



**HAL**  
open science

# Computer Algebra for Lattice Path Combinatorics

Alin Bostan

► **To cite this version:**

Alin Bostan. Computer Algebra for Lattice Path Combinatorics. Doctoral. Ellwangen, Germany. 2015. cel-01242698

**HAL Id: cel-01242698**

**<https://hal.science/cel-01242698v1>**

Submitted on 13 Dec 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Computer Algebra for Lattice Path Combinatorics

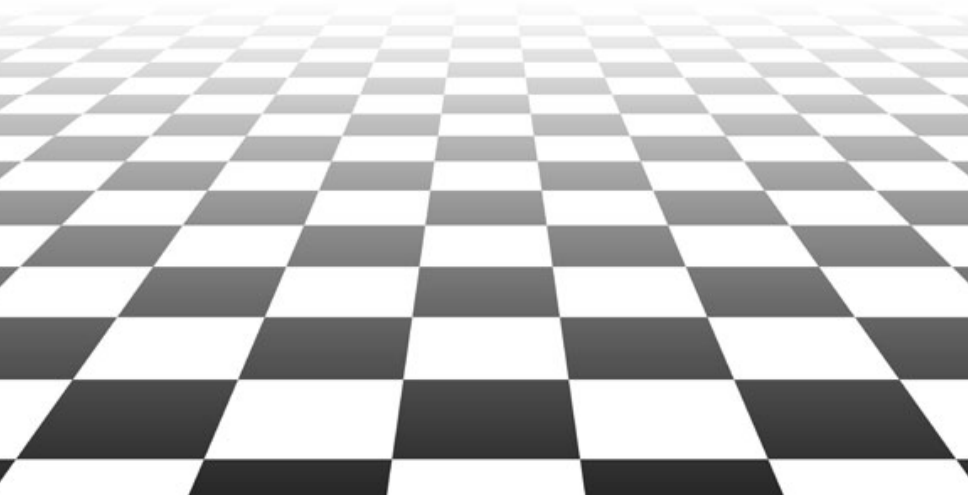
Alin Bostan



The 74th Séminaire Lotharingien de Combinatoire  
Ellwangen, March 23–25, 2015

- ① Monday: General presentation
- ② Tuesday: Guess'n'Prove
- ③ Wednesday: Creative telescoping

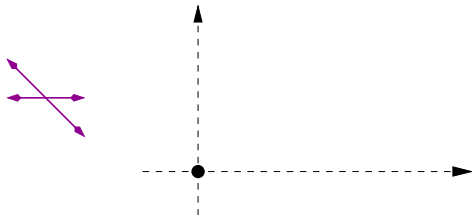
## Part I: General presentation



## General context: lattice paths confined to cones

Let  $\mathfrak{S}$  be a subset of  $\mathbb{Z}^d$  (**step set**, or **model**) and  $p_0 \in \mathbb{Z}^d$  (**starting point**).

**Example:**  $\mathfrak{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$ ,  $p_0 = (0,0)$







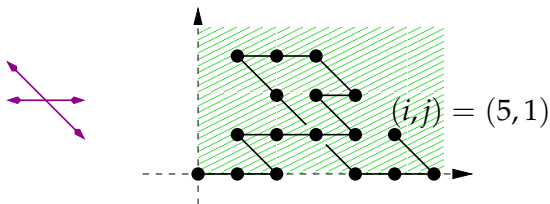
## General context: lattice paths confined to cones

Let  $\mathfrak{S}$  be a subset of  $\mathbb{Z}^d$  (**step set**, or **model**) and  $p_0 \in \mathbb{Z}^d$  (**starting point**).

A **path (walk) of length  $n$  starting at  $p_0$**  is a sequence  $(p_0, p_1, \dots, p_n)$  of elements in  $\mathbb{Z}^d$  such that  $p_{i+1} - p_i \in \mathfrak{S}$  for all  $i$ .

Let  $C$  be a **cone** of  $\mathbb{R}^d$  (if  $x \in C$  and  $r \geq 0$  then  $r \cdot x \in C$ ).

**Example:**  $\mathfrak{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$ ,  $p_0 = (0,0)$  and  $C = \mathbb{R}_+^2$



### Questions

- What is the number  $a(n)$  of  $n$ -step walks contained in  $C$ ?
- For  $i \in C$ , what is the number  $a(n; i)$  of such walks that end at  $i$ ?
- What about their generating series  $A(t)$ , resp.  $A(t; x)$ ?



## Why count walks in cones?

Many discrete objects can be encoded in that way:

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

# Why count walks in cones?

Many discrete objects can be encoded in that way:

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

Journal of Statistical Planning and Inference 140 (2010) 2237–2254



Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: [www.elsevier.com/locate/jspi](http://www.elsevier.com/locate/jspi)



## A history and a survey of lattice path enumeration

Katherine Humphreys

Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA

### ARTICLE INFO

Available online 21 January 2010

Keywords:  
Lattice path  
Reflection principle  
Method of images

### ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

© 2010 Elsevier B.V. All rights reserved.

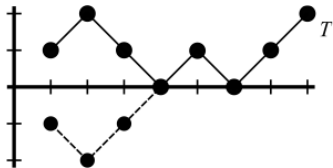
# An old topic: The ballot problem and the reflection principle

## Ballot problem [Bertrand, 1887]

Suppose that candidates  $A$  and  $B$  are running in an election. If  $a$  votes are cast for  $A$  and  $b$  votes are cast for  $B$ , where  $a > b$ , then the probability that  $A$  stays ahead of  $B$  throughout the counting of the ballots is  $(a - b)/(a + b)$ .

**Lattice path reformulation:** given positive integers  $a, b$  with  $a > b$ , find the number of Dyck paths starting at the origin and consisting of  $a$  upsteps  $\nearrow$  and  $b$  downsteps  $\searrow$  such that no step ends on the  $x$ -axis.

**Reflection principle:** Dyck paths from  $(1, 1)$  to  $T(a + b, a - b)$  that touch the  $x$ -axis  $\equiv$  Dyck paths from  $(1, -1)$  to  $T$



**Answer:** good paths = paths from  $(1, 1)$  to  $T$  that never touch the  $x$ -axis

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b} \binom{a+b}{a}$$

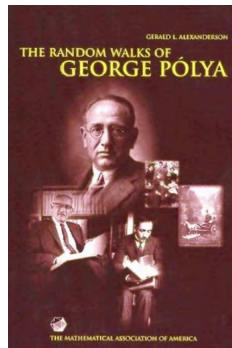
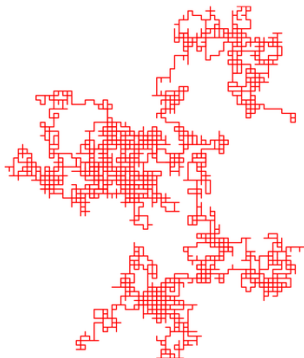
# An old topic: Pólya's "promenade au hasard" / "Irrfahrt"

*Motto:* Drunkard: "Will I ever, ever get home again?"

Polya (1921): "You can't miss; just keep going and stay out of 3D!"  
(Adam and Delbruck, 1968)

[Pólya, 1921] The simple random walk on  $\mathbb{Z}^d$  is **recurrent** in dimensions  $d = 1, 2$  ("Alle Wege fuhren nach Rom"), and **transient** in dimension  $d \geq 3$

Über eine Aufgabe der Wahrscheinlichkeitsrechnung  
betreffend die Irrfahrt im Straßennetz.



Many recent contributions:

Adan, Banderier, Bernardi, Bostan, Bousquet-Mélou, Chyzak, Cori, Denisov, Duchon, Dulucq, Fayolle, Fisher, Flajolet, Garbit, Gessel, Guttmann, Guy, Gouyou-Beauchamps, van Hoeij, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kreweras, Kurkova, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

*etc.*

Many recent contributions:

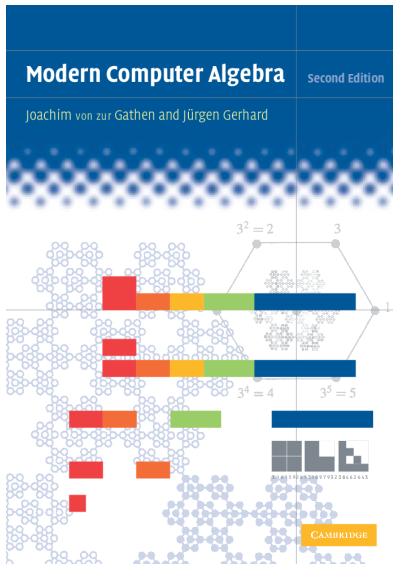
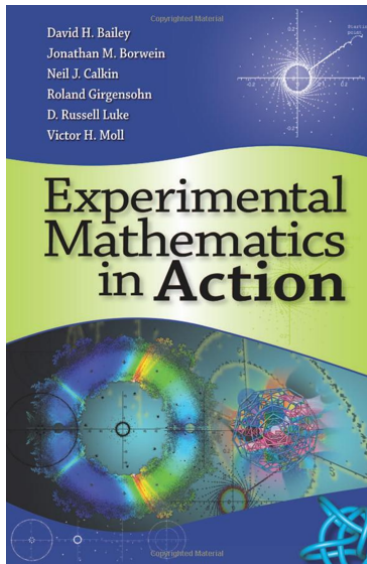
Adan, Banderier, Bernardi, Bostan, Bousquet-Mélou, Chyzak, Cori, Denisov, Duchon, Dulucq, Fayolle, Fisher, Flajolet, Garbit, Gessel, Guttmann, Guy, Gouyou-Beauchamps, van Hoeij, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kreweras, Kurkova, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

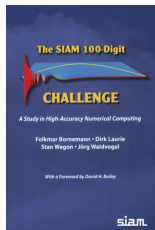
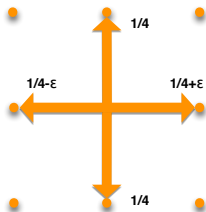
etc.

~~Specific question  
Ad hoc solution~~



**Systematic approach**





## Problem 6

*A flea starts at  $(0,0)$  on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability  $1/4$ , east with probability  $1/4 + \epsilon$ , and west with probability  $1/4 - \epsilon$ . The probability that the flea returns to  $(0,0)$  sometime during its wanderings is  $1/2$ . What is  $\epsilon$ ?*

► Computer algebra **conjectures** and **proves**

$$p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot {}_2F_1 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \frac{2\sqrt{1-16\epsilon^2}}{A} \right)^{-1}, \quad \text{with } A = 1 + 8\epsilon^2 + \sqrt{1-16\epsilon^2}.$$



## A (very) basic cone: the full space

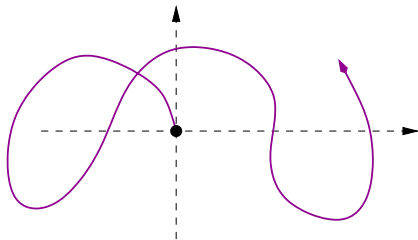
### Rational series

If  $\mathfrak{S} \subset \mathbb{Z}^d$  is finite and  $C = \mathbb{R}^d$ , then  $A(t; \mathbf{x})$  is rational:

$$a(n) = |\mathfrak{S}|^n \Leftrightarrow A(t) = \sum_{n \geq 0} a(n)t^n = \frac{1}{1 - |\mathfrak{S}|t}$$

More generally:

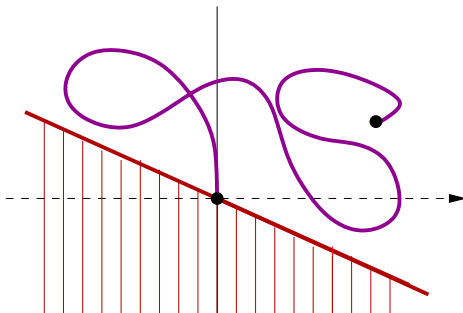
$$A(t; \mathbf{x}) = \frac{1}{1 - t \sum_{s \in \mathfrak{S}} \mathbf{x}^s}.$$



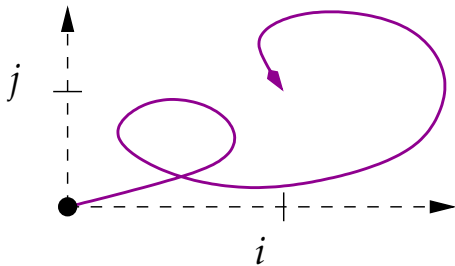
## Also well-known: a (rational) half-space

Algebraic series [Bousquet-Mélou-Petkovšek 00]

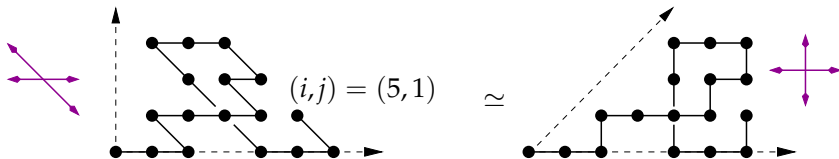
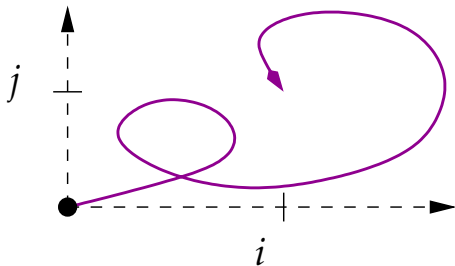
If  $\mathfrak{S} \subset \mathbb{Z}^d$  is finite and  $C$  is a rational half-space, then  $A(t; x)$  is algebraic, given by an explicit system of polynomial equations.



## The “next” case: intersection of two half-spaces



# The “next” case: intersection of two half-spaces





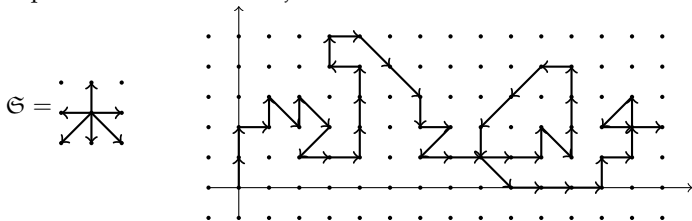


# Lattice walks with small steps in the quarter plane

- ▶ From now on: we focus on **nearest-neighbor walks in the quarter plane**, i.e. walks in  $\mathbb{N}^2$  starting at  $(0,0)$  and using steps in a prefixed subset  $\mathfrak{S}$  of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

- ▶ Example with  $n = 45$ ,  $i = 14$ ,  $j = 2$  for:



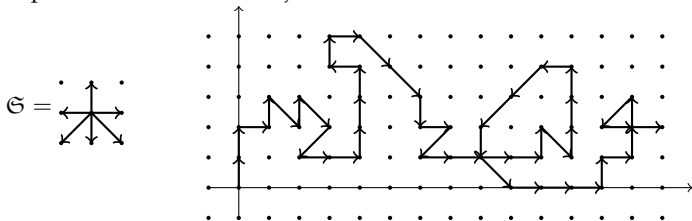
- ▶  $f_{n;i,j}$  = number of walks of length  $n$  ending at  $(i,j)$ .
- ▶  $f_{n;0,0}$  = number of walks returning to  $(0,0)$ , a.k.a. “excursions”, of length  $n$ .

# Lattice walks with small steps in the quarter plane

- ▶ From now on: we focus on **nearest-neighbor walks in the quarter plane**, i.e. walks in  $\mathbb{N}^2$  starting at  $(0,0)$  and using steps in a prefixed subset  $\mathfrak{S}$  of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

- ▶ Example with  $n = 45$ ,  $i = 14$ ,  $j = 2$  for:



- ▶  $f_{n;i,j}$  = number of walks of length  $n$  ending at  $(i,j)$ .
- ▶  $f_{n;0,0}$  = number of walks returning to  $(0,0)$ , a.k.a. “excursions”, of length  $n$ .
- ▶  $f_n = \sum_{i,j \geq 0} f_{n;i,j}$  = number of total walks with length  $n$ .



- ▶ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbf{Q}[x, y][[t]].$$

- ▶ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbf{Q}[x, y][[t]].$$

- ▶ Special, combinatorially meaningful specializations:
  - $F(t; 0, 0)$  counts excursions;
  - $F(t; 1, 1) = \sum_{n \geq 0} f_n t^n$  counts walks with prescribed length;
  - $F(t; 1, 0)$  counts walks ending on the horizontal axis.

- ▶ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbf{Q}[x, y][[t]].$$

- ▶ Special, combinatorially meaningful specializations:
  - $F(t; 0, 0)$  counts excursions;
  - $F(t; 1, 1) = \sum_{n \geq 0} f_n t^n$  counts walks with prescribed length;
  - $F(t; 1, 0)$  counts walks ending on the horizontal axis.

Combinatorial questions: Given  $\mathfrak{S}$ , what can be said about  $F(t; x, y)$ , resp.  $f_{n,i,j}$ , and their variants?

- Properties of  $F$ : algebraic? transcendental? D-finite?
- Explicit form: of  $F$ ? of  $f$ ?
- Asymptotics of  $f$ ?

- ▶ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbf{Q}[x, y][[t]].$$

- ▶ Special, combinatorially meaningful specializations:

- $F(t; 0, 0)$  counts excursions;
- $F(t; 1, 1) = \sum_{n \geq 0} f_n t^n$  counts walks with prescribed length;
- $F(t; 1, 0)$  counts walks ending on the horizontal axis.

Combinatorial questions: Given  $\mathfrak{S}$ , what can be said about  $F(t; x, y)$ , resp.  $f_{n,i,j}$ , and their variants?

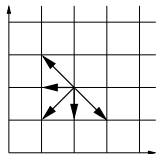
- Properties of  $F$ : algebraic? transcendental? D-finite?
- Explicit form: of  $F$ ? of  $f$ ?
- Asymptotics of  $f$ ?

Our goal: Use computer algebra to give computational answers.

From the  $2^8$  step sets  $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , some are:

## Small-step walks of interest

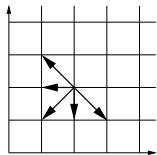
From the  $2^8$  step sets  $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , some are:



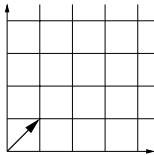
trivial,

## Small-step walks of interest

From the  $2^8$  step sets  $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , some are:



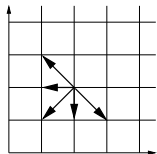
trivial,



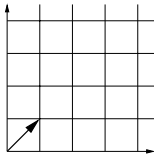
simple,

## Small-step walks of interest

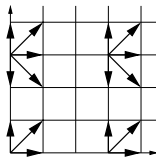
From the  $2^8$  step sets  $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , some are:



trivial,



simple,

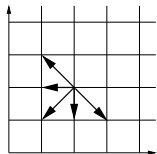


intrinsic to the  
half plane,

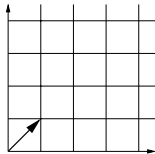


# Small-step walks of interest

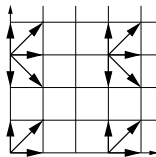
From the  $2^8$  step sets  $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , some are:



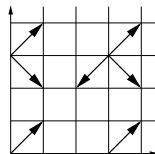
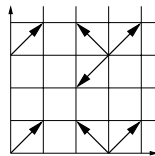
trivial,



simple,



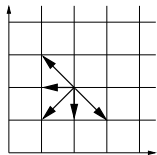
intrinsic to the  
half plane,



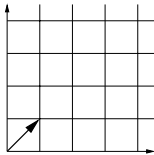
symmetrical.

# Small-step walks of interest

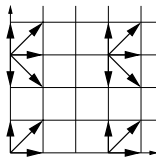
From the  $2^8$  step sets  $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , some are:



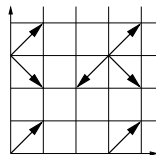
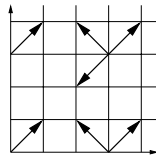
trivial,



simple,



intrinsic to the  
half plane,



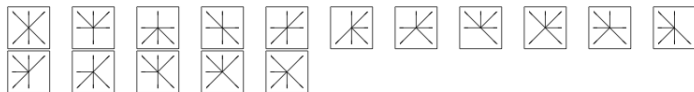
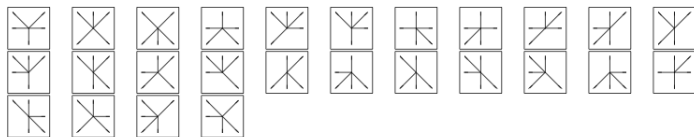
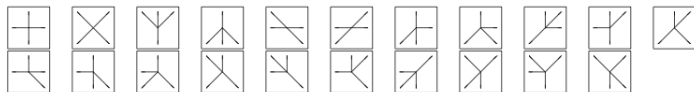
symmetrical.

One is left with [79 interesting distinct models](#).

# The 79 models



Non-singular



Singular

# The 79 models




Non-singular





Singular



# “Special” models


Dyck: 

Motzkin: 


Pólya: 

Kreweras: 

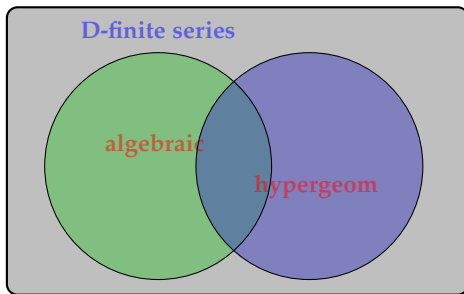
Gessel: 

Gouyou-Beauchamps: 

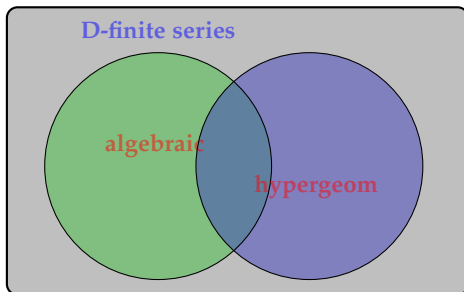
King: 

Exercise: 

# Important classes of univariate power series



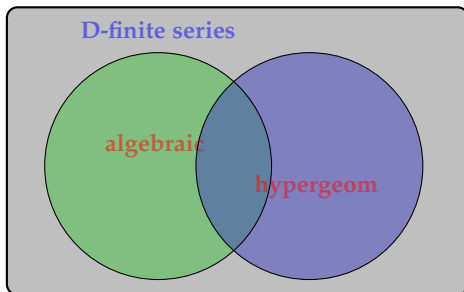
# Important classes of univariate power series



**Algebraic:**  $S(t) \in \mathbf{Q}[[t]]$  root of a polynomial  $P \in \mathbf{Q}[t, T]$ , i.e.,  $P(t, S(t)) = 0$ .



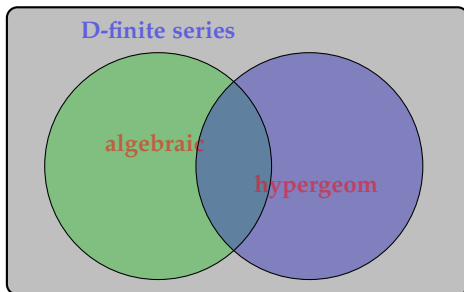
# Important classes of univariate power series



**Algebraic:**  $S(t) \in \mathbf{Q}[[t]]$  root of a polynomial  $P \in \mathbf{Q}[t, T]$ , i.e.,  $P(t, S(t)) = 0$ .

**D-finite:**  $S(t) \in \mathbf{Q}[[t]]$  satisfying a linear differential equation with polynomial coefficients  $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$ .

# Important classes of univariate power series

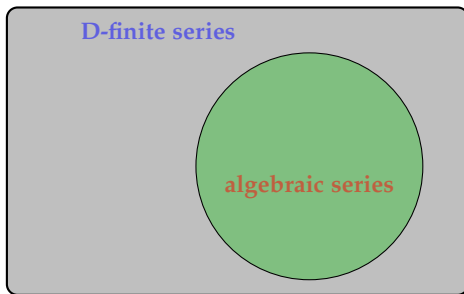


**Algebraic:**  $S(t) \in \mathbf{Q}[[t]]$  root of a polynomial  $P \in \mathbf{Q}[t, T]$ , i.e.,  $P(t, S(t)) = 0$ .

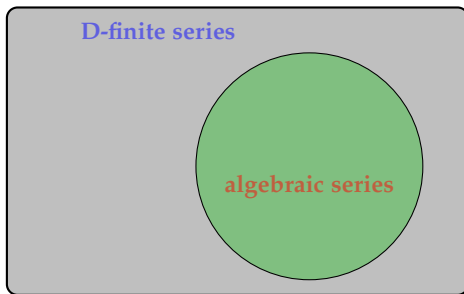
**D-finite:**  $S(t) \in \mathbf{Q}[[t]]$  satisfying a linear differential equation with polynomial coefficients  $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$ .

**Hypergeometric:**  $S(t) = \sum_{n=0}^{\infty} s_n t^n$  such that  $\frac{s_{n+1}}{s_n} \in \mathbf{Q}(n)$ . E.g.,

$${}_2F_1 \left( \begin{matrix} a & b \\ c \end{matrix} \middle| t \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1).$$



$S \in \mathbb{Q}[[x, y, t]]$  is **algebraic** if it is the root of a  $P \in \mathbb{Q}[x, y, t, T]$ .



$S \in \mathbb{Q}[[x, y, t]]$  is **algebraic** if it is the root of a  $P \in \mathbb{Q}[x, y, t, T]$ .

$S \in \mathbb{Q}[[x, y, t]]$  is **D-finite** if the set of all partial derivatives of  $S$  spans a finite-dimensional vector space over  $\mathbb{Q}(x, y, t)$ .

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2\left(\begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

## Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

## Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

**Question:** What about  $K(t; x, y)$  and  $G(t; x, y)$ ?

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

**Question:** What about  $K(t; x, y)$  and  $G(t; x, y)$ ?

**Theorem** [Gessel 1986, Bousquet-Mélou 2005]  $K(t; x, y)$  is algebraic.



**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

**Question:** What about  $K(t; x, y)$  and  $G(t; x, y)$ ?

**Theorem** [Gessel 1986, Bousquet-Mélou 2005]  $K(t; x, y)$  is algebraic.

►  $G(t; x, y)$  had been conjectured to be non-D-finite.

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ & 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

**Question:** What about  $K(t; x, y)$  and  $G(t; x, y)$ ?

**Theorem** [Gessel 1986, Bousquet-Mélou 2005]  $K(t; x, y)$  is algebraic.

►  $G(t; x, y)$  had been conjectured to be non-D-finite.

**Theorem** [B. & Kauers 2010]  $G(t; x, y)$  is D-finite, even algebraic.

► Computer-driven discovery and proof; no human proof yet.

## Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

**Question:** What about  $K(t; x, y)$  and  $G(t; x, y)$ ?

**Theorem** [Gessel 1986, Bousquet-Mélou 2005]  $K(t; x, y)$  is algebraic.

►  $G(t; x, y)$  had been conjectured to be non-D-finite.

**Theorem** [B. & Kauers 2010]  $G(t; x, y)$  is D-finite, even algebraic.

► Computer-driven discovery and proof; ~~no human proof yet.~~

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

**Question:** What about  $K(t; x, y)$  and  $G(t; x, y)$ ?

**Theorem** [Gessel 1986, Bousquet-Mélou 2005]  $K(t; x, y)$  is algebraic.

►  $G(t; x, y)$  had been conjectured to be non-D-finite.

**Theorem** [B. & Kauers 2010]  $G(t; x, y)$  is D-finite, even algebraic.

► Fresh news: recent human proof [B., Kurkova & Raschel 2015].

## Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 1/3 & 2/3 & 1 \\ & 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

**Theorem** [Gessel's conjecture; Kauers, Koutschan & Zeilberger 2009]

$$G(t; 0, 0) = {}_3F_2 \left( \begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

**Question:** What about  $K(t; x, y)$  and  $G(t; x, y)$ ?

**Theorem** [Gessel 1986, Bousquet-Mélou 2005]  $K(t; x, y)$  is algebraic.

►  $G(t; x, y)$  had been conjectured to be non-D-finite.

**Theorem** [B. & Kauers 2010]  $G(t; x, y)$  is D-finite, even algebraic.

► **Guess'n'Prove** method, using **Hermite-Padé approximants** → **Tuesday**

## Main results (II): Explicit form for $G(t; x, y)$

**Theorem** [B., Kauers & van Hoeij 2010]

Let  $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$  be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let  $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$  be a root of

$$\begin{aligned} x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ - xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let  $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$  be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then  $G(t; x, y)$  is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t-xy} - \frac{1}{tx(y+1)}.$$

► Computer-driven discovery and proof; no human proof yet.

## Main results (II): Explicit form for $G(t; x, y)$

**Theorem** [B., Kauers & van Hoeij 2010]

Let  $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$  be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let  $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$  be a root of

$$\begin{aligned} &x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ &- xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let  $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$  be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then  $G(t; x, y)$  is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t-xy} - \frac{1}{tx(y+1)}.$$

- ▶ Computer-driven discovery and proof; no human proof yet.
- ▶ Proof uses **guessed minimal polynomials** for  $G(t; x, 0)$  &  $G(t; 0, y)$

# Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{S}$	Pol size	ODE size		OEIS	$\mathfrak{S}$	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@{algeq, diffeq}

- ▶ Computerized discovery by enumeration + Hermite–Padé
- ▶ 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- ▶ 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]



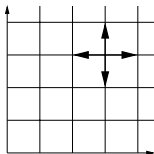
# Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{G}$	alg	asympt		OEIS	$\mathfrak{G}$	alg	asympt
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{3/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{3/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{3/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{3/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{3/2}}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$					

$$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}, \lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

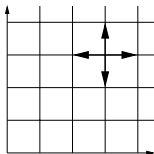
► Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

## The group of a model: the simple walk case



The characteristic polynomial  $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

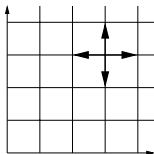
## The group of a model: the simple walk case



The characteristic polynomial  $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$  is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

## The group of a model: the simple walk case



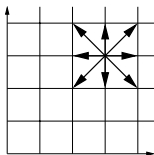
The characteristic polynomial  $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$  is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

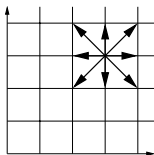
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

## The group of a model: the general case

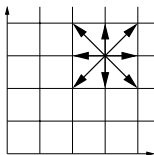


The polynomial  $\chi_{\mathfrak{G}} := \sum_{(i,j) \in \mathfrak{G}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$



The polynomial  $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$  is left invariant under

$$\psi(x, y) = \left( x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$



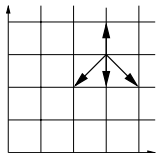
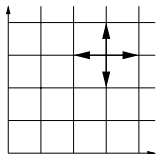
The polynomial  $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$  is left invariant under

$$\psi(x, y) = \left( x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

and thus under any element of the group

$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

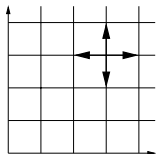
# Examples of groups



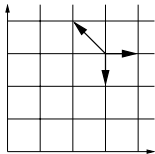
Order 4,



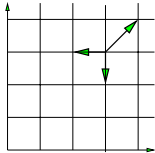
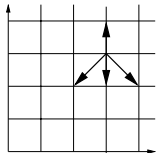
# Examples of groups



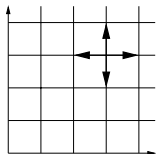
Order 4,



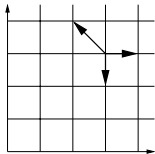
order 6,



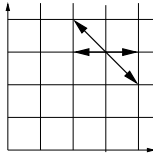
# Examples of groups



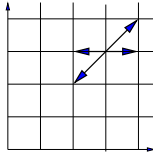
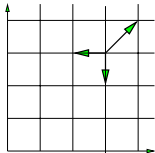
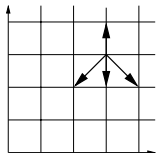
Order 4,



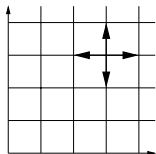
order 6,



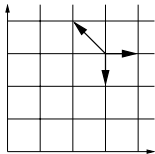
order 8,



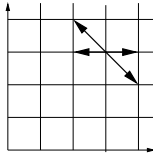
# Examples of groups



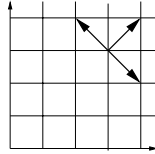
Order 4,



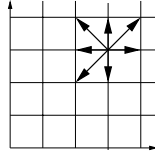
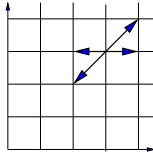
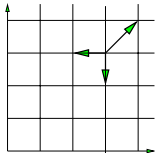
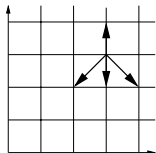
order 6,



order 8,



order  $\infty$ .



## An important object: the orbit sum (OS)

The **orbit sum of a model**  $\mathfrak{G}$  is the following polynomial in  $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$ :

$$\text{OrbitSum}(\mathfrak{G}) := \sum_{\theta \in \mathcal{G}_{\mathfrak{G}}} (-1)^{\theta} \theta(xy)$$

► E.g., for the simple walk:

$$\text{OS} \begin{array}{c} \nearrow \\ \leftarrow \\ \rightarrow \\ \searrow \\ \uparrow \\ \downarrow \end{array} = x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

► For 4 models, the orbit sum is zero:

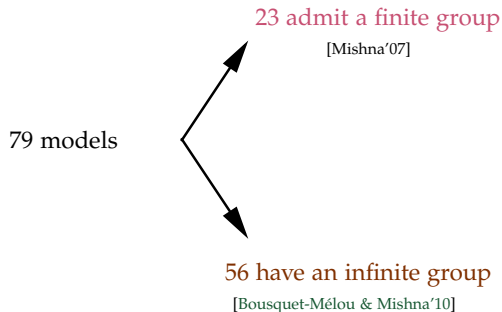


E.g. for the **Kreweras** model:

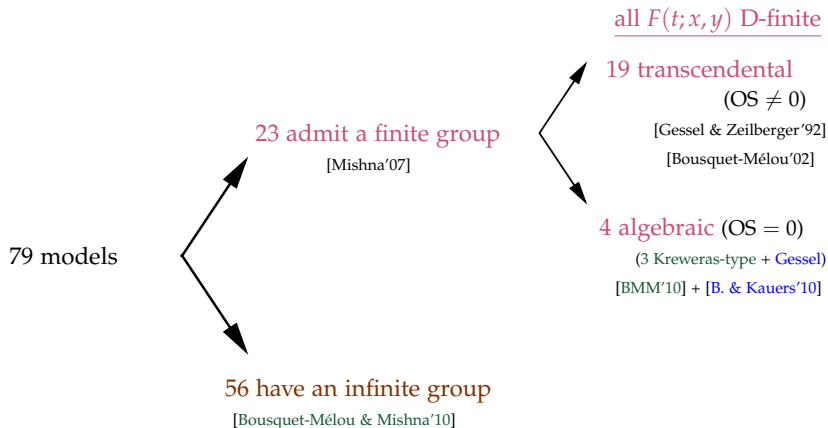
$$\text{OS} \begin{array}{c} \nearrow \\ \leftarrow \\ \rightarrow \\ \searrow \\ \uparrow \\ \downarrow \end{array} = x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} = 0$$

79 models

# The 79 models: finite and infinite groups



# The 79 models: finite and infinite groups



# The 79 models: finite and infinite groups

79 models

23 admit a finite group

[Mishna'07]

56 have an infinite group

[Bousquet-Mélou & Mishna'10]

all  $F(t; x, y)$  D-finite

19 transcendental

(OS  $\neq 0$ )

[Gessel & Zeilberger'92]

[Bousquet-Mélou'02]

4 algebraic (OS = 0)

(3 Kreweras-type + Gessel)

[BMM'10] + [B. & Kauers'10]

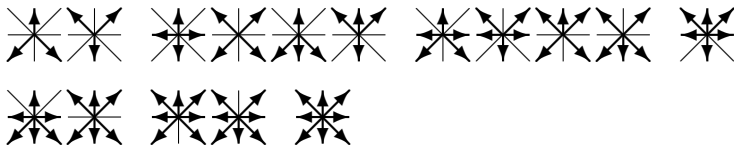
→ all non-D-finite

- [Mishna & Rechnitzer'07] and [Melczer & Mishna'13] for 5 singular models
- [Kurkova & Raschel'13] and [B., Raschel & Salvy'13] for all others



# The 23 models with a finite group

- (i) 16 with a **vertical symmetry**, and group isomorphic to  $D_2$



- (ii) 5 with a **diagonal** or **anti-diagonal symmetry**, and group isomorphic to  $D_3$



- (iii) 2 with group isomorphic to  $D_4$



(i): vertical symmetry; (ii)+(iii): zero drift  $\sum_{s \in \mathfrak{G}} s$

In **red**, models with  $OS = 0$  and **algebraic GF**

## Main results (IV): explicit expressions for the 19 D-finite transcendental models

**Theorem** [B., Chyzak, van Hoeij, Kauers & Pech 2015]

Let  $\mathfrak{S}$  be one of the 19 models with finite group  $\mathcal{G}_{\mathfrak{S}}$ , and non-zero orbit sum. Then  $F$  is expressible using iterated integrals of  ${}_2F_1$  expressions.

# Main results (IV): explicit expressions for the 19 D-finite transcendental models

**Theorem** [B., Chyzak, van Hoeij, Kauers & Pech 2015]

Let  $\mathfrak{S}$  be one of the 19 models with finite group  $\mathcal{G}_{\mathfrak{S}}$ , and non-zero orbit sum. Then  $F$  is expressible using iterated integrals of  ${}_2F_1$  expressions.

**Example** (King walks in the quarter plane, A025595)

$$F_{\begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \\ \leftarrow \\ \swarrow \\ \downarrow \end{array}}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2}, \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

# Main results (IV): explicit expressions for the 19 D-finite transcendental models

**Theorem** [B., Chyzak, van Hoeij, Kauers & Pech 2015]

Let  $\mathfrak{S}$  be one of the 19 models with finite group  $\mathcal{G}_{\mathfrak{S}}$ , and non-zero orbit sum. Then  $F$  is expressible using iterated integrals of  ${}_2F_1$  expressions.

**Example** (King walks in the quarter plane, A025595)

$$F_{\begin{matrix} \nearrow \\ \rightarrow \\ \searrow \\ \leftarrow \\ \swarrow \\ \downarrow \end{matrix}}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2}, \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

- ▶ Computer-driven discovery and proof; no human proof yet.
- ▶ Proof uses **creative telescoping**, **ODE factorization**, **ODE solving**.  $\longrightarrow$

Wednesday

# Hypergeometric Series Occurring in Explicit Expressions for $F(t; 1, 1)$

	hyp <sub>1</sub>	hyp <sub>2</sub>	$w$		hyp <sub>1</sub>	hyp <sub>2</sub>	$w$
1	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle  w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle  w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle  w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{5}{2} \\ 3 \end{matrix} \middle  w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle  w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 1 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle  w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle  w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle  w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{5}{4} \\ 1 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle  w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle  w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle  w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ 1 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{3}{4} & \frac{5}{4} \\ 2 \end{matrix} \middle  w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 1 \end{matrix} \middle  w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle  w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle  w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle  w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle  w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(-\frac{1}{2} \middle  \frac{1}{2} \middle  w\right)$	${}_2F_1\left(\frac{1}{2} \middle  \frac{1}{2} \middle  w\right)$	$16t^2$

**Theorem** [B., Rachel & Salvy 2013]

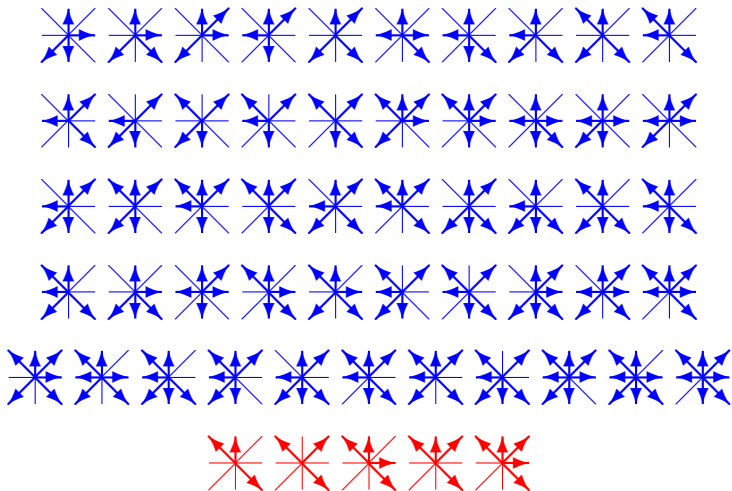
Let  $\mathfrak{G}$  be one of the 51 non-singular models with infinite group  $\mathcal{G}_{\mathfrak{G}}$ .  
Then  $F_{\mathfrak{G}}(t; 0, 0)$ , and in particular  $F_{\mathfrak{G}}(t; x, y)$ , are non-D-finite.

**Theorem** [B., Rachel & Salvy 2013]

Let  $\mathfrak{G}$  be one of the 51 non-singular models with infinite group  $\mathcal{G}_{\mathfrak{G}}$ .  
Then  $F_{\mathfrak{G}}(t;0,0)$ , and in particular  $F_{\mathfrak{G}}(t;x,y)$ , are non-D-finite.

- ▶ **Algorithmic proof.** Uses **Gröbner basis computations, polynomial factorization, cyclotomy testing.**
- ▶ Based on **two ingredients: asymptotics + irrationality.**
  
- ▶ [Kurkova & Raschel 2013] Human proof that  $F_{\mathfrak{G}}(t;x,y)$  is non-D-finite.
- ▶ No human proof yet for  $F_{\mathfrak{G}}(t;0,0)$  **non-D-finite.**

# The 56 models with infinite group



In **blue**, non-singular models, solved by [B., Raschel & Salvy 2013]  
In **red**, singular models, solved by [Melczer & Mishna 2013]



## Example: the scarecrows

[B., Raschel & Salvy 2013]:  $F_{\mathfrak{S}}(t;0,0)$  is not D-finite for the models



For the 1st and the 3rd, the excursions sequence  $[t^n] F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \dots$$

is  $\sim K \cdot 5^n \cdot n^{-\alpha}$ , with  $\alpha = 1 + \pi / \arccos(1/4) = 3.383396 \dots$

The **irrationality** of  $\alpha$  prevents  $F_{\mathfrak{S}}(t;0,0)$  from being D-finite.

**The Main Theorem** Let  $\mathfrak{S}$  be one of the 74 non-singular models. The following assertions are equivalent:

- (1) The full generating series  $F_{\mathfrak{S}}(t; x, y)$  is D-finite
- (2) the excursions generating series  $F_{\mathfrak{S}}(t; 0, 0)$  is D-finite
- (3) the excursions sequence  $[t^n] F_{\mathfrak{S}}(t; 0, 0)$  is  $\sim K \cdot \rho^n \cdot n^\alpha$ , with  $\alpha \in \mathbb{Q}$
- (4) the group  $\mathcal{G}_{\mathfrak{S}}$  is finite (and  $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$ )
- (5) the step set  $\mathfrak{S}$  has either an axial symmetry, or zero drift and cardinal different from 5.

**The Main Theorem** Let  $\mathfrak{S}$  be one of the 74 non-singular models. The following assertions are equivalent:

- (1) The full generating series  $F_{\mathfrak{S}}(t; x, y)$  is **D-finite**
- (2) the excursions generating series  $F_{\mathfrak{S}}(t; 0, 0)$  is **D-finite**
- (3) the excursions sequence  $[t^n] F_{\mathfrak{S}}(t; 0, 0)$  is  $\sim K \cdot \rho^n \cdot n^\alpha$ , with  $\alpha \in \mathbb{Q}$
- (4) the group  $\mathcal{G}_{\mathfrak{S}}$  is **finite** (and  $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$ )
- (5) the step set  $\mathfrak{S}$  has either an **axial symmetry**, or **zero drift and cardinal different from 5**.

Moreover, under (1)–(5),  $F_{\mathfrak{S}}(t; x, y)$  is **algebraic** if and only if the model  $\mathfrak{S}$  has **positive covariance**  $\sum_{(i,j) \in \mathfrak{S}} ij - \sum_{(i,j) \in \mathfrak{S}} i \cdot \sum_{(i,j) \in \mathfrak{S}} j > 0$ , and iff it has **OS = 0**.

## Summary: Classification of 2D non-singular walks

**The Main Theorem** Let  $\mathfrak{S}$  be one of the 74 non-singular models. The following assertions are equivalent:

- (1) The full generating series  $F_{\mathfrak{S}}(t; x, y)$  is **D-finite**
- (2) the excursions generating series  $F_{\mathfrak{S}}(t; 0, 0)$  is **D-finite**
- (3) the excursions sequence  $[t^n] F_{\mathfrak{S}}(t; 0, 0)$  is  $\sim K \cdot \rho^n \cdot n^\alpha$ , with  $\alpha \in \mathbb{Q}$
- (4) the group  $\mathcal{G}_{\mathfrak{S}}$  is **finite** (and  $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$ )
- (5) the step set  $\mathfrak{S}$  has either an **axial symmetry**, or **zero drift and cardinal different from 5**.

Moreover, under (1)–(5),  $F_{\mathfrak{S}}(t; x, y)$  is **algebraic** if and only if the model  $\mathfrak{S}$  has **positive covariance**  $\sum_{(i,j) \in \mathfrak{S}} ij - \sum_{(i,j) \in \mathfrak{S}} i \cdot \sum_{(i,j) \in \mathfrak{S}} j > 0$ , and iff it has **OS = 0**.

In this case,  $F_{\mathfrak{S}}(t; x, y)$  is expressible using **nested radicals**.

If not,  $F_{\mathfrak{S}}(t; x, y)$  is expressible using **iterated integrals of  ${}_2F_1$  expressions**.

- (1) for proving algebraicity / D-finiteness
  - (1a) Guess'n'Prove
  - (1b) Creative telescoping
  
- (2) for proving non-D-finiteness
  - (2a) Infinite number of singularities, or lacunary
  - (2b) Asymptotics

(1) for proving algebraicity / D-finiteness

(1a) Guess'n'Prove

(1b) Creative telescoping

Hermite-Padé approximants  
Diagonals of rational functions

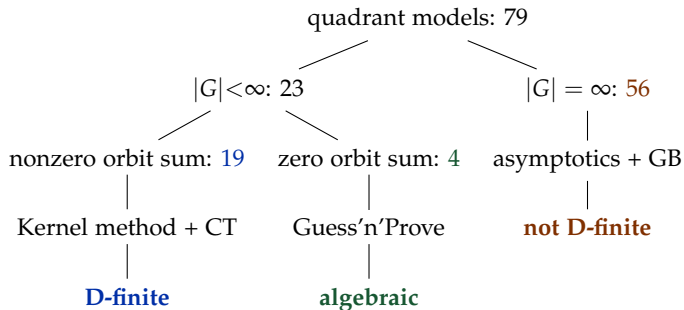
(2) for proving non-D-finiteness

(2a) Infinite number of singularities, or lacunary

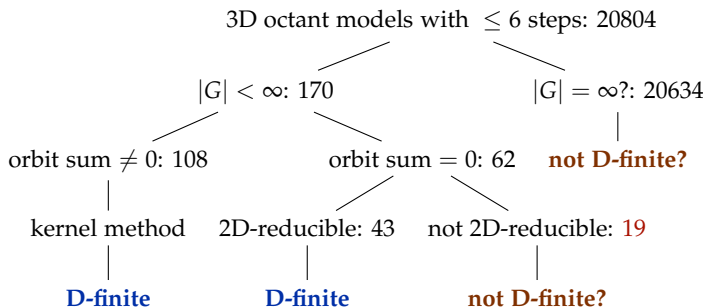
(2b) Asymptotics

► All methods are algorithmic.

## Summary: Walks with unit steps in $\mathbb{N}^2$



11 074 225 distinct interesting models

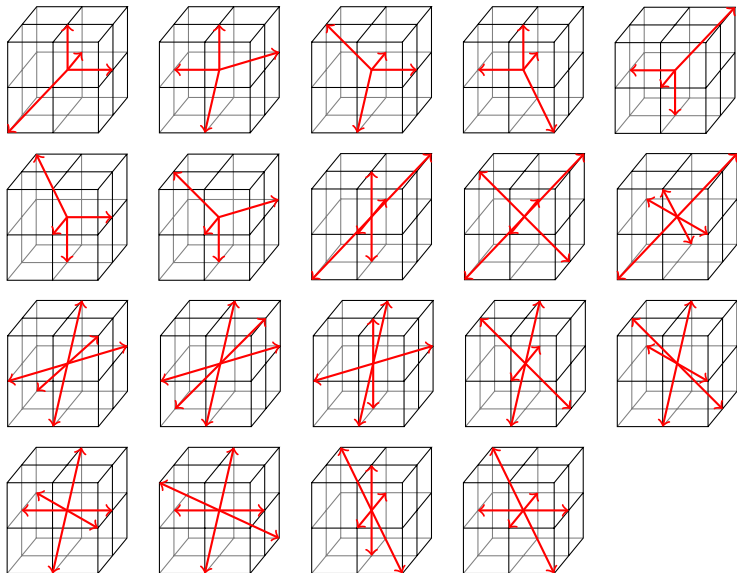


[B., Bousquet-Mélou, Kauers, Melczer 2015]

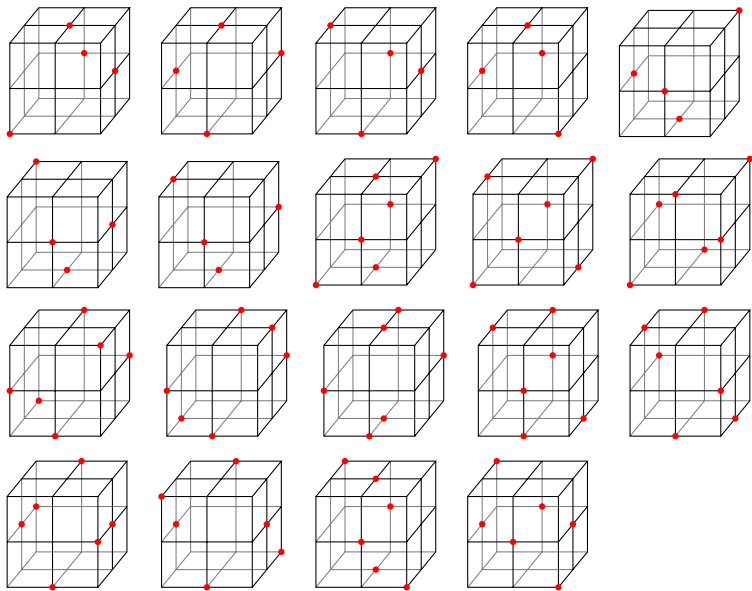
- Open question: **some non-D-finite models with a finite group?**



# The 19 mysterious 3D-models



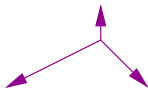
# The 19 mysterious 3D-models



## Extensions: Walks in $\mathbb{N}^2$ with longer steps

- Define (and use) a group  $\mathcal{G}$  for models with larger steps?
- **Example:** When  $\mathfrak{S} = \{(0,1), (1,-1), (-2,-1)\}$ , there is an underlying group that is finite and

$$xyF(t; x, y) = [x^{>0}y^{>0}] \frac{(x - 2x^{-2})(y - (x - x^{-2})y^{-1})}{1 - t(xy^{-1} + y + x^{-2}y^{-1})}$$



[B., Bousquet-Mélou & Melczer, in progress]

- ▶ Current status:
  - 680 models with one large step, 643 **proved non D-finite**, 32 of 37 have differential equations **guessed**.
  - 5910 models with two large steps, 5754 **proved non D-finite**, 69 of 156 have differential equations **guessed**.

- Automatic classification of restricted lattice walks, with M. Kauers. Proc. FPSAC, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. Proc. Amer. Math. Soc., 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2013.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2015.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, 2015.
- Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, 2015.



Let  $\mathfrak{S} = \{N, W, SE\}$ . A  $\mathfrak{S}$ -walk is a path in  $\mathbb{Z}^2$  using only steps from  $\mathfrak{S}$ . Show that, for any integer  $n$ , the following quantities are equal:

- (i) the number of  $\mathfrak{S}$ -walks of length  $n$  confined to the upper half plane  $\mathbb{Z} \times \mathbb{N}$  that start and end at the origin  $(0, 0)$ ;
- (ii) the number of  $\mathfrak{S}$ -walks of length  $n$  confined to the quarter plane  $\mathbb{N}^2$  that start at the origin  $(0, 0)$  and finish on the diagonal  $x = y$ .



Let  $\mathfrak{S} = \{N, W, SE\}$ . A  $\mathfrak{S}$ -walk is a path in  $\mathbb{Z}^2$  using only steps from  $\mathfrak{S}$ . Show that, for any integer  $n$ , the following quantities are equal:

- (i) the number of  $\mathfrak{S}$ -walks of length  $n$  confined to the upper half plane  $\mathbb{Z} \times \mathbb{N}$  that start and end at the origin  $(0, 0)$ ;
- (ii) the number of  $\mathfrak{S}$ -walks of length  $n$  confined to the quarter plane  $\mathbb{N}^2$  that start at the origin  $(0, 0)$  and finish on the diagonal  $x = y$ .

For instance, for  $n = 3$ , this common value is 3:

- (i)  $(0, 0) \mapsto (-1, 0) \mapsto (-1, 1) \mapsto (0, 0)$ ,  $(0, 0) \mapsto (0, 1) \mapsto (-1, 1) \mapsto (0, 0)$  and  $(0, 0) \mapsto (0, 1) \mapsto (1, 0) \mapsto (0, 0)$ , i.e., **W-N-SE**, **N-W-SE**, **N-SE-W**
- (ii)  $(0, 0) \mapsto (0, 1) \mapsto (1, 0) \mapsto (0, 0)$ ,  $(0, 0) \mapsto (0, 1) \mapsto (0, 2) \mapsto (1, 1)$  and  $(0, 0) \mapsto (0, 1) \mapsto (1, 0) \mapsto (1, 1)$ , i.e., **N-SE-W**, **N-N-SE**, **N-SE-N**

Thanks for your attention!