Long Range Ferromagnets: Renormalization Group Analysis

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Outline

The following dialogue actually took place many years ago. It went something like this (paraphrase)

SALAM: We then remove infinities by renormalisation theory.

DIRAC: I don't like this way of doing things. You should put in a cutoff and get well defined equations. You should solve these equations and then take limits.

SALAM (student of Dirac at St. John's, Cambridge): Professor Dirac , with all due respect, we don't do things that way any more.

What would Ken Wilson have said? Wilson by this time had his own particular way of looking at this: the Renormalization Group (his formulation). It does employ ultraviolet cutoffs (lattice or continuum) with well defined functional integral, analyzes the RG flow and takes limits. No infinities are ever met. The functional integral with cutoffs of course does solve the cutoff quantum field equations and in this sense Wilson is closer to Dirac than to say Salam.

Wilson also realised (and this was a revolutionary achievement) that in a statistical system near a second order phase transition when the correlation length approaches infinity the critical exponents are those of a continuum (no ultraviolet cutoff limit) scale invariant field theory. The two problems, that of approaching criticality and that of removing ultraviolet cutoffs to produce scale invariant field theories are related via a special scaling limit which we may call the Wilson scaling limit which we will explain later.

In this talk I will try to illustrate Wilson's way of doing things in the light of modern developments by looking at an interesting statistical model which near the critical point is also a full fledged scale invariant field theory. This is the theory of the critical ferromagnet with long range interactions when we consider large wavelengths. The critical theory was analyzed in [BMS] and later in [A]. A supersymmetric version is related to a class of self-avoiding Lévy walks, [MS]. Wilson's ideas are fully expressed in his 1972 Princeton lectures (reference in next frame).

Based on :

1. [BMS] : Brydges, Mitter and Scoppola: Commun.Math.Phys. (2003) 240: 281-327

2. [BGM]: Brydges, Mitter and Guadagni: J Stat Phys (2004) 115: 415-449

3. [MS] : Mitter and Scoppola: J Stat Phys (2008) 133: 921-1011

4. [BM] : Brydges and Mitter: J Stat Phys (2012) 147: 716-727

5. [W1]: K. G. Wilson and J. Kogut: Phys. Rep. (1974) 12 (No.2): 75-200

6. [A]: A. Abdesselam: Commun. Math. Phys. (2007), 276: 727-772

7. [D]: F.J. Dyson: Commun.Math.Phys. (1969) 12: 91, 212

8. [AF] : Aizenman and Fernandez: Lett.Math.Phys (1988) 16: 39-49

9. [FMN]: M. Fisher, Ma and Nickel: Phys. Rev. Lett (1972) 29: 917-920

Long range ferromagnets have a long history going back to Dyson: Commun.Math.Phys. (1969) 12: 91, 212. Dyson considered Ising spins with long range interaction on a lattice in $d = 1$:

$$J(x-y) = \frac{1}{|x-y|^{1+\alpha}}$$

where $\alpha > 0$ with Hamiltonian and partition function

$$H(\sigma) = - \sum_{x,y \in \mathbb{Z},\, x \neq y} \sigma(x) J(x-y) \sigma(y)$$

$$Z = \sum_{\sigma \in \{-1,1\}, \forall \sigma} e^{-\beta H(\hat{\sigma})}$$

For the system to be well defined we should restrict ourselves to a finite interval $\Lambda \subset \mathbb{Z}$. Take an increasing sequence of intervals $\{\Lambda_N\}$. The existence of the thermodynamic limit $N \to \infty$ is assured because

$$\sum_{x \neq 0} J(x) < \infty$$

since $\alpha > 0$ as first observed by Gallavotti and Miracle-Sole.

Dyson then proved the existence of long range order (spontaneous magnetization) for sufficiently high β (inverse temperature) provided $0 < \alpha < 1$ and no long range order for $\alpha > 1$. I shall use the notation α instead of the usual notation σ for the decay parameter and reserve σ for spin fields.

More general rigorous results have been established by Aizenman and Fernandez [AF] in higher dimensions. They consider the long range interaction on the lattice $\mathbb{Z}^d$:

$$J(x-y) = \frac{1}{|x-y|^{d+\alpha}}$$

and prove the existence of a critical value $\beta_c < \infty$ provided $d \geq 2$ or $d = 1$ and $0 < \alpha \leq 1$. For $\beta < \beta_c$ the Gibbs state is unique and the two point function has power decay with exponent $d - \alpha$. For $\beta > \beta_c$ there is spontaneous magnetization.

Aizenman and Fernandez also establish rigorously the upper critical dimension. We have mean field critical exponents if either $d \geq 4$ or $d \geq 2\alpha$. For $0 < \alpha < 2$ the upper critical dimension is

$$d_c = 2\alpha$$

We will be interested in the case $d = 2, 3$ and $d < d_c = 2\alpha$ where $0 < \alpha < 2$. where we expect nontrivial critical behaviour.

These results also go through for continuous spin ferromagnets on $\mathbb{Z}^d$ with Hamiltonian

$$H(\sigma) = - \sum_{x,y\in\mathbb{Z},\, x\neq y} \sigma(x) J(x-y) \sigma(y) + \sum_{x\in\mathbb{Z}} \left(g\, \sigma^4(x) + \mu\, \sigma^2(x) \right)$$

where $\sigma(x) \in \mathbb{R}$, $\forall x$ and the constant $g > 0$. These continuous spin models fall in the so called Griffiths-Simon class and can be considered as suitable limits of discrete spin ferromagnetic models very much as in Simon and Griffiths paper [SG].

We are interested in nontrivial behaviour for $d = 2, 3$ near β_c below the critical dimension: $d < d_c = 2\alpha$ and $0 < \alpha < 2$. We consider the Hamilonian in the long wave length (low Fourier mode) approximation. This was first done by Michael Fisher, Ma, and Nickel [FMN] long ago (1972). They considered these systems using Wilson's approximate recursion relation in the $\epsilon = d_c - d = 2\alpha - d$ expansion. Many of their results can be established by rigorous RG methods outside of the ϵ expansion (but $\epsilon > 0$ held very small). More later.

The Fourier transform of $J(x)$ in the continuum in the sense of distributions is $\hat{J}(p) = -c_{\alpha,d}\,|p|^{\alpha}$ where $c_{\alpha,d}$ is a positive constant for $0 < \alpha < 2$. For low Fourier modes this can be replaced by the lattice expression

$$\hat{J}(p) = -(-\hat{\Delta}(p))^{\frac{\alpha}{2}}$$

where $p \in [-\pi, \pi]^d$, the first Brillouin zone of the dual lattice and

$$\hat{\Delta}(p) = 2\sum_{\mu=1}^{d}(\cos p_\mu - 1)$$

This leads us to the unit lattice Hamiltonian in the long wave length approximation

$$H(\sigma) = -\sum_{x\in\mathbb{Z}^d} \sigma(x)((-\Delta)^{\frac{\alpha}{2}}\sigma)(x) + \sum_{x,\mathbb{Z}^d}\left(g\,\sigma^4(x) + \mu\,\sigma^2(x)\right)$$

and we have scaled away the constant $c_{\alpha,d}$ by redefining the other constants.

σ is a Gaussian random variable with covariance C

$$C(x-y) = (-\Delta)^{-\alpha/2}(x-y)$$

$$= \int_{[-\pi,\pi]^d} \frac{d^d p}{(2\pi)^d} e^{ip.(x-y)} (-\hat{\Delta}(p))^{-\alpha/2}$$

and μ_C is the corresponding Gaussian measure on the space of fields $\sigma : \mathbb{Z}^d \to \mathbb{R}$. Let $\Lambda_N = [-\frac{L^N}{2}, \frac{L^N}{2}]^d \subset \mathbb{Z}^d$ be a large cube with L a triadic integer (power of 3) and N a large positive integer. The interactions will be restricted to Λ_N and eventually $N \to \infty$.

Corresponding to the Hamiltonian H with the local interaction in Λ_N we have the normalized probability measure

$$d\mu_N(\sigma) = \frac{1}{Z_N} d\mu_C(\sigma) e^{\sum_{x\in\Lambda_N}\left(g\,\sigma^4(x)+\mu\,\sigma^2(x)\right)}$$

The partition function Z_N is defined by requiring that μ_N is a probability measure. The spin correlation functions are moments (marginals) of μ_N. The temperature appears (because we have rescaled) linearly in the mass parameter μ. It appears quadratically in the coupling constant g, but that is not of any importance since the coupling constant will hit a fixed point of the RG.

Let $d = 2$ or 3. Define

$$\epsilon = d_c - d = 2\alpha - d > 0$$

$$\eta = 2 - \alpha$$

$$[\sigma] = \frac{d-\alpha}{2} = \frac{d-2}{2} + \frac{\eta}{2}$$

and define new fields σ_N by

$$\sigma_N(x) = L^{N[\sigma]}\sigma(L^N x)$$

Theorem 1

Let j be a test function of compact support which we can restrict to the lattice. Then j depends only on a finite number of points. Let $\epsilon = d_c - d = 2\alpha - d > 0$ be sufficiently small. The thermodynamic limit

$$\lim_{N\to\infty} \int d\mu_N(\sigma) e^{i\sigma(j)}$$

exists and $\mu_N \to \mu$ exists as weak convergence of measures. Then we have the scaling limit

$$\lim_{N\to\infty} \int d\mu(\sigma)\sigma_N(j_1)\sigma_N(j_2) = \text{const.}C_{\text{cont}}(j_1, j_2)) + R(j_1, j_2)$$

Here C_{cont} is the continuum limit of the covariance of the Gaussian measure $\mu_{C_{\epsilon_N}}$ on the space of spin fields on the lattice $(\epsilon_N \underline{Z})^d$ as the lattice spacing $\epsilon_N = L^{-N} \to 0$. For $x \neq y$,

$$C_{\text{cont}}(x-y) = const. \frac{1}{|x-y|^{d-\alpha}}$$

$$|R(x,y)| \leq \text{const.} \epsilon^{\frac{1}{2}} \frac{1}{|x-y|^{d-\alpha+\delta}}$$

for some $\delta > 0$. From this we can easily deduce that the classical value $\eta = 2 - \alpha$ is the 2-point function critical exponent (see next frame)

Let $\lambda \geq 2$ be a real parameter. Define the scale transformation S_λ by

$$S_\lambda f(x, y) = \lambda^{2[\sigma]} f(\lambda x, \lambda y)$$

Observe that $C_{\text{cont}}(x - y)$ is scale invariant whereas

$$|S_\lambda R(x, y)| \leq \text{const.} \epsilon^{\frac{1}{2}} \lambda^{-[\sigma]\delta} . \frac{1}{|x - y|^{d-\alpha+\delta}}$$

and this $\to 0$ as $\lambda \to \infty$ whence our statement that η is the 2-point function critical exponent. The $\epsilon^{1/2}$ factor comes from a non-trivial fixed point of $O(\epsilon)$ the $1/2$ in the exponent being due to a loss from (rough) estimates (formally it is $O(\epsilon^2)$).

We will prove a simplified version of Theorem 1 in the *Wilson scaling limit*. The Wilsonian scaling limit is a special way of taking the continuum limit in finite volume. The parameters (coupling constants, mass) are made to depend on the lattice spacing dictated by dimensional analysis. *In our case the bare coupling constant will go to infinity at a fixed rate*. This drives the dimensionless coupling constant to the attractive RG fixed point which in our case is the infrared fixed point (the Gaussian fixed point being unstable). The dimensionless mass is fine tuned to be critical. As we shall see *this is the same as being on on the unit lattice and taking the scaling limit as well as the thermodynamic limit at the same time.*

Let $(\epsilon_N \mathbb{Z})^d$ be a fine lattice with spacing $\epsilon_N = L^{-N}$. Let Λ_M be a fixed cube in $\mathbb{R}^d$ be a fixed continuum cube with edge length $L^M$and $\Lambda_{M,N} = \Lambda_M \cap (\epsilon_N \mathbb{Z})^d$ be its restriction to the lattice $(\epsilon_N \mathbb{Z})^d$.

Let ϕ be the Gaussian spin field on $(\epsilon_N \mathbb{Z})^d$ with covariance

$$C_{\epsilon_N}(x-y) = (-\Delta_{\epsilon_N})^{-\alpha/2}(x-y)$$

$$= \int_{[-\frac{\pi}{\epsilon_N}, \frac{\pi}{\epsilon_N}]^d} \frac{d^d p}{(2\pi)^d} e^{ip.(x-y)} (-\hat{\Delta}_{\epsilon_N}(p))^{-\alpha/2}$$

$$\hat{\Delta}_{\epsilon_N}(p) = \frac{2}{{\epsilon_N}^2} \sum_{\mu=1}^{d} (\cos \epsilon_N p_\mu - 1)$$

Our measure on the random fields ϕ on the ϵ_N lattice is now

$$d\mu_{M,N}(\phi) = \frac{1}{Z_{M,N}} d\mu_{C_{\epsilon_N}}(\phi) e^{-V(\tilde{g}_N, \tilde{\mu}_N, \Lambda_{M,N}, \phi))}$$

$$V(\tilde{g}_N, \tilde{\mu}_N, \Lambda_{M,N}, \phi)) = \epsilon_N^d \sum_{x \in \Lambda_{M,N}} \left(\tilde{g}_N \, \phi^4(x) + \tilde{\mu}_N \, \phi^2(x) \right)$$

$$\tilde{g}_N = \epsilon_N^{-\epsilon} g_0 = L^{N\epsilon} g_0$$

$$\tilde{\mu}_N = \epsilon_N^{-\alpha} \mu_0 = L^{N\alpha} \mu_0$$

The partition function $Z_{M,N}$ is defined by requiring that $\mu_{M,N}$ is a probability measure. The spin correlation functions are moments (marginals) of $\mu_{M,N}$.

Dimensionless analysis gives $[g] = \epsilon$ and $[\mu] = \alpha$. Define unit lattice fields σ with covariance C_{ϵ_0} by

$$\phi(x) = \epsilon_N^{-[\sigma]}\sigma(\epsilon_N^{-1}x) = L^{N[\sigma]}\sigma(L^N x) = \sigma_N(x)$$

Then a straight forward change of variables gives for the correlation functions

$$< \phi(x_1).....\phi(x_m) >_{\mu_{M,N},\tilde{g}_N,\tilde{\mu}_N} = < \sigma_N(x_1).....\sigma_N(x_m) >_{\mu_{M+N,0},g_0,\mu_0}$$

The ϵ_N lattice points x_j are restricted to the unit lattice (always possible) and thus non-coinciding. In the following M is fixed whereas we will take $N \to \infty$. This gives the relation between the continuum limit and the scaling limit.

In the following I will give for the purposes of this exposition the RG analysis directly in the continuum following Brydges, Mitter and Scoppola (2003). This is much less cumbersome than the RG analysis on the lattice but the methods are the same. The same results however can be proved on the lattice using the lattice Finite Range RG (instead of Kadanoff-Wilson block spins) of Brydges, Mitter and Guadagni [BGM] very much as in Mitter and Scoppola (2008) (or by more recent methods of Brydges and Slade (to be published)) where in addition fermionic fields are also present.

Continuum RG program:

Instead of the ϵ_N-lattice we take $\mathbb{R}^d$, put in a volume (cube) cutoff on the interaction and an ultraviolet cutoff ϵ_N directly in the covariance as follows. We convert this to the unit cutoff (ϵ_0) problem by a change of scale employing new variables and as before the the size of the cube becomes very large, sides being rescaled by a factor $\epsilon_N^{-1} = L^N$. Then RG step : Integrate out a high frequency slice (in a special way) and rescale fields. This reduces the volume (side $L^N \to L^{N-1}$). Do this N times:

At each step the interaction evolves: we get expanding variables (local couplings) and contracting variables (irrelevant terms). The irrelevant terms are gathered in objects called polymer activities. They are measured in RG stable norms. After N steps the volume has shrunk to a cube (block) of side length L. Then take $N \to \infty$. We have a non-linear dynamical system in a Banach space. This system has a non-trivial hyperbolic fixed point and a stable (critical) manifold which solves the mass fine tuning problem. Once the RG analysis has been performed on the measure we obtain easily the results on the correlation functions. This is now explained.

The continuum covariance C_{cont} is given (upto constants) by

$$C_{\text{cont}}(x-y) = \text{const.}\frac{1}{|x-y|^{d-\alpha}} = \text{const.}\frac{1}{|x-y|^{2[\sigma]}}$$

Let g be a smooth rotation invariant function of compact support (in fact finite range) : $g(x) = 0 : \ |x| \geq 1/2$. Let $u = g * g$ (convolution). Then u is positive definite, C^∞, finite range $u(x) = 0 : \ |x| \geq 1$, and rotation invariant. Observe that (upto constants)

$$C_{\text{cont}}(x-y) = \int_0^\infty \frac{dl}{l}\, l^{-2[\sigma]}\, u(\frac{x-y}{l})$$

Convergence at the upper end point since $[\sigma] > 0$. If $x \neq y$ we also have convergence at the lower end point since $l < |x - y|$ is outside the support of u. Ultraviolet singularity at the lower end point as $x \to y$. We introduce the ultraviolet cutoff covariance

$$C_{\epsilon_N}(x-y) = \int_{\epsilon_N}^{\infty} \frac{dl}{l}\, l^{-2[\sigma]}\, u(\frac{x-y}{l})$$

$C_{\epsilon_N}(x)$ is now C^∞ because of the lower end point cutoff. Observe that

$$C_{\epsilon_N}(x-y) = {\epsilon_N}^{-2[\sigma]} C_{\epsilon_0}(\frac{x-y}{\epsilon_N})$$

$C_{\epsilon_0} = C_1$ is the unit cutoff covariance.

Let $j = (j_1, ...j_m)$ be test functions of compact support. Let $j(s) = \sum_{i=1}^{m} s_i\, j_i$, $s = (s_1, ..., s_m)$ real numbers. Let ϕ be the random fields with measure

$$d\mu_{M,N}(\phi) = d\mu_{C_{\epsilon_N}}(\phi)\; e^{-V(\tilde{g}_N, \tilde{\mu}_N, C_{\epsilon_N}, \Lambda_M, \phi)}$$

$$Z(\epsilon_n, \Lambda_M, j(s)) = \int d\mu_{M,N}(\phi) e^{i\phi(j(s)}$$

Then the truncated connected correlation functions are given by

$$< \phi(x_1).....\phi(x_m) >_{\Lambda_M, \epsilon_N}$$

$$= \frac{1}{Z(\epsilon_n, \Lambda_M, 0)} (-i)^m \prod_{i=1}^{m} \partial_{s_i} \log\; Z(\epsilon_n, \Lambda_M, j(s))\big|_{s_i=0, \forall i}$$

Finally we pass from the ϵ_N cutoff theory to the $\epsilon_0 = 1$ cutoff theory with enlarged volume Λ_{M+N} by making the substitution $\phi(x) = \sigma_N(x) = \epsilon_N^{-[\sigma]}\sigma(\epsilon_N^{-1}x)$. This gives

$$Z(\epsilon_n, \Lambda_M, j(s)) = \int d\mu_{C_{\epsilon_0}}(\sigma)\, e^{-V(g_0,\mu_0,C_{\epsilon_0},\Lambda_{M+N},\sigma)+i\sigma(j_N(s))}$$

where

$$j_N(s)(x) = \epsilon_N^{d-[\sigma]} j(s)(\epsilon_N x)$$

Hence forth we write C instead of $C_{\epsilon_0} = C_1$ for unit cutoff covariance. Divide up $[1, \infty) = [1, L] \cup [L, \infty)$ and define

$$\Gamma_L(x-y) = \int_1^L \frac{dl}{l}\, l^{-2[\sigma]}\, u(\frac{x-y}{l})$$

Then

$$C(x-y) = \Gamma_L(x-y) + L^{-2[\sigma]} C(\frac{x-y}{L})$$

$\Gamma_L(x)$ is a smooth positive definite function (because u is positive definite) and of finite range

$$\Gamma_L(x-y) = 0 : \ |x-y| \geq L$$

Iterating we get a convergent (in L^∞) expansion

$$C(x-y) = \sum \Gamma_n(x-y)$$

$$\Gamma_n(x-y) = L^{-2n[\sigma]} \Gamma_L(\frac{x-y}{L^n})$$

Each Γ_n is positive definite, finite range L^n and C^∞.

Correspondingly we have the field σ is a sum of independent Gaussian random fields (fluctuation fields) ζ_n, almost surely C^∞, with covariances Γ_n

$$\sigma = \sum \zeta_n$$

$$E(\zeta_n(x)\zeta_m(y)) = 0 : \ n \neq m$$
$$E(\zeta_n(x)\zeta_n(y)) = \Gamma_n(x-y)$$

implies finite range correlations:

$$E(\zeta_n(x)\zeta_n(y)) = 0 : \ |x-y| \geq L^n$$

This is a simple example of a *finite range multi scale expansion*.

Important property:

$$|x - y| \leq L^n \Rightarrow \mu_C(|\zeta_n(x) - \zeta_n(y)| \geq \gamma) \leq \text{const.}\gamma^{-2}$$

which show that the fluctuation fields ζ_n are slowly varying on scale L^n. This captures an essential idea of Wilson.

The decomposition into a sum of independent fluctuation fields means that we have to do a multiple integral over the fluctuation fields. Do this step by step each step. RG transformation: For any functional $F(\sigma)$

$$(T_L F)(\sigma) = S_L \mu_{\Gamma_L} * F(\sigma) = \int d\mu_{\Gamma_L}(\zeta) F(\zeta + S_L \sigma)$$

where S_L is the scale transformation

$$S_L \sigma(x) = L^{-[\sigma]} \sigma(\frac{x}{L})$$

Semigroup property:

$$T_L T_{L^n} = T_{L^{n+1}}$$

μ_c is the (unique) invariant measure of T_L:

$$\int d\mu_C(\sigma) T_L F(\sigma) = \int d\mu_C F(\sigma)$$

We apply this to our problem. Put the external source $j = 0$. We will put it in later. Define

$$\mathcal{Z}_0(\Lambda_N, \sigma) = e^{-V_0(g_0, \mu_0, \Lambda_N, \sigma)}$$

Then

$$\int d\mu_C(\sigma) \mathcal{Z}_0(\Lambda_N, \sigma) = \int d\mu_C(\sigma) \mathcal{Z}_1(\Lambda_{N-1}, \sigma)$$

where

$$\mathcal{Z}_1(\Lambda_{N-1}, \sigma) = \int d\mu_{\Gamma_L}(\zeta) \mathcal{Z}_0(\Lambda_N, \zeta + S_L \sigma)$$

Iterating n times we get:

$$\int d\mu_C(\sigma)\mathcal{Z}_0(\Lambda_N,\sigma) = \int d\mu_C(\sigma)\mathcal{Z}_n(\Lambda_{N-n},\sigma)$$

where

$$\mathcal{Z}_n(\Lambda_{N-n},\sigma) = \int d\mu_{\Gamma_L}(\zeta)\mathcal{Z}_{n-1}(\Lambda_{N-n+1},\zeta + S_L\sigma)$$

At the end of N steps the volume reduces to the unit cube (unit block) and we take the $N \to \infty$.

We want to analyze the generic step. The first problem that we want to take care of is nonlocality. The starting density has a locality property which we lose after one iteration. Thus if X, Y are two subsets with disjoint interiors then

$$\mathcal{Z}_0(X \cup Y)) = \mathcal{Z}_0(X)\mathcal{Z}_0(Y))$$

but after one iteration this is no longer true

$$\mathcal{Z}_1(X \cup Y)) \neq \mathcal{Z}_1(X)\mathcal{Z}_1(Y))$$

This is resolved by introducing a new representation for the densities-the so called polymer gas representation- and the finite range of the fluctuation field correlations.

Pave $\mathbb{R}^d$ with unit blocks (unit cubes). Then $\Lambda \subset \mathbb{R}^d$ has the induced paving. A *polymer* X is a connected subset of blocks. A *polymer activity* K is a map $(X, \sigma) \to K(X, \sigma) \in \mathbb{R}$. The field σ has been restricted to X.

At any given step n of the sequence of RG transformations the densities will be given coordinates g_n, μ_n, K_n. Here g_n, μ_n are the evolved parameters of the local potential V_n. and K_n is a so called irrelevant (contracting) term characterized as a polymer activity. The density $\mathcal{Z}_n(\Lambda_{N-n}, \sigma)$ can be expressed in terms of these coordinates in a polymer gas representation.

Polymer Gas Representation

$$\mathcal{Z}_n(\Lambda_{N-n},\sigma) = \sum_{N\geq 0} \frac{1}{N!} e^{-V_n(X_c,\sigma)} \sum_{X_1,..X_N} \prod_{j=1}^{N} K_n(X_j,\sigma)$$

where $X_c = \Lambda_{N-n} / \cup_{j=1}^{n} X_j$, and the sum is over mutually disjoint connected polymers X_j in Λ_{N-n}.

Note that the local potential V_n depends on (evolved) coupling g_n and mass parameter μ_n.

This representation is *stable* under RG.

The renormalization group step T_L involves fluctuation integration and then rescaling. The density depends on $\zeta + \sigma$ and we integrate with measure $d\mu_{\Gamma_L}(\zeta)$. Replace σ by $\sigma + \zeta$ in V_n, K_n. We now prepare the integrand a bit before actually doing the integral.

$V_n(X_c, \zeta + \sigma)$ is local. We consider also an *arbitrary* local potential $\tilde{V}_n(X_c, \sigma)$ *which depends only on* σ. We can write

$$\exp - V_n(X_c, \zeta + \sigma) = \prod_{\Delta \subset X_c} \exp - V_n(\Delta) =$$

$$\prod_{\Delta \subset X_c} [P_n(\Delta, \zeta, \sigma) + \exp - \tilde{V}_n(\Delta, \sigma)]$$

where

$$P_n(\Delta, \zeta, \sigma) = \exp - V_n(\Delta, \zeta + \sigma) - \exp - \tilde{V}_n(\Delta, \sigma)$$

Expand out and glue together the $P_n(\Delta)$ with the polymer activities K_n. This will create new polymer activities. Finally remember that the fluctuation covariance Γ_L has finite range L. So we should glue together these new $1-$ polymers into disjoint connected $L-$ polymers built out of connected $L-$ blocks (cubes of side length L).

The net result is a new representation:

$$\mathcal{Z}_n(\Lambda_{N-n}, \zeta + \sigma) = \sum_{N \geq 0} \frac{1}{N!} e^{-\tilde{V}_n(Y_c, \sigma)} \sum_{Y_1, .. Y_N} \prod_{j=1}^{N} \mathcal{B}K_n(Y_j, \zeta, \sigma)$$

The sum is now over mutually disjoint connected L- polymers in Λ_{N-n} . $\mathcal{B}K_n$ is a non-linear functional of K_n, $\tilde{V}_n$ which depends on σ, ζ. $\tilde{V}_n$ is a yet to be chosen local potential *which depends only on* σ .

The *fluctuation* map $S_L \mu_{\Gamma_n} *$ integrates out the ζ and then rescales. The integral sails through $\exp - \tilde{V}_n(Y_c, \sigma)$ which is independent of ζ. Then it *factorizes* over the product of polymer activities because of the finite range property of Γ_n since the connected L- polymers are separated by a distance $\geq L$. Thus the polymer representation is preserved after fluctuation integration. Then we rescale to get back to $1-$ polymers.

The fluctuation integration plus rescaling has given a map

$$V_n \to \tilde{V}_{n,L} = S_L \tilde{V}_n \quad \tilde{V}_{n,L}(\Delta, \sigma) = \tilde{V}_n(L\Delta, S_L\sigma)$$

$$K_n \to \mathcal{F}K_n \quad \mathcal{F}K_n(X, \sigma) = \int d\mu_{\Gamma_L}(\zeta)\mathcal{B}K(LX, \zeta, S_L\sigma)$$

We shall now do one more *crucial* step to produce our final *renormalization map* .

The representation that we have given is not unique because $\tilde{V}_n$ is upto us to choose. A change in $\tilde{V}_n$ changes $\mathcal{F}K_n$. For example choose $\tilde{V}_n = V_n$. Then subtract out the (localized) expanding parts of $\mathcal{F}K_n$, and absorb them in $V_{n,L}$ thus producing a flow of parameters. The new subtracted polymer activities have good contraction properties measured in appropriate norms.

This subtraction operation on $\mathcal{F}K_n(X,\sigma)$ needs only to be be done for *small* sets X : $|X| \leq 2^d$, because *large* sets provide contracting contributions measured in appropriate norms. The new subtracted polymer activities have good contraction properties (*irrelevant terms*).

This procedure produces our final RG map f_{n+1}

$$f_V(V_n, K_n) = V_{n+1}, \quad f_K(V_n, K_n) = K_{n+1}$$

Using second order perturbation theory,

$$K_n = e^{-V_n} Q_n + R_n$$

Q_n is a second order contribution. It is form invariant and depends on g_n, μ_n and a non-local kernel w_n which converges fast to a fixed point kernel w_*. R_n is a remainder (formally of third order).

To speed things up we just set $w = w_*$. Then $u_n = (g_n, \mu_n, R_n)$ represents a point on the RG trajectory. The RG map produces a discrete flow:

$$u_{n+1} = f(u_n)$$

The flow map in components is:

$$g_{n+1} = f_g(u_n) = L^\epsilon g_n(1 - L^\epsilon a g_n) + \xi_n(u_n)$$
$$\mu_{n+1} = f_\mu(u_n) = L^\alpha \mu_n - L^{2\epsilon} b g_n^2 + \rho_n(u_n)$$
$$R_{n+1} = f_R(u_n) =: U_{n+1}(u_n)$$

The coefficient a is positive. We have an approximate flow $\bar{g}_n$ obtained by ignoring the remainder ξ_n. This approximate flow generated by second order perturbation theory has an attractive fixed point $\bar{g} = O(\epsilon)$, for ϵ sufficiently small.

Let $\tilde{g}_n = g_n - \bar{g}$. Then $v_n = (\tilde{g}_n, \mu_n, R_n)$ are the new coordinates. Then

$$\tilde{g}_{n+1} = f_g(v_n) = (2 - L^\epsilon)\tilde{g}_n + \tilde{\xi}_n(v_n)$$

$$\mu_{n+1} = f_\mu(v_n) = L^\alpha \mu_n + \tilde{\rho}_n(v_n)$$

$$R_{n+1} = f_R(v_n) =: U(v_n)$$

are the new flow equations.

$\gamma(\epsilon) = 2 - L^\epsilon = 1 - O(\log L)\epsilon < 1$, for sufficiently small ϵ (with L fixed. $\gamma(\epsilon)$ is a contraction factor. Also we will see that the R evolution has a contraction factor. The μ evolution is dangerous because of the L^α factor.

Banach spaces:

1. We endow polymer activities $K(X)$ with a Banach space norm $\|K(X)\|$. This norm measures large σ field growth and a finite number of partial (Fréchet) derivatives in σ (functional derivatives). Under this norm we have

$$||K(X)K(Y)|| \leq ||K(X)||\,||K(Y)||$$

$$||\mathcal{F}(K)(X)|| \leq c^{|X|}||K(X)||$$

Our final norm is

$$||K|| = \sup_{\Delta} \sum_{X \supset \Delta} L^{(d+2)|X|} ||K(X)||$$

This norm gives us a Banach space of Polymer activities. It says that that bigger the polymer the smaller is its contribution.

It has the *important property* : large sets contribute contracting (by a factor $L^{-(d+1)}$) contributions to the fluctuation map. Hence the relevant (expanding) parts have to be only extracted from small sets : $|X| \leq 2^d$.

We introduce a second norm $|\cdot|$. This is the same as the previous norm except that we evaluate poymer activities at $\sigma = 0$, and therefore no large field growth to be measured. We measure the polymer activity R_n in a norm $|||\cdot|||$, where

$$|||R_n|| = max\{|R_n|, \epsilon^2||R_n||\}$$

Define a Banach space E consisting of elements $v = (\tilde{g},\ \mu,\ R)$ with norm

$$||v|| = max\{(\epsilon)^{-3/2}|\tilde{g}|, \epsilon^{-(2-\delta)}|\mu|, \epsilon^{-(11/4-\eta)}|||R|||\}$$

where $\delta, \eta > 0$ are very small numbers and $0 < \nu < 1/2$.

Let $v_n = (\tilde{g}_n, \mu_n, R_n)$ and let $E(r) \subset E$ be an open ball of radius r centered at the origin. Then our next theorem says

Theorem 2 (stability): Let $v_n \in E(1)$. Then

$$|\tilde{\xi}(v_n)| \leq C_L \epsilon^{11/4-\eta}, \quad |\tilde{\rho}(v_n)| \leq C_L \epsilon^{11/4-\eta}$$

These are estimates for the error terms in the g_n, μ_n flow. Moreover $R_{n+1} = U_{n+1}(v_n)$ has the bound

$$|||U_{n+1}(v_n)||| \leq L^{-1/4} \epsilon^{11/4-\eta}$$

On the right hand side we have a contraction factor.

Lipshitz continuity

Let $v,\ v' \in E(1/4)$. Then we have Lipshitz continuity:

$$|\tilde{\xi}(v) - \tilde{\xi}(v')| \leq \epsilon^{11/4-\eta}||v - v'||$$

$$|\tilde{\rho}(v) - \tilde{\rho}(v')| \leq \epsilon^{5/2-\eta}||v - v'||$$

$$|||U(v) - U(v')||| \leq L^{-1/4}\epsilon^{11/4-\eta}||v - v'||$$

Write the flow equations in integral form: after n steps of the renormalization map we get

$$\tilde{g}_k = \gamma(\epsilon)^k \tilde{g}_0 + \sum_{j=0}^{k-1} \gamma(\epsilon)^{k-1-j} \tilde{\xi}(v_j), \;\; 1 \leq k \leq n$$

and the reversed flow for μ

$$\mu_k = L^{-\alpha(n-k)} \mu_n - \sum_{j=k}^{n-1} L^{-\alpha(j+1-k)} \tilde{\rho}(v_j), \;\; 0 \leq k \leq n-1$$

We want to solve for a bounded flow. So fix $\mu_n = f$ and let $n \to \infty$ in the reversed μ flow equation. We must show that such a flow exists.

Therefore we have to solve

$$\tilde{g}_k = \gamma(\epsilon)^k \tilde{g}_0 + \sum_{j=0}^{k-1} \gamma(\epsilon)^{k-1-j} \tilde{\xi}(v_j), \;\; 1 \leq k \leq n$$

$$\mu_k = -\sum_{j=k}^{n-1} L^{-\alpha(j+1-k)} \tilde{\rho}(v_j), \;\; 0 \leq k \leq n-1$$

$$R_k = U(v_{k-1})$$

Existence of bounded RG flow

We consider a Banach space **E** of sequences $\mathbf{v} = (v_0, v_1, v_2, ...)$, with $v_n \in E$, supplied with the norm

$$||\mathbf{v}|| = \sup_{n \geq 0} ||v_n||$$

$\mathbf{E}(r) \subset \mathbf{E}$ is an open ball of radius r. Let $v_0 = (\tilde{g}_0, \mu_0, 0)$.

Theorem 3 Existence of global bounded RG trajectory: There exists an initial mass μ_0 such that for $v_0 \in E(1/32)$, $v_k = f(v_{k-1}) \in E(1/4)$ for all $k \geq 1$.

Write the RG flow in the integral form in the space of sequences $\mathbf{E}$:

$$v_k = F_k(\mathbf{v}) : \; F_k = (F_k^g, F_k^\mu, F_k^R)$$

where the right hand side side is defined by the right hand side of the integral flow equations. If we define the sequence

$$\mathbf{F}(\mathbf{v}) = (F_0(\mathbf{v}), F_1(\mathbf{v}), ...)$$

then the integral flow equation can be written as a fixed point equation:

$$\mathbf{v} = \mathbf{F}(\mathbf{v})$$

This fixed point equation

$$\mathbf{v} = \mathbf{F}(\mathbf{v})$$

has a unique bounded solution under the conditions of Theorem 3 by virtue of Liphsitz continuity.

Stable manifold and non-trivial fixed point.
Let f^k be the $k-$ *fold* composition of the map f. The *stable (critical) manifold* of f is defined by

$$W^s(f) = v \in E(1/32) : \ f^k(v) \in E(1/4) \, \forall k \geq 0$$

Write $v = (v_1, v_2)$ with $v_1 = (\tilde{g}, R, 0)$ and $v_2 = \mu$. Initially $v_{1,0} = (\tilde{g}_0, 0, 0)$ and $v_{2,0} = \mu_0$. Theorem 3 says that for $v \in E(1/32)$, there exists v_2 such that $f^k(v) \in E(1/4) : \ \forall k \geq 0$.

Theorem 4 (Stable manifold theorem)

$W^s(f)$ is the graph $\{v_1, h(v_1)\}$ of a function $v_2 = h(v_1)$ with h Lipshitz continuous. Moreover iterations of f restricted to $W^s(f)$ contracts distances and ttherefore has a unique fixed point which attracts all points of $W^s(f)$.

Corollary: The theorem has established that the critical mass $\mu_0 = h(\tilde{g}_0) = \mu_c(g_0)$ is a Lipshitz continuous function. Moreover $v_n \to v_*$ in the ball $E(1/4)$. If $\tilde{g}_* = g * -\bar{g}$ is one of the coordinates of v_* then $g_* \neq 0$ since $v_* \in E(1/4)$ and therefore

$$|g_* - \bar{g}| \leq \frac{1}{4}\epsilon^{3/2}$$

and we know that $\bar{g} = O(\epsilon)$. So our fixed point is nontrivial.

Correlation functions

We have seen the convergence of the coordinates $v_n = (g_n, \mu_n, R_n$ of the RG trajectory to the fixed point $v_* = (g_*, \mu_*, R_*)$ in the Banach space E provided the initial mass μ_0 is chosen to lie on the critical curve $\mu_0 = \mu_c(g_0)$ with initial $R_0 = 0$. This is sufficient to prove the existence of the ultraviolet (scaling limit) for correlation functions.

Recall: $C_{\epsilon_0} = C$ is the unit cutoff cocariance and

$$Z(\epsilon_n, \Lambda_0, j(s)) = \int d\mu_C(\sigma)\, \mathcal{Z}_0(\Lambda_N.\sigma) e^{i\sigma(j_N(s))}$$

where

$$\mathcal{Z}_0(\Lambda_N.\sigma) = e^{-V(g_0,\mu_0,C_{\epsilon_0},\Lambda_N,\sigma)}$$

and

$$j_N(s)(x) = \epsilon_N^{d-[\sigma]} j(s)(\epsilon_N x) = L^{-N(d-[\sigma])} j(s)(L^{-N} x)$$

Translating in the field σ gives

$$Z(\epsilon_n, \Lambda_0, j(s)) = e^{-1/2(j_N, C*j_N)} \int d\mu_C(\sigma)\, \mathcal{Z}_0(\Lambda_N,\, \sigma + iC * j_N)$$

Applying the RG transformation once gives:

$$Z(\epsilon_n, \Lambda_0, j(s)) = e^{-1/2(j_N(s), C*j_N(s))} \int d\mu_C(\sigma) \mathcal{Z}_1(\Lambda_{N-1}, \sigma + iS_{L^{-1}} C*j_N(s)$$

where

$$S_{L^{-1}}(C * j_N(s))(x) = L^{[\sigma]}(C * j_N(s))(Lx)$$

Iterating N times gives

$$Z(\epsilon_n, \Lambda_0, j(s)) = e^{-1/2(j_N(s), C*j_N)(s))} \int d\mu_C(\sigma) \mathcal{Z}_N(\Delta, \sigma + iS_{L^{-N}} C*j_N(s))$$

where Δ is a unit block (unit cube). Easy to check

$$(j_N(s), C*j_N)(s)) = (j(s), C_{\epsilon_N} j(s))$$

$$S_{L^{-N}} C*j_N(s)(x) = (C_{\epsilon_N} * j)(x)$$

We will look at the 2-point function: Let $j(s) = s_1 j_1 + s_2 j_2$. Taking partial derivatives with respect to s_1, s_2 at $s_1 = s_2 = 0$ and dividing out by the normalizing factor (vacuum energy) gives

$$< \sigma(j_1)\sigma(j_2) >_{\epsilon_N,\Lambda_0} = (j_1, C_{\epsilon_N} * j_2) -$$

$$\frac{1}{Z_N(0)} \int d\mu_C(\sigma)\, (D^2 \mathcal{Z}_N)(\Delta, \sigma; C_{\epsilon_N} * j_1, C_{\epsilon_N} * j_2)$$

$$Z_N(0) = \int d\mu_C(\sigma) \mathcal{Z}_N(\Delta, \sigma)$$

Since we are on a unit block we have simple expressions

$$\mathcal{Z}_N(\Delta, \sigma) = e^{\Omega_N}\, \bar{\mathcal{Z}}_N(\Delta, \sigma)$$

$$\bar{\mathcal{Z}}_N(\Delta, \sigma) = e^{-V_N(\Delta,\sigma)} + K_N(\Delta, \sigma)$$

and Ω_N is the total extracted vacuum energy which divides out in the normalized Schwinger functions.

As $N \to \infty$, we have $C_{\epsilon_N} \to C_{\text{cont}}$ the free continuum covariance with no cutoff. Moreover $V_N \to V_*$ where $V_*(\Delta, \sigma) = V(\Delta, \sigma, g_*, \mu_*)$ and $K_N \to K_*$ in an open ball in the Banach space E. The norms are such that the two functional derivatives of K_* smeared with test functions are easily estimated. The derivatives on V_* are estimated explicitly by integration with the Gaussian measure. The upshot is that the ultraviolet cutoff limit $N \to \infty$ or $\epsilon_N \to 0$ exists for the connected truncated Schwinger functions. By arranging the supports of the test functions appropriately one easily obtains the estimate for the correction term.