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# Portfolio Management

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# Overview

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## Laws of Returns of Financial Assets

# Returns and Rate of Returns

- Return of an investment  $W_0$  :  $R_T = \frac{W_T}{W_0} - 1$
- Return of an (investment in an) asset  $P_t$ :  $R_T = \frac{P_T + Div(0, T)}{P_0} - 1$
- Different ways to define a rate of return:
  - monetary rate:  $1 + r \times T = 1 + R_T$
  - actuarial rate:  $(1 + r)^T = 1 + R_T$
  - exponential rate:  $\exp(r \times T) = 1 + R_T$
- Different ways to define  $T$ : 30/360, Act/360, Act/Act, Act/365
- For modelization, exponential rates make calculations simpler:
  - time scale properties:  $\exp(\frac{r}{\lambda} \lambda T) = \exp(rT)$
  - simplicity to compound interests:  $\exp(rT_1)\exp(rT_2) = \exp(r(T_1 + T_2))$
  - well suited for modelization in continuous time  
 $dP_t = rP_t dt \implies P_T = P_0 e^{rT}$
  - enable to conserve the property of normality when assuming the independence of the returns

- **Probabilist** definitions for a r.v  $X$

- Expectation:  $E[X]$
- Variance:  $Var[X] = E[X^2] - E[X]^2$
- Standard deviation:  $\sigma[X] = \sqrt{Var[X]}$
- Skew:  $Skew[X] = E \left[ \left( \frac{X - E(X)}{\sigma(X)} \right)^3 \right]$
- Kurtosis:  $Kur[X] = E \left[ \left( \frac{X - E(X)}{\sigma(X)} \right)^4 \right]$

- **Empirical** definitions for a sample  $x = (x_1, x_2, \dots, x_n)$

- Sample Mean:  $\bar{x} = \hat{E}(x) = \frac{1}{n} \sum_{i=1}^{i=n} x_i$

- Sample Variance:  $\widehat{Var}(x) = \frac{1}{n} \sum_{i=1}^{i=n} (x_i - \bar{x})^2$

- Sample Standard Deviation:  $\hat{\sigma}(x) = \sqrt{\widehat{Var}(x)}$

- Sample Skew:  $\widehat{Skew}(x) = \frac{1}{n} \sum_{i=1}^{i=n} \left( \frac{x_i - \bar{x}}{\hat{\sigma}(x)} \right)^3$

- Sample Kurtosis:  $\widehat{Kur}(x) = \frac{1}{n} \sum_{i=1}^{i=n} \left( \frac{x_i - \bar{x}}{\hat{\sigma}(x)} \right)^4$

- **Remark:** the empirical quantities can be obtained from the probabilist definitions by taking  $P(X_i = x_i) = \frac{1}{n}$

- **Properties** skewness:

- anti-symmetry:  $Skew(-X) = -Skew(X)$
- scale invariance: if  $\lambda > 0$ ,  $Skew(\lambda X) = Skew(X)$
- location invariance:  $\forall \lambda$ ,  $Skew(X + \lambda) = Skew(X)$
- for  $X \sim N(m, \sigma^2)$ ,  $Skew(X) = 0$

- **Properties** kurtosis:

- symmetry:  $Kur(-X) = Kur(X)$
- scale invariance: if  $\lambda \neq 0$ ,  $Kur(\lambda X) = Kur(X)$
- location invariance:  $\forall \lambda$ ,  $Kur(X + \lambda) = Kur(X)$
- for  $X \sim N(m, \sigma^2)$ ,  $Kur(X) = 3$

- **Definition:** if  $Kur(X) > 3$  the distribution is leptokurtic if  $Kur(X) < 3$  the distribution is platykurtic



- **Remark:** the properties above are true for the empirical quantities as well
- **Exercise:** demonstrate the properties above
- **Remark:** for a mixture of normal distributions we have: kurtosis  $> 3$

## Theorem (admitted) and the Berra and Jarque test

Let  $X_1, X_2, \dots, X_n$  be i.i.d  $N(m, \sigma^2)$  and  $X = (X_1, X_2, \dots, X_n)$  then asymptotically:

- $\sqrt{n}\widehat{Skew}(X) \sim N(0, 6)$
- $\sqrt{n}[\widehat{Kur}(X) - 3] \sim N(0, 24)$

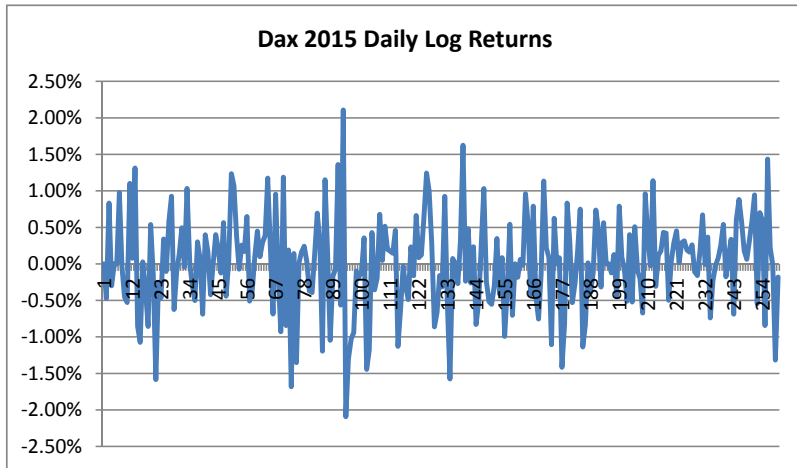
Let  $\widehat{BJ}(X) = \frac{n}{6}[\widehat{Skew}(X)]^2 + \frac{n}{24}[\widehat{Kur}(X) - 3]^2$  then asymptotically:

- $\widehat{BJ}(X) \sim \chi^2(2)$

Let  $\chi_{1-\alpha}^2(2)$  be such that  $P(\chi^2 > \chi_{1-\alpha}^2(2)) = 1 - \alpha$

Then for  $n$  large enough the Berra and Jarque test rejects at confidence level  $\alpha$  the normality hypothesis iff:  $\widehat{BJ}(x) > \chi_{1-\alpha}^2(2)$

# Goodness of Fit Tests



**Example:** We calculate the daily log returns of the GDAX for year 2015

- number of daily returns 255,  $x = (x_1, x_2, \dots, x_{252})$
- $\bar{x} = 0.02\%$ ,  $\hat{\sigma}(x) = 0.63\%$
- $\widehat{Skew}(x) = -0.15$ ,  $\widehat{Kur}(x) = 3.56$  (fat tails)
- $\widehat{BJ}(x) = 4.32$ ,  $\chi_{5\%}^2(2) = 5.99$

So we accept the normality hypothesis at confidence level 95%

**Remark:** The Berra and Jarque's test is very sensitive to outliers

## Further Statistical Tests

## Theorem of Parzen Rosenblatt (admitted)

Let  $X$  be a random variable of density function  $f(x)$

Let  $X_i$  be i.i.d variables of the same law as  $X$

Let  $K$  be positive of integral 1 and  $(h_n)_{n \in \mathbb{N}}$  such that  $h_n \rightarrow 0$  and  $nh_n \rightarrow \infty$

Let  $f_n(x)$  be defined by  $f_n(x) = \frac{1}{n} \sum_{i=1}^n K\left(\frac{X_i - x}{h_n}\right)$

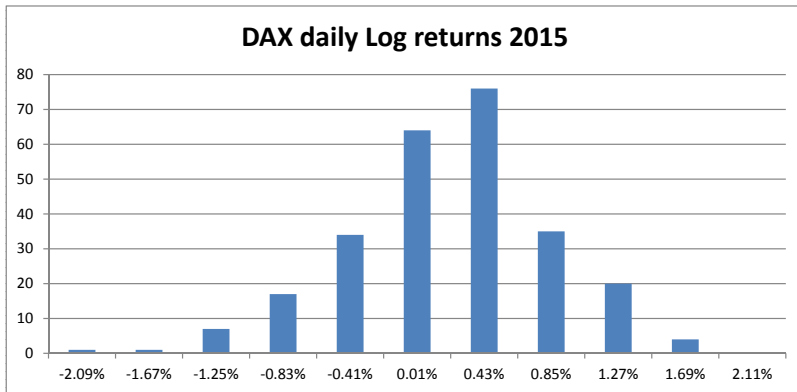
then  $\sqrt{nh_n}[f_n(x) - f(x)] \rightarrow N(0, f(x) \int_{-\infty}^{+\infty} K^2(x) dx)$

## Exemples of Kernels

- Rectangular Kernel  $K(u) = \frac{1}{2} \mathbf{1}_{|u| < 1}$
- Gaussian Kernel  $K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$

**Remark:** "visual test" where the estimated density is usually compared to the density of a normal distribution with the same mean and variance as the empirical mean and variance of the sample.

# Tests Based on Density Function Estimates



# Tests Based on Cumulative Distribution Function Estimates

Let  $X$  and  $(X_i)_{i \in [1, n]}$  be i.i.d r.v with the same laws

Let  $F_n(x) = \frac{1}{n} \sum_{i=1}^{i=n} 1_{X_i \leq x}$  and  $\|F_n(x) - F(x)\|_\infty = \sup_x |F_n(x) - F(x)|$

Central Limit Theorem (admitted)

$\forall x, F_n(x) \rightarrow F(x)$  p.s

Glivenko Cantelli Theorem (admitted)

$\|F_n(x) - F(x)\|_\infty \rightarrow 0$  p.s

Kolmogorov Smirnov Theorem (admitted)

$\sqrt{n} \|F_n(x) - F(x)\|_\infty \sim K$  when  $n$  is large

$K$  is the Kolmogorov's law and is independant from  $F$



# Tests Based on Cumulative Distribution Function Estimates

Several Goodness of fit tests are based on Kolmogorov Smirnov's theorem with  $F$  being a normal cdf with the same mean and variance as the sample observed. Amongst these tests:

- Kolmogorov Smirnov's test
- Cramer von Mises's test
- Anderson Darling's test

## Definition:

if  $Z$  is a random variable the  $\alpha$  quantile of  $Z$  is the smallest number  $q_Z(\alpha)$  such that  $P(Z \leq q_Z(\alpha)) \geq \alpha$

if  $z = (z_1, z_2, \dots, z_n)$  is a sample with frequency  $(f_1, f_2, \dots, f_n)$  the  $\alpha$  quantile of  $z$ ,  $\hat{q}_z(\alpha)$  is the quantile of the r.v  $Z$  defined by  $P(Z = z_i) = f_i$

**Remark 1:** if we have 10 observations:  $\{z_i = i\}_{i \in [1,10]}$ :

$$\hat{q}_z(1) = 10 \text{ and } \forall i \in \{1, \dots, 9\} \forall \alpha \in ]\frac{i}{10}, \frac{i+1}{10}] \hat{q}_z(\alpha) = i + 1$$

**Remark 2:** if  $Z \sim N(0, 1)$ :  $q_Z(0.5) = 0$  and  $q_Z(97.5\%) = 1.96$

# Tests Based on Order Statistics

**Exercise 1:** Show that if  $F_Z$  is invertible then:  $q_Z(\alpha) = F_Z^{-1}(\alpha)$

**Exercise 2:** Show that if  $F_Z$  is invertible then:  $F_Z(Z) \sim U([0, 1])$

**Exercise 3:** Show that if  $Z_1 \sim N(m_1, \sigma_1^2)$  and  $Z_2 \sim N(m_2, \sigma_2^2)$  then:  $\{(q_{Z_1}(\alpha), q_{Z_2}(\alpha)), \alpha \in [0, 1]\}$  is a line.

## Theorem: order statistics (admitted) and QQ-Plot test

Let  $N \sim N(0, 1)$  and  $z(n) = (z_1, z_2, \dots, z_n)$  be a sample for the r.v  $Z$  we want to test. Then when  $n$  is large:

- $\hat{q}_{z(n)}(\frac{i}{n}) \sim q_Z(\frac{i}{n})$  and

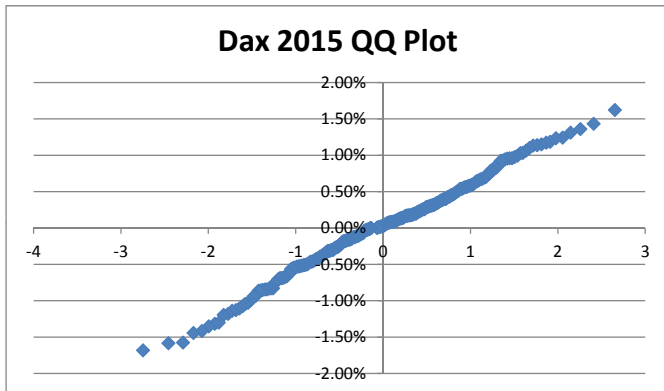
if  $Z$  is a normal Law:

- The points  $(q_N(\frac{i}{n}), \hat{q}_{z(n)}(\frac{i}{n}))$  should be "concentrated" around a line (called the Henry's line)

# Tests Based on Order Statistics

**Remark 1:** The test can be used "visually" but can also be quantified through the Shapiro-Wilk test

**Example:** QQ plot GDAX returns for 2015



## (Reminder) Definition

- if  $X \sim N(0, 1)$  then  $X^2 \sim \chi^2(1)$
- if  $X_i \sim N(0, 1)$  i.i.d then  $\sum_{i=1}^{i=n} X_i^2 \sim \chi^2(n)$

## (Reminder) Theorem and Definition

If  $X_1 \sim N(M_1, Id_{\mathbb{R}^d})$  and  $X_2 \sim N(M_2, Id_{\mathbb{R}^d})$  then:

$\|M_1\| = \|M_2\| \implies \|X_1\|^2$  and  $\|X_2\|^2$  have the same law and this law is called  $\chi^2(d, \|M_1\|^2)$

demonstration:

$\|M_1\| = \|M_2\| \implies \exists A$  orthonormal such that  $AM_1 = M_2$

Let's consider  $AX_1$  then:

- $X_1$  has a Normal law  $\implies AX_1$  has a Normal law
- $E[AX_1] = AE[X_1]$
- $Var[AX_1] = Cov(AX_1, AX_1) = ACov(X_1, X_1)A' = AId_{\mathbb{R}^d}A' = Id_{\mathbb{R}^d}$

so,  $AX_1$  and  $X_2$  are both normal with the same mean and variance

so,  $AX_1 \sim_{Law} X_2$  and consequently  $\|AX_1\|^2 \sim_{Law} \|X_2\|^2$ .

As  $\|AX_1\|^2 = \|X_1\|^2$  (because  $A$  is orthonormal) by transitivity

$\|X_1\|^2 \sim_{Law} \|X_2\|^2$ . Q.E.D

## (Reminder) Definition

- if  $X \sim N(0, 1)$  and  $Z \sim \chi^2(d)$  then  $\frac{X}{\sqrt{\frac{Z}{d}}}$  is called a Student Law and is noted  $t(m, d)$ . If  $m = 0$  we simply note  $t(d)$

## Theorem: Student Law for Confidence Intervals

Let  $X = (X^1, X^2, \dots, X^n)$  with  $X_i \sim N(m, \sigma^2)$  i.i.d

Let  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$  and  $\hat{\sigma}(X) = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}$  then:

- $\bar{X}$  and  $\hat{\sigma}(X)$  are independant
- $\sqrt{n}(\frac{\bar{X}-m}{\sigma}) \sim N(0, 1)$
- $n(\frac{\hat{\sigma}(X)}{\sigma})^2 \sim \chi^2(n-1)$
- $\frac{\bar{X}-m}{\frac{\hat{\sigma}(X)}{\sqrt{n-1}}} \sim t(n-1)$

# Parameter Estimations and Confidence Intervals

Sketch of the proof: we just need to show the result for  $X^i \sim N(0, 1)$  i.i.d

Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  be defined by  $h : \begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix} \rightarrow \begin{pmatrix} \bar{X}1_n \\ \vdots \\ X - \bar{X}1_n \end{pmatrix}$  then:

- $X$  Gaussian and  $h$  linear  $\Rightarrow h(X)$  is Gaussian
- $\forall i \text{ Cov}(\bar{X}, X^i - \bar{X}) = 0 \Rightarrow \bar{X}1_n$  and  $X - \bar{X}1_n$  are independent because for Gaussian vectors zero covariance means independence
- $\langle X - \bar{X}1_n, 1_n \rangle = 0 \Rightarrow X - \bar{X}1_n \in (\mathbb{R}1_n)^\perp$

Note that  $\forall u \in (\mathbb{R}1_n)^\perp$

$$\begin{aligned} \text{Var}(\langle u, X - \bar{X}1_n \rangle) &= \text{Var}(\langle u, X \rangle - \bar{X} \langle u, 1_n \rangle) \\ &= \text{Var}(\langle u, X \rangle) = u' \text{Var}(X) u = \|u\|^2 \end{aligned}$$

so if we call  $\begin{pmatrix} 0 \\ Z \end{pmatrix}$  the component of  $X - \bar{X}1_n$  in an orthonormal basis of

$\mathbb{R}^n$  whose first vector is in  $\mathbb{R}1_n$  we have  $Z \sim N(0, Id_{\mathbb{R}^{n-1}})$

and so  $\|X - \bar{X}1_n\|_n^2 = \|Z\|_{n-1}^2 \sim_{Law} t(n-1)$  Q.E.D.



**Exemple:** GDAX returns for 2015

- $\bar{X} = 0.02\%$
- $\frac{\hat{\sigma}(X)}{\sqrt{n}} = 0.04\%$

So at confidence level 95% the expected daily rate of return is in the interval  $[-0.06\%, 0.09\%]$

# Utility Functions

## Definition: Utility functions

A utility function is any function  $u : \mathbb{R} \rightarrow \mathbb{R}$  such that  $u$  is continuous, strictly increasing and two times differentiable.

**Remark:** based on the definition  $u$  is invertible

## Definition: Preferred Investment and Risk Premium

If  $X$  and  $Y$  are two random payoffs:

- $X \succ Y$  ( $X$  is preferred to  $Y$  for  $u$ )  $\Leftrightarrow E[u(X)] > E[u(Y)]$

Let  $C_u(X)$  be the constant defined by  $u[C_u(X)] = E[u(X)]$

- $C_u(X)$  is called the certain equivalent to  $X$
- $\Pi^u(X) = E[X] - C_u(X)$  is called the risk premium for  $X$

## Properties (exercise)

- $u$  convex  $\Leftrightarrow \forall X, u(E[X]) \leq E[u(X)] \Leftrightarrow \forall X, \Pi^u(X) \leq 0 \Leftrightarrow$  risk taker
- $u$  concave  $\Leftrightarrow \forall X, u(E[X]) \geq E[u(X)] \Leftrightarrow \forall X, \Pi^u(X) \geq 0 \Leftrightarrow$  risk adverse
- $u$  affine  $\Leftrightarrow \forall X, u(E[X]) = E[u(X)] \Leftrightarrow \forall X, \Pi^u(X) = 0 \Leftrightarrow$  risk neutral

## Theorem: Risk Aversion Measure

Let  $u$  and  $v$  be two utility functions strictly concave or convex then:

$$(\forall X \text{ discrete r.v.}, \Pi^u(X) \geq \Pi^v(X)) \Leftrightarrow (\forall a, -\frac{u''(a)}{u'(a)} \geq -\frac{v''(a)}{v'(a)})$$

## Remarks:

- $-\frac{u''}{u'}$  defines the risk aversion/concavity of  $u$
- $\forall \lambda \neq 0, \lambda u$  and  $u$  have the same risk aversion

## Demonstration theorem:

Let  $u$  be a utility function two times differentiable and strictly convex or concave (i.e.  $u'' \neq 0$ ).

Demonstration  $\Rightarrow$ :

Let  $X_h^a$  be defined by:  $P(X_h^a = a) = \frac{1}{2}$  and  $P(X_h^a = a + h) = \frac{1}{2}$

Let  $C_u^a(h) = C_u(X_h^a)$  i.e.  $u(C_u^a(h)) = E[u(X_h^a)] = \frac{1}{2}u(a) + \frac{1}{2}u(a + h)$

by derivation on  $h$  of  $u(C_u^a(h)) = \frac{1}{2}u(a) + \frac{1}{2}u(a + h)$  we obtain:

- $C_u^a(0) = a$
- $C_u^{a'}(0) = \frac{1}{2}$
- $C_u^{a''}(0) = \frac{1}{4} \frac{u''(a)}{u'(a)}$

# Preferred investments

$$\begin{aligned} \text{Now, } \Pi^u \geq \Pi^v &\Rightarrow \forall a, \forall h, \Pi^u(X_h^a) \geq \Pi^v(X_h^a) \\ &\Rightarrow \forall h, E[X] - C_u^a(h) \geq E[X] - C_v^a(h) \\ &\Rightarrow -C_u^{a''}(0) \geq -C_v^{a''}(0) \text{ (as in 0 the value and first derivatives are equal)} \\ &\Rightarrow -\frac{u''(a)}{u'(a)} \geq -\frac{v''(a)}{v'(a)} \text{ (and this is true for all } a\text{). Q.E.D} \end{aligned}$$

Demonstration  $\Leftarrow$

Let  $X$  be a discrete variable with  $P(X = a_i) = p_i$

$$\Pi^u(X) \geq \Pi^v(X) \Leftrightarrow C_v(X) - C_u(X) \geq 0$$

$$\Leftrightarrow v^{-1}[E(v(X))] - u^{-1}[E(u(X))] \geq 0$$

$$\Leftrightarrow v^{-1}\left(\sum_{i=1}^{i=n} p_i v(a_i)\right) - u^{-1}\left(\sum_{i=1}^{i=n} p_i u(a_i)\right) \geq 0$$

$$\Leftrightarrow \sum_{i=1}^{i=n} p_i v(a_i) \geq v \circ u^{-1}\left(\sum_{i=1}^{i=n} p_i u(a_i)\right) \text{ (as } v \text{ is increasing)}$$

which should be true if  $v \circ u^{-1}$  is convex.

# Preferred investments

Let's calculate  $(v \circ u^{-1})''$  to prove that  $v \circ u^{-1}$  is convex

$$(v \circ u^{-1})' = (v' \circ u^{-1})(u^{-1})'$$

$$(v \circ u^{-1})'' = (v'' \circ u^{-1})[(u^{-1})']^2 + (v' \circ u^{-1})(u^{-1})''$$

$$\text{Now, } (u^{-1})' = \frac{1}{u' \circ u^{-1}} \text{ and } \left(\frac{1}{u' \circ u^{-1}}\right)' = -\frac{(u'' \circ u^{-1})(u^{-1})'}{(u' \circ u^{-1})^2} = -\frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})^3}$$

$$\text{so, } (v \circ u^{-1})'' \geq 0 \Leftrightarrow \frac{(v'' \circ u^{-1})}{(u' \circ u^{-1})^2} - (v' \circ u^{-1}) \frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})^3} \geq 0$$

$$\Leftrightarrow (v'' \circ u^{-1}) \geq (v' \circ u^{-1}) \frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})}$$

$$\Leftrightarrow -\frac{(v'' \circ u^{-1})}{(v' \circ u^{-1})} \leq -\frac{(u'' \circ u^{-1})}{(u' \circ u^{-1})} \text{ Q.E.D as we assumed here } -\frac{v''}{v'} \leq -\frac{u''}{u'}$$

## Remarks:

If there is aversion to risk we use  $u$  concave to modelize

For  $u(a) = \exp(-\lambda a)$  (which is often used)  $-\frac{u''}{u'} = \lambda$

# Gaussian Laws and Mean Variance Implications

When investing  $W_0$  the utility function criteria picks strategies  $\pi$  which maximizes:  $E[u(W_T^\pi)]$  where  $\frac{W_T^\pi}{W_0} = 1 + R_T^\pi$

If we assume  $R_T^\pi \sim N(m_\pi, \sigma_\pi^2)$  then:  $E[u(W_T^\pi)] = E[u(W_0 + W_0 R_T^\pi)]$   
 $= E[u(W_0 + m_\pi W_0 + \sigma_\pi W_0 Z)]$  with  $Z \sim N(0, 1)$

If we define  $U(m, \sigma) = E[u(W_0 + mW_0 + \sigma W_0 Z)]$  then:

$$\max_{\pi} E[u(W_T^\pi)] = \max_{\pi} U(m_\pi, \sigma_\pi)$$

So in practice the model consists in maximizing  $\max_{\pi} U(m_\pi, \sigma_\pi)$  where  $U$  is derived from a utility function.



# Gaussian Laws and Mean Variance Implications

From now on we will modelize directly with a function  $U$  and we will assume that  $U$  is increasing in  $m$  and decreasing in  $\sigma$  which means that:

- if the investor has the choice between two strategies with the same expected returns he prefers the one with the less variance on the returns
- if the investor has the choice between two strategies with the same variance on the returns he prefers the one with the higher expected returns

In practice an investor will define the level of risks he accepts and based on this will find the efficient portfolio (that maximizes the expected return). Based on this we make the following definition

# Efficient Investment Strategies

## Definition: Efficient investment strategy

An investment strategy  $\pi_e$  is efficient iff for all  $\pi$ :

$$E[R^\pi] \geq E[R^{\pi_e}] \Rightarrow \sigma(R^\pi) \geq \sigma(R_e^\pi)$$

## Proposition:

$$\pi_e \text{ is a solution of } \begin{cases} \min_{\pi} \sigma(R^\pi) \\ E[R^\pi] = E[R^{\pi_e}] \end{cases}$$

**Demonstration:** If the solution of the constraint minimization was less than  $\sigma(R^{\pi_e})$  then we could find  $\pi$  such that  $\sigma(R^\pi) < \sigma(R^{\pi_e})$  and

$E[R^\pi] = E[R^{\pi_e}]$  but the contraposition of

$E[R^\pi] \geq E[R^{\pi_e}] \Rightarrow \sigma(R^\pi) \geq \sigma(R_e^\pi)$  implies that

$\sigma(R^\pi) < \sigma(R_e^\pi) \Rightarrow E[R^\pi] < E[R^{\pi_e}]$

So  $\pi_e$  is solution of the constraint problem. QED

## Remarks: Definitions and Geometric interpretation

If we note:

- $\mathcal{E} = \{\pi_e \text{ efficient}\}$  and
- $\mathcal{E}(\sigma, m) = \left\{ \begin{pmatrix} \sigma(R^{\pi_e}) \\ E(R^{\pi_e}) \end{pmatrix}, \pi_e \in \mathcal{E} \right\}$

Then any point  $\begin{pmatrix} \sigma(R^\pi) \\ E(R^\pi) \end{pmatrix}$  is on the right of  $\mathcal{E}(\sigma, m)$

## Markowitz: The mean variance framework

# Notations and Definitions

## We note:

- $S_t^i$  : the value of share  $i$  at time  $t$
- $q_i$  : the number of shares  $i$  held ( $q_i > 0$ ) or shorted ( $q_i < 0$ ) at time 0
- $x_0$  : the notional
- $\pi_i = \frac{q_i S_0^i}{x_0}$  the allocation in asset  $i$  at time 0.
- $\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_n \end{pmatrix}$  the allocation at time 0 and  $R_T = \begin{pmatrix} R_T^1 \\ \vdots \\ R_T^n \end{pmatrix}$  the vector of returns of the assets  $S^i$

**Remark:** A portfolio is defined by  $(x_0, \pi)$  and we note:

- $W_0(x_0, \pi) = \sum_{i=1}^{i=n} x_0 \pi_i$  the Value at time 0 of portfolio  $(x_0, \pi)$
- $W_t(x_0, \pi) = \sum_{i=1}^{i=n} x_0 \pi_i \frac{S_t^i}{S_0^i}$  the Value at time  $t$  of portfolio  $(x_0, \pi)$

## Definition:

- $(x_0, \pi)$  is an investment portfolio iff  $\sum_{i=1}^{i=n} \pi_i = 1$
- $(x_0, \pi)$  is a self financing portfolio iff  $\sum_{i=1}^{i=n} \pi_i = 0$

## Exercise:

Show that A if  $(x_0, \pi)$  is an investment portfolio  $W_0^\pi = x_0$

Show that A if  $(x_0, \pi)$  is a self financing portfolio  $W_0^\pi = 0$

## Remark:

$\pi^i < 0$  means that the strategy has a short position in asset  $i$

$S^i$  is called a risky asset iff  $R_T^i$  is not determinist i.e  $Var(R_T^i) \neq 0$

# Notations and Definitions

**Example:** when we consider various allocations  $(x_0, \pi)$  between two shares  $S_0^1 = 100$   $S_0^2 = 50$  we get the following table:

$x_0$	$\pi_1$	$\pi_2$	$q^1$	$q^2$	$W_0$
100	1	-1	1	-2	0
1000	1	-1	10	-20	0
100	0.5	0.5	0.5	1	100
1000	0.5	0.5	5	10	1000

# Notations and Definitions

## Definition Return:

We define the return of a portfolio  $(x_0, \pi)$  as:  $R_T = \frac{W_T(x_0, \pi) - W_0(x_0, \pi)}{x_0}$

## Proposition

For an investment portfolio this definition corresponds to the definition of return of an asset as:  $x_0 = W_0(x_0, \pi)$

For a self financing portfolio the definition corresponds to the quantity  $\frac{W_T(x_0, \pi)}{x_0}$

We note  $R_T^i$  the return of asset  $S^i$ , i.e  $R_T^i = \frac{S_T^i}{S_0^i} - 1$

## Proposition

For any investment or self financing portfolio  $(x_0, \pi)$  the return  $R_T$  verifies:

$$R_T = \sum_{i=1}^{i=n} \pi_i R_T^i.$$

As this expression depends only on  $\pi$  and not on  $x_0$  we note it  $R_T^\pi$ .



## Proposition:

For any investment or self-financing portfolio  $(x_0, \pi)$  we have:

- $R_T^\pi = \pi' R_T$
- $E(R_T^\pi) = E(\pi' R_T) = \pi' E[R_T]$
- $Cov(R_T^{\pi_1}, R_T^{\pi_2}) = Cov(\pi_1' R_T, \pi_2' R_T) = \pi_1' Cov[R_T, R_T] \pi_2'$

**Exercise:** demonstrate the proposition

**We note:**

- $\sigma^\pi = \sigma(R_T^\pi)$
- $m^\pi = E(R_T^\pi)$

# Markowitz's Framework

We are going to consider two cases, each based on the existence or not of a risk-free investment strategy:

## First case:

There are  $n$  risky assets  $(S^i)_{i \in [1, n]}$  and no risk-free asset.

## Second case:

There are  $n$  risky assets  $(S^i)_{i \in [1, n]}$  and one risk-free asset  $S^0$ .

## Remarks:

In both cases it is natural to assume that the  $R_T^i$  are such as :

- (H1) there is no way to build a risk-free investment portfolio based on the  $(S^i)_{i \in [1, n]}$
- (H2) there is no way to build a risk-free self-financing portfolio (other than zero) based on the  $(S^i)_{i \in [1, n]}$

## Comments:

The reason why we assume (H1) in the first case is because otherwise we could build a risk-free asset. The reason why we assume (H1) in the second case is because otherwise we could build a second risk-free asset and then:

- either this second risk-free asset has the same return as the risk-free asset and in this case one risky asset could be replicated (and then should be disregarded).
- or this second risk-free asset has a different return from the risk-free asset, in which case there would be some arbitrage opportunities.

The reason why we assume (H2) is because for a risk-free self-financing strategy:

- if the return is different from zero there are some arbitrage opportunities
- if the return is different is zero one risky asset can be replicated (and then should be be disregarded)

## We Note :

- $\Sigma = \text{Cov}[R_T, R_T]$
- $M = E[R_T]$

## Proposition:

(H1) and (H2)  $\implies \Sigma$  is invertible

## Demonstration:

If  $\Sigma$  was not invertible we could find  $\pi \neq 0$  such that  $\pi' \Sigma \pi = 0$  and then there would be two cases:

either  $\pi' 1_n = 0$  in which case we could find a self-financing portfolio made of the risky assets without risk or

$\pi' 1_n \neq 0$  in which case we could find an investment portfolio made of the risky assets without risk.

in both cases this would be in contradiction with (H1) or (H2).

## Markowitz: Optimization without a risk-free asset

# Optimization without a risk-free asset

From the utility framework and in the context of normal distributions assumptions the efficient investment portfolios  $(x_0, \pi)$  are the solutions of

$$(P) \begin{cases} \min_{\pi} \pi' \Sigma \pi \\ \pi' M = m \text{ where } 1_n \text{ is the vector of } \mathbb{R}^n \text{ with components equal to } 1 \\ \pi' 1_n = 1 \end{cases}$$

We define a new scalar product in  $\mathbb{R}^n$  by  $\langle x, y \rangle_{\Sigma^{-1}} = x' \Sigma^{-1} y$

We can write (P) as

$$(P) \begin{cases} \min_{\pi} \langle \Sigma \pi, \Sigma \pi \rangle_{\Sigma^{-1}} \\ \langle \Sigma \pi, M \rangle_{\Sigma^{-1}} = m \\ \langle \Sigma \pi, 1_n \rangle_{\Sigma^{-1}} = 1 \end{cases}$$

(P) can be solved either by writing its Lagrangien or geometrically by noticing that  $\Sigma \pi$  must be in  $\text{Vect}(M, 1_n)$  in order to minimize its  $\langle, \rangle_{\Sigma^{-1}}$  norm while satisfying the constraints. We note  $\mathcal{F}$  all the investment portfolios solutions of (P) for any possible value of  $m$ .

## We note:

- $a = \langle 1_n, 1_n \rangle_{\Sigma^{-1}}$  and
- $b = \langle M, 1_n \rangle_{\Sigma^{-1}}$

Excluding for now the case  $M - \frac{b}{a}1_n = 0$  and noticing that  $\langle M - \frac{b}{a}1_n, 1_n \rangle_{\Sigma^{-1}} = 0$  we get that  $M - \frac{b}{a}1_n$  and  $1_n$  form an orthogonal basis of  $\text{Vect}(M, 1_n)$ . So we can write:

$$\Sigma \pi_e = \lambda 1_n + \nu (M - \frac{b}{a}1_n) \text{ or } \pi_e = \lambda \Sigma^{-1} 1_n + \nu \Sigma^{-1} (M - \frac{b}{a}1_n)$$

We can now renormalize this decomposition.

## We note:

- $\pi_a = \frac{1}{a} \Sigma^{-1} 1_n$  which satisfies  $\pi_a' 1_n = 1$  and
- $\omega_{a,b} = \frac{\Sigma^{-1} (M - \frac{b}{a}1_n)}{\|M - \frac{b}{a}1_n\|_{\Sigma^{-1}}}$  which satisfies  $\omega_{a,b}' 1_n = 0$  and  $\text{var}[R_T^{\omega_{a,b}}] = 1$

## Proposition: Building Blocks

- $\pi_a' \mathbf{1}_n = 1$  (so  $\pi_a$  is an investment portfolio).
- $m^{\pi_a} = \frac{b}{a}$
- $\sigma^{\pi_a} = \frac{1}{\sqrt{a}}$
- $\omega_{a,b}' \mathbf{1}_n = 0$  (so  $\omega_{a,b}$  is a self-financing portfolio).
- $m^{\omega_{a,b}} = \|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}$
- $\sigma^{\omega_{a,b}} = 1$
- $\text{cov}(R^{\pi_a}, R^{\omega_{a,b}}) = 0$



## Demonstration:

$$\pi_a' \mathbf{1}_n = \mathbf{1}_n' \pi_a = \frac{1}{a} \mathbf{1}_n' \Sigma^{-1} \mathbf{1}_n = 1$$

$$m^{\pi_a} = M' \pi_a = \frac{1}{a} M' \Sigma^{-1} \mathbf{1}_n = \frac{b}{a}$$

$$\sigma^{\pi_a} = [\pi_a' \Sigma \pi_a]^{\frac{1}{2}} = \left[ \frac{1}{a^2} \mathbf{1}_n' \Sigma^{-1} \Sigma \Sigma^{-1} \mathbf{1}_n \right]^{\frac{1}{2}} = \left[ \frac{a}{a^2} \right]^{\frac{1}{2}} = \frac{1}{\sqrt{a}}$$

$$\omega_{a,b}' \mathbf{1}_n = \frac{1}{\|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}} \mathbf{1}_n' \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) = 0$$

$$m^{\omega_{a,b}} = M' \omega_{a,b} = M' \frac{\Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n)}{\|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}}$$

using  $\mathbf{1}_n' \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) = 0$  we get:

$$M' \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) = (M - \frac{b}{a} \mathbf{1}_n)' \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) \text{ and so}$$

$$m^{\omega_{a,b}} = \frac{\|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}^2}{\|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}} = \|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}$$

$$(\sigma^{\omega_{a,b}})^2 = \frac{1}{\|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}^2} (M - \frac{b}{a} \mathbf{1}_n)' \Sigma^{-1} \Sigma \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) = 1$$

$$\text{cov}(R^{\pi_a}, R^{\omega_{a,b}}) = \frac{1}{a} \mathbf{1}_n' \Sigma^{-1} \Sigma \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) \frac{1}{\|M - \frac{b}{a} \mathbf{1}_n\|_{\Sigma^{-1}}^2} = 0 \text{ Q.E.D}$$

## Proposition

- $\mathcal{F} = \{\pi_a + \lambda\omega_{a,b}, \lambda \in \mathbb{R}\}$
- $R^{\pi_a + \lambda\omega_{a,b}} = R^{\pi_a} + \lambda R^{\omega_{a,b}}$
- $m^{\pi_a + \lambda\omega_{a,b}} = m^{\pi_a} + \lambda m^{\omega_{a,b}}$
- $(\sigma^{\pi_a + \lambda\omega_{a,b}})^2 = (\sigma^{\pi_a})^2 + \lambda^2$
- $\min_{\pi \in \mathcal{E}} \sigma^\pi = \sigma^{\pi_a}$

## Demonstration:

We know that a solution of  $(P)$  is of the form  $\alpha\pi_a + \beta\omega_{a,b}$  as it must be in  $\text{Vect}(\pi_a, \omega_{a,b})$ . The condition  $\pi' \mathbf{1}_n = 1$  implies  $\alpha\pi_a' \mathbf{1}_n = 1$  which implies  $\alpha = 1$ . So the solutions of  $(P)$  are all of the forms  $\pi_a + \lambda\omega_{a,b}$ . Reciprocally the investment portfolio  $\pi_a + \lambda\omega_{a,b}$  is the solution of  $(P)$  for  $m = m^{\pi_a} + \lambda m^{\omega_{a,b}}$ . The other conditions are trivial to verify.

## Corollary

If  $\pi_e \in \mathcal{F}$

- $(\sigma^{\pi_e})^2 = (\sigma^{\pi_a})^2 + \left(\frac{m^{\pi_e} - m^{\pi_a}}{m^{\omega_{a,b}}}\right)^2$
- $m^{\pi_e} = m^{\pi_a} + \sqrt{(\sigma^{\pi_e})^2 - (\sigma^{\pi_a})^2}$  if  $m^{\pi_e} > m^{\pi_a}$
- $m^{\pi_e} = m^{\pi_a} - \sqrt{(\sigma^{\pi_e})^2 - (\sigma^{\pi_a})^2}$  if  $m^{\pi_e} < m^{\pi_a}$
- $\pi_e = \pi_a + \frac{m^{\pi_e} - m^{\pi_a}}{m^{\omega_{a,b}}} \omega_{a,b}$

## Demonstration:

Trivial when writing  $\pi_e$  as  $\pi_a + \lambda \omega_{a,b}$  and eliminating  $\lambda$  between the equations.

# Optimization without a risk-free asset

## Remarks:

- The frontier  $\mathcal{F}(\sigma, m)$  is an hyperbole
- For all assets and investment portfolios:  $(\sigma, m)$  is on or inside the hyperbole
- Geometrically: the line  $y = a + bx$  is above the hyperbole  $y = a + b\sqrt{x^2 - 1}$  and is asymptotically tangent as:
  - $y = a + bx - (a + b\sqrt{x^2 - 1}) \xrightarrow{x \rightarrow \infty} 0$  and
  - $b / [b \frac{x}{\sqrt{x^2 - 1}}] \xrightarrow{x \rightarrow \infty} 1$

## Definitions:

We note:

- $\mathcal{F}^+ = \{\pi_a + \lambda \omega_{a,b}, \lambda \geq 0\}$
- $\mathcal{F}^- = \{\pi_a + \lambda \omega_{a,b}, \lambda \leq 0\}$
- $\mathcal{F}^+(\sigma, m) = \{(\sigma^\pi, m^\pi), \pi \in \mathcal{F}^+\}$  and call it the Efficient Frontier
- $\mathcal{F}^-(\sigma, m) = \{(\sigma^\pi, m^\pi), \pi \in \mathcal{F}^-\}$  and call it the Inefficient Frontier

## Two Funds Theorem:

All the efficient portfolios can be built from two fixed distinct efficient investment portfolios  $\pi_1$  and  $\pi_2$ .

For this reason to analyse the efficient frontier we just need to analyse a model based on two risky assets!

### Demonstration:

Let  $\pi_1 = \pi_a + \lambda_1 \omega_{a,b}$  and  $\pi_2 = \pi_a + \lambda_2 \omega_{a,b}$  be distinct on  $\mathcal{F}$

Let  $\pi = \pi_a + \lambda \omega_{a,b}$  be on  $\mathcal{F}$ .

For any  $\alpha \in \mathbb{R}$ ,  $\alpha \pi_1 + (1 - \alpha) \pi_2$  is an investment portfolio (as  $\alpha \pi_1' \mathbf{1}_n + (1 - \alpha) \pi_2' \mathbf{1}_n = 1$ ) and if we choose  $\alpha = \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2}$  we get  $\alpha \pi_1 + (1 - \alpha) \pi_2 = \pi_a + \lambda \omega_{a,b} = \pi$ . Q.E.D

## Exercise 1:

We consider 2 risky assets  $S^1, S^2$ :

- $m_1 = 5\%, \sigma_1 = 15\%$
- $m_2 = 20\%, \sigma_2 = 30\%$

We note  $\rho$  the correlation between  $R^1$  and  $R^2$ .

a) Plot  $\mathcal{F}$  for:  $\rho = -1, \rho = 0, \rho = 0.5, \rho = 1$ .

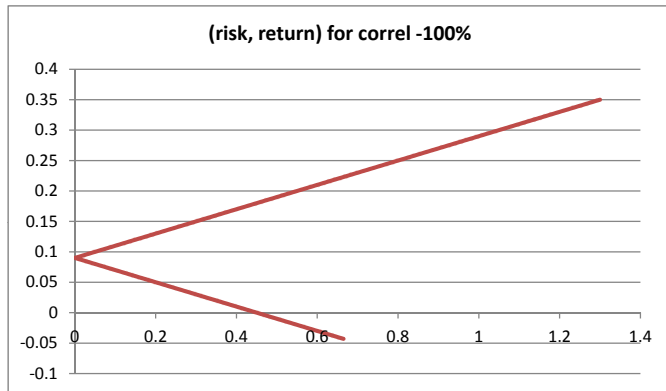
b) What is the portfolio of minimum standard deviation in each case?

If we consider the portfolio  $\pi_\alpha = \alpha\pi_1 + (1 - \alpha)\pi_2$

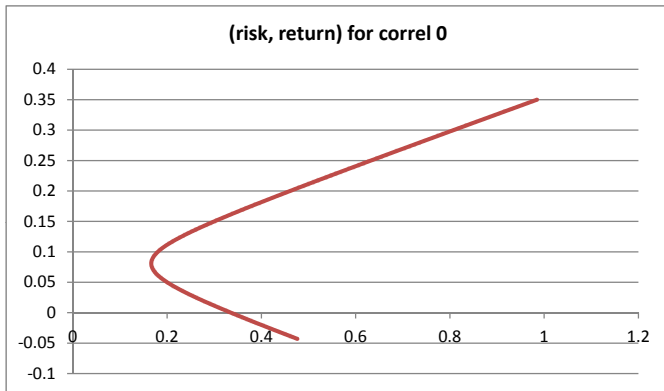
c) In which region of  $\mathcal{F}(\sigma, m)$  is  $(\sigma^{\pi_\alpha}, m^{\pi_\alpha})$  in the following cases:

- $\alpha < 0$
- $0 \leq \alpha \leq 1$
- $\alpha > 1$

# Optimization without a risk-free asset

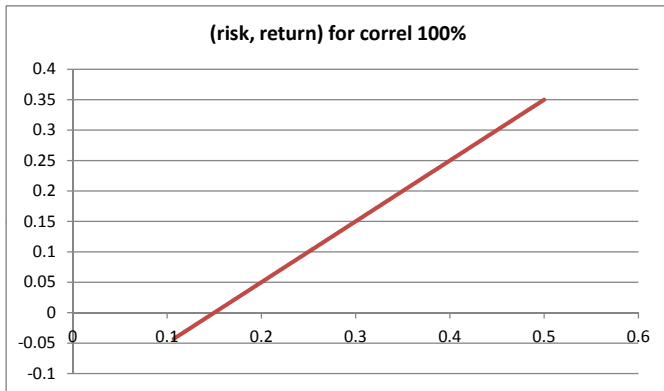


# Optimization without a risk-free asset





# Optimization without a risk-free asset



## Exercise 2:

Show that:  $\mathcal{F} = \left\{ \frac{1}{b-ma} \Sigma^{-1} (M - m\mathbf{1}_n), m \neq \frac{b}{a} \right\} \cup \left\{ \frac{1}{a} \Sigma^{-1} \mathbf{1}_n \right\}$

**Remarks:** We have assumed so far that  $M - \frac{b}{a}1_n \neq 0$ . We analyse here what would happen if this was not the case.

If  $M - \frac{b}{a}1_n = 0$  then all portfolios would have the same returns equal to  $\frac{b}{a}$ . In this case (P) would be a problem of minimizing, for an investment portfolio, the standard deviation of the return, i.e to solve:

$$(P) \begin{cases} \min_{\pi} \langle \Sigma \pi, \Sigma \pi \rangle_{\Sigma^{-1}} \\ \langle \Sigma \pi, 1_n \rangle_{\Sigma^{-1}} = 1 \end{cases}$$

As previously, geometrically we see that the solution should verify

$\Sigma \pi_e \in \text{Vect}(1_n)$ , so  $\pi_e = \lambda \Sigma^{-1} 1_n$ . The only  $\pi_e$  of this form satisfying  $(\pi_e)' 1_n = 1$  is  $\pi_e = \frac{1}{a} \Sigma^{-1} 1_n$

## Markowitz: Optimization with a risk-free asset

# Optimization with a risk-free asset

We note  $\Pi = \begin{pmatrix} \pi^0 \\ \pi \end{pmatrix}$  the allocation between the risk-free asset and the  $n$  risky assets.

We note  $\Pi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  the risk-free asset of return  $r_0$ .

For any investment portfolios  $\Pi$  we must have  $\pi^0 + (\pi)' \mathbf{1}_n = 1$ .

Replacing  $\pi^0$  by  $1 - \pi' \mathbf{1}_n$  the problem we have to solve, to find the efficient portfolios, can now be written as:

$$(Q) \begin{cases} \min_{\pi} \pi' \Sigma \pi \\ \pi' M + (1 - \pi' \mathbf{1}_n) r_0 = m \end{cases}$$

Which we can also write as:

$$(Q) \begin{cases} \min_{\pi} \langle \Sigma \pi, \Sigma \pi \rangle_{\Sigma^{-1}} \\ \langle \Sigma \pi, M - r_0 \mathbf{1}_n \rangle_{\Sigma^{-1}} = m - r_0 \end{cases}$$

Geometrically, the solution has to be of the form  $\Sigma\pi = \lambda(M - r_0\mathbf{1}_n)$  or equivalently  $\pi = \lambda\Sigma^{-1}(M - r_0\mathbf{1}_n)$ .

Note that  $\mathbf{1}'_n\Sigma^{-1}(M - r_0\mathbf{1}_n) = (b - r_0a)$ .

Until the end of this section we will consider that  $b - r_0a \neq 0$

To renormalize the problem we define:  $\pi_M = \frac{1}{b - r_0a}\Sigma^{-1}(M - r_0\mathbf{1}_n)$

Note that  $\Pi_M = \begin{pmatrix} 0 \\ \pi_M \end{pmatrix}$  is an investment portfolio as  $\pi'_M\mathbf{1}_n = 1$

With these notations we now state:

## Theorem: Capital Market Line

The portfolios solutions of (Q) are the portfolios:  $\lambda\Pi_M + (1 - \lambda)\Pi_0$  with  $\lambda \in \mathbb{R}$ . We note  $\mathcal{C} = \{\lambda\Pi_M + (1 - \lambda)\Pi_0, \lambda \in \mathbb{R}\}$

## Corollaries and "Market Portfolio"

① The portfolios  $\Pi$  of  $\mathcal{C}$  (i.e the solutions of  $(Q)$ ) verify:

- $m^\Pi = \lambda m^{\Pi_M} + (1 - \lambda)m^{\Pi_0}$
- $\sigma^\Pi = |\lambda|\sigma^{\Pi_M}$

so their risk parameters  $(\sigma, m)$  are on a cone (i.e  $\mathcal{C}(\sigma, m)$  is a cone).

② All the efficient portfolios are built by allocating money only between  $\Pi_0$  and  $\Pi_M$ . For this reason to study optimal investments we just need to use a model with one single risky asset!

③ CAPM/MEDAF: if all the market participants are allocating efficiently and with the same parameters, then:

- all the risky investments are in  $\Pi^M$  and
- in  $\Pi^M$  the weight in risky asset  $i$  is the % of the total Market Capitalization of risky assets that asset  $i$  represents.

For this reason  $\Pi^M$  should be/is called the (risky) "Market Portfolio".

## Corollaries:

- 1 Usually in the model  $\frac{b}{a} > r_0$  otherwise all the efficient portfolios would short  $\Pi_M$  which practically would not make sense.
- 2 Assuming  $\frac{b}{a} > r_0$  we define:
  - $\mathcal{C}^+ = \{\lambda m^{\Pi_M} + (1 - \lambda)m^{\Pi_0}, \lambda \geq 0\}$  and call it the Cone Efficient Frontier (or Capital Market Line)
  - $\mathcal{C}^- = \{\lambda m^{\Pi_M} + (1 - \lambda)m^{\Pi_0}, \lambda \leq 0\}$  and call it the Cone Inefficient Frontier
  - All the assets and portfolios we can build have their risk parameters  $(\sigma, m)$  within the cone  $\mathcal{C}(\sigma, m)$  and in particular  $\mathcal{F}(\sigma, m)$  is included in  $\mathcal{C}(\sigma, m)$ .

## Demonstration:

Straightforward when writing  $\Pi_e = \lambda \Pi_M + (1 - \lambda) \Pi_0$

## Exercise:

Draw  $\mathcal{F}$  and  $\mathcal{C}$  in a model where there are two risky assets and a risk free asset and see what happens when you are changing the correlation



## Lemma: Tangent Portfolio

The "Market Portfolio"  $\Pi_M$ , which has no allocation in the risk-free asset, is also a solution of the mean/variance optimization problem ( $P$ ) where there was no risk-free asset. So we can write  $\Pi_M \in \mathcal{F}$

### Demonstration:

Geometrically: it is obvious as otherwise there would be some portfolios more efficient (above) than those on the Capital Market Line.

Algebraically:

$$\begin{aligned}\pi_M &= \frac{1}{b-r_0a} \Sigma^{-1} (M - r_0 \mathbf{1}_n) = \frac{1}{b-r_0a} \Sigma^{-1} [(M - \frac{b}{a} \mathbf{1}_n) + (\frac{b}{a} \mathbf{1}_n - r_0 \mathbf{1}_n)] \\ &= \frac{1}{b-r_0a} (\frac{b}{a} - r_0) \Sigma^{-1} \mathbf{1}_n + \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) \\ &= \frac{1}{a} \Sigma^{-1} \mathbf{1}_n + \frac{1}{b-r_0a} \Sigma^{-1} (M - \frac{b}{a} \mathbf{1}_n) \\ &= \pi_a + \lambda \omega_{a,b} \text{ which is the form of the portfolios of } \mathcal{F}. \text{ Q.E.D}\end{aligned}$$

Until the end of this section we assume that  $\frac{b}{a} > r_0$

## Theorem: Tangent Portfolio

$\mathcal{C}(\sigma, m)$  is tangent to  $\mathcal{F}(\sigma, m)$  at the point  $(\sigma^{\Pi_M}, m^{\Pi_M})$

### Demonstration:

We know that  $\Pi_M$  is on  $\mathcal{C}$  and that  $\Pi_M$  is on  $\mathcal{F}$ .

Geometrically: If a line and an hyperbole have a contact point either they are tangent on this contact point or they cross each other. The situation where they cross each other is not possible here as it would imply that some portfolios of  $\mathcal{F}$  are more efficient than any portfolios of  $\mathcal{C}$ .

### Remark:

Equivalently, we can say that the tangent to the Efficient Frontier  $\mathcal{F}$  at the point  $\Pi_M$  intersects the  $\sigma = 0$  axis at the point  $\begin{pmatrix} 0 \\ r_0 \end{pmatrix}$

## Corollary: Geometry of the Efficient Frontier

- For any risky efficient investment portfolio  $\pi = \frac{1}{b-ma}\Sigma^{-1}(M - m\mathbf{1}_n)$  of  $\mathcal{F}$ , the tangent to  $\mathcal{F}(\sigma, m)$  at  $(\sigma^\pi, m^\pi)$  intersects the  $\{\sigma = 0\}$  axis at the point  $\begin{pmatrix} 0 \\ m \end{pmatrix}$ .
- For the risky efficient investment portfolio  $\pi = \frac{1}{a}\Sigma^{-1}\mathbf{1}_n$  of  $\mathcal{F}$ , the tangent to  $\mathcal{F}(\sigma, m)$  at  $(\sigma^\pi, m^\pi)$  is parallel to the  $\{\sigma = 0\}$  axis.

These results mean that there is a bijection between  $\mathcal{F} - \{\frac{1}{a}\Sigma^{-1}\mathbf{1}_n\}$  and  $\mathbb{R} - \{\frac{b}{a}\}$  and that is it the tangents to the the  $\mathcal{F}(\sigma, m)$  curve which establish the bijection between the portfolios and their parameters  $m$ .

### Demonstration:

This result for a portfolio of parameter  $m$  corresponds to the result of the preceding corollary when taking  $r_0 = m$ . Q.E.D

## Security Market Line

## Theorem: Security Market Line

Let  $\Pi_M$  be the Market Portfolio as defined previously

Let  $\Pi_P$  be any investment portfolio composed of the risk-free and risky assets. Then:

- $m_P - r_0 = (m_M - r_0)\rho(R_P, R_M)\frac{\sigma_P}{\sigma_M}$  (SML equation) and
- $R_P - r_0 = (R_M - r_0)\rho(R_P, R_M)\frac{\sigma_P}{\sigma_M} + \epsilon$  with  $\epsilon$  normal independent from  $R_M$  and centered.

## Demonstration:

Let  $\Pi_P$  be an investment portfolio then:

$$\text{cov}(R_M, R_P) = \pi_M' \Sigma \pi_P = \frac{1}{b-r_0a} (M - r_0 \mathbf{1}_n)' \pi_P = \frac{m_P - r_0}{b - r_0 a}$$

If we apply the same calculation to  $\Pi_P$  then:

$$\text{cov}(R_M, R_M) = \frac{m_M - r_0}{b - r_0 a}$$

From this we get:

$$\text{cov}(R_M, R_P) = \frac{m_P - r_0}{m_M - r_0} \text{cov}(R_M, R_M)$$

From which we get :  $m_P - r_0 = (m_M - r_0)\rho(R_M, R_P)\frac{\sigma_P}{\sigma_M}$

We now want to show a relationship for the r.v and not only for their expectations.

$\left( \begin{array}{c} (R_P - r_0) - (R_M - r_0)\rho(R_M, R_P)\frac{\sigma_P}{\sigma_M} \\ R_M \end{array} \right)$  is Gaussian because it is an affine

transformation of the vector of the returns of the risky assets which is assumed to be a Gaussian vector. Thus, to show that the first variable that we call  $\epsilon$  is independent from the second one we just need to show that the covariance is zero.

$$\begin{aligned} \text{Indeed, } \text{cov}(\epsilon, R_M) &= \text{cov}(R_P - R_M\rho(R_M, R_P)\frac{\sigma_P}{\sigma_M}, R_M) \\ &= \text{cov}(R_P, R_M) - \rho(R_M, R_P)\frac{\sigma_P}{\sigma_M}\text{cov}(R_M, R_M) = 0 \end{aligned}$$

The fact that  $\epsilon$  is centered i.e  $E(\epsilon) = 0$ , results from the previous result.

Q.E.D

## Definition:

The quantity  $\rho(R_M, R_P) \frac{\sigma_P}{\sigma_M}$  is noted  $\beta_{P,M}$  and is called the beta of the asset  $\Pi_P$  (in respect to the Market Portfolio  $\Pi_M$ ).

## Remarks:

The equation:  $m_P - r_0 = (m_M - r_0)\beta_{P,M}$  (SML)

- is valid for all investment portfolios and not only for efficient ones.
- shows that only the risk correlated with the "Market Portfolio/Market Risk" is remunerated.
- is used in capital budgeting / CAPM to determine the price an asset should have based on its expected returns and beta with the sector.

## Definition:

When we write  $R_P = r_0 + (R_P - r_0)\beta_{R_P, R_M} + \epsilon$  we have

$$\sigma_M^2 = \sigma_P^2 \beta_{P, M}^2 + \sigma_\epsilon^2$$

- $\sigma_P |\beta_{P, M}|$  is called the systematic risk. It cannot be reduced by diversification in the model and thus is remunerated.
- $\sigma_\epsilon$  is called the idiosyncratic risk. It can be reduced by diversification in the model, and thus is not remunerated.



# Security Market Line and "Arbitrage" Detections

**Remarks:** The  $(\beta, m)$  of the assets we consider investing it according to the SML should be on a line.

- The beta will usually be calculated in relation to a broader index to which these stocks belong
- The expected returns will be based either on some historical estimates or some analysts predictions

In practice, the points will not be perfectly aligned and a regression line will be calculated.

- The assets over the line will look cheap
- The assets below the line will look expensive

In "pair-trading", strategies will be considered consisting in:

- Selling assets lying below the line
- Buying assets lying above the line (usually in the same sector)

**Exercise:** Show that if we consider an investment portfolio  $\Pi_P$  with risk parameters  $(\sigma_P, m_P)$  and if we call in the  $\{(\sigma, m)\}$  plane,  $(x, m_P)$  the intersection of the SML and the line  $\{m = m_P\}$  then  $x = \beta_{P,M}\sigma_M$ . Conclude that we can read in the  $\{(\sigma, m)\}$  plane the decomposition between systematic risk and idiosyncratic risk.

# Performance Indicators

## Definition: Sharp Ratio

The Sharpe Ratio of an investment portfolio  $P$  is defined as:  $\frac{m_P - r_0}{\sigma_P}$

### Remarks:

Under the Markowitz's framework:

- The Ratio should be maximal for portfolios belonging to the CML
- All the portfolios belonging to the CML have the same Ratio
- Wealth should be allocated:
  - First by determining a portfolio with the maximum Sharpe Ratio that can be built
  - Then by allocating all the wealth between this portfolio and the risk free asset
- The Sharpe Ratio is independent from the leverage has  $\lambda \Pi_P + (1 - \lambda) \Pi_0$  has the same Sharpe Ratio as  $\Pi_P$  for any  $\lambda > 0$ . So the indicator is really "intrinsic to the fund".
- The Sharpe Ratio is usually estimated by:  $\frac{\hat{m}_P - r_0}{\hat{\sigma}_P}$

**Remarks:** An investor choosing a mutual fund to represent a large portion of his/her wealth should be concerned by the full risk of the fund and should look at the Sharpe Ratio.

**Exercise 1:** Show that the Sharpe Ratio is independent from the leverage

**Exercise 2:** How do you read in a  $\{(\sigma, m)\}$  representation the Sharpe Ratio of a fund as the slope of a particular line?

## Definition: Jensen Index

The Jensen Index of an investment portfolio  $P$  is defined as:

$$m_P - [r_0 + \beta_{M,P}(m_M - r_0)]$$

## Remarks:

Under the Markowitz's framework this quantity should be zero according to the SML

In practice all the portfolios considered for investment are represented in the  $(\beta, m)$  plane where:

- the *beta* are estimated historically
- the expected returns are either historical estimates or analyst predictions

Then a regression line is calculated and the portfolios above the line could be considered for addition to the investment portfolio as:

- their systematic risk is remunerated more than expected
- their idiosyncratic risk should disappear via diversification

## Remarks:

A large pension fund which allocates money amongst many asset managers may assume that the idiosyncratic risk is going to be reduced/cancelled through diversification and in this case may be concerned only by the remuneration of the non diversifiable risk and by the Jensen Index of each Asset Manager's funds.

**Exercise 1:** Show that the Jensen Ratio is dependent on leverage

**Exercise 2:** How do you read in a  $\{(\beta, m)\}$  representation the Jensen Ratio of a fund as the distance above a particular line ?

## Definition: Treynor Index

The Treynor Index of an investment portfolio  $P$  is defined as:  $\frac{m_P - r_0}{\beta_{P,M}}$

### Remarks:

Under the Markowitz's framework the Treynor Index should be constant according to the SML.

The Treynor Index is similar to the Jensen index in its objectives to detect funds for which there is an excess of remuneration of the systematic risk.

The excess is usually called the  $\alpha$  !

Compared to the Jensen Index the advantage of the Treynor Index is that it does not depend on leverage and thus is a more intrinsic measure.

### Remarks:

Show that the Treynor Index does not depend on leverage.



# Factor Model

# Factor Model

We revisit here the SML equation:

$$r^i(t) = a^i + b^i(r_M(t) - r_0) + \epsilon^i(t)$$

because it appears that in practice it is difficult to identify accurately what the market portfolio is and because the  $\epsilon^i(t)$  appear to represent a significant portion of the variance of the assets and to be correlated.

By adding additional factors we aim at identifying better the common sources of risks (even when they may not be remunerated) and to end up with smaller non explained residual idiosyncratic risks.

We will identify what should be the remuneration of each individual source of risks through a non arbitrage argument.

The factor decomposition of the returns in a K-factor model is expressed as:

$$r^i(t) = a^i + \sum_{j=1}^{j=K} \beta_j^i f^j(t) + \epsilon^i(t) \text{ for all assets } i \text{ in } \{1, \dots, N\} \text{ and all instants } t \text{ in } \{1, \dots, T\}$$

# Factor Model

We can write matricially :  $R(t) = A + BF(t) + E(T)$

$$\text{with } R(T) = \begin{pmatrix} r^1(t) \\ \vdots \\ r^N(t) \end{pmatrix} \quad A = \begin{pmatrix} a^1 \\ \vdots \\ a^N \end{pmatrix} \quad B = \begin{pmatrix} \beta_1^1 & \vdots & \beta_K^1 \\ \vdots & \vdots & \vdots \\ \beta_1^N & \vdots & \beta_K^N \end{pmatrix}$$

$$F(t) = \begin{pmatrix} f^1(t) \\ \vdots \\ f^K(t) \end{pmatrix} \quad \text{and} \quad E(t) = \begin{pmatrix} \epsilon^1(t) \\ \vdots \\ \epsilon^N(t) \end{pmatrix}$$

We make the following assumptions:

$F(t)$  normal centered,  $\text{Var}[F(t)] = \Sigma$  def positive independent from  $t$ .

$F(t)$  independent from  $F(t')$  for  $t \neq t'$ .

$A$  and  $B$  independent from  $t$

$E(t)$  normal centered and  $\text{Var}[E(t)] = \text{diag}(\sigma_i^2)$  independent from  $t$ .

$E(t)$  independent from  $E(t')$  for  $t \neq t'$ .

$E(\cdot)$  independent from  $F(\cdot)$

## Remark 1 :

We assume that  $\Sigma$  is symmetric strictly positive.

If  $\Sigma$  was not strictly positive we could find  $u = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix} \neq 0$  such that

$u' \text{Var}[F(t)]u = 0$  and in this case we would have  $\text{Var}[u'F(t)] = 0$  and so  $u'F(t) = \text{Cte}$  which would mean that some of the factors would be redundant (could be "cointegrated").

## Remark 2:

$$\begin{aligned} \text{Var}(R(t)) &= \text{cov}(BF(t) + E(t), BF(t) + E(t)) \\ &= \text{cov}(BF(t), BF(t)) + \text{cov}(E(t), E(t)) \\ &= B\text{cov}(F(t), F(t))B' + \text{diag}(\sigma_i^2) \\ &= B\Sigma B' + \text{diag}(\sigma_i^2) \end{aligned}$$

## Exercice 2:

Prove that  $\text{cov}(AX, BY) = A\text{cov}(X, Y)B'$

## Remark 2:

Factor models were introduced by Charles Spearman in 1904 in psychometrics.

## Remark 3:

In financial econometrics, the factors used are either:

- Macroeconomics factors: ex GDP, inflation rate, unemployment rate..etc, in this case the  $F(t)$  are "exogene" i.e given and observable.
- Fundamental factors: ex market cap, leverage, book/price ...etc which are as well exogene.
- Statistical factors: in this case the  $F(t)$  are "endogene" / hidden factors and the aim is to determine these  $F(t)$  as well as the corresponding sensibilities (i.e B).

## Reminder: Diagonalisation Theorem

If  $\Sigma$  is symmetric definite positive in  $(\mathbb{R}^K, \langle . \rangle)$  we can find  $V_1, V_2, \dots, V_K$  in  $\mathbb{R}^K$  such that:

- $\langle V_i, V_j \rangle = \delta_{i,j}$
- $\Sigma V_i = \lambda_i V_i$

and if we note  $V$  the matrix whose vectors columns are the  $V_i$  then:

- $V' \Sigma V = \text{diag}(\lambda_i)$
- $V' V = Id_K$

Here the  $\lambda_i$  are the (positive) eigenvalues of  $\Sigma$  and the  $V_i$  are the eigenvectors.

## Standard Form Theorem

We can re-write the factor model in the form:

$$R(t) = A + DH(t) + E(t)$$

with  $\text{Var}[H(t)] = Id_K$

This form is called the Standard Form of the Factor Model

**demonstration:**

$$A + BF(t) + E(t) = A + BV \text{diag}(\sqrt{\lambda_i}) \text{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right) V' F(t) + E(t)$$

and  $\text{Var}[V' F(t)] = V' \Sigma V = \text{diag}(\lambda_i)$

so  $\text{Var}\left[\text{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right) V' F(t)\right] = \text{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right) \text{diag}(\lambda_i) \text{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right) = Id_K$ . Q.E.D

## Fundamental APT Theorem

Let  $r^i(t) = a^i + \sum_{j=1}^K b_j^i f^j(t) + \epsilon^i(t)$  or in matrix terms

$R(t) = A + BF(t) + E(t)$  be a  $K$ -factor model.

If there is no arbitrage in the reduced model  $R(t) = A + BF(t)$  where the "diversifiable" risk  $E(t)$  is neglected then:

there exist  $\alpha^0, \lambda_1, \lambda_2, \dots, \lambda_K$  such that

$$\forall i \in \{1, 2, \dots, N\}, a^i = \alpha^0 + \sum_{j=1}^K b_j^i \lambda^j(t)$$

or matrixially  $A = \alpha^0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + B \begin{pmatrix} \lambda_1(t) \\ \vdots \\ \lambda_K(t) \end{pmatrix}$



## Remarks:

No arbitrage means that:

- all investment portfolios with no risk (i.e fixed returns) should have the same return. We note  $r_0$  this return.
- all self financing portfolios with no risk (i.e fixed returns) should have a return of zero.

## Lemma 1

If there is no arbitrage and if we call  $r_0$  the risk free rate then:

$$\forall \pi \in \mathbb{R}^K \quad \pi' B = [0, 0, \dots, 0] \implies \pi' (A - r_0 \mathbf{1}_K) = 0$$

## Demonstration Lemma 1:

Let  $\pi \in \mathbb{R}^K$  be such that  $\pi' B = [0, 0, \dots, 0]$ .

- if  $\pi' 1_K \neq 0$  then  $\tilde{\pi} = \frac{\pi}{\pi' 1_K}$  is an investment portfolio which is without risk in the reduced model as  $R^{\tilde{\pi}} = \tilde{\pi}' A + \tilde{\pi}' B = \tilde{\pi}' A$  therefore we should have  $\tilde{\pi}' A = r_0$  that we can also write  $\tilde{\pi}' (A - r_0 1_K) = 0$  as  $\tilde{\pi}' 1_K = 0$
- if  $\pi' 1_K = 0$  then  $\pi$  is a self-financing portfolio without risk which should therefore satisfy  $\pi' A = 0$  that we can also write  $\tilde{\pi}' (A - r_0 1_K) = 0$  as  $\tilde{\pi}' 1_K = 0$ . Q.E.D

## Lemma 2:

If

$$\forall \pi \in \mathbb{R}^K \quad \pi' B = [0, 0, \dots, 0] \implies \pi' (A - r_0 \mathbf{1}_K) = 0 \quad (1)$$

then

$$\exists \lambda_1, \lambda_2, \dots, \lambda_K, \quad A - r_0 \mathbf{1}_K = \sum_{i=1}^{i=K} \lambda_i b_i \quad (2)$$

## Demonstration Lemma 2:

We note  $B = [b_1, b_2, \dots, b_K]$  and  $\text{Vect}\{b_1, b_2, \dots, b_K\}$  the vector space generated by  $b_1, b_2, \dots, b_K$ .

$$(1) \iff \text{Vect}\{b_1, b_2, \dots, b_K\}^\perp \subset \text{Vect}\{A - r_0 \mathbf{1}_K\}^\perp$$

$$\implies \text{Vect}\{A - r_0 \mathbf{1}_K\} \subset \text{Vect}\{b_1, b_2, \dots, b_K\}$$

$$\implies A - r_0 \mathbf{1}_K \in \text{Vect}\{b_1, b_2, \dots, b_K\}$$

This proves Lemma 2.

Lemma 1 and Lemma 2 together proves the theorem.

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# The End