \theta_j} \right]$$

The Jeffreys' prior distribution is $f(\theta) \propto |I(\theta)|^{1/2}$

where $|I(\theta)|$ is the determinant of the Fisher Information matrix

Exponential family: if $f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta))$ then

$$I(\theta) = \nabla^2 \Psi(\theta) \text{ and } f(\theta) \propto \left(\frac{\partial^2 \Psi(\theta)}{\partial \theta_i^2} \right)^{1/2}$$

Key feature: Reparameterization invariance

Assume $f(\theta) \propto |I(\theta)|^{1/2}$ and $\eta = g(\theta)$ for a 1-to-1 mapping g

$$\begin{aligned} f(\eta) &= f(\theta) \left| \frac{\partial \theta}{\partial \eta} \right| \propto \sqrt{|I(\theta)| \left(\frac{\partial \theta}{\partial \eta} \right)^2} \\ &\propto \sqrt{E_X \left[\left(\frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \left(\frac{\partial \theta}{\partial \eta} \right)^2 \right]} \\ &\propto \sqrt{E_X \left[\left(\frac{\partial \log f(X|\theta)}{\partial \eta} \right)^2 \right]} \\ &\propto |I(\eta)|^{1/2} \end{aligned}$$

Other features

- ▶ **Information based:** $I(\theta)$ corresponds to the amount of information brought by the model on θ .
Noninformative: Minimize the effect of the prior which is in accordance with the model.
- ▶ Violates the likelihood principle
- ▶ Usually improper
- ▶ **May lead to incoherences in multidimensional case**
- ▶ Have been generalized into reference priors (Berger and Bernardo) by distinguishing between nuisance and interest parameters

Example: $x \sim \mathcal{N}(\mu, \sigma)$

- ▶ $\theta = (\mu, \sigma)$ unknown: $f(\theta) \propto 1/\sigma^2$

$$\begin{aligned} \text{because } I(\theta) &= E_X \left[\begin{pmatrix} 1/\sigma^2 & 2(x-\mu)/\sigma^3 \\ 2(x-\mu)/\sigma^3 & 3(x-\mu)^2/\sigma^4 - 1/\sigma^2 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix} \end{aligned}$$

- ▶ $\theta = \mu, \sigma$ fixed: $f(\mu) \propto 1$
- ▶ $\theta = \sigma, \mu$ fixed: $f(\sigma) \propto 1/\sigma$
- ▶ μ and σ a priori independent: $f(\theta) = f(\mu)f(\sigma) \propto 1/\sigma$

Hierarchical modelling

Consider a conjugate prior for $f(x|\theta)$: $f_1(\theta|\lambda)$

$f_1(\theta|\lambda)$ may be too restrictive and require specification of λ .

λ unknown \longrightarrow add a noninformative prior on λ :

$$\begin{aligned}\lambda &\sim f_2(\lambda) \\ \theta|\lambda &\sim f_1(\theta|\lambda) \\ x|\theta &\sim f(x|\theta)\end{aligned}$$

The prior on θ is then $f(\theta) = \int f_1(\theta|\lambda)f_2(\lambda) d\lambda$

- ▶ not conjugate anymore
- ▶ heavier tails (eg. Student distributions or Gaussian scale mixtures)
- ▶ **Computationally flexible**

Posterior distributions

Computing posterior distributions

Posteriors are not always tractable...

Observed data: $x = \{x_1, \dots, x_N\}$ eg. a discretized signal

Hidden variables: $z = \{z_1, \dots, z_M\}$. eg. a segmentation or a clean version of x

Add prior knowledge on z but if the dependence structure in z is too complex (eg an image), $f(z|x)$ can't be obtained analytically

Solution: "Approximate" the dependence structure

- ▶ Sampling methods (Gibbs sampler, MCMC)
- ▶ Approximations (Laplace, Variational Bayes, EP)

Conclusion

- ▶ Maximum likelihood for large training data. Risk of overfitting for small data set.
- ▶ Bayesian framework to incorporate prior information (eg temporal dynamics, spatial relationships) and prevent overfitting
- ▶ MMSE and MAP provide point estimates that use prior information
- ▶ For fully Bayesian treatment, use predictive distributions
- ▶ If posterior distributions are not tractable, use sampling methods (eg MCMC) or approximate inference (eg Variational Bayes)

Main references

- ▶ C. P. Robert. The Bayesian Choice. Springer texts in statistics.
- ▶ A. Gelman, J. B. Carlin, H. S. Stern, D. Rubin. Bayesian data analysis. Texts in statistical science. Chapman & Hall CRC.